

Quantum non-Gaussian high Fock states of light pulses and their superpositions

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The generation of high Fock states of light pulses and their superpositions with provable quantum non-Gaussian features is still very challenging, although the power of conditional methods to herald the approximate state from the available Gaussian states is growing. The atom-light interaction in the high- Q cavity has been considered a viable alternative to the heralded Fock states of the light pulses from nonlinear optics limited to three-photon Fock states for the last decade. Here, by optimizing the realistic protocol combining it with available optical delay elements, we conclusively predict filtering of Fock states up to ten photons with a high success rate of 20% using a hierarchy of quantum non-Gaussian criteria. Moreover, the filtering protocol enables the preparation of superposition of Fock states, and we analyze this emerging case up to two photons with provable quantum non-Gaussian coherence and high success rate of 50%. To demonstrate their quality for applications, we evaluate the robustness of such features, the bunching capability in a linear network, and the sensing capability to estimate the magnitude of unknown force, noise, and phase. These assessments outline the essential conditions and application criteria for realistic optical cavity QED interaction on light pulses to outperform photon detection methods.

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I. INTRODUCTION

Quantum optics and photonic technology developed broadly over the last decade, mainly increasing the number of entangled and interfering modes in Gaussian states of light. It allowed the experimental test of Gaussian boson sampling [1–3], the construction of Gaussian cluster states for measurement-induced quantum computing [4–7], and the broad use of Gaussian squeezed light for quantum sensing [8–12] and quantum key distribution [13–16]. However, to investigate unexplored physics beyond Gaussian states and their classical mixtures and advance the applications further and turn them into more universal [17] and fault-tolerant forms [18,19], either quantum nonlinearities beyond the quadratic or quantum non-Gaussian states need to be involved.

In quantum optics with Gaussian states, nonlinear single-photon detection typically provides such quantum non-Gaussianity. A low-noise recognition of any number of nonzero photons from a vacuum allows a trigger for conditional single-photon states from a split single-mode squeezed light [20–22], two-mode squeezed light [23–27], or interfering multimode squeezed states of light [28–33]. Such conditional

states limitedly exhibit quantum non-Gaussian aspects. It happens even if the trigger efficiency is low and collection and verification efficiency is limited, however, with an extremely low probability of success. When the probability of such a heralded single-photon generation overcomes 50%, the single-photon state further exhibits a negative Wigner function [34]. This can be measured directly using unbalanced homodyning [35,36] or eventually estimated indirectly from homodyne tomography [37].

Negativity of Wigner function is a necessary condition for universal quantum processing [38,39]. However, the negativity is insufficient for already simpler effects and applications, and for breeding of more complex states [40]. Therefore, further hierarchical criteria of quantum non-Gaussianity and subsequent analysis are highly required [41,42].

Recent approach is to use low-noise multiplexed single-photon detectors [43,44] or, ideally, photon-number-resolving detectors (PNRDs) [45,46]. However thresholds for hierarchical criteria increases due to inefficiency of the detectors and noise [41]. Simultaneously, the heralding rate does not increase sufficiently high. Therefore, for a decade, Fock-state generation is practically limited to the preparation of three-photon states witnessed by three annuli of the negative Wigner function [43,44]. Moreover, for the superposition up to two-photon Fock states of light, more fragile quantum coherences represented by the individual off-diagonal elements in the Fock basis are still not ultimately quantum non-Gaussian.

This bottleneck needs to be overcome to fully explore the photonic world and applications mentioned above, namely, in quantum sensing [47,48], quantum error correction [40],

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non-Clifford gates for quantum computing [49], and nonlinear optical measurements [50]. Recently, attempts to overcome this limiting bottleneck have been presented using improved generation and detection techniques [51–53]. However, such generations still use only linearized couplings between photons, and all nonlinearity is provided only in the triggering detection.

A viable approach can be based on optical interaction with atomic, molecular, and solid-state emitter levels [54–56]. Single-photon light deterministically generated from a single emitter in a free space can already be quantum non-Gaussian [57–59]. However, it does not exhibit a negative Wigner function due to optical losses. It is also hard to redirect such emitted light in free space to the same or another emitter and use stimulated emission for higher photon generation in a well-defined mode. It complicates even more if many free space emitters should contribute coherently to such protocols. Therefore, a prospective solution is to use a cascaded interaction with a single emitter [60–62] or even more emitters [63–65] in the high-quality optical cavity. For optical frequencies, such interaction is either a strong and resonant Jaynes-Cummings type within rotating wave approximation [56] or a weak, largely off resonant of the dispersive type [66]. The dispersive protocols have been experimentally tested [67,68] and further elaborated [69]. Conversely, superconducting systems with intracavity microwave states can employ different ECD and SNAP gates [70,71] or pulse engineering [72–74] beyond dispersive interaction to achieve fully deterministic state preparation.

Here, we utilize the emitter in the optical cavity not just for heralding but dominantly for high-fidelity filtering of the desired photon numbers of optical pulses and surprisingly, their binary superpositions. We keep such conditional preparation of the Fock states and superpositions from the initial Gaussian states of the pulses for its advantages. However, we use the power of optical emitters and their dynamics in a cavity to extend the cavity QED experiment [68] by using the modern tools of optical delays [75,76] to significantly outperform photon detection methods on Gaussian states.

Recent work proposed a repetitive single-atom protocol for distilling high Fock states to explore the limits of hybrid optical technologies [77], with a superconducting experiment demonstrating the feasibility of an intracavity Fock-state filtration [78]. Conversely, in optical pulse experiments, generating higher Fock states and binary Fock-state superpositions remains a significant challenge. Moreover, an essential evaluation of the quantum non-Gaussian rank—a definitive hierarchical measure—is needed to rigorously justify claims of enhanced quantum non-Gaussianity. Crucially, while Wigner negativities observed in the experiment [78] serve as a witness of nonclassicality, they do not certify the QNG rank, as such negativities persist in principle for any loss below 50%. On the other hand, the QNG rank is loss sensitive [41]. Furthermore, Wigner function negativities provide no direct insight into quantum coherences governed by off-diagonal elements in the Fock basis.

For the first time, we employ definitive hierarchical criteria to rigorously evaluate the quantum non-Gaussian rank of Fock states and their superpositions generated in optical cavity QED, encompassing both diagonal and off-diagonal elements

in the Fock basis. This approach enables us to predict the achievable limits of QNG features for states up to ten photons while also paving the way for binary superpositions. To fully understand the prospects of optical cavity QED for that, we compare a fully conclusive hierarchy of measures, including the QNG rank [41], quantum non-Gaussian depth [79], photon bunching signatures [42], and force sensing applicability [80].

Based on these criteria and measures, we carefully optimize the most feasible sequential configuration of the pulsed optical cavity QED experiment [68] to a realistic protocol, implementable in the current laboratories. Our evaluations establish the experimental conditions required to produce ten-photon Fock states of the optical pulses with a corresponding QNG rank and a probability of 20%. Therefore, it surpasses Gaussian methods using photon-number-resolving detectors. Moreover, as an immediate extension, a binary superposition of up to two photons with a genuine QNG coherence and probability 50% is proposed to enhance the Gaussian method with photon detectors [81]. Therefore, their experimental verification will provide full access to the quantum non-Gaussian regime in quantum optics, on par with state-of-the-art mechanical and microwave experiments [74,82,83] that already verified high QNG rank of Fock states and their superposition, conclusively. All these quantitative results [80] confirm the feasibility of high-Fock-state generation of optical pulses in a delay-based optical cavity QED and open further investigation toward their superpositions and applications in quantum sensing, as well as future uses in quantum error correction and quantum computing.

II. EVALUATION AND APPLICATION

A. QNG hierarchy and QNG depth

A pure state is said to show genuine n -photon quantum non-Gaussianity (QNG) if it cannot be expressed as a Gaussian transformation of a superposition of states containing up to $n - 1$ photons [41], i.e.,

$$\Psi_n \neq \hat{D}(\alpha)\hat{S}(r)\psi_{n-1}, \quad \text{with } \psi_{n-1} = \sum_k^{n-1} c_k |k\rangle, \quad (1)$$

where $\hat{S}(r) = \exp[\frac{1}{2}(r^*\hat{a}^2 - r\hat{a}^{\dagger 2})]$ and $\hat{D}(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$ are the general single-mode squeezing and displacement operators, respectively. Genuine n -photon QNG of impure states also rejects any processes that affect the amplitude α , squeezing r and amplitudes c_k of the core state ψ_{n-1} in Eq. (1). We certify the genuine n -photon QNG for a mixed state ρ using a sufficient condition: The probability $p_n(\rho) = \langle n|\rho|n\rangle$ must satisfy $p_n > \mathcal{T}_n$, where the threshold

$$\mathcal{T}_n = \max_{\alpha, r, c_k} |\langle n|\hat{D}(\alpha)\hat{S}(r)\psi_{n-1}|^2| \quad (2)$$

covers values of the probability p_n that any Gaussian evolution of the state Ψ_n induces. For example, numerical calculations show that $\mathcal{T}_{10} = 0.65$ [80]. For $n > 1$, thresholds are above the conditions to observe the negativity of the Wigner function.

For cases when we fail in achieving $p_n > \mathcal{T}_n$, we use a relative but still hierarchical criterion (r-QNG) considering the probability of the multiphoton errors $p_{>n} = \sum_{k=n+1}^{\infty} \langle k|\rho|k\rangle$ [41]. Figure 4 depicts the example of such a threshold derived

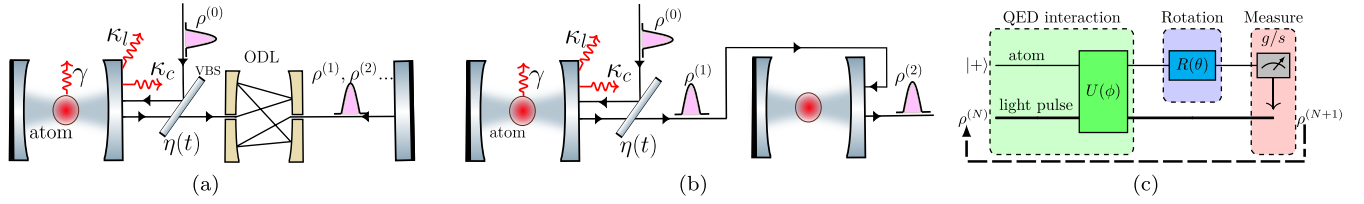


FIG. 1. (a) Single-cavity QED setup is used for the atom controlled phase shift optical operation [68], combined with a classical delay circuit [75] to implement the iterative protocol for the high Fock states. A VBS with time-controlled transmission $[\eta(t)]$ separates the input and output pulses [75,76]. The ODL is introduced by employing an advanced Herriott cell feasible with current technology [50,75,76,99]. Here, γ and κ_c represent the decay rates of the atom and cavity, respectively, while κ_l denotes the scattering rate, and g is the atom-cavity coupling strength. (b) Multiple atom-cavity setup architecture permits filtration without delay loops by implementing atomic operations following reflection processes, which is particularly crucial for waveguide-based implementations. Here, the second atom-cavity parameters are identical to the first atom-cavity setup. (c) Circuit of a single round of conditional Fock-state filtration. The three steps of the protocol are highlighted. Each step of the iterative protocol starts with the state $\rho^{(N)}$ of the input light mode and produces the state $\rho^{(N+1)}$ of the output mode, which can then be used for the next step either in the same or subsequent cavity. $U(\phi)$ denotes the controlled phase shift operation between light pulse (bold line) and two-level atom (thin line) in the cavity. $R(\theta)$ is unitary rotation on the two-level atom and $(|g\rangle, |s\rangle)$ are the measurements on the two-level atom. For the Fock states, the robustness against arbitrary dephasing noise (with a jump operator, $\hat{a}^\dagger \hat{a}$) acting on light mode outside the cavity means that no phase locking in the optical delay is needed.

for the probabilities p_{10} and $p_{>10}$. It shows that the values of p_{10} drop substantially for small values of $p_{>10}$, and therefore the criterion certifies genuine n -photon r-QNG of the filtered states.

Detecting genuine quantum non-Gaussian features using this relative criterion under photon loss is advantageous, which is the main imperfection of optical experiments. To explore the correlation between QNG, r-QNG, and noise, we investigate the depth of tolerable loss, following the approach used for nonclassical optical states [79]. The additional attenuation testing the robustness of QNG and r-QNG features is modeled as

$$p'_n = \sum_{m=n}^{\infty} \binom{m}{n} T^n (1-T)^{m-n} p_m, \quad (3)$$

where $p_m = \langle m | \rho | m \rangle$ and T is the transmittance of the noisy channel. The QNG depth (r-QNG depth) is defined as amount of T tolerable until p_n falls below the QNG threshold $\mathcal{T}_n$. This methodology allows us to compare quantum non-Gaussian states conclusively by their robustness.

To access the quantum non-Gaussian coherences, a criterion is developed for binary superpositions of the Fock states of the form $\frac{1}{\sqrt{2}}(|n_1\rangle + e^{i\phi}|n_2\rangle)$ [83,84]. The coherence measure is defined as [81]

$$C_{n_1, n_2}(\rho) = \frac{1}{2} \left(\max_{\phi} \text{Tr}[O(\phi)\rho] - \min_{\phi} \text{Tr}[O(\phi)\rho] \right), \quad (4)$$

where $O(\phi) = \cos \phi X_{n_1, n_2} + \sin \phi Y_{n_1, n_2}$, with $X_{n_1, n_2} = |n_1\rangle\langle n_2| + \text{H.c.}$ and $Y_{n_1, n_2} = -i|n_1\rangle\langle n_2| + \text{H.c.}$ Since to demonstrate that open direction we will be working with superposition states of $|0\rangle$ and $|2\rangle$, we focus on the coherence measure C_{02} , for which the threshold is numerically obtained to be 0.86. Furthermore, the sensitivity of QNG to loss is investigated via QNG coherence depth following an approach similar to Fock states in Eq. (3).

B. Bunching capability of Fock states

Further essential evaluation of filtered Fock states as the main building blocks must consider their multiple replicas.

The Fock state $|n\rangle$ can conditionally bunch with another such state and form a higher Fock state $|2n\rangle$. Such bunching phenomena are at the core of complex bosonic behaviors [85–90] and are also considered in quantum technology [40,91–94]. Fock states are typically distinguished by the number of negative annuli in the phase space, but this method is not sufficiently sensitive to contributions from higher Fock states and is prone to noise. To overcome these issues, the Fock-state bunching capability was proposed as a signature of the high quality of non-Gaussianity of the generated states [42]. The bunching capability will be evaluated by the sensitive QNG hierarchy in Eq. (1). First, we produce a near Fock state using the filtration process as depicted in Fig. 1. Subsequently, N copies of these states are combined in a energy-preserving balanced linear-optics network. In order to test their potential bunching performance, we consider an extreme rare event when all the photons eventually appear in one mode. If that happens correctly, we consider all other lower bunching events to perform sufficiently well. To emulate that, all modes except the first mode are projected onto the vacuum state

$$\rho_{\text{out}} = \left\langle U_N \left[\bigotimes_{k=1}^N \rho_k \right] U_N^\dagger \right\rangle 0_2 0_3 \cdots 0_N, \quad (5)$$

where ρ_k denotes the state in the k th mode. U_N denotes the unitary symmetrical interference network on N modes. $|0_2 0_3 \cdots 0_N\rangle$ denotes vacuum in all modes except the first mode. Finally, we evaluate the quantum non-Gaussian feature of the final conditional state in the optical mode 1.

C. Sensing with Fock states and their superpositions

In quantum metrology, precise sensing of an unknown parameter (θ) is achieved by probing with a quantum state. Cramer Rao bound states that estimation error (variance) is bounded by the inverse of quantum Fisher information ($\mathcal{F}$), i.e., $\Delta^2 \theta \geq 1/M\mathcal{F}$ where M is the number of trails. Fock states offer a distinct advantage as optimal probes for detecting small magnitudes of phase-random optical displacement and squeezing [47]. This approach demonstrates how

quantum non-Gaussianity can enhance performance, similar to findings previously analyzed for phonons [80].

A Gaussian external field can be decomposed into displacement, phase shift, and squeezing operations. Before detailed phase-sensitive aspects of measured external fields are analyzed, we often estimate first just squared amplitude N_d of displacement and amount N_s of the squeezing, assuming uniform distribution for phase averaging the transformation is expressed by the map $\mathcal{M}$:

$$\mathcal{M}(\rho) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} d\phi_1 d\phi_2 \hat{D}(\sqrt{N_d}e^{i\phi_1}) \hat{S}(\sqrt{N_s}e^{i\phi_2}) \rho \times \hat{S}^\dagger(\sqrt{N_s}e^{i\phi_2}) \hat{D}^\dagger(\sqrt{N_d}e^{i\phi_1}), \quad (6)$$

where $\langle n_d \rangle = N_d$ and $\langle n_s \rangle = \sinh^2 \sqrt{N_s}$ are average photon numbers added to a vacuum state by the displacement $\hat{D}(\sqrt{N_d}e^{i\phi_1})$ and squeezing operations $\hat{S}(\sqrt{N_s}e^{i\phi_2})$, respectively. Classical Fisher information (CFI) for the ideal detection of m photons is defined as

$$\mathcal{F}(\theta) = \sum_m \frac{1}{p(m|\theta)} \left(\frac{\partial p(m|\theta)}{\partial \theta} \right)^2, \quad (7)$$

where $p(m|\theta) = \langle \mathcal{M}(\rho) \rangle m$ and θ represents the unknown parameter we aim to measure, such as the coherent amplitude (N_d) or the squeezing parameter (N_s). We focus on estimating either the coherent amplitude ($N_s = 0$) or the squeezing amplitude ($N_d = 0$). For an ideal Fock state $|m\rangle$, Eq. (7) gives CFIs $\mathcal{F}_D = \frac{2m+1}{N_d}$ and $\mathcal{F}_S = \frac{m^2+m+1}{2N_s}$ for displacement and squeezing operations, respectively. Advantageously, in both the cases, the classical Fisher information saturates the quantum Fisher information (QFI) and the scheme is optimal [47].

We employ the CFI of ideal Fock states to benchmark the phase sensing capabilities of optical states $|\theta\rangle = \cos\theta|0\rangle + \sin\theta|2\rangle$. These superpositions of Fock states are particularly suitable for phase sensing, as previously established for the motional states of ions [95]. To validate this for our optical states, we model the sensing operation as Eq. (6):

$$\mathcal{M}(\rho) = e^{i\Theta\hat{n}} \rho e^{-i\Theta\hat{n}}, \quad (8)$$

and using Eq. (7) with the simplest binary projective measurements $|\phi\rangle\langle\phi|, I - |\phi\rangle\langle\phi|$, where $|\phi\rangle = \cos\phi|0\rangle + \sin\phi|2\rangle$, gives CFI for pure states as (Appendix F)

$$\mathcal{F}(\Theta) = \frac{\sin^2(2\theta)\sin^2(2\phi)\sin^2(2\Theta)}{p(\phi, \theta, \Theta)(1 - p(\phi, \theta, \Theta))}, \quad (9)$$

where $p(\phi, \theta, \Theta) = \frac{\sin(2\theta)\sin(2\phi)\cos(2\Theta)}{2} = (\cos\theta\cos\phi)^2 + (\sin\theta\sin\phi)^2 + \frac{\sin(2\theta)\sin(2\phi)\cos(2\Theta)}{2}$. Note that $p(\phi, \theta, \Theta)$ is symmetric with respect to θ and ϕ , and therefore CFI as well. The maximum QFI ($\mathcal{F} = 4$) is achieved when $\phi = \theta = \pi/4$. For superpositions limited to states $|0\rangle$ and $|1\rangle$, the maximal QFI is only unity. Exceeding this value demonstrates enhanced phase sensitivity using higher two-photon Fock states in the binary superposition with the vacuum.

III. PROTOCOL

We optimize the filtration protocol for filtering non-Gaussian states, ideally Fock states, from broadly available

Gaussian states [77]. Jaynes-Cummings interactions are a natural choice for achieving controlled phase flip (CPF) between an atomic qubit and an optical mode [96–98]. Recently, dispersive interactions in superconducting circuits have also been exploited to achieve CPF [78]. The protocol involves three main steps (see Fig. 1):

- (1) QED interaction: to achieve the CPF on atom and light interacting in a cavity.
- (2) Rotation: unitary operation on the two levels of atom.
- (3) Measurement: atomic-level measurement and subsequent optical output selection.

After the atomic measurement, the output light is used as an input state to repeat the steps using single-cavity QED setup or multiple QED setup, both depicted in Figs. 1(a) and 1(b).

In terms of experimental implementation, the single-cavity protocol can be divided into two main components: cavity QED interaction and the classical optics control part. The classical part involves directing the pulses using a time-dependent variable beam splitter (VBS) and an optical delay line (ODL), which is necessary to ensure adequate time for atomic measurements on the atom. The optical delay can be implemented using Herriott cells or optical fibers.

Recently, the time-dependent VBSs are used in quantum optics experiments [75,76]. Also, ODLs have demonstrated the ability to provide microsecond-scale delays, making them ideal for cavity QED experiments with atoms [68]. Low-loss ODLs have also been utilized in recent quantum experiments [50,75,76,99]. Alternatively, the ODL can be replaced with a quantum memory [100,101], enabling the storage and retrieval of microsecond light pulses at desired times. Another promising approach for reducing delay time involves utilizing multiple atom-cavity systems visualized in Fig. 1(b), where atomic measurements are performed after the interaction. This method is particularly well suited for artificial atoms operating on optical nanosecond timescales, such as quantum dots embedded in photonic crystal waveguides [102–104].

The controlled operation between the atomic states $|g\rangle$ and $|s\rangle$ and light field can be effectively written as [68]

$$\hat{U}(\phi) = |g\rangle\langle g| \otimes I + |s\rangle\langle s| \otimes \exp i\phi \hat{a}^\dagger \hat{a}, \quad (10a)$$

$$e^{i\phi} \approx 1 - \frac{2}{1 - 2i(\Delta/\kappa)}, \quad (10b)$$

where $|g\rangle\langle g| + |s\rangle\langle s| = I$. Here, $\hat{a}$ and κ are the cavity mode annihilation operator and its decay rate, respectively. $\Delta = \omega_c - \omega_a$ is the atom-cavity detuning, where ω_c (ω_a) represent atom (cavity) frequencies; further, Δ can be adjusted by changing the cavity frequency. Note that the approximation in Eq. (10) is obtained under strong atom-cavity coupling strength ($g \gg \kappa$) and the mean frequency of input light (ω_L) is in resonance with atom ($\omega_c = \omega_L$). The atom is prepared in the state $|\psi_a\rangle = (|g\rangle + |s\rangle)/\sqrt{2}$ and the light field in a Gaussian state $\rho^{(0)}(\alpha, r, \bar{n})$ [Eq. (15)]. The circuit in Fig. 1 will transform the input state as

$$\rho'^{(N)} = \hat{R}(\theta) \hat{U}(\phi) [|\psi_a\rangle\langle\psi_a| \otimes \rho^{(N)}] \hat{U}^\dagger(\phi) \hat{R}^\dagger(\theta),$$

$$\rho^{(N+1)} = \frac{\hat{M} \rho'^{(N)} \hat{M}^\dagger}{\text{Tr}[\hat{M}^\dagger \hat{M} \rho'^{(N)}]}, \quad \hat{M} \in \{|g\rangle\langle g|, |s\rangle\langle s|\}, \quad (11)$$

where $\hat{R}(\theta)$ and $\hat{M}$ denote the two-level general unitary and atomic measurement operators, respectively. By starting with Gaussian states, such as thermal and squeezed coherent states, and by repeating the ideal circuit, one can approach a Fock state [77]. All numerical simulations in this work were performed using the QuTiP [105].

However, for the realistic physical scenario with noisy state, optical dephasing, and atom-cavity scattering, the quality of Fock-state preparation and its quantum non-Gaussian aspects remained unclear and challenging to predict without analyzing the QNG depth. Also, the strength of such QNG states in potential applications based on their sensitivity is unclear without analysis. Moreover, the QED circuit must be sensitively optimized to fulfil such demands under realistic conditions.

A. Thermal state and optical dephasing

We examine the detrimental effects of photon noise and optical phase noise (decoherence) jointly within the filtration protocol. We start with the atom in $|\psi_a\rangle = 1/\sqrt{2}(|g\rangle + |s\rangle)$ and input light in thermal state. After atom-cavity gate, the state is written as

$$U(\phi)[|\psi_a\rangle\langle\psi_a| \otimes \rho_{\text{th}}]U(\phi)^\dagger = \frac{1}{2} \sum_{n=0}^{\infty} p_n [I + e^{-i\phi n} |g\rangle\langle s| + e^{i\phi n} |s\rangle\langle g|] \otimes |n\rangle\langle n|, \quad (12)$$

where $p_n = \frac{\bar{n}^n}{(1+\bar{n})^{n+1}}$ represents the initial thermal Bose-Einstein statistics of a single-mode light field. Further performing atomic state rotation $R(\theta) = \sigma_z - \sigma_x$ and measuring the atom projects the optical field to

$$\rho = N_g \sum_{n=0}^{\infty} c_n \cos^2\left(\frac{n\phi}{2}\right) |n\rangle\langle n|, \quad \text{for } |g\rangle, \quad (13a)$$

$$\rho = N_s \sum_{n=0}^{\infty} c_n \sin^2\left(\frac{n\phi}{2}\right) |n\rangle\langle n|, \quad \text{for } |s\rangle, \quad (13b)$$

where $N_{\{g,s\}}(\phi)$ is the normalization constant. By adjusting Δ/κ to change the controlled phase ϕ , we can asymptotically reach any desired Fock state with high fidelity. It is important to note that Eq. (13) always results in a diagonal matrix (in Fock basis). Therefore, by repeating the above steps we can sequentially remove Fock numbers to filter a given Fock state without locking phase of light pulse outside the cavity. As shown above, it is evident that for the Fock filtration process, only the diagonal terms of the input light are relevant. Hence, the filtration will be immune to dephasing of the form $\mathcal{D}(\rho) = \hat{L}\rho\hat{L}^\dagger - \frac{1}{2}\hat{L}^\dagger\hat{L}\rho - \frac{1}{2}\rho\hat{L}^\dagger\hat{L}$, where the jump operator $L = \hat{a}^\dagger\hat{a}$. Although dephasing in the light pulse outside the cavity does not affect the filtration of Fock states, dephasing during the atom-photon interaction will impact the process (see Appendix D). Identical results to those in Fig. 2 and Table I are obtained even with optical dephasing noise. It is advantageous for future experimental tests of the Fock states as the optical field does not have to be phase locked if direct photon number detection or multiplexed single-photon detectors are used to measure photon number statistics.

B. Atom-cavity loss

The dynamics of atom-field interactions are best captured using virtual cavities [106,107]; however, this approach makes it very difficult to numerically solve for higher Fock states. Hence, we will use this approach only when the dimensions of the density matrix in the Fock basis are manageable. Another approach is to account for noise using Kraus operators [108]. This method is suitable for steady-state dynamics and is effective for simulating higher Fock states. Since we will be using high Fock states, we will adopt the Kraus operator approach. The controlled operation between atom and light depends on two important metrics [108], cooperativity C and cavity efficiency β :

$$C = \frac{g^2}{\kappa\gamma}, \quad \beta = \frac{\kappa_c}{\kappa}. \quad (14)$$

Here, g and γ are the atom-cavity coupling strength and the atomic decay rate outside the cavity, respectively. The total cavity decay rate $\kappa = \kappa_c + \kappa_l$ is split into cavity mode decay and scattering losses (κ_l). The efficiency β denotes the ratio of light emission into cavity over total emission. In all our simulations, unless stated otherwise, we use the numbers targeted by current technology: $C = 250$ and $\beta = 0.99$.

In conventional cavities, cooperativity on the order of $C \sim 10 - 100$ can be achieved by increasing the finesse more than 10^5 [56], and $C \sim 10^4$ can be achieved by coupling atoms to photonic crystal cavities [109]. In conventional Fabry-Pérot cavities, the typical values are around $\beta \approx 0.92$ [68], while β factors close to unity are demonstrated in semiconductor cavities [103] and photonic crystal waveguides [98]. Furthermore, the ODL can also result in photon losses that effectively can be viewed as increasing the scattering losses κ_l , which reduces both the cooperativity C and the cavity efficiency β , as discussed in Appendix A 1.

IV. RESULTS

A. Filtering Fock states and their superpositions from Gaussian states

The input Gaussian states are generally represented by a displaced squeezed thermal noise:

$$\rho(\alpha, r, \bar{n}) = \hat{\mathcal{D}}(\alpha)\hat{\mathcal{S}}(r)\rho_a^{\text{th}}(\bar{n})\hat{\mathcal{S}}^\dagger(r)\hat{\mathcal{D}}^\dagger(\alpha), \quad (15)$$

where $\rho_{\text{th}} = \frac{1}{1+\bar{n}} \sum_{n=0}^{\infty} \left(\frac{\bar{n}}{1+\bar{n}}\right)^n |n\rangle\langle n|$ denotes the thermal state with a mean photon number of $\bar{n}$. $\hat{\mathcal{D}}(\alpha)$ and $\hat{\mathcal{S}}(r)$ are the displacement and squeezing operators, respectively, as before.

Figure 2 demonstrates how different Gaussian input states produce targeted Fock states within a maximum of $N = 4$ protocol rounds. Notably, even states with large photon noise like thermal states efficiently yield Fock states. However, the probability of success remains limited to $\mathcal{P} = 0.01$. By optimization, as shown in Fig. 2 and Table I, the success probability of atomic measurements can reach up to $\mathcal{P} = 0.17$ using just Gaussian input states. Already, Poissonian distribution of coherent states increases the success rate by one order ($\mathcal{P} > 0.1$) when optimized to have a probability maximum at the desired Fock state $|10\rangle$, as visible in Fig. 2(a). Further improvement is possible by Gaussian squeezing concentrating

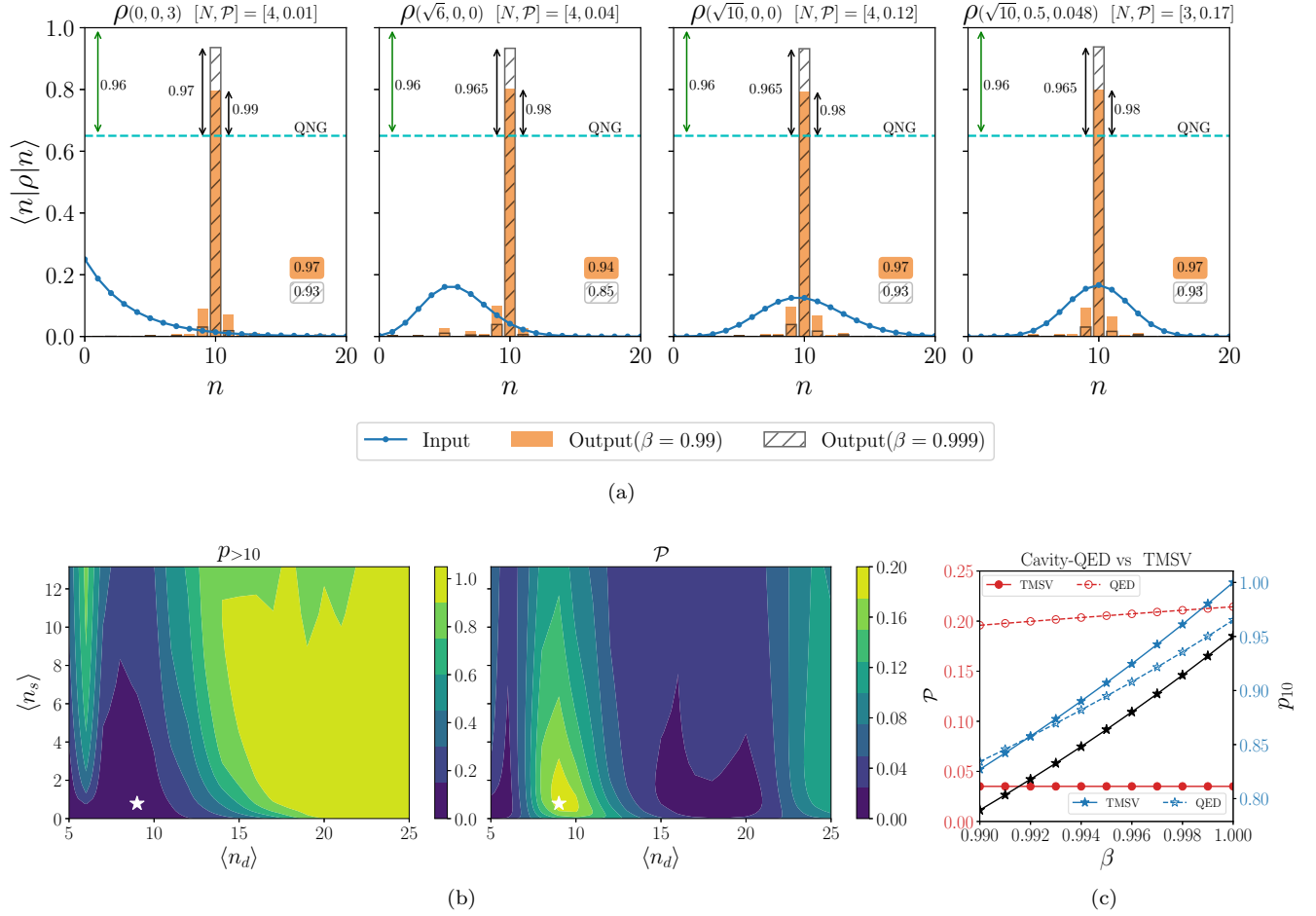


FIG. 2. (a) Filtration of Fock state $|10\rangle$ from various Gaussian sources [Eq. (15)] for different cavity noises $[\beta]$ in Eq. (14). The input states are denoted as $\rho(\alpha, r, \bar{n})$, as defined in Eq. (15), with the blue lines representing the photon distribution of these input states. The required number of access to the cavity and the success rate of atomic measurements is denoted by $[N, \mathcal{P}]$. The QNG line represents the threshold $\mathcal{T}_{10}$ [Eq. (1)]. The arrows denote quantum depth for the filtered Fock states and ideal state ($|10\rangle$) before they drop below the QNG threshold. The lower numbers in boxes show the noise of the Fock state before they drop below the r-QNG criterion. Panel (b) shows the higher Fock-state contributions (left) and atomic measurement success ($\mathcal{P}$) as a function of average photon numbers corresponding to displacement and squeezing in the input optical state parametrized as $\langle n_d \rangle = |\alpha|^2$ and $\langle n_s \rangle = \sinh^2 r$. The marked point ($\star$) represents the squeezed coherent state $\rho(\sqrt{9}, 0.8, 0)$ that is compared with an example of TMSV in panel (c). (c) Comparison of cavity QED filtration and TMSV protocol for Fock-state generation. $\mathcal{P}$ denotes success probability and p_{10} denotes fidelity. All TMSV plots are simulated with $\beta = 0.99$ and unit efficiency for PNRDs (see Sec. B), except the black line that shows the result for PNRDs with efficiency of 0.995. The cavity QED parameters in panels (b) and (c) are set to $\{C, \beta\} = \{250, 0.99\}$.

the photon number distribution more around that maximum. The maximum success probability ($\mathcal{P} \approx 0.20$) can be reached when the Gaussian squeezing is optimally adjusted to the dis-

placement of the coherent state [see Fig. 2(b)]. It demonstrates the power of the emitters in cavities with the delay elements to be used as filters rather than heralding detectors.

TABLE I. Comparison of atomic measurements in the filtration protocol for various Gaussian input states. N denotes the measurement after n th reflection and $\mathcal{P}$ is the product of atomic measurements. $\rho(\alpha, r, \bar{n})$ denotes the displaced squeezed thermal state in Eq. (15). The atom-cavity parameters are set to $\{C, \beta\} = \{250, 0.99\}$ (see Fig. 2).

N/input	$\rho(\sqrt{10}, 0, 0)$	$\rho(\sqrt{6}, 0, 0)$	$\rho(0, 0, 3)$	$\rho(\sqrt{10}, 0.5, 0.0485)$	$\rho(\sqrt{4.5}, 0.5, 0.04)$	$\rho(\sqrt{50}, 0, 0)$
1	0.50	0.50	0.57	0.50	0.50	0.50
2	0.50	0.50	0.36	0.50	0.53	0.50
3	0.52	0.36	0.76	0.68	0.89	0.50
4	0.90	0.40	0.09	—	—	0.50
5	—	—	—	—	—	0.75
$\mathcal{P}$	0.12	0.04	0.01	0.17	0.236	0.047

To use an advantage of a loss-tolerant r-QNG hierarchy, it is crucial to minimize contributions from higher Fock states. For instance, when filtering state $|10\rangle$, the probability $p_{>10} = \sum_{n=11}^{\infty} \langle n|\rho|n\rangle$ should be minimal. Higher Fock-state contributions can be reduced by lowering the mean of the input light, as shown with the states $\rho(0, 0, 3)$ and $\rho(\sqrt{6}, 0, 0)$. However, this reduction comes at the cost of the success rate of atomic measurements, resulting in $\mathcal{P} < 0.05$. By allowing for small contributions from higher Fock states, the success rate can be improved by an order of magnitude, achieving $\mathcal{P} > 0.12$, as illustrated with the states $\rho(\sqrt{10}, 0, 0)$ and $\rho(\sqrt{10}, 0.5, 0.048)$. This trade-off is beneficial for experiments that need to tolerate more significant losses.

In Fig. 2(b), we show the contribution from higher Fock states ($p_{>10}$) and atomic success rate ($\mathcal{P}$) as a function of photon numbers corresponding to displacement ($\langle n_d \rangle = |\alpha|^2$) and squeezing ($\langle n_s \rangle = \sinh^2 r$), and it is clear that the optimized range to distil state $|10\rangle$ is $\langle n_d \rangle \approx 10$ and $\langle n_s \rangle < 2$. It is advantageous, therefore, to use coherent states with $\langle n_s \rangle \sim 0$ from suitable lasers tunable to the atomic transition for the first tests of such iterative protocols.

In Fig. 2(c), we compare the filtration protocol using the squeezed coherent state $\rho(\sqrt{9}, 0.8, 0)$ with the state-of-the-art Fock-state generation utilizing ideal two-mode squeezed vacuum (TMSV) and ideal PNRDs [41,43,44,110]. We chose the TMSV with the maximum probability for state $|10, 10\rangle$, i.e., $\lambda = \tanh(1.85)$ (see Appendix B). The success probability (red lines) of the QED filtration protocol ($\mathcal{P} \approx 0.20$) is 6 times higher than that of the ideal TMSV and PNRD protocols ($\mathcal{P} \sim 0.03$). Although the fidelities (p_{10}) of ideal TMSV with ideal PNRDs (blue lines) are slightly better than those of the filtration protocol, the addition of already slight inefficiency of PNRD ($\beta = 0.995$) [111] reduces the fidelities below those achieved by the QED filtration protocol (black line). Overall, this demonstrates that the QED filtration protocol offers higher success rate ($\sim 20\%$) without sacrificing the quality of generated Fock states. This improvement without compromise must be confirmed in the experiments, which can stimulate further developments of the cavity-delay techniques. The filtration protocol repeatedly exploits CPF in sequence 1-2-3 to filter out a Fock state from a given state and, therefore, it might look that it lacks flexibility in preparing a binary superposition of the Fock states. However, we show that by optimizing the input state, the light filtration technique allows for the preparation of states of the type $\cos \theta |0\rangle + \sin \theta |2\rangle$. It is an essential example that this method is capable of higher and ultimate QNG coherence, success rate, and is more variable for the optical state preparation than was considered in Ref. [77].

Displaced squeezed vacuum states in Eq. (15) with low coherent and squeezing amplitudes (α, r) can be approximated as (see Appendix E)

$$\psi_{sc} \approx C_0[|0\rangle + C_1|1\rangle + C_2|2\rangle + C_3|3\rangle], \quad (16a)$$

$$\sum_{k=0}^3 |C_k|^2 \sim 1, \quad (16b)$$

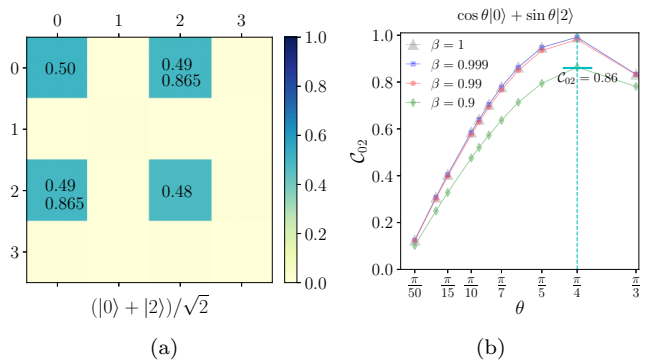


FIG. 3. (a) Density matrix for the filtered superposition of the Fock states $(|0\rangle + |2\rangle)/\sqrt{2}$. Optimized displaced squeezed vacuum state $\rho(1.035, 0.247, 0)$ is used as input. Cavity parameters: $\{C, \beta\} = \{205, 0.99\}$ yields an atomic success probability of 0.515 and depth of QNG coherence of 0.865. (b) Coherence measure of filtered states $\cos \theta |0\rangle + \sin \theta |2\rangle$, with the threshold ($C_{02} = 0.86$, green dash) indicating genuine QNG coherence for $\theta = \pi/4$ [83]. All states are generated via single atom-cavity interaction. Already $\beta = 0.99$ is sufficient to approach ideal performance $\beta = 1$.

after atomic rotation and measurement, projects the light state to

$$|\theta\rangle \approx \cos \theta |0\rangle + \sin \theta |2\rangle, \quad (17a)$$

$$\theta \approx \tan^{-1} \left(\frac{\tanh r}{2\sqrt{2}} H_2 \left[\frac{\alpha e^r}{\sqrt{\sinh 2r}} \right] \right). \quad (17b)$$

For a given θ , we numerically optimize α and r such that Eqs. (16b) and (17b) are optimized, but the figures are computed without any truncation. For example, to filter state $(|0\rangle + |2\rangle)/\sqrt{2}$, the optimized input $\rho(1.035, 0.247, 0)$ gives $\theta = \pi/4$ and $\sum_{k=0}^3 |C_k|^2 = 0.935$ with coherence C_{02} represented by the off-diagonal elements over the ultimate QNG threshold 0.86 and close to maximal value, as visible in Fig. 3(b).

Our results, shown in Fig. 3, demonstrate that cavity QED filtered states $(|0\rangle + |2\rangle)/\sqrt{2}$ achieve high quality and surpass the absolute QNG coherence threshold, even with substantial cavity losses ($\beta = 0.9$). Furthermore, the coherence QNG depth with $\beta = 0.99$ reaches $T = 0.875$, and for $\beta = 0.999$ even $T = 0.866$, close to the ideal cavity ($\beta = 1$) depth of $T = 0.865$. Therefore, for $\beta = 0.99$ we already reach 92.5% of the maximal loss tolerance for QNG coherence.

Beyond generating Fock states, this establishes cavity QED as an effective platform for producing superpositions of Fock states with unprecedented quality using only single atomic measurements. Such balanced superpositions $(|0\rangle + |2\rangle)/\sqrt{2}$ surpass cat-state preparation capabilities [67] and open avenues for advancing optical sensing protocols, building on mechanical implementations already demonstrated [95]. Note that in Fig. 3(b), $\theta = \pi/2$ is excluded since achieving high-quality state $|2\rangle$ requires two reflections.

B. QNG depth and bunching capability of high Fock states

To understand the distance of the predicted bars in Fig. 2(a) from the quantum non-Gaussianity threshold, we examine in

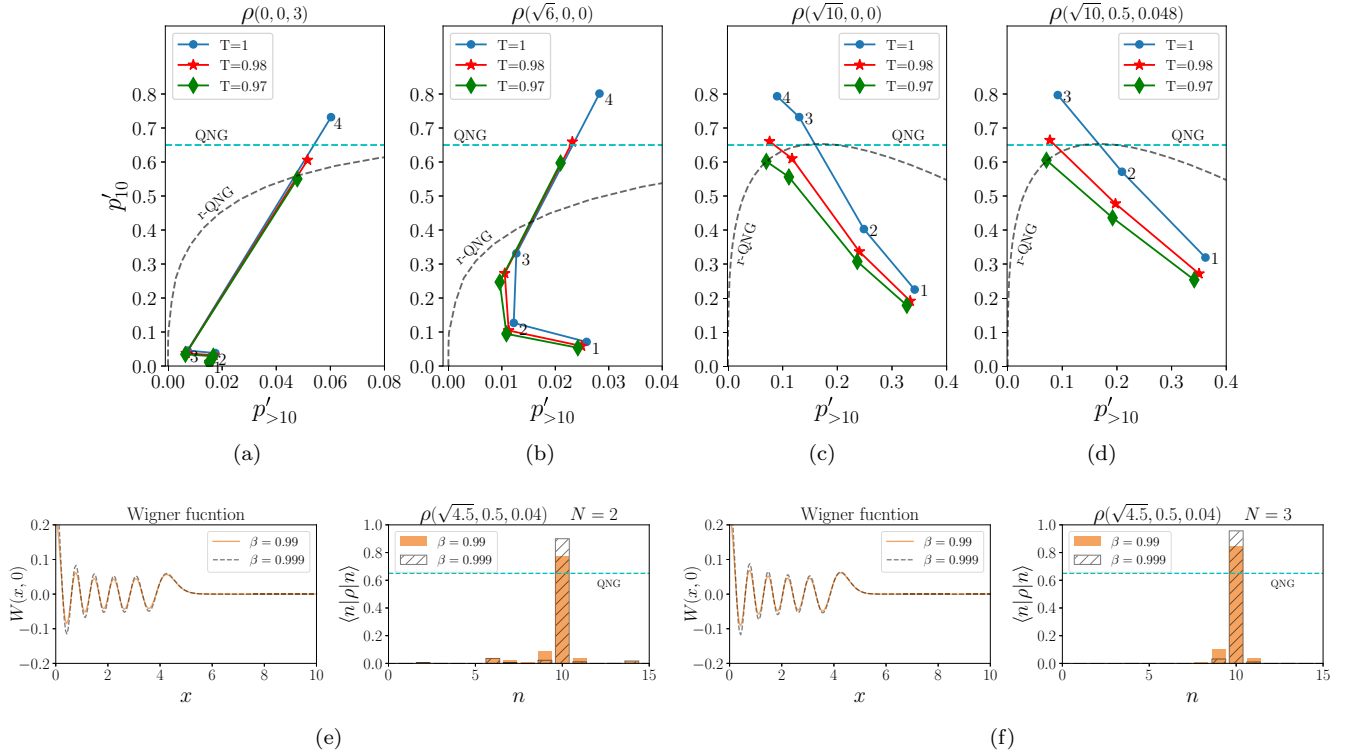


FIG. 4. (a)–(d) QNG depth of the Fock states of light. Top labels show the various states used in filtration. The numbers 1 – 4 denote the ordering of atom-cavity reflections. The probabilities p'_{10} and $p'_{>10}$ are calculated using Eq. (3), for each state after atom-cavity reflection. T is the transmittance of the output lossy channel. The r-QNG (black dashed curve) represents the relative criterion, while the dashed line represents the absolute criterion (T_{10}), Eq. (2). The cavity parameters are set to $\{C, \beta\} = \{250, 0.99\}$; see description in the text. (e)–(f) Wigner distribution and Fock-state bunching capability. Fock state $|5\rangle$ filtered from the state $\rho(\sqrt{4.5}, 0.5, 0.04)$ is bunched in two modes to generate $|10\rangle$. Panels (e) and (f) show the state after two and three atomic measurements. QNG denotes the absolute threshold [Eq. (1)] for the Fock state $|10\rangle$. The cavity parameters are set to $\{C, \beta\} = \{250, 0.99\}$.

detail the depth of the approximate high Fock states obtained from the filtration protocol. We calculate p'_{10} [Eq. (3)] for states after each atom-cavity reflection (plus measurement), and then compare it with T_{10} . In Fig. 2(a), we observe that Fock states filtered with $\beta = 0.999$ from various Gaussian inputs exhibit a QNG depth ranging from $T = 0.97$ to 0.965 . For comparison, lowering β to 0.99 reduces the QNG depth to $T = 0.98$. As a reference point, the ideal Fock state $|10\rangle$ has a QNG depth of $T = 0.96$ (indicated by the green arrow). Therefore, even with practical scattering losses ($\beta = 0.999$ and $\beta = 0.99$), we achieve 75% and 50% of the ideal QNG depth, respectively.

In Fig. 4, we show the QNG and r-QNG depths of filtration protocol after each atom-cavity reflection (marked by numbers). The dashed lines show the QNG criterion obtained by numerically solving Eq. (1) for state $|10\rangle$. From Figs. 4(b) to 4(c), we see that three atomic measurements are required for coherent states to surpass the threshold in the hierarchy in Eq. (1), while the thermal state requires four atomic measurements [Fig. 4(a)]. Figure 4(d) shows that by optimizing the squeezing and displacement parameters, three atomic measurements are sufficient for the state $\rho(\sqrt{10}, 0.5, 0.048)$ to surpass the QNG threshold. Also note that the states after the first filtration step (marked 1) are experimentally already demonstrated Schrödinger cat states formed by a superposi-

tion of phase randomized Gaussian states [68,112]. Although they are non-Gaussian states, contributions from higher photon numbers prevent them from surpassing even the relative QNG criterion for the Fock states. More filtration steps are needed to reach the r-QNG threshold and eventually the absolute one. However, a positive step can be visible after the first delay and second reflection. During the process, the multiphoton contributions $p_{>10}$ can be compromised with the probability of success $\mathcal{P}$.

In Fig. 4, although the filtered Fock states are similar, the QNG and r-QNG depths are highly sensitive to the photon distribution. To achieve a high depth, it is crucial to have high p_{10} probabilities and low contributions from higher Fock states $p_{>10}$. While QNG and r-QNG depths characterize the noise tolerance of Fock states obtained through filtration, the total efficiency of the process is determined by the atomic measurements. As shown in Fig. 2(b) and Table I, there is a trade-off between low contributions from higher Fock states and the success rate of atomic measurements. Similarly, there is an inverse correlation between atomic measurements and QNG depth; for example, in Fig. 4(c) the r-QNG depth with $T = 0.97$ using four measurements is identical to the depth with $T = 0.98$ using three measurements.

The quality of generated Fock states is not only quantified by their robustness to loss and, moreover, also their possibility

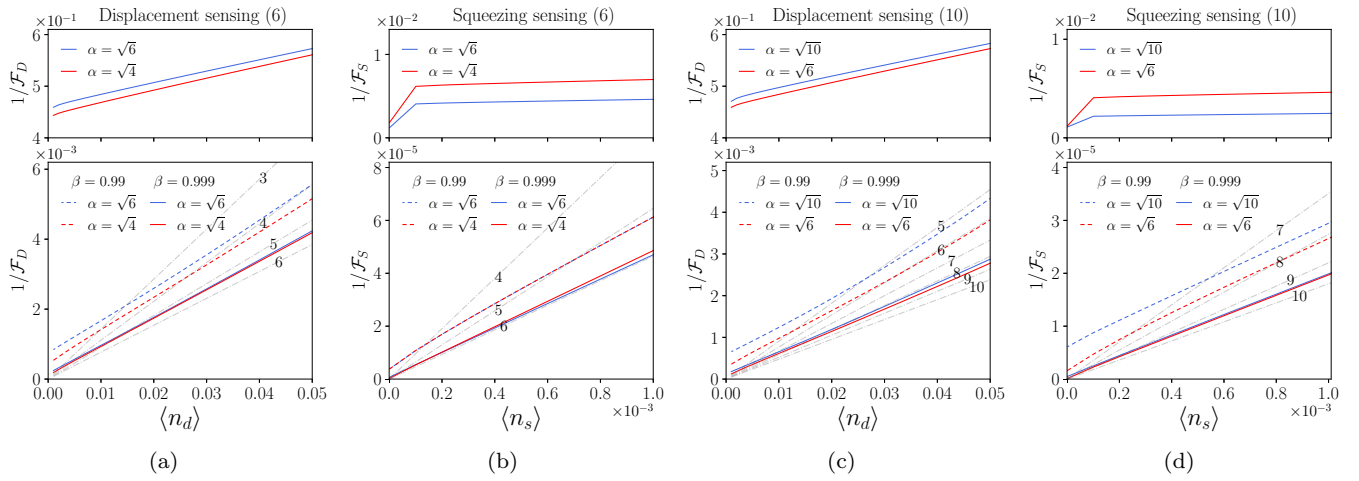


FIG. 5. Sensing photon numbers corresponding to displacement [panels (a) and (c)] and squeezing [panels (b) and (d)] using filtered states derived from coherent states is presented. The numbers (10, 6) in the top labels indicate the targeted Fock states. The average photon numbers are given by $\langle n_d \rangle = N_d$ and $\langle n_s \rangle = \sinh^2 \sqrt{N_s}$, while $\mathcal{F}_D$ and $\mathcal{F}_S$ denote the Fisher information related to displacement and squeezing operations, respectively (see Sec. II C). The top plots depict the estimation using the input coherent states with $\alpha = \{\sqrt{4}, \sqrt{6}, \sqrt{10}\}$, while the bottom plots illustrate the estimation of the output filtered states from the corresponding inputs for various β [Eq. (14)] values. Estimations for ideal Fock states ($|5\rangle - |10\rangle$) are shown by the dot-dashed (black) lines. The cooperativity for the QED filtration is set to $C = 250$.

of combining with resources similar to more capable ones. In Figs. 4(e) and 4(f), we show the bunching capability [42] of the Fock state with five photons obtained from the squeezed coherent state after two and three atomic measurements. We simultaneously use continuous-variable interference effects in the Wigner function exhibiting adequate negative regions and discrete variable characteristics in photon number distribution. Both confirm the high quality of the bunching capability of two copies of five-photon state. It allows them to be used for advanced bosonic applications, like gates on binomial quantum error correction codes [19].

V. SENSING

To perform further analysis, we advantageously use that the Fock states of light are already resources for the advanced quantum sensing of the amplitude of the external displacement (and squeezing) [47,48]. We analyze an estimation error to verify whether quantum non-Gaussian states provide its reduction conclusively.

In Fig. 5, we numerically calculate the Fisher information in Eq. (7), with ρ being the state obtained from the cavity QED filtration. Although the statistical properties of the filtered states are close to the Fock states $|6\rangle$ and $|10\rangle$ [Fig. 2(a)], their sensing capabilities differ. When sensing small values of $\langle n_d \rangle$ and $\langle n_s \rangle$, noise introduces an offset from the thresholds determined by the ideal Fock states [80]. Interestingly, squeezing values exhibit better performance than displacement by a factor of 100. Furthermore, improving β not only minimizes the offset but also enhances sensing capabilities.

In Fig. 4(a), displacement sensing with the targeted state $|6\rangle$ surpasses the performance of the ideal Fock state $|5\rangle$ for $\beta = 0.999$, after an initial offset where the Fisher information drops below that of lower Fock states. For lower $\beta = 0.99$, the Fisher information from the generated states surpasses that of the ideal Fock state $|3\rangle$, emphasizing the sensitivity of Fisher

information to β . A similar trend is observed in Fig. 4(b) for squeezing sensing, where for $\beta = 0.99$, the performance exceeds the threshold of Fock state $|4\rangle$, and for $\beta = 0.999$, it approaches the performance of the ideal Fock state $|6\rangle$.

In Fig. 4(c), targeting the Fock state $|10\rangle$ shows that Fisher information for the generated states surpasses that of the ideal states $|5\rangle$ for $\beta = 0.99$ and $|8\rangle$ for $\beta = 0.999$, after an initial offset. Similarly, in squeezing sensing shown in Fig. 4(d), the performance surpasses the Fock state $|7\rangle$ for $\beta = 0.99$ and $|9\rangle$ for $\beta = 0.999$.

These results demonstrate that targeting higher Fock states provably improves the sensing protocol. Moreover, lowering the initial coherent state from $\alpha = 10$ to $\alpha = 6$ proves advantageous for sensing applications, albeit at the cost of a reduced generation rate. The phase sensing capabilities are analyzed using filtered states $|\theta\rangle = \cos \theta |0\rangle + \sin \theta |2\rangle$, the sensing operation in Eq. (8), the Fisher information from Eq. (7), and projective measurements of the form $|\phi\rangle = \cos \phi |0\rangle + \sin \phi |2\rangle$. Similarly, as for the sensing with Fock states, we project on the same class of states we generate [95]. As demonstrated in Fig. 6(a), balanced filtered states exceed the sensing capabilities of input displaced squeezed states and also any superposition up to single-photon state having $\mathcal{F} = 1$, despite substantial cavity losses ($\beta = 0.99$). Furthermore, they approach the quantum Fisher information bound ($\mathcal{F} = 4$) for loss values up to $\beta = 0.99$. Figure 6(b) demonstrates that detector loss reduces the Fisher information, yet the system nearly attains the QFI limit ($\mathcal{F} = 4$) despite 0.97 loss. Collectively, the results in Fig. 6 establish cavity QED as a promising platform for generating Fock superposition states for phase sensing.

VI. CONCLUSION

The generation of quantum non-Gaussian states at optical frequencies has long been a desirable goal for applications in

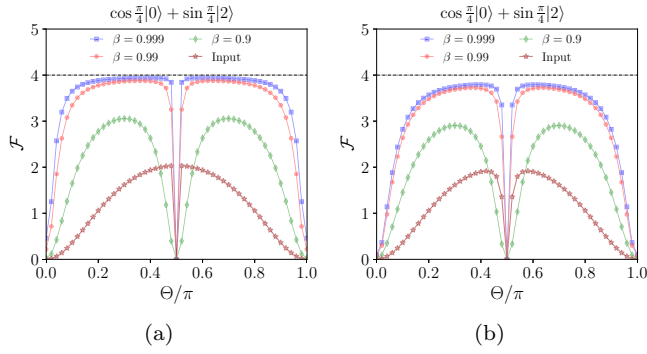


FIG. 6. Sensing the phase Θ in the operation $e^{i\Theta\hat{n}}$ [Eq. (8)], using filtered optical states from the single-round cavity QED setup. β represents cavity efficiency used to generate the states. The line ($\mathcal{F} = 4$) shows the QFI of ideal states. (a) Displaced squeezed vacuum state $\rho(1.035, 0.247, 0)$ is used as an input to filter the state $\cos(\pi/4)|0\rangle + \sin(\pi/4)|2\rangle$. The atomic success rates for the cavity efficiencies $\{0.999, 0.99, 0.9\} = \{0.515, 0.515, 0.525\}$. (b) Effect of noisy detection on the filtered state $\cos(\pi/4)|0\rangle + \sin(\pi/4)|2\rangle$. Projective measurements with 0.97 efficiency are considered, while other parameters match those in panel (a).

communication, sensing, and recently, quantum error correction for quantum computing. While various protocols have been proposed, most have been limited to the heralded generation of such states using TMSV from nonlinear optics and PNRDs with still nonunity efficiency.

As a viable alternative, we explored the merits of the cavity QED filtration protocol, which enables the generation of non-Gaussian states in travelling pulses with a success rate of 20%, by an order higher than conventional methods based on TMSV and PNRD, without compromising the quality of photon number statistics. Our scheme allows using available coherent states of light to herald high-quality large Fock states, which is challenging for conventional methods. We assessed the advantages of the filtered state $|10\rangle$ using a hierarchy of QNG criteria, bunching capability, and Fisher information for sensing of the optical displacement and squeezing.

As an attractive high target, our study shows that input squeezed coherent states, with optimized parameters and two atomic measurements, can filter state $|10\rangle$ under realistic atom-cavity conditions. The filtered state has QNG depth of $T = 0.97$, which is remarkably close (by 75%) to the ideal Fock-state depth of $T = 0.96$. Additionally, the bunching capability analysis confirms that the filtration protocol produces noise tolerant Fock states with minimal contributions from higher Fock states. Such states can be used for displacement and squeezing sensing, which requires high-quality sub-Poissonian light.

Beyond generating non-Gaussian states at each iteration, filtration also serves as a powerful tool for quantum state engineering. While the process is inherently limited in its ability to adjust coefficients, this constraint can be mitigated by optimizing input displaced squeezed vacuum states. This allows for the creation of binary superposition of Fock states up to two photons, having coherence QNG depth of $T = 0.875$, which is nearly identical (by 92.5%) to the ideal superposition state depth of $T = 0.865$. Remarkably, such QNG

superposition states can be used for optical phase sensing to approach the ultimate Heisenberg limits, as demonstrated for mechanical states [95], and can be further extended after the proof-of-principle experimental verifications.

In Appendix C, we compare the filtration protocol using a single atom with that using two atoms. Introducing two atoms maintains crucial parameters $\{C, \beta\}$ unchanged, offering no significant advantages over single-atom filtration for the generation of Fock states. Although a slight advantage may arise in the two-atom case due to stronger interaction ($\sqrt{2}g$), increasing g in the single-atom scenario yields results similar to those in the two-atom scenario.

An alternative technique for the deterministic generation of Fock states involves controlling atom-cavity interactions [113]. However, this control becomes cumbersome, and the fidelity of the generated Fock states is time dependent and decreases with higher photon number states [102,114]. Another approach to generate Fock states utilizes atom-cavity evolutions and qubit measurements, but this method forms the Fock states as standing waves rather than travelling pulses [115–117]. This approach has recently been demonstrated in superconducting circuits using dispersive Hamiltonians ($\hat{\sigma}_z \hat{a}^\dagger \hat{a}$) [78].

This work simultaneously employs different concepts of QNG rank and depth, bunching, and sensing capabilities to establish a definitive framework for optimizing optical cavity QED protocol experiments for propagating pulses. By this, they will achieve the same conclusively justified quality as mechanical [80,82,83] and microwave intracavity experiments [74], which have already been tested for their QNG rank. By analyzing the necessary conditions for atomic (or quantum dot) cavity QED systems [68,97,102,103] alongside available optical delay elements, we highlight the feasibility of the sequential filtration protocol in generating non-Gaussian states in traveling optical pulses. This approach not only surpasses the limitations of photon-number-resolving detectors on the Gaussian states but also provides the first conclusive justification for a feasible optical cavity QED protocol capable of producing high-rank optical Fock states and their superpositions. These results offer a valuable tool for advancing quantum sensing and computing technologies.

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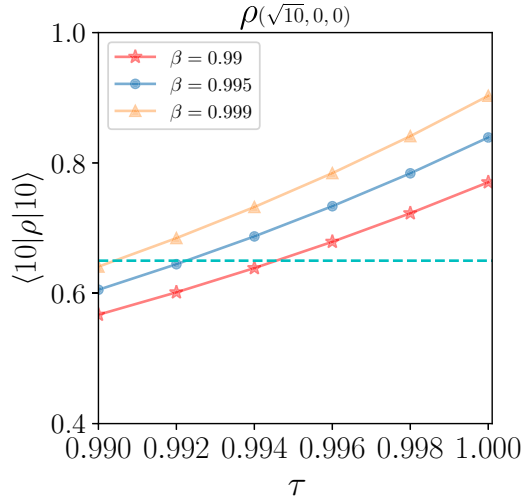


FIG. 7. Delay loss effects on state filtration. State $|10\rangle$ is filtered from the coherent state $\rho(\sqrt{10}, 0, 0)$. Delay loss is modeled as a beam splitter with transmissivity τ . It is clear that the losses ($\beta = 0.999$, $\tau = 0.994$), ($\beta = 0.995$, $\tau = 0.996$) and ($\beta = 0.99$, $\tau = 0.998$) are equivalent for filtration.

DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: THERMAL STATE FILTRATION

1. Success rates and delay-loop noise

Although squeezed coherent and thermal states yield similar Fock states, the success probability, success rate (atomic measurements), varies depending on the input states as shown in Table I.

From the atomic measurement probabilities, it is evident that the squeezed coherent states are beneficial. Though shifted coherent states exhibit lower success rates, they have the advantage of low contribution from higher Fock states, i.e., $\langle \rho \rangle_{11} \approx 0$. The thermal states demonstrate that the coherence (off diagonal) in the input light state is irrelevant and the filtration of Fock states is immune to dephasing noise.

Moreover, residual small losses in the loop only affect the optical states, not light-matter interaction. In general, this will further reduce the efficiency of the protocol because, intuitively, it can be viewed as increasing the scattering losses κ_l (see Fig. 7), which reduces both the cooperativity C and the cavity efficiency β .

APPENDIX B: TWO-MODE SQUEEZED STATES

Here, we derive probability and fidelity of Fock states generated using TMSV and PNRDs. In order to have a faithful comparison and to accommodate cavity scattering losses, we pass both the modes through a beam splitter with transmittances (β) identical to cavity losses. We consider a TMSV state generated by the action of two-mode squeezing operator on two-mode vacuum state as follows:

$$\begin{aligned} |\psi\rangle_{AB} &= \exp[r(\hat{a}_A^\dagger \hat{a}_B^\dagger - \hat{a}_A \hat{a}_B)] |0\rangle_A |0\rangle_B \\ &= \sqrt{1 - \lambda^2} \sum_{n=0}^{\infty} (-\lambda)^n |n\rangle_A |n\rangle_B, \end{aligned} \quad (\text{B1})$$

where A and B denote two modes and $\lambda = \tanh r$. To generate Fock state $|n\rangle$, we perform n -photon detection on one of the modes of the TMSV state. The probability of n -photon detection is given by

$$P = \frac{(1 - \lambda^2)}{(\lambda^2(\beta - 1) + 1)} \left(\frac{\lambda^2 \beta}{\lambda^2(\beta - 1) + 1} \right)^n. \quad (\text{B2})$$

For low loss, i.e., $\beta \approx 1$, the probability of n -photon detection reduces to $(1 - \lambda^2)\lambda^{2n}$ and for $n = 10$, $P_{\max} = 0.036$ at $\lambda = \tanh(1.85)$.

Due to the decay of the TMSV state, the generated state is not identical to Fock state $|n\rangle$. The fidelity of the generated state with the state $|n\rangle$ is

$$F = \frac{1}{P} \frac{(1 - \lambda^2)}{\mathcal{D}} \left(\frac{\lambda^2 \beta^2}{\mathcal{D}} \right)^n P_n \left[\frac{\lambda^2(\beta - 1)^2 + 1}{\mathcal{D}} \right], \quad (\text{B3})$$

where $\mathcal{D} = 1 - \lambda^2(\beta - 1)^2$ and $P_n(x)$ is n th Legendre polynomial.

APPENDIX C: PHASE OPERATION WITH TWO ATOMS

In the context of generating non-Gaussian states, our focus has been on filtration with single atoms. However, we now explore potential advantages that may emerge from employing two atoms in the process. Here, we focus only the atom-cavity phase operation as that will be major change due to more atoms. It is intuitive to start the atoms in the state $\frac{|g\rangle + |s\rangle}{\sqrt{2}} \otimes \frac{|g\rangle + |s\rangle}{\sqrt{2}}$ since atoms are indistinguishable. The $|gg\rangle$, $|gs\rangle$, and $|sg\rangle$ states will all interact with the light. Only the $|ss\rangle$ state is forbidden. To keep it analogous with the single-atom scenario, we begin by initializing the atom-light system in a state

$$|\Psi\rangle = \frac{1}{2}(|gg\rangle + |ss\rangle) \otimes |\psi_l\rangle, \quad (\text{C1})$$

and atom-cavity reflection will result in

$$|\Psi_r\rangle = \frac{1}{\sqrt{2}}(|gg\rangle|\psi_l\rangle + |ss\rangle e^{i\pi\hat{n}}|\psi_l\rangle). \quad (\text{C2})$$

Introducing two atoms indeed changes the atom-cavity coupling strength as $g^2 \rightarrow 2g^2$ and the total spontaneous decay rate to 2γ . However, crucial factors like $C = \frac{g^2}{\kappa\gamma}$ and $\eta = \frac{\kappa_l}{\kappa}$ remain unaffected, offering no significant advantages over single-atom filtration for the Fock states.

It is noteworthy that Eq. (C2) is derived under the assumption $\frac{\partial \sigma_{gg}}{\partial t} = 0$; i.e., the atomic population is unchanged. From this standpoint, potential advantages with two atoms may emerge to maintain the atomic population. In Fig. 8, we compare filtration outcomes with one and two atoms for parameters $\{g, \kappa, \gamma\} = 2\pi\{25, 10, 3\}$. We perform numerical simulations utilizing virtual cavities [106], to determine the necessary quantities $\langle gg|\rho|gg\rangle$, $\langle g|\rho|g\rangle$ and compare the fidelities ($\langle \Psi_r|\rho|\Psi_r\rangle$) with Eq. (C2).

The input state $|\psi_l\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ gives fidelities of 0.97 (two atoms) and 0.93 (one atom), primarily due to slight excitation to $|e\rangle$, i.e., $\frac{\partial \langle g|\rho|g\rangle}{\partial t} \neq 0$ and $\frac{\partial \langle gg|\rho|gg\rangle}{\partial t} \neq 0$. This excitation can be mitigated by increasing g . For instance, with $g = \sqrt{2} \times 25$, excitation to $|e\rangle$ decreases, and the fidelity is similar to that of two atoms. While the coherent state ($|\psi_l\rangle = |\sqrt{0.2}\rangle$)

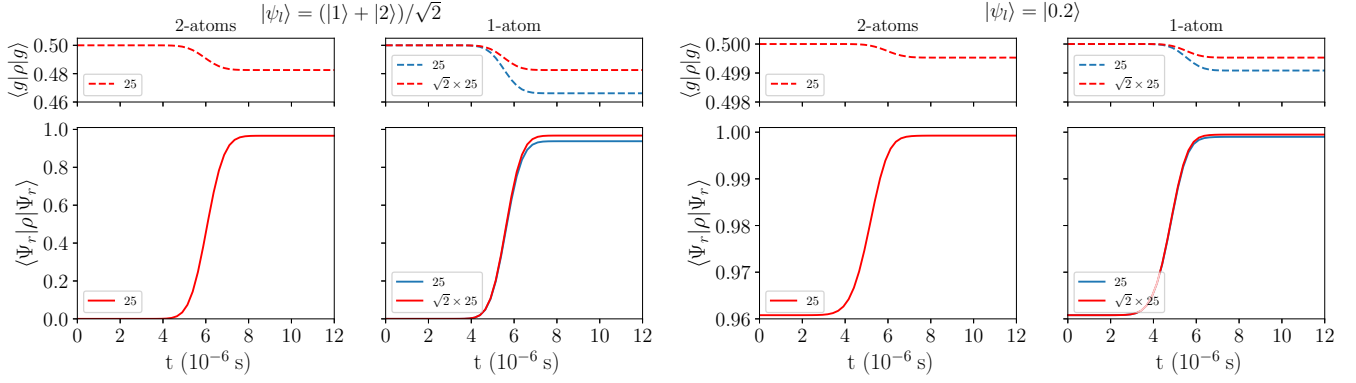


FIG. 8. Comparison of atom-cavity phase operation $U(\phi)$ using two atoms with one atom. $\{g, \kappa, \gamma\} = 2\pi\{25, 10, 3\}$ MHz. The initial states of light are $|\psi_l\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$ and coherent state. Fidelity is calculated with state in Eq. (C2)

gives similar results, due to its low amplitude the differences in fidelities are negligible.

APPENDIX D: PHASE OPERATION WITH DEPHASING

Here, we study the effect of dephasing during the atom-cavity phase operation $U(\phi)$. Although this type of noise is common in superconducting circuits [78], in typical optical resonators, it can occur due to jittering of the cavity mirrors [118]. As above, we simulate the phase operation using virtual cavities along with dephasing noise $\mathcal{D}(\rho) = \gamma_d/2[2\hat{n}\rho\hat{n}^\dagger - \hat{n}^2\rho - \rho\hat{n}^2]$, added during dynamical evolution of atom-cavity system. For the single atom, Eq. (C2) modifies to

$$|\Psi_r\rangle = \frac{1}{2}[(|g\rangle - |s\rangle) \otimes |1\rangle + (|g\rangle + |s\rangle) \otimes |2\rangle]. \quad (\text{D1})$$

Here, the initial state of the light field is given by $|\psi_l\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$. In the absence of dephasing noise, measuring the atom in the $|g\rangle + |s\rangle$ basis should project the light field onto state $|2\rangle$. In Fig. 9(a), we compare the phase opera-

tion with and without dephasing. As expected the unitary operation without dephasing outperforms the noisy operation. Furthermore, with atomic measurement, the ideal scenario ($\gamma_d = 0$ MHz) results in state $|2\rangle$, as shown in Fig. 9(b). However, when dephasing noise is introduced, it disrupts the phase operation, resulting in a mixed state after filtration, as illustrated in Fig. 9(c).

APPENDIX E: SQUEEZED COHERENT STATE

The displaced squeezed vacuum state can be written as $|\alpha e^{i\phi}, re^{i\theta}\rangle = \hat{\mathcal{D}}(\alpha e^{i\phi})\hat{\mathcal{S}}(re^{i\theta})|0\rangle$:

$$|\alpha e^{i\phi}, re^{i\theta}\rangle = \frac{e^{-\frac{\alpha^2}{2}(1+e^{i(\theta-2\phi)}\tanh r)}}{\sqrt{\cosh r}} \sum_n \frac{(\frac{1}{2}e^{i\theta}\tanh r)^{n/2}}{\sqrt{n!}} \times H_n\left[\frac{\alpha(e^{i\phi}\cosh r + e^{i(\theta-\phi)}\sinh r)}{\sqrt{e^{i\theta}\sinh 2r}}\right]|n\rangle,$$

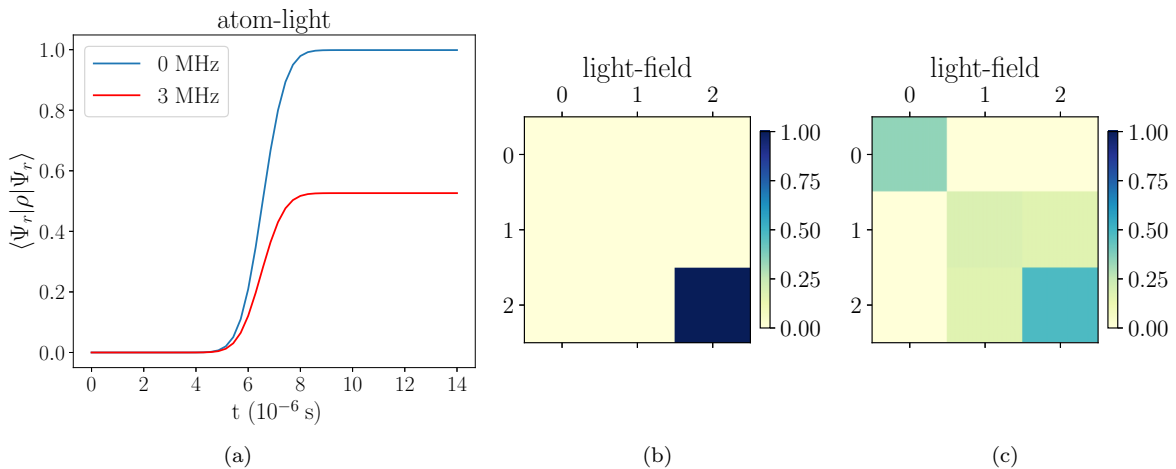


FIG. 9. Comparison of the atom-cavity phase operation $U(\phi)$ with dephasing. Panel (a) shows the phase operation under dephasing noise with $\gamma_d = 2\pi \times \{0, 3\}$ MHz. The initial state of the light field is $\frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$, and fidelity is calculated with the state in Eq. (D1). (b) ($\gamma_d = 0$) and (c) ($\gamma_d = 2\pi \times 3$ MHz) display the states of the light field at $t = 14 \times 10^{-6}$ s after the measurement on the atom. Other atom-cavity parameters are set to $\{g, \kappa, \gamma\} = 2\pi \times \{25, 10, 0\}$ MHz.

TABLE II. Comparison of Fisher information and average photon numbers for states $|\theta\rangle = \cos\theta|0\rangle + \sin\theta|N\rangle$.

State	FI ($\mathcal{F}$)	$\langle\hat{n}\rangle$
$\cos\theta 0\rangle + \sin\theta N\rangle$	QFI = $N^2 \sin^2 2\theta$	$N \sin^2 \theta$
$\cos\theta 0\rangle + \sin\theta 1\rangle$	CFI = $\frac{\sin^2(2\theta)\sin^2(\Theta)}{1-\sin^2(2\theta)\cos^2(\Theta)}$	$\sin^2 \theta$
$\cos\theta 0\rangle + \sin\theta 2\rangle$	CFI = $\frac{4\sin^2(2\theta)\cos^2(\Theta)}{1-\sin^2(2\theta)\sin^2(\Theta)}$	$2\sin^2 \theta$

$$\begin{aligned}
 |\alpha e^{i\phi}, re^{i2\phi}\rangle &= \frac{e^{-\frac{\alpha^2}{2}(1+\tanh r)}}{\sqrt{\cosh r}} \sum_n \sqrt{\frac{(\tanh r)^n}{2^n n!}} e^{in\phi} \\
 &\quad \times H_n \left[\frac{\alpha e^r}{\sqrt{\sinh 2r}} \right] |n\rangle, \\
 p_n &\equiv |\langle n | \alpha e^{i\phi}, re^{i2\phi} \rangle|^2 \\
 &= p_0 \frac{(\tanh r)^n}{n! 2^n} \left| H_n \left[\frac{\alpha e^r}{\sqrt{\sinh 2r}} \right] \right|^2, \quad (E1)
 \end{aligned}$$

where $p_0 = \frac{e^{-\alpha^2(1+\tanh r)}}{\cosh r}$ and H_n represents the n th-order Hermite polynomial. $p_n \equiv |\langle n | \alpha e^{i\phi}, re^{i2\phi} \rangle|^2$ are n th-photon probabilities. On expanding, Eq. (E1) gives

$$\begin{aligned}
 |\alpha e^{i\phi}, re^{i2\phi}\rangle &= C_0[|0\rangle + e^{i\phi}C_1|1\rangle + e^{i2\phi}C_2|2\rangle \\
 &\quad + e^{i3\phi}C_3|3\rangle + \sum_{n=4} \mathcal{O}_n|n\rangle]. \quad (E2)
 \end{aligned}$$

To improve the fidelity of the output states, we choose α, r such that $\sum_{n=0}^3 p_n \approx 1$; thus, we can drop the higher order terms and rewrite

$$\psi_{sc} \approx C_0[|0\rangle + C_1|1\rangle + C_2|2\rangle + C_3|3\rangle]. \quad (E3)$$

After the interaction with an atom in the cavity, the atom-light state is written as

$$\begin{aligned}
 \psi_{sc} &\approx C_0[|0\rangle + C_1|1\rangle + C_2|2\rangle + C_3|3\rangle]|g\rangle \\
 &\quad + C_0[|0\rangle - C_1|1\rangle + C_2|2\rangle - C_3|3\rangle]|s\rangle, \quad (E4)
 \end{aligned}$$

and Hadamard operation gives

$$\psi_{sc} \approx C_0[|0\rangle + C_2|2\rangle]|g\rangle + C_0[C_1|1\rangle + C_3|3\rangle]|s\rangle, \quad (E5)$$

and measuring the atoms in the state $|g\rangle$ gives $\psi_{02} \propto C_0[|0\rangle + \tan\theta|2\rangle]$, where $\theta = \tan^{-1}(\frac{\tanh r}{2\sqrt{2}} H_2[\frac{\alpha e^r}{\sqrt{\sinh 2r}}])$. All numerical calculations are performed with states in Eq. (E1) using sufficiently large basis.

APPENDIX F: FISHER INFORMATION FOR PHASE SENSING

The quantum Fisher information for pure states under the action $e^{i\Theta\hat{n}}$ is simplified as

$$\mathcal{F} = 4[\langle\hat{n}^2\rangle - \langle\hat{n}\rangle^2]. \quad (F1)$$

It is clear from Table II that the QFI can be achieved using binary projections with $\{|\pi/4\rangle\langle\pi/4|, I - |\pi/4\rangle\langle\pi/4|\}$ and balanced superposition states ($\theta = \pi/4$). Furthermore, superpositions up to $|1\rangle$ have a maximum CFI of 1, whereas states with $|0\rangle$ and $|2\rangle$ superpositions yield four times the CFI. For a general state with positive operator-valued measure (POVM) measurements Π_m , the Fisher information is expressed as

$$\begin{aligned}
 \mathcal{F}_\Theta &= \sum_m \frac{1}{p(m|\Theta)} \left(\frac{\partial \Theta}{\partial p} (m|\Theta) \right)^2, \quad \text{with} \\
 (m|\Theta) &= \text{Tr}[\Pi_m e^{-i\Theta\hat{n}} \rho e^{i\Theta\hat{n}}], \quad (F2)
 \end{aligned}$$

where ρ is the filtered state from the cavity. Using the state $|\theta\rangle = \cos\theta|0\rangle + \sin\theta|2\rangle$ and the projective measurements $\Pi_1 = |\phi\rangle\langle\phi|$ and $\Pi_2 = I - \Pi_1$, the probabilities are given as

$$\begin{aligned}
 p_1(\Theta) &\equiv |\langle\theta|\phi\rangle|^2 = \cos^2\theta \cos^2\phi + \sin^2\theta \sin^2\phi \\
 &\quad + \frac{1}{2} \sin(2\theta) \sin(2\phi) \cos(2\Theta), \quad (F3a)
 \end{aligned}$$

$$F(\Theta) = \frac{\sin^2(2\theta) \sin^2(2\phi) \sin^2(2\Theta)}{p_1(\Theta, \theta, \phi) p_2(\Theta, \theta, \phi)}, \quad (F3b)$$

where we used $p_2(\Theta) = 1 - p_1(\Theta, \theta, \phi)$. Note that $p_1(\Theta, \theta, \phi)$ is symmetric with respect to θ and ϕ . For a given Θ , the optimized Fisher information can be obtained with $\phi = \theta = \pi/4$.

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