

Optimal work control for an odd-diffusive tracer under active fluctuations

Koushik Goswami ^{1,*} Hong-Yan Shih (施宏燕) ^{1,2,†} Cheng-Hung Chang (張正宏) ^{1,3}Hsuan-Yi Chen (陳宣毅) ^{1,2,4} and Pik-Yin Lai (黎璧賢) ^{1,4,‡}¹Physics Division, National Center for Theoretical Sciences, National Taiwan University, Taipei 106319, Taiwan²Institute of Physics, Academia Sinica, Nankang, Taipei 115201, Taiwan³Institute of Physics, National Yang Ming Chiao Tung University, 1001 Daxue Road, Hsinchu City 300093, Taiwan⁴Department of Physics and Center for Complex Systems, National Central University, Chung-Li District, Taoyuan City 320317, Taiwan

(Received 18 August 2025; accepted 23 November 2025; published 17 December 2025)

We investigate optimal work protocols for a tracer particle with odd mobility, confined in a rotationally asymmetric parabolic potential, and driven by temporally correlated (active) noise originating from an active process. Two control strategies are considered: open loop (without measurement of the initial active force) and closed loop (with measurement-based feedback). In the open-loop case, neither activity nor odd mobility affects the average optimal work; however, the potential asymmetry enables the work extraction, and the oddness parameter κ shapes the protocol structure—introducing asymmetries and even backward jumps at the initial and final times. In the closed-loop setting, both activity and odd mobility shape the protocol design to enable work extraction. Optimal protocols exhibit directional asymmetries, including discontinuities and reversals, which depend on the alignment of the initial active force and the persistence time. Maximum work extraction occurs when the initial force aligns with the target at an intermediate persistence time, even in symmetric potentials. Furthermore, in regimes where such alignment disfavors work extraction for a normal tracer, odd mobility facilitates it. These findings highlight the potential for energy harvesting in active systems, driven by feedback and asymmetric interactions between different degrees of freedom.

DOI: [10.1103/qrvi-mng6](https://doi.org/10.1103/qrvi-mng6)

I. INTRODUCTION

The construction of efficient machines has long been a central goal in physics and engineering, aiming to optimize energy utilization and maximize work output [1–3]. In recent decades, advances in nanotechnology and microscale fabrication have enabled the design of machines operating at microscopic scales [4,5]. Unlike their macroscopic counterparts, which are governed by deterministic thermodynamic cycles, these small-scale systems are dominated by stochastic fluctuations arising from both thermal and nonthermal sources [6–10]. Consequently, energy transduction at this scale inherently involves a *thermodynamic cost of dissipation* [11–13].

A prototypical example is that of biomolecular motors, which perform directed motion and extract usable work by consuming chemical fuel (e.g., ATP), despite operating in highly noisy environments [14]. Recent theoretical and experimental progress has demonstrated that dissipation can be minimized to improve finite-time performance [15], and that

fluctuations—rather than being merely detrimental—can be strategically harnessed to improve energy efficiency [16–20]. However, realizing this potential typically requires carefully designed control strategies, including feedback mechanisms [21–24], confinement modulation, and precise tuning of fluctuation strengths [10,25–36]. A landmark contribution in this context was made by Schmiedl and Seifert [37], who analytically derived the *optimal protocol* that minimizes work input over a finite duration for a Brownian particle in a harmonic trap with a time-dependent center. In particular, the optimal protocol exhibits discontinuities at the beginning and end of the driving process, corresponding to abrupt system reconfigurations that reduce excess dissipation.

In recent years, there has been growing interest in extending such optimization frameworks to active matter systems, which are driven out of equilibrium by internally generated or environmentally induced forces [38–45]. These systems encompass a broad spectrum, including biological agents such as bacteria and cells, as well as synthetic microswimmers [46–50]. Optimization in such nonequilibrium settings can proceed under different levels of information about internal states of the entire system [42,44]. Two primary control paradigms are typically considered. In the *open-loop* setting—where the control protocol is fixed in advance and is not conditioned on any measurements, the presence of activity has been shown to have no influence on the average optimal work, thus effectively making it equivalent to the passive case. In contrast, the *closed-loop* setting incorporates measurement-based feedback, where the protocol is adapted according to

*Contact author: koushik.goswami@phys.ncts.ntu.edu.tw

†Contact author: hongyan@as.edu.tw

‡Contact author: pylai@phy.ncu.edu.tw

the information obtained from the system, such as the initial value of the active forces. In this case, the mutual information gained through the measurement can be exploited to lower the thermodynamic cost and, under suitable conditions, even enable the extraction of net work [44].

Beyond activity, thermodynamic features of small-scale systems are also shaped by additional factors such as non-reciprocal interactions, anomalous transport, and memory effects arising from complex environments [20,51,52]. In this work, we focus on the impact of *symmetry-breaking mechanisms* on optimal control strategies under active fluctuations. A particularly intriguing class of systems includes those with *odd (or nonreciprocal) diffusion*, where the mobility tensor includes antisymmetric components [53–58]. These give rise to a *transverse response* to perturbations, often manifested as mutual rolling and resulting in distinctive transport phenomena, such as enhanced self-diffusion and oscillatory force correlations [56]. Examples include charged tracers in magnetic fields [59], spinning colloidal particles [60], biological swimmers in chiral environments, and active chiral fluids [61,62]. In two-dimensional systems, odd diffusion can be quantitatively characterized by introducing an odd diffusion parameter κ , which is coupled to the antisymmetric Levi-Civita symbol ϵ_{ij} , defined as $\epsilon_{X_1 X_2} = +1$, $\epsilon_{X_2 X_1} = -1$, and zero otherwise. The effective diffusivity tensor D_{ij} then takes the form [54,58]

$$D_{ij} = D_0(\delta_{ij} + \kappa \epsilon_{ij}), \quad (1)$$

where D_0 is the bare (symmetric) diffusion coefficient. The presence of the antisymmetric term $\kappa \epsilon_{ij}$ leads to rotational fluxes and anisotropic spreading. Odd mobility is also important in modeling diffusion through porous media [63], the dynamics of spinning particles subject to Magnus forces in viscoelastic fluids [64,65], and transport phenomena in astrophysical and interstellar media [59,66].

Motivated by these developments, we revisit the classical problem of optimal work extraction for a Brownian tracer, generalizing it to a two-dimensional setting with a rotationally asymmetric parabolic potential and an odd (nonreciprocal) mobility tensor. The tracer is embedded in an active medium modeled by an Ornstein-Uhlenbeck process, introducing exponentially correlated noise [67,68]. Owing to the linearity of the dynamics, the system admits exact analytical solutions, which are corroborated by numerical simulations. We explore two control strategies applied independently along each spatial direction: an open-loop protocol (without access to the initial active force) and a closed-loop protocol (with initial force measurement). Based on these protocols, we determine the corresponding optimal strategies and compute the associated work. This, in turn, allows us to identify parameter regimes that are favorable for work extraction.

The remainder of the paper is organized as follows. In Sec. II, we introduce the model and its governing dynamics. Section III outlines the classes of control protocols, the thermodynamic work functional, and the overall control framework. In Sec. IV, we derive the optimal protocols and work under both open- and closed-loop strategies. Section V summarizes the key findings and discusses their potential applicability. Detailed derivations and supplementary calculations are provided in the Appendixes.

II. MODEL

We consider a charged Brownian tracer confined within a two-dimensional potential $V(X_1, X_2)$ and subjected to a nonconservative force—specifically, a static magnetic field oriented along the Z axis. Consequently, the Lorentz force manifests only within the (X_1, X_2) plane, effectively restricting the tracer’s motion to two dimensions. In the overdamped limit, the tracer’s dynamics are governed by the Langevin equation

$$\Gamma_{\text{tr}} \dot{\mathbf{X}}(t) = -\frac{\partial}{\partial \mathbf{X}} V(\mathbf{X}, \boldsymbol{\lambda}) + \boldsymbol{\eta}(t) + \mathbf{F}(t), \quad (2)$$

where $\mathbf{X} = (X_1, X_2)^T$ denotes the position vector of the tracer. Such a system is commonly referred to as an odd tracer, as the magnetic field introduces an antisymmetric component to the friction tensor [58]

$$\Gamma_{\text{tr}} = \begin{bmatrix} 1 & -\kappa \\ \kappa & 1 \end{bmatrix}, \quad (3)$$

where $\kappa \geq 0$ is the oddness parameter, characterizing the nonreciprocal effects induced by the magnetic field. The stochastic term $\boldsymbol{\eta}(t)$ represents Gaussian white noise, whose statistical properties are given by

$$\langle \eta_i(t) \rangle = 0, \quad (4)$$

$$\langle \eta_i(t_1) \eta_j(t_2) \rangle = 2D_0 \delta_{ij} \delta(t_1 - t_2), \quad (5)$$

where D_0 is the thermal diffusivity and δ_{ij} is the Kronecker delta ensuring that noise components in orthogonal directions are uncorrelated. The vector $\mathbf{F}(t)$ denotes an active stochastic force, which can either arise from the tracer’s intrinsic self-propulsion mechanisms or be externally imposed [69,70]. In this study, we examine an odd tracer confined in a two-dimensional harmonic potential $V(\mathbf{X}, \boldsymbol{\lambda})$, modulated by a time-dependent control protocol $\boldsymbol{\lambda}(t)$. The work done on the tracer over a finite time interval t_f is defined as [6,71]

$$W[\mathbf{X}, \boldsymbol{\lambda}] = \int_0^{t_f} dt \frac{\partial V(\mathbf{X}, \boldsymbol{\lambda})}{\partial \boldsymbol{\lambda}} \cdot \dot{\boldsymbol{\lambda}}(t). \quad (6)$$

As a representative and analytically tractable case, we consider an odd tracer initially confined in a rotationally asymmetric parabolic potential. For $t > 0$, the potential minimum is dragged in both spatial directions according to prescribed protocols $\lambda_1(t)$ and $\lambda_2(t)$. Thus, the potential at any time t is written as

$$V(\mathbf{X}, \boldsymbol{\lambda}) = \frac{k_0}{2} (X_1 - \lambda_1(t))^2 + \frac{k_0}{2} (X_2 - \lambda_2(t))^2 + k_0 \gamma X_1 X_2, \quad (7)$$

where k_0 is the stiffness of the harmonic confinement and γ quantifies the coupling between the X_1 and X_2 coordinates. The parameter γ must satisfy $-1/k_0 < \gamma < 1/k_0$ to ensure a confining potential. A nonzero value of γ ($\gamma \neq 0$) introduces rotational asymmetry into the potential at the initial time and is therefore referred to as the asymmetry parameter. Using this potential into Eq. (2), the equations of motion for the tracer

components can be written explicitly as

$$\begin{aligned}\dot{X}_1(t) = & -\alpha_k[X_1(t) - \lambda_1(t) + \gamma X_2(t) + \kappa X_2(t) - \kappa \lambda_2(t) \\ & + \kappa \gamma X_1(t)] + \alpha[\eta_1(t) + \kappa \eta_2(t)] \\ & + \alpha[F_1(t) + \kappa F_2(t)],\end{aligned}\quad (8)$$

$$\begin{aligned}\dot{X}_2(t) = & \alpha_k[\kappa(X_1(t) - \lambda_1(t)) + \kappa \gamma X_2(t) - X_2(t) + \lambda_2(t) \\ & - \gamma X_1(t)] - \alpha[\kappa \eta_1(t) - \eta_2(t)] - \alpha[\kappa F_1(t) - F_2(t)],\end{aligned}\quad (9)$$

where $\alpha_k = k_0/(1 + \kappa^2)$ and $\alpha = 1/(1 + \kappa^2)$. Here, the subscript $i = 1, 2$ refers to the two Cartesian components of the respective variables.

To model activity, we consider the active force $\mathbf{F}(t)$ as an Ornstein-Uhlenbeck process, which serves as a standard and analytically tractable representation of temporally correlated noise [72–75]. Each spatial component of the active force

evolves according to

$$\dot{F}_i(t) = -\frac{1}{\tau}F_i(t) + \frac{1}{\tau}\eta_{A,i}(t),\quad (10)$$

where τ denotes the correlation time and $\eta_{A,i}(t)$ is a Gaussian white noise with zero mean and strength D_A . The autocorrelation function of the active noise decays exponentially,

$$\langle F_i(t)F_j(t') \rangle = \delta_{ij} \frac{D_A}{\tau} \exp\left(-\frac{|t - t'|}{\tau}\right).\quad (11)$$

We define the squared amplitude of the active force (or, equivalently, the self-propulsion speed for an active particle) as $f_{\text{av}}^2 = \frac{D_A}{\tau}$, such that $\langle F_i(t)^2 \rangle = f_{\text{av}}^2$.

When a time-dependent protocol $\lambda_i(t)$ is imposed on the tracer, work is performed on the system. Following Sekimoto's convention [71], the work done on the tracer along a single stochastic trajectory is given by

$$\begin{aligned}W[X_1, X_2, \lambda_1, \lambda_2] = & k_0 \int_0^{t_f} dt' \dot{\lambda}_1(t')[\lambda_1(t') - X_1(t')] + k_0 \int_0^{t_f} dt' \dot{\lambda}_2(t')[\lambda_2(t') - X_2(t')] \\ = & \frac{k_0}{2}[L_1^2 + L_2^2 - 2L_1X_1(t_f) - 2L_2X_2(t_f)] + k_0 \int_0^{t_f} dt' [\lambda_1(t')\dot{X}_1(t') + \lambda_2(t')\dot{X}_2(t')].\end{aligned}\quad (12)$$

Here, the protocols in the two directions are initiated at $t = 0$, with initial values $\lambda_1(0^-) = \lambda_2(0^-) = 0$, and evolve to final values $\lambda_1(t_f^+) = L_1$ and $\lambda_2(t_f^+) = L_2$, respectively.

III. PROTOCOLS AND AVERAGE WORK

We now consider the dynamics of average quantities, conditioned on the initial state of the system. Specifically, we define the initial conditions as $\mathbf{X}(0) = \mathbf{X}_0 = (X_{10}, X_{20})$ and $\mathbf{F}(0) = \mathbf{f}_0 = (f_{10}, f_{20})$. The average position and force components at time t are then defined as

$$x_i^a(t) = \langle X_i(t) \rangle_a, \quad f_i^a(t) = \langle F_i(t) \rangle_a, \quad (13)$$

where the angular brackets $\langle \cdot \rangle_a$ denote an average over all trajectories of thermal noise $\boldsymbol{\eta}(t)$, conditioned on the initial values $a = \{\mathbf{X}_0, \mathbf{f}_0\}$. In cases where the system is initialized in thermal equilibrium or a nonequilibrium steady state, we denote the initial condition as $a = \text{eq}$ or $a = \text{ss}$, respectively. Under such conditions, the initial averages vanish: $\langle \mathbf{X}(0) \rangle_a = 0$ and $\langle \mathbf{F}(0) \rangle_a = \langle \mathbf{f}_0 \rangle = 0$. The governing equations for the average position dynamics are then given by

$$\begin{aligned}\dot{x}_1^a(t) = & -\alpha_k[x_1^a(t) - \lambda_1(t) + \gamma x_2^a(t) + \kappa x_2^a(t) - \kappa \lambda_2(t) \\ & + \kappa \gamma x_1^a(t)] + \alpha[f_1^a(t) + \kappa f_2^a(t)],\end{aligned}\quad (14)$$

$$\begin{aligned}\dot{x}_2^a(t) = & \alpha_k[\kappa x_1^a(t) - \kappa \lambda_1(t) + \kappa \gamma x_2^a(t) - x_2^a(t) + \lambda_2(t) \\ & - \gamma x_1^a(t)] + \alpha[-\kappa f_1^a(t) + f_2^a(t)].\end{aligned}\quad (15)$$

From these equations, the control protocols $\lambda_1(t)$ and $\lambda_2(t)$ can be expressed in terms of the average dynamical variables as

$$\lambda_1(t) = \frac{1}{k_0}[\kappa \dot{x}_1^a(t) - \kappa \dot{x}_2^a(t)] + x_1^a(t) + \gamma x_2^a(t) - \frac{f_1^a(t)}{k_0},\quad (16)$$

$$\lambda_2(t) = \frac{1}{k_0}[\kappa \dot{x}_1^a(t) + \dot{x}_2^a(t)] + \gamma x_1^a(t) + x_2^a(t) - \frac{f_2^a(t)}{k_0},\quad (17)$$

with boundary conditions $\lambda_i(0^-) = 0$ and $\lambda_i(t_f^+) = L_i$. The control protocol can be implemented in two distinct ways [42]:

(1) *Open-loop control.* The protocol $\boldsymbol{\lambda}(t)$ is predetermined and does not depend on the instantaneous state of the system. The system is initialized from a known distribution, typically corresponding to thermal equilibrium or a nonequilibrium steady state (here, a steady state under active forcing). Consequently, $x_{10} = x_{20} = 0$ and $\langle f_{10} \rangle = \langle f_{20} \rangle = 0$.

(2) *Closed-loop (feedback) control.* In this case, the protocol $\boldsymbol{\lambda}(t)$ is updated based on measurements of the system's state. Here, we assume that only the initial active force $\mathbf{f}_0$ is measured (e.g., by a Maxwell-demon-like controller), and the corresponding average initial positions $\mathbf{x}_0$ are then determined via their statistical dependence on $\mathbf{f}_0$.

Substituting Eqs. (16) and (17) into the work definition [see Eq. (12)], and averaging over all realizations of thermal noise conditioned on the initial state a , the mean work over the total

duration t_f can be written as

$$\begin{aligned}\omega(t_f) &= k_0 \int_0^{t_f} dt' \dot{\lambda}_1(t') [\lambda_1(t') - x_1^a(t')] + k_0 \int_0^{t_f} dt' \dot{\lambda}_2(t') [\lambda_2(t') - x_2^a(t')] \\ &= \Delta\tilde{V} + \int_0^{t_f} dt' [\dot{x}_1^a(t')^2 + \dot{x}_2^a(t')^2 - \dot{x}_1^a(t')f_1^a(t') - \dot{x}_2^a(t')f_2^a(t')],\end{aligned}\quad (18)$$

where

$$\begin{aligned}V(x_1^a, x_2^a, t_f) &= \frac{k_0}{2}(L_1 - x_1^a(t_f))^2 + \frac{k_0}{2}(L_2 - x_2^a(t_f))^2 + k_0\gamma x_1^a(t_f)x_2^a(t_f), \\ V(x_1^a, x_2^a, 0) &= \frac{k_0}{2}x_{10}^2 + \frac{k_0}{2}x_{20}^2 + k_0\gamma x_{10}x_{20}, \quad \Delta\tilde{V} = V(x_1^a, x_2^a, t_f) - V(x_1^a, x_2^a, 0).\end{aligned}\quad (19)$$

We can also define the cumulative work up to time $t \leq t_f$ as

$$\omega(t) = k_0 \int_0^t dt' \dot{\lambda}_1(t') [\lambda_1(t') - x_1^a(t')] + k_0 \int_0^t dt' \dot{\lambda}_2(t') [\lambda_2(t') - x_2^a(t')]. \quad (20)$$

IV. OPTIMAL CONTROL

We now turn to the core optimization problem, where the objective is to minimize the mean work defined in Eq. (18). To this end, we employ the standard Euler-Lagrange formalism, incorporating the appropriate initial condition a , and analyze the two cases as outlined below.

(1) *Open-loop control.* The optimal protocols in the open-loop setting, while remaining linear in time as in the case of a passive Brownian tracer, are fundamentally modified by the presence of odd mobility (κ) and asymmetry (γ). These parameters introduce anisotropy in the temporal evolution of the protocols along the two spatial directions. The analytical expressions for the optimal protocols are given by (see Appendix C for derivation)

$$\lambda_1^*(t) = \frac{c_1^o}{k_0}(1 + k_0 t) + c_2^o \left(\gamma t - \frac{\kappa}{k_0} \right), \quad (21)$$

$$\lambda_2^*(t) = c_1^o \left(\gamma t + \frac{\kappa}{k_0} \right) + \frac{c_2^o}{k_0}(1 + k_0 t), \quad (22)$$

with the coefficients c_1^o and c_2^o given by

$$c_1^o = \frac{k_0 L_1 (2 + k_0 t_f) - \gamma k_0^2 t_f L_2}{4 + 4k_0 t_f + k_0^2 t_f^2 (1 - \gamma^2)}, \quad (23)$$

$$c_2^o = \frac{-\gamma k_0^2 t_f L_1 + k_0 L_2 (2 + k_0 t_f)}{4 + 4k_0 t_f + k_0^2 t_f^2 (1 - \gamma^2)}. \quad (24)$$

Notably, the discontinuities in the protocols—denoted by $\Delta\lambda_i^*(0) = \lambda_i^*(0^+) - 0$ at $t = 0$ and $\Delta\lambda_i^*(t_f) = L_i - \lambda_i^*(t_f^-)$ just before the final time—depend explicitly on κ , leading to asymmetric jumps even when $L_1 = L_2$. Interestingly, in a given direction, the initial and final jumps themselves are unequal in magnitude. This behavior is illustrated in Fig. 1, which shows that only for $\kappa = 0$ do the protocol discontinuities $\Delta\lambda_1^*(0)$ and $\Delta\lambda_2^*(0)$ coincide for $L_1 = L_2$. As κ increases, a reversal in the direction of the initial jump—opposite to the final displacement L_1 —is observed for $\lambda_1(t)$ (associated with the X_1 direction), whereas in the X_2 direction, the jumps become progressively larger in the forward direction, aligned with L_2 . The critical value of the oddness parameter κ , at which the initial discontinuity in $\lambda_1^*(t)$ vanishes

[i.e., $\lambda_1^*(0^+) = 0$], is denoted by κ_o^* and can be analytically obtained as

$$\kappa_o^* = \frac{k_0 L_1 (2 + k_0 t_f) - \gamma k_0^2 t_f L_2}{k_0 L_2 (2 + k_0 t_f) - \gamma k_0^2 t_f L_1}. \quad (25)$$

In the special case $L_1 = L_2$, it simplifies to $\kappa_o^* = 1$. Moreover, we note a directional asymmetry in the magnitude of jumps: for $\lambda_1(t)$, the jumps satisfy $\Delta\lambda_1^*(0) < \Delta\lambda_1^*(t_f)$, whereas for $\lambda_2(t)$, the opposite holds— $\Delta\lambda_2^*(0) > \Delta\lambda_2^*(t_f)$. Despite this asymmetry, both protocols preserve a generalized form of time-reversal symmetry [20], captured by the relation

$$\lambda_i^*(t) + \lambda_i^*(t_f - t) + (\Delta\lambda_i^*(t_f) - \Delta\lambda_i^*(0)) = L_i, \quad (26)$$

for $i = 1, 2$, provided the jumps at $t = 0$ and $t = t_f$ are properly included.

The optimal average work in the open-loop protocol is computed as [see Eq. (C24)]

$$\begin{aligned}\omega^*(t_f) &= (c_1^o)^2 t_f + (c_2^o)^2 t_f \\ &\quad + \frac{k_0}{2} [(L_1 - c_1^o t_f)^2 + (L_2 - c_2^o t_f)^2 + 2\gamma c_1^o c_2^o t_f^2].\end{aligned}\quad (27)$$

For any protocol except the optimal one, the mean work $\omega(t_f)$ in the open-loop setting increases with κ (see Fig. S1 in the Supplemental Material [76]). However, this increase is purely geometric and unrelated to the presence of activity, which remains ineffectual in the absence of measurement. Notably, under optimal conditions, the average work remains invariant with respect to variations in κ (see Fig. 2) and is unaffected by the strength of activity, f_{av} . However, the effect of asymmetry is retained, as it enters the expression through the change in potential energy, thus implicitly influencing the work via coupling between the directions. As illustrated in Fig. 2, for positive coupling ($\gamma > 0$), the mean work $\omega^*(t_f)$ remains strictly positive and increases monotonically with γ . In contrast, for negative coupling, beyond a threshold $\gamma = \gamma^*$, the optimal mean work becomes negative, indicating a regime of net work extraction (see also Fig. S2 in the Supplemental Material [76]).

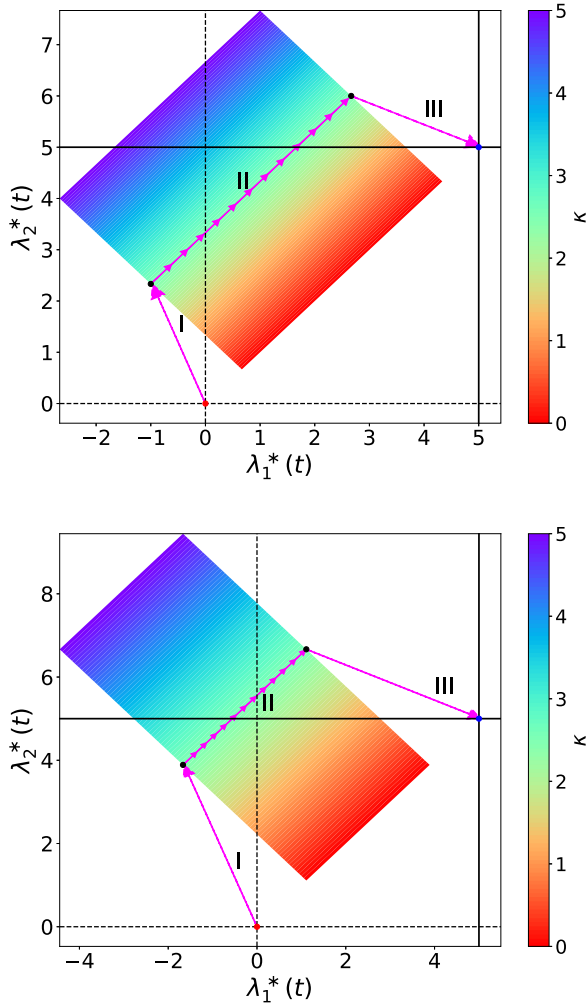


FIG. 1. Parametric plots of the optimal open-loop protocols $\lambda_1^*(t)$ and $\lambda_2^*(t)$ for different values of the oddness parameter $\kappa \in [0, 5]$, shown by the color gradient. Panel (a) corresponds to $\gamma = 0.1$, and panel (b) corresponds to $\gamma = -0.5$, with $k_0 = 1$, $t_f = 5$, and $L_1 = L_2 = 5$. The protocols satisfy $\lambda_1^*(0^-) = \lambda_2^*(0^-) = 0$ and $\lambda_1^*(t_f^+) = L_1$, $\lambda_2^*(t_f^+) = L_2$. Bold black lines mark the final values $\lambda_i^* = 5$, and dashed black lines indicate the initial points $\lambda_i^* = 0$. For $\kappa = 2.5$, a representative protocol trajectory is shown (in magenta), with arrows indicating the key stages: (I) an initial jump, (II) linear evolution toward L_i , and (III) a final jump to reach the target value L_i .

To gain a more detailed understanding of this behavior, we calculate the cumulative work $\omega^*(t)$ using Eq. (20) under the optimal protocols defined in Eqs. (21) and (22), as shown in Fig. 3. For $\gamma > 0$, the cumulative work exhibits an initial jump corresponding to the initial jump in the control protocol, followed by a monotonic increase and a final jump at $t = t_f$, reaching $\omega^*(t_f)$. In contrast, for $\gamma < \gamma^* < 0$, the cumulative work begins with a larger initial jump, which is more energy intensive due to a pronounced initial reverse motion in $\lambda_1(t)$. However, it subsequently decreases nonmonotonically over time, with a final drop near $t = t_f$, ultimately reaching a lower value than in the previous ($\gamma > 0$) case. Consistent with the preservation of the time-reversal symmetry in optimal protocols [see Eq. (26)], the initial work cost increases with κ due to larger initial jumps. However,

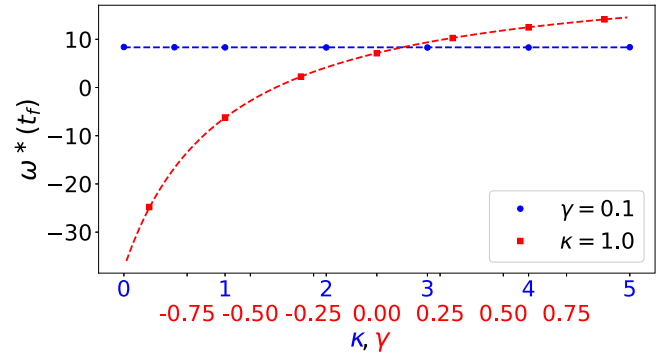


FIG. 2. Optimal average work for open-loop protocols: simulations (symbols) and analytical predictions (dashed lines) from Eq. (27), shown as a function of the oddness parameter κ and the asymmetry parameter γ . Both plots share a consistent axis width for direct visual comparison, though with different tick intervals. Other parameters are as in Fig. 1.

this is compensated by a favorable energetic contribution at the final stage, rendering the total work independent of κ . Overall, this analysis reveals that even in an open-loop setting, asymmetric coupling can facilitate net work extraction, highlighting its nontrivial role in nonequilibrium thermodynamic control.

(2) *Closed-loop control.* In this control strategy, only the initial value of the active force (or velocity), denoted by $\mathbf{f}_0 = (f_{10}, f_{20})$, is measured. However, this measurement inherently provides information about the average initial position of the particle, which can be inferred from the statistical correlations between force and position. The resulting average positions can be expressed in terms of the initial force components as [see Eqs. (A14), (A18), and (A19)]

$$x_{10} = \tilde{x}_{11}f_{10} + \tilde{x}_{12}f_{20}, \quad x_{20} = \tilde{x}_{21}f_{10} + \tilde{x}_{22}f_{20}, \quad (28)$$

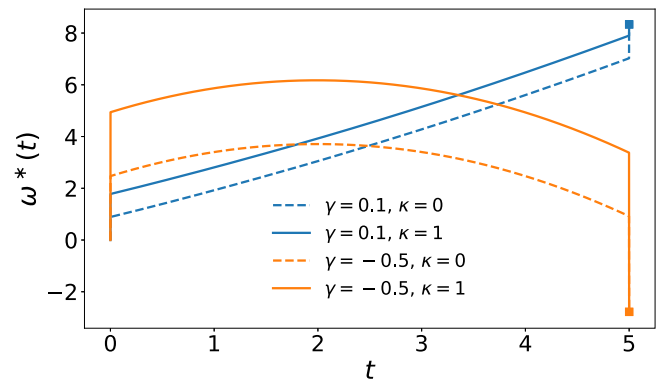


FIG. 3. Cumulative work $\omega^*(t)$ under the optimal open-loop protocols as a function of time t for two values of the asymmetry parameter: $\gamma = 0.1$ and $\gamma = -0.5$, and for two oddness parameters: $\kappa = 0$ (dashed lines) and $\kappa = 1$ (solid lines). All curves are obtained from numerical simulations averaged over 10 000 trajectories. Square symbols indicate the corresponding analytical values of $\omega^*(t_f)$ computed using Eq. (27). All other parameters are the same as in Fig. 1.

where the coefficients $\tilde{x}_{ij}$ are given by

$$\tilde{x}_{11} = \tilde{x}_{22} = \frac{\tau(1 + k_0\tau)}{(1 - \gamma^2)k_0^2\tau^2 + 2k_0\tau + 1 + \kappa^2}, \quad (29a)$$

$$\tilde{x}_{12} = \frac{\kappa - \gamma k_0\tau}{1 + k_0\tau} \tilde{x}_{11}, \quad \tilde{x}_{21} = -\frac{\kappa + \gamma k_0\tau}{1 + k_0\tau} \tilde{x}_{11}. \quad (29b)$$

Using these relations, the optimal protocol in the closed-loop case can be written as [see Eqs. (C19) and (C20)]

$$\lambda_1^*(t) = \lambda_1^o(t) + f_{10}\lambda_{11}^*(t) + f_{20}\lambda_{12}^*(t), \quad (30)$$

$$\lambda_2^*(t) = \lambda_2^o(t) + f_{10}\lambda_{21}^*(t) + f_{20}\lambda_{22}^*(t), \quad (31)$$

where $\lambda_i^o(t)$ and $\lambda_2^o(t)$ denote optimal open-loop protocols [cf. Fig. 1; see also Eqs. (C6) and (C7)], which are linear in time. In contrast, the components $\lambda_{ij}^*(t)$ vary nonlinearly in time and depend on system parameters such as κ , γ , L_i , and t_f . Their explicit forms, while not essential to the present discussion, are provided in Eq. (C22). As a representative example, consider the case $\gamma = 0$; however, the following analysis remains qualitatively valid in the regime $1 \gg \gamma \geq 0$. In the limit $\gamma \rightarrow 0$, the optimal protocols retain the same functional form as Eqs. (30) and (31), with modified expressions given by

$$\begin{aligned} \lambda_1^o(t) &= \frac{L_1}{2 + k_0t_f}(1 + k_0t) - \frac{\kappa L_2}{2 + k_0t_f}, \\ \lambda_2^o(t) &= \frac{\kappa L_1}{2 + k_0t_f} + \frac{L_2}{2 + k_0t_f}(1 + k_0t) \end{aligned} \quad (32)$$

and coefficients given by

$$\begin{aligned} \lambda_{11}^*(t) &= -\frac{1 + k_0t}{2 + k_0t_f} \left[\tilde{x}_{11} + \frac{\tau}{2} \rho(t) \right] + \frac{\kappa \tilde{x}_{21}}{2 + k_0t_f} \\ &\quad + \frac{\tau}{2} \rho(t) - \frac{e^{-\frac{t}{\tau}}}{2k_0} + \tilde{x}_{11}, \end{aligned} \quad (33)$$

$$\begin{aligned} \lambda_{12}^*(t) &= -\frac{\tilde{x}_{12}(1 + k_0t)}{2 + k_0t_f} + \frac{\kappa}{2 + k_0t_f} \left[\tilde{x}_{22} + \frac{\tau}{2} \rho(t) \right] \\ &\quad - \frac{\kappa e^{-\frac{t}{\tau}}}{2k_0} + \tilde{x}_{12}, \end{aligned} \quad (34)$$

$$\begin{aligned} \lambda_{21}^*(t) &= -\frac{\kappa}{2 + k_0t_f} \left[\tilde{x}_{11} + \frac{\tau}{2} \rho(t) \right] - \frac{\tilde{x}_{21}(1 + k_0t)}{2 + k_0t_f} \\ &\quad + \frac{\kappa e^{-\frac{t}{\tau}}}{2k_0} + \tilde{x}_{21}, \end{aligned} \quad (35)$$

$$\begin{aligned} \lambda_{22}^*(t) &= -\frac{\kappa \tilde{x}_{12}}{2 + k_0t_f} - \frac{1 + k_0t}{2 + k_0t_f} \left[\tilde{x}_{22} + \frac{\tau}{2} \rho(t) \right] \\ &\quad + \frac{\tau}{2} \rho(t) - \frac{e^{-\frac{t}{\tau}}}{2k_0} + \tilde{x}_{22}, \end{aligned} \quad (36)$$

where $\rho(t) = (1 - e^{-\frac{t}{\tau}})$. The resulting protocol structure exhibits a rich dependence on the orientation and magnitude of the initial active forces, as well as the oddness parameter κ ; see Fig. 4 and also Figs. S5 and S6 in the Supplemental Material [76] for detailed variations with τ and κ . The optimal protocols primarily follow two distinct strategies between the initial jump at $t = 0^+$ and the final jump at $t = t_f^-$, as schematically depicted in Fig. 4. In *Strategy I*, the protocol $\lambda_i^*(t)$ typically advances forward, i.e., $\lambda_i(0^+) > 0$ after the

initial jump, may reach a local maximum at time t^m , and then terminates with a final jump to reach the target L_i (e.g., $f_{10} = f_{20} = 10$ in Fig. 4). In contrast, *Strategy II* features an initial reversal, i.e., $\lambda_i(0^+) < 0$ following the jump, possibly reaching a minimum at time t^r , and subsequently progressing forward before executing the final jump to L_i (e.g., $f_{10} = f_{20} = -10$ in Fig. 4; see also Figs. S5 and S6 in the Supplemental Material [76]). Both characteristic times t^r and t^m scale with τ . Notably, such nonmonotonic protocol evolution is absent in the open-loop setting.

Let us now analyze the impact of the initial force set (f_{10}, f_{20}) on the strategy adopted by optimal protocols, based on Eqs. (30) and (31) for the simplified case $\gamma = 0$. Specifically, we focus on the slope of the protocol $\lambda_i^*(t)$ immediately after the initial jump. These slopes are given by

$$\begin{aligned} \dot{\lambda}_i^*(0^+) &= \frac{L_i k_0}{2 + k_0t_f} + \frac{1}{2} \left[1 + \frac{1}{k_0\tau} - \frac{1 - 2k_0\tilde{x}_{11}}{2 + k_0t_f} \right] f_{i0} \\ &\quad + \kappa \frac{(-1)^{i+1}}{2} \left[\frac{1 - \frac{k_0\tilde{x}_{11}}{1 + k_0\tau}}{2 + k_0t_f} + \frac{1}{k_0\tau} \right] f_{j0}, \end{aligned} \quad (37)$$

where $i \neq j$ and $i, j \in \{1, 2\}$. From Eq. (37), a critical threshold f_{i0}^* can be defined by setting $\dot{\lambda}_i^*(0^+) = 0$, which yields

$$f_{i0}^* = -2 \frac{\frac{L_i k_0}{2 + k_0t_f} + \kappa \frac{(-1)^{i+1}}{2} \left[\frac{1 - \frac{k_0\tilde{x}_{11}}{1 + k_0\tau}}{2 + k_0t_f} + \frac{1}{k_0\tau} \right] f_{j0}}{1 + \frac{1}{k_0\tau} - \frac{1 - 2k_0\tilde{x}_{11}}{2 + k_0t_f}}. \quad (38)$$

For $f_{i0} > f_{i0}^*$, the protocol $\lambda_i(t)$ follows *Strategy I*, while for $f_{i0} < f_{i0}^*$, it follows *Strategy II*. For $\kappa = 0$, the last term in Eq. (37) vanishes, rendering f_{i0}^* independent of its orthogonal component f_{j0} . In this case, $f_{i0}^* \leq 0$, so the protocol $\lambda_i(t)$ always follows *Strategy I* for any $f_{i0} \geq 0$ and transitions to *Strategy II* only when $f_{i0} < f_{i0}^*$. When $\gamma \neq 0$, however, f_{i0}^* becomes dependent on f_{j0} . Specifically, the sign of the threshold follows $f_{i0}^* \sim -\text{sgn}(f_{j0})|f_{j0}^*|$, while the magnitude $|f_{i0}^*|$ increases with both $|f_{j0}|$ and γ . These trends are similar for both protocols $\lambda_1(t)$ and $\lambda_2(t)$, as illustrated in the Supplemental Material [76]; see Figs. S7 and S8 for the case $\kappa = 0$. For $\kappa \neq 0$, even when $\gamma = 0$, the threshold f_{i0}^* retains a dependence on the orthogonal component f_{j0} , as evident from Eq. (37). Moreover, f_{i0}^* is also modulated by the correlation time τ ; see Figs. S7 and S8 in the Supplemental Material for comparison [76]. In this regime, the dependence of f_{i0}^* on f_{j0} becomes asymmetric in the two directions for $i = 1, 2$ with $i \neq j$ [see Eq. (38)]. Specifically, the trends are given as $f_{10}^* \sim -\text{sign}(f_{20})|f_{10}^*|$ and $f_{20}^* \sim \text{sign}(f_{10})|f_{20}^*|$, with the magnitude $|f_{i0}^*|$ increasing with $|f_{j0}|$, consistent with earlier observations. This asymmetry is further amplified as either κ , γ , or both increase.

The initial jumps of the protocols are derived from Eqs. (30) and (31), and are given by

$$\begin{aligned} \Delta \lambda_i^*(0) &= \lambda_i^*(0^+) = \frac{L_i}{2 + k_0t_f} - \frac{(-1)^{i+1} \kappa L_j}{2 + k_0t_f} - \frac{f_{i0}}{2k_0} \\ &\quad + \frac{\tilde{x}_{11} f_{i0}}{2 + k_0t_f} \left[1 + k_0t_f - \frac{\kappa^2}{1 + k_0\tau} \right] - \frac{(-1)^{i+1} \kappa f_{j0}}{2k_0} \\ &\quad - \frac{(-1)^{i+1} \kappa \tilde{x}_{11} f_{j0}}{2 + k_0t_f} \left[\frac{k_0(t_f - \tau)}{1 + k_0\tau} \right], \end{aligned} \quad (39)$$

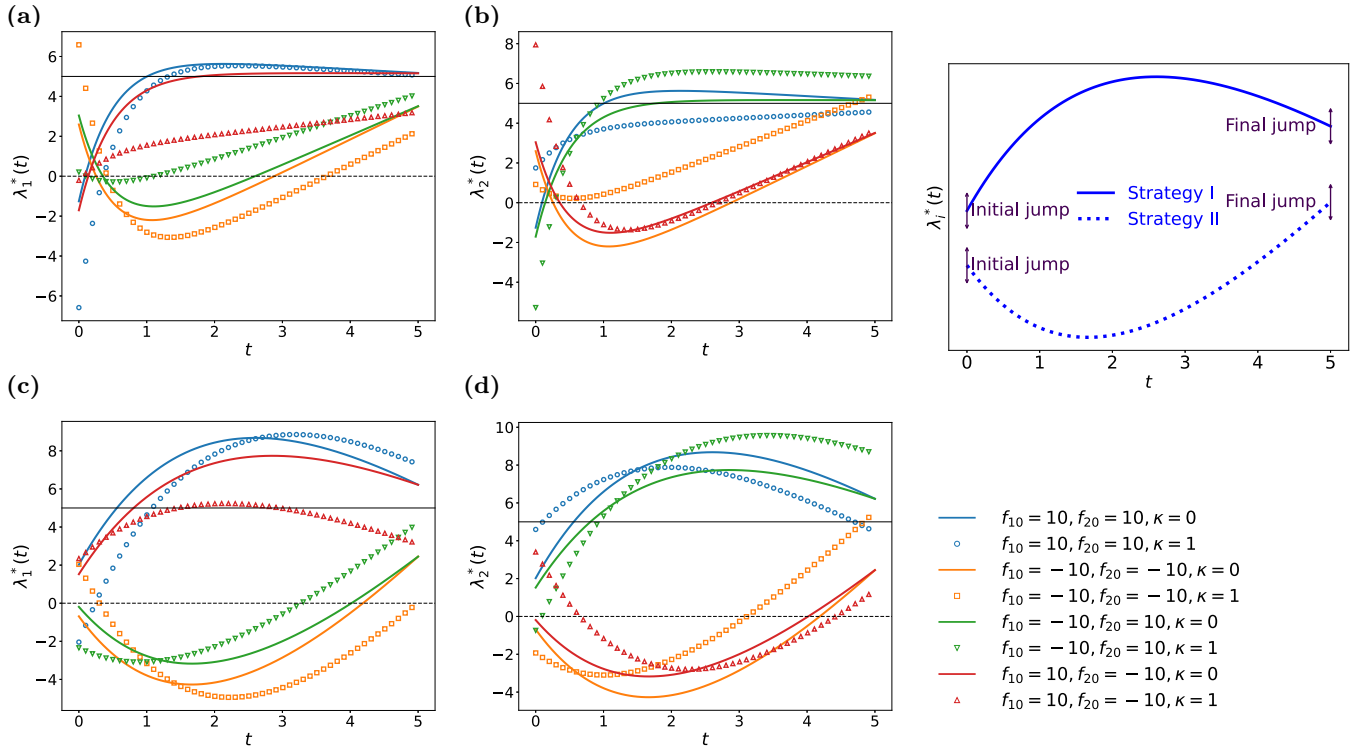


FIG. 4. Time evolution of the optimal closed-loop protocols $\lambda_1^*(t)$ and $\lambda_2^*(t)$ for different pairs of initial active forces (f_{10}, f_{20}) , comparing two noise correlation times. Panel (a) shows $\lambda_1^*(t)$ for $\tau = 0.5$; panel (b) shows $\lambda_2^*(t)$ for $\tau = 0.5$; panel (c) shows $\lambda_1^*(t)$ for $\tau = 2.5$; and panel (d) shows $\lambda_2^*(t)$ for $\tau = 2.5$. Each colored curve corresponds to a distinct pair (f_{10}, f_{20}) , with the effect of odd mobility characterized by the parameter κ . Horizontal dashed and solid lines denote $\lambda_i = 0$ and $\lambda_i = 5$, respectively. In the rightmost column, the upper panel schematically illustrates two distinct protocol strategies adopted for specific force pairs, while the lower panel displays the legend shared by panels (a)–(d). Other parameters are $k_0 = 1$, $t_f = 5$, $L_1 = L_2 = 5$, and $\gamma = 0.1$.

where $i \neq j$ and $i, j \in \{1, 2\}$. To gain some intuition about the jump direction, let us first consider the simple case $\kappa = 0$. In this limit, the no-jump condition $\lambda_i^*(0^+) = 0$ is satisfied at a specific correlation time $\tau = \tau_{\text{nojump}}^i$ (abbreviated as τ_{nj}^i), which obeys

$$\lambda_i^*(0^+) \big|_{\tau=\tau_{\text{nj}}^i} = \frac{L_i}{2 + k_0 t_f} - \frac{f_{i0}}{2k_0} \left(1 - \frac{2k_0 \tau_{\text{nj}}^i}{1 + k_0 \tau_{\text{nj}}^i} \cdot \frac{1 + k_0 t_f}{2 + k_0 t_f} \right) = 0, \quad (40)$$

where we used $\tilde{x}_{11} = \frac{\tau}{1+k_0\tau}$. Note that τ_{nj}^i remains unchanged for any f_{j0} , given a fixed f_{i0} . As established earlier, $f_{i0}^* < 0$ in the case $\gamma = 0$ and $\kappa = 0$. Thus, for $f_{i0} < f_{i0}^*$ (Strategy II), the initial jump follows what we refer to as *jump case 2*: $\lambda_i^*(0^+) > 0$ (initial forward jump) when $\tau < \tau_{\text{nj}}^i$ and $\lambda_i^*(0^+) < 0$ (initial backward jump) when $\tau > \tau_{\text{nj}}^i$ [see Eq. (40)]. Conversely, for $f_{i0} > f_{i0}^*$ (Strategy I), the trend is reversed, corresponding to *jump case 1*: $\lambda_i^*(0^+) < 0$ for $\tau < \tau_{\text{nj}}^i$ and $\lambda_i^*(0^+) > 0$ for $\tau > \tau_{\text{nj}}^i$. For $\kappa \neq 0$, one can also define τ_{nj}^i , which now depends on the orthogonal component f_{j0} for a given f_{i0} . However, τ_{nj}^i is not defined for all finite values of f_{j0} ; that is, in certain ranges of f_{j0} , the protocol $\lambda_i^*(t)$ exhibits only either an initial forward or backward jump, irrespective of the value of τ . From Eq. (39), one can qualitatively infer the no-jump regions in the (f_{10}, f_{20}) space (for details, see Fig. S9 in the Supplemental Material [76]), which differ for $\lambda_1^*(t)$ and $\lambda_2^*(t)$. For Strategy I ($f_{i0} > f_{i0}^*$), the no-jump region corresponds to

$\Delta\lambda_i(0) < 0$, occurring when $f_{20} \gg 0$ for $\lambda_1^*(t)$ and $f_{10} \ll 0$ for $\lambda_2^*(t)$. For Strategy II ($f_{i0} < f_{i0}^*$), $\Delta\lambda_i(0) > 0$ and the no-jump regions occur in the opposite limits [i.e., $f_{20} \ll 0$ for $\lambda_1^*(t)$ and $f_{10} \gg 0$ for $\lambda_2^*(t)$]. In parameter regimes where τ_{nj}^i is well defined, the two distinct jump patterns (*jump case 1* and *jump case 2*) discussed earlier can again be recognized for both protocols. A schematic illustration of the protocol strategies corresponding to different combinations of (f_{10}, f_{20}) is presented in Fig. 5 (see caption), where four distinct regions—11, 22, 21, and 12—are identified based on the above analysis.

The optimal mean work in the closed-loop setting is given by

$$\omega^*(t_f) = \Delta\tilde{V} + (c_1^2 + c_2^2)t_f - \frac{\tau}{8}(f_{10}^2 + f_{20}^2)(1 - e^{-2t_f/\tau}), \quad (41)$$

where this expression is obtained by substituting Eqs. (B4) and (B5) into Eq. (C23). For $\gamma = 0$, the coefficients c_1 and c_2 are given by

$$c_1(t) = \frac{k_0}{2 + k_0 t_f} \left[L_1 - f_{10} \left(\tilde{x}_{11} + \frac{\tau}{2} \rho(t) \right) - \frac{\kappa \tilde{x}_{11}}{1 + k_0 \tau} f_{20} \right], \quad (42)$$

$$c_2(t) = \frac{k_0}{2 + k_0 t_f} \left[L_2 - f_{20} \left(\tilde{x}_{11} + \frac{\tau}{2} \rho(t) \right) + \frac{\kappa \tilde{x}_{11}}{1 + k_0 \tau} f_{10} \right] \quad (43)$$

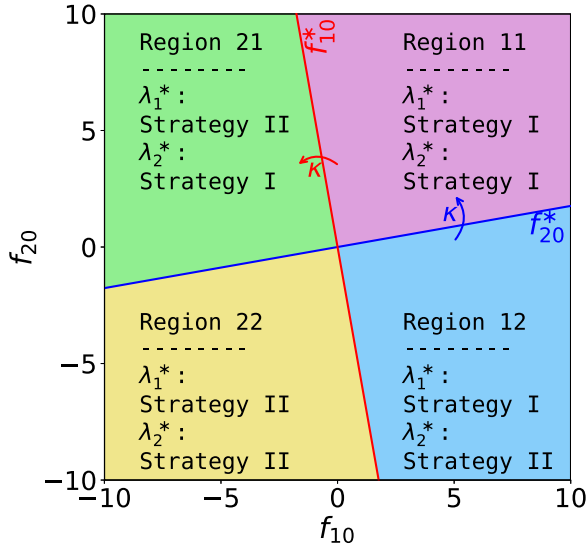


FIG. 5. Schematic illustration in the f_{10} - f_{20} plane showing the protocol strategies adopted by the optimal closed-loop protocols (λ_1^* , λ_2^*). Each protocol λ_i^* adopts either Strategy I or Strategy II depending on whether f_{i0} lies above or below the corresponding threshold f_{i0}^* , resulting in four distinct regions, which are labeled and shaded with different colors in the plot. The bold lines, slightly slanted along the vertical and horizontal directions, represent the threshold values f_{10}^* and f_{20}^* , respectively, which demarcate the strategic regions. Curved arrows indicate how these thresholds shift with increasing κ .

[see also Eqs. (C13) and (C14) for the more general expressions]. Unlike the open-loop case, $\omega^*(t_f)$ in the closed-loop setup exhibits an explicit dependence on κ as well as f_{10} and f_{20} . Here, we focus on the regime $\gamma \geq 0$, where work extraction is not possible under open-loop control (see Fig. 6). For a more detailed analysis of the parametric dependence, refer to Figs. S10–S12 in the Supplemental Material [76].

Figures 6 and 7 demonstrate that for various combinations of initial forces (f_{10} , f_{20}), the mean optimal work $\omega^*(t_f)$ exhibits a nonmonotonic dependence on the persistence time τ . Specifically, $\omega^*(t_f)$ attains a minimum value $\omega_{\min}^*$ at a persistence time τ^* , beyond which it increases rapidly. For the perspicuous case $\kappa = 0$ with $f_{10}, f_{20} \ll 0$, the minimum occurs at $\tau^* = 0$, where $\omega_{\min}^* > 0$. As κ increases, τ^* shifts from zero to a finite value; see Fig. 7 (also Fig. S13 in the Supplemental Material [76] for further details). In contrast, for $f_{10}, f_{20} \gg 0$, the optimal work becomes negative, i.e., $\omega_{\min}^* < 0$, as is also evident from Eq. (41). Negative work input is further observed in mixed configurations where $f_{i0} > 0$ and $f_{j0} < 0$ ($i \neq j$); see Fig. 6 and Fig. S12 in the Supplemental Material [76]. These observations suggest that when the initial forces in one or both directions are aligned toward the target, net work can be extracted. In particular, if both f_{10} and f_{20} are aligned in that direction, the extracted work is maximized for $\kappa = 0$, and the magnitude $|\omega_{\min}^*|$ decreases with increasing κ , reflecting reduced output; see panel (a) of Fig. 6. Thus, for fixed values of κ and τ , the extractable work follows the trend $\omega^*(f_{10} \gg 0, f_{20} \gg 0) > \omega^*(f_{10} \gg 0, f_{20} \ll 0) \approx \omega^*(f_{10} \ll 0, f_{20} \gg 0) > \omega^*(f_{10} \ll 0, f_{20} \ll 0)$.

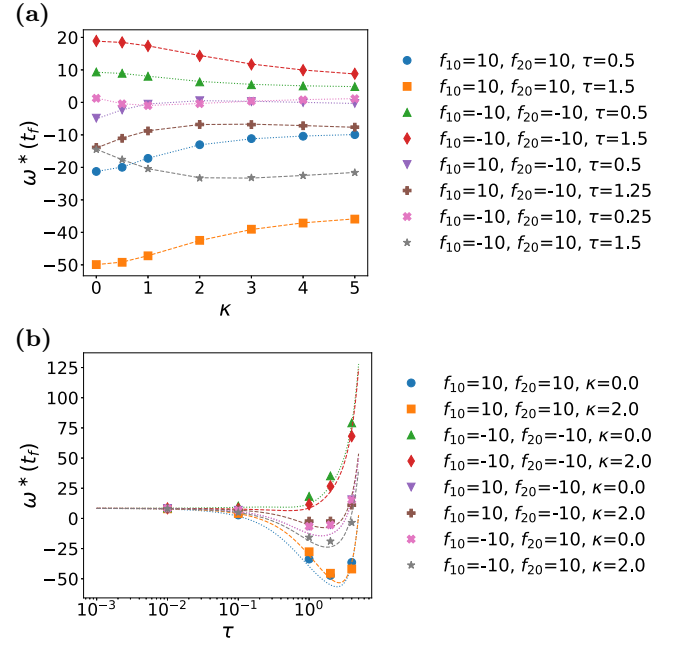


FIG. 6. Dependence of the optimal average work $\omega^*(t_f)$ [Eq. (41)] on (a) the oddness parameter κ and (b) the persistence time τ , for various combinations of initial forces (f_{10} , f_{20}). Markers denote simulation data, while lines indicate analytical predictions. Panel (b) shows a log-linear plot to emphasize trends at small and intermediate values of τ . All simulations use the following parameters: $\gamma = 0.1$, $k_0 = 1$, $L_1 = L_2 = 5$, and $t_f = 5$.

The cumulative work under the optimal protocol, $\omega^*(t)$, evaluated as a function of time for various initial force configurations (f_{10} , f_{20}), is shown in Fig. 8. While the initial and final jumps contribute significantly to the total energetic cost in all cases, the intermediate-time evolution of $\omega^*(t)$ depends sensitively on the specific protocol strategy determined by the initial force pair. For $\kappa = 0$, different initial force configurations correspond to distinct regions in the f_{10} - f_{20} plane as indicated in Fig. 5: $(f_{10}, f_{20}) \gg 0$ lies in region 11 (both protocols follow Strategy I), $(f_{10} \gg 0, f_{20} \ll 0)$ in region 12, $(f_{10} \ll 0, f_{20} \gg 0)$ in region 21, and $(f_{10}, f_{20}) \ll 0$ in region 22 (both protocols follow Strategy II). In region 22, $\omega^*(t)$

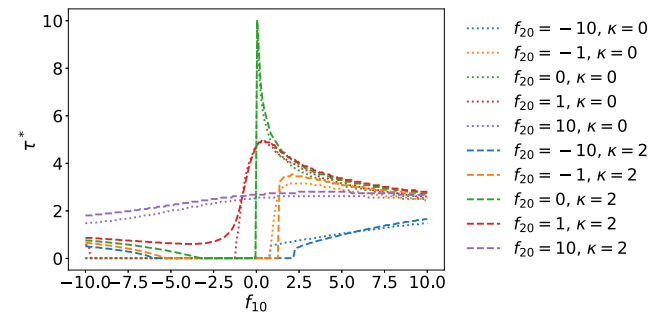


FIG. 7. Plot of the optimal relaxation time τ^* as a function of f_{10} for fixed values of f_{20} . Dotted lines correspond to $\kappa = 0$, while dashed lines represent $\kappa = 2$. The optimal values τ^* minimize the total work $\omega^*(\tau)$ in each case and are obtained from analytical result given by Eq. (41). Other parameters are the same as in Fig. 6.

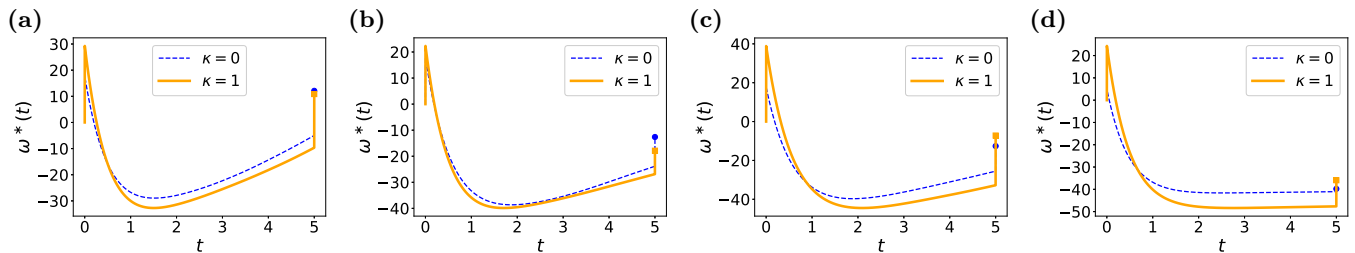


FIG. 8. Time evolution of the cumulative work $\omega^*(t)$ [Eq. (20)] under optimal closed-loop protocols for different combinations of initial active forces (f_{10}, f_{20}) at $\tau = 1$. Panel (a) shows results for $f_{10} = -10$, $f_{20} = -10$; panel (b) for $f_{10} = -10$, $f_{20} = 10$; panel (c) for $f_{10} = 10$, $f_{20} = -10$; and panel (d) for $f_{10} = 10$, $f_{20} = 10$. Panels (a)–(d) correspond, respectively, to protocols belonging to regions 22, 21, 12, and 11. Other simulation parameters are the same as in Fig. 6.

initially decreases and reaches a minimum on a timescale of order τ , followed by a steep rise that leads to a positive final work, $\omega(t_f) > 0$ [panel (a), Fig. 8]. In contrast, for regions 12 and 21, the postminimum growth of $\omega^*(t)$ is more gradual, resulting in net negative work output, $\omega(t_f) < 0$ [panels (b) and (c)]. In region 11, $\omega^*(t)$ decays nearly monotonically throughout the protocol, yielding the largest negative final work, $\omega(t_f) < 0$ [panel (d)]. These trends establish a hierarchy in the magnitude of extractable work under optimal protocols, approximately ordered as region 11 $>$ region 12 or 21 $>$ region 22. With increasing κ , the protocol strategies adopted in the (f_{10}, f_{20}) plane shift, as schematically indicated in Fig. 5 and further quantitatively supported by the parametric variations shown in Fig. S12 of the Supplemental Material [76]. Specifically, the following transitions occur: $(f_{10}, f_{20}) \gg 0$ moves from region 11 to region 12, $(f_{10} \gg 0, f_{20} \ll 0)$ from region 12 to region 22, $(f_{10} \ll 0, f_{20} \gg 0)$ from region 21 to region 11, and $(f_{10}, f_{20}) \ll 0$ from region 22 to region 21. As a result, for configurations such as $(f_{10}, f_{20}) \gg 0$ or $(f_{10} \gg 0, f_{20} \ll 0)$, the extractable work decreases with increasing κ . In contrast, force configurations like $(f_{10} \ll 0, f_{20} \gg 0)$ and $(f_{10}, f_{20}) \ll 0$ transition into more favorable strategic regimes as κ increases, which may enhance the amount of extractable work or reduce the overall energy input required [see again panel (a) of Fig. 6].

V. CONCLUSION

In this work, we investigated the optimal conditions for minimizing thermodynamic work in driving an odd tracer in an active bath. Due to the asymmetry in mobility along different directions—quantified by the oddness parameter κ —we imposed two independent protocols along each spatial direction for a tracer confined in a rotationally asymmetric parabolic potential. Two control strategies were considered: an open-loop approach, where the initial active force is not measured, and a closed-loop approach, where the optimal protocol is designed based on a measurement of the initial active force or velocity.

In the open-loop case, the mean optimal work is unaffected by the active noise statistics and coincides with that of a passive Brownian tracer. However, unlike the passive case, the optimal protocols for an odd tracer explicitly depend on κ , leading to discontinuous jumps at the initial and final times. Despite the presence of asymmetric jumps at these times, we demonstrate that the overall protocol structure retains

time-reversal symmetry in a generalized sense. While such symmetry has been established for normal tracers in nonequilibrium environments [20], we extend this result to the case of an odd tracer. Notably, while the optimal protocols themselves are κ dependent, the resulting mean optimal work remains invariant with respect to κ . However, introducing rotational asymmetry in the potential structure can reduce the mean work and, in certain parameter regimes, even facilitate work extraction.

In contrast, the closed-loop scenario exhibits significantly richer behavior. Here, activity strongly influences both the protocol design and the optimal work, with the protocols predominantly adopting either Strategy I or Strategy II. Remarkably, even in a symmetric potential, work extraction becomes possible by appropriately tuning the initial active forces and κ . When the initial active forces are aligned with the direction of the target displacement (i.e., both protocols follow Strategy I), optimal work extraction is achieved at an intermediate persistence time τ^* ; persistence times larger than τ^* are generally less favorable. Similar trends were reported in recent studies [42,44], where only the effect of active fluctuations on a normal (even) tracer was considered. Here, in addition, we incorporate the influence of odd mobility, thereby extending and generalizing those results. Interestingly, in this regime, normal tracers ($\kappa = 0$) can outperform odd tracers in minimizing work. However, when the initial active forces in the two directions are oriented oppositely—one aligned with and the other against the target displacement (that is, one protocol follows Strategy I and the other Strategy II)—odd tracers outperform normal ones in terms of extractable work within certain ranges of τ . In situations where both components of the initial active force are misaligned with the target displacement (i.e., both protocols follow Strategy II), work extraction becomes unfavorable. Nonetheless, the presence of odd mobility in such configurations can still mitigate the overall energetic cost, effectively reducing the required input work relative to the normal tracer case.

Overall, our findings contribute to the growing body of research on optimal control in active systems [36,41–44]. They reveal how odd mobility and directional asymmetry can be harnessed to extract work from active environments. More broadly, our work highlights the intricate interplay between time-reversal asymmetry, nonreciprocal couplings, and optimal protocol design in systems with multiple interacting degrees of freedom. The theoretical predictions presented here are experimentally accessible: for instance, one could

realize an odd tracer as a charged Janus particle trapped by optical tweezers under an external magnetic field. Similarly, chiral active particles, often referred to as circular swimmers, naturally exhibit odd diffusion [77]. Such systems arise in biological contexts such as motile *Escherichia coli* near solid interfaces [78] or can be artificially engineered, for example, using asymmetric colloidal disks or rotors equipped with anisotropic arms [79]. Alternatively, the effect could be probed by immersing a passive colloid in a bath of chiral swimming bacteria [80]. Together, these scenarios provide feasible routes to test our predictions and establish a foundation for exploring optimal thermodynamic control in chiral and nonreciprocal systems. From a technological standpoint, understanding and optimizing negative work at the mesoscale may inform the design of autonomous soft robots, nanoscale engines, and synthetic swimmers capable of harnessing ambient fluctuations for self-sustained operation in complex media.

ACKNOWLEDGMENTS

K.G. gratefully acknowledges support from the National Science and Technology Council, the Ministry of Education (Higher Education Sprout Project No. NTU-113L104022), and the National Center for Theoretical Sciences, Taiwan. This work has been supported by the National Science and Technology Council of Taiwan under Grants No. 111-

2112-M-001-027-MY3 (H.-Y.S.), No. 114-2112-M-001-062 (H.-Y.S.), No. 114-2112-M-008-015 (H.-Y.C.), No. 113-2112-M-A49-019 (C.-H.C.), and No. 113-2112-M008-018-MY2 (P.-Y.L.), and Academia Sinica Career Development Award under No. AS-CDA-114-M02 (H.-Y.S.). We thank the anonymous referees for their constructive comments and for suggesting additional references on chiral active particles, which substantially improved our manuscript.

DATA AVAILABILITY

The data that support the findings of this article are openly available [81]. The simulation codes used for analysis are available from the corresponding authors upon reasonable request.

APPENDIX A: AVERAGE DYNAMICAL VARIABLES

From the active force dynamics [see Eq. (10)], and for an initial condition $a = \{x_0, f_0\}$, the average active force evolves as

$$f_i^a(t) = f_{i0} e^{-t/\tau}. \quad (\text{A1})$$

Taking the Laplace transform of Eqs. (14) and (15), we obtain the following algebraic equations in Laplace space:

$$s\tilde{x}_1^a(s) - x_1^a(0) = -\alpha_k [\tilde{x}_1^a(s) - \tilde{\lambda}_1(s) + \gamma\tilde{x}_2^a(s) + \kappa\tilde{x}_2^a(s) - \kappa\tilde{\lambda}_2(s) + \kappa\gamma\tilde{x}_1^a(s)] + \alpha [\tilde{f}_1^a(s) + \kappa\tilde{f}_2^a(s)], \quad (\text{A2})$$

$$s\tilde{x}_2^a(s) - x_2^a(0) = \alpha_k [\kappa\tilde{x}_1^a(s) - \kappa\tilde{\lambda}_1(s) + \kappa\gamma\tilde{x}_2^a(s) - \tilde{x}_2^a(s) + \tilde{\lambda}_2(s) - \gamma\tilde{x}_1^a(s)] + \alpha [-\kappa\tilde{f}_1^a(s) + \tilde{f}_2^a(s)]. \quad (\text{A3})$$

The solutions to these coupled algebraic equations in the Laplace domain are

$$\begin{aligned} \tilde{x}_1^a(s) &= \frac{1}{\beta_0} [\beta_{22}(x_{10} + \alpha_k\tilde{\lambda}_1(s) + \alpha_k\kappa\tilde{\lambda}_2(s) + \alpha\tilde{f}_1^a(s) + \alpha\kappa\tilde{f}_2^a(s)) - \beta_{12}(x_{20} - \alpha_k\kappa\tilde{\lambda}_1(s) + \alpha_k\tilde{\lambda}_2(s) - \alpha\kappa\tilde{f}_1^a(s) + \alpha\tilde{f}_2^a(s))] \\ &= \frac{1}{\beta_0} [\beta_{22}x_{10} - \beta_{12}x_{20} + \alpha_k(s + k_0)\tilde{\lambda}_1(s) + \alpha_k(s\kappa - \gamma k_0)\tilde{\lambda}_2(s) + \alpha(s + k_0)\tilde{f}_1^a(s) + \alpha(s\kappa - \gamma k_0)\tilde{f}_2^a(s)] \end{aligned} \quad (\text{A4})$$

and

$$\begin{aligned} \tilde{x}_2^a(s) &= \frac{1}{\beta_0} [\beta_{11}(x_{20} - \alpha_k\kappa\tilde{\lambda}_1(s) + \alpha_k\tilde{\lambda}_2(s) - \alpha\kappa\tilde{f}_1^a(s) + \alpha\tilde{f}_2^a(s)) - \beta_{21}(x_{10} + \alpha_k\tilde{\lambda}_1(s) + \alpha_k\kappa\tilde{\lambda}_2(s) + \alpha\tilde{f}_1^a(s) + \alpha\kappa\tilde{f}_2^a(s))] \\ &= \frac{1}{\beta_0} [-\beta_{21}x_{10} + \beta_{11}x_{20} - \alpha_k(s\kappa + \gamma k_0)\tilde{\lambda}_1(s) + \alpha_k(s + k_0)\tilde{\lambda}_2(s) - \alpha(s\kappa + \gamma k_0)\tilde{f}_1^a(s) + \alpha(s + k_0)\tilde{f}_2^a(s)], \end{aligned} \quad (\text{A5})$$

where

$$\begin{aligned} \beta_{11} &= s + \alpha_k + \alpha_k\kappa\gamma, & \beta_{12} &= \alpha_k\gamma + \alpha_k\kappa, & \beta_{21} &= -\alpha_k\kappa + \alpha_k\gamma, & \beta_{22} &= s - \alpha_k\kappa\gamma + \alpha_k, \\ \beta_0 &= \beta_{11}\beta_{22} - \beta_{12}\beta_{21} = s^2 + 2s\alpha_k + \alpha_k k_0(1 - \gamma^2). \end{aligned} \quad (\text{A6})$$

After performing the inverse Laplace transform, we obtain

$$\begin{aligned} x_1^a(t) &= x_{10} \exp(-\alpha_k t) \cosh(\alpha_s t) - \frac{\exp(-\alpha_k t)}{\alpha_s} \alpha_k [x_{10}\kappa\gamma \sinh(\alpha_s t) + x_{20}(\gamma + \kappa) \sinh(\alpha_s t)] \\ &\quad + \alpha \int_0^t dt_1 (k_0\lambda_1(t_1) + f_1^a(t_1)) \exp(-\alpha_k(t - t_1)) \left[\cosh(\alpha_s(t - t_1)) + \frac{k_0 - \alpha_k}{\alpha_s} \sinh(\alpha_s(t - t_1)) \right] \\ &\quad + \alpha \int_0^t dt_1 (k_0\lambda_2(t_1) + f_2^a(t_1)) \exp(-\alpha_k(t - t_1)) \left[\kappa \cosh(\alpha_s(t - t_1)) - \frac{\gamma k_0 + \kappa\alpha_k}{\alpha_s} \sinh(\alpha_s(t - t_1)) \right] \end{aligned} \quad (\text{A7})$$

and

$$x_2^a(t) = x_{20} \exp(-\alpha_k t) \cosh(\alpha_s t) + \frac{\exp(-\alpha_k t)}{\alpha_s} \alpha_k [x_{20} \kappa \gamma \sinh(\alpha_s t) + x_{10}(\kappa - \gamma) \sinh(\alpha_s t)] \\ + \alpha \int_0^t dt_1 (k_0 \lambda_2(t_1) + f_2^a(t_1)) \exp(-\alpha_k(t-t_1)) \left[\cosh(\alpha_s(t-t_1)) + \frac{k_0 - \alpha_k}{\alpha_s} \sinh(\alpha_s(t-t_1)) \right] \\ - \alpha \int_0^t dt_1 (k_0 \lambda_1(t_1) + f_1^a(t_1)) \exp(-\alpha_k(t-t_1)) \left[\kappa \cosh(\alpha_s(t-t_1)) + \frac{\gamma k_0 - \kappa \alpha_k}{\alpha_s} \sinh(\alpha_s(t-t_1)) \right], \quad (\text{A8})$$

where

$$\alpha_s = \alpha_k \sqrt{1 - \frac{k_0}{\alpha_k} (1 - \gamma^2)}. \quad (\text{A9})$$

From Eqs. (A7) and (A8), one can compute the following correlations between the average position and active force components in the steady state:

$$\langle x_1^a(t) f_1^a(t) \rangle_{ss} = \alpha \int_0^t dt_1 \langle f_1^a(t) f_1^a(t_1) \rangle \exp(-\alpha_k(t-t_1)) \left[\cosh(\alpha_s(t-t_1)) + \frac{k_0 - \alpha_k}{\alpha_s} \sinh(\alpha_s(t-t_1)) \right] \\ = D_A \frac{1 + k_0 \tau}{(1 - \gamma^2) k_0^2 \tau^2 + 2k_0 \tau + \alpha^{-1}} = f_{av}^2 \tilde{x}_{11}, \quad (\text{A10})$$

$$\langle x_1^a(t) f_2^a(t) \rangle_{ss} = \alpha \int_0^t dt_1 \langle f_2^a(t) f_2^a(t_1) \rangle \exp(-\alpha_k(t-t_1)) \left[\kappa \cosh(\alpha_s(t-t_1)) - \frac{\gamma k_0 + \kappa \alpha_k}{\alpha_s} \sinh(\alpha_s(t-t_1)) \right] \\ = D_A \frac{\kappa - \gamma k_0 \tau}{(1 - \gamma^2) k_0^2 \tau^2 + 2k_0 \tau + \alpha^{-1}} = f_{av}^2 \tilde{x}_{12}, \quad (\text{A11})$$

$$\langle x_2^a(t) f_1^a(t) \rangle_{ss} = -\alpha \int_0^t dt_1 \langle f_1^a(t) f_1^a(t_1) \rangle \exp(-\alpha_k(t-t_1)) \left[\kappa \cosh(\alpha_s(t-t_1)) + \frac{\gamma k_0 - \kappa \alpha_k}{\alpha_s} \sinh(\alpha_s(t-t_1)) \right] \\ = -D_A \frac{\kappa + \gamma k_0 \tau}{(1 - \gamma^2) k_0^2 \tau^2 + 2k_0 \tau + \alpha^{-1}} = f_{av}^2 \tilde{x}_{21}, \quad (\text{A12})$$

$$\langle x_2^a(t) f_2^a(t) \rangle_{ss} = \alpha \int_0^t dt_1 \langle f_2^a(t) f_2^a(t_1) \rangle \exp(-\alpha_k(t-t_1)) \left[\cosh(\alpha_s(t-t_1)) + \frac{k_0 - \alpha_k}{\alpha_s} \sinh(\alpha_s(t-t_1)) \right] \\ = D_A \frac{1 + k_0 \tau}{(1 - \gamma^2) k_0^2 \tau^2 + 2k_0 \tau + \alpha^{-1}} = f_{av}^2 \tilde{x}_{22}. \quad (\text{A13})$$

Here,

$$\tilde{x}_{11} = \tilde{x}_{22} = \frac{\tau(1 + k_0 \tau)}{(1 - \gamma^2) k_0^2 \tau^2 + 2k_0 \tau + 1 + \kappa^2}, \quad \tilde{x}_{12} = \frac{\kappa - \gamma k_0 \tau}{1 + k_0 \tau} \tilde{x}_{11}, \quad \tilde{x}_{21} = \frac{-\kappa - \gamma k_0 \tau}{1 + k_0 \tau} \tilde{x}_{11}. \quad (\text{A14})$$

These results allow one to construct the steady-state cross-covariance matrix between the average positions and active forces:

$$\Sigma_{xf} = \begin{bmatrix} \langle x_1^a f_1^a \rangle_{ss} & \langle x_1^a f_2^a \rangle_{ss} \\ \langle x_2^a f_1^a \rangle_{ss} & \langle x_2^a f_2^a \rangle_{ss} \end{bmatrix}. \quad (\text{A15})$$

Similarly, using Eq. (11), the covariance matrix of the active forces at steady state is given by

$$\Sigma_{ff} = \begin{bmatrix} \frac{D_A}{\tau} & 0 \\ 0 & \frac{D_A}{\tau} \end{bmatrix}. \quad (\text{A16})$$

If the initial force is measured as $\mathbf{f}_0 = (f_{10}, f_{20})$, then the corresponding average initial position is automatically determined as [82]

$$\mathbf{x}_0^\top = \Sigma_{xf} \Sigma_{ff}^{-1} \mathbf{f}_0^\top. \quad (\text{A17})$$

This gives explicit expressions for the initial conditions:

$$x_{10} = \frac{\tau}{D_A} [\langle x_1^a f_1^a \rangle_{ss} f_{10} + \langle x_1^a f_2^a \rangle_{ss} f_{20}] = \tilde{x}_{11} f_{10} + \tilde{x}_{12} f_{20}, \quad (\text{A18})$$

$$x_{20} = \frac{\tau}{D_A} [\langle x_2^a f_1^a \rangle_{ss} f_{10} + \langle x_2^a f_2^a \rangle_{ss} f_{20}] = \tilde{x}_{21} f_{10} + \tilde{x}_{22} f_{20}. \quad (\text{A19})$$

APPENDIX B: OPTIMIZATION OF WORK

From Eq. (18), the mean work can be expressed as a functional of the average trajectories $x_i^a(t')$. Accordingly, we define the Lagrangian as

$$\mathcal{L}(x_i^a, \dot{x}_i^a) = \dot{x}_1^a(t')^2 + \dot{x}_2^a(t')^2 - \dot{x}_1^a(t')f_1^a(t') - \dot{x}_2^a(t')f_2^a(t'). \quad (\text{B1})$$

Applying the Euler-Lagrange equation [83]

$$\frac{d}{dt'} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}_i^a} \right) - \frac{\partial \mathcal{L}}{\partial x_i^a} = 0,$$

we obtain the following equations of motion for the optimal average trajectories:

$$2\ddot{x}_1^a(t') = \dot{f}_1^a(t'), \quad (\text{B2})$$

$$2\ddot{x}_2^a(t') = \dot{f}_2^a(t'). \quad (\text{B3})$$

Using the explicit form of the active force from Eq. (A1), the solutions to the above differential equations are computed as

$$x_1^a(t) = x_{10} + c_1 t + \frac{f_{10}\tau}{2}(1 - e^{-t/\tau}), \quad (\text{B4})$$

$$x_2^a(t) = x_{20} + c_2 t + \frac{f_{20}\tau}{2}(1 - e^{-t/\tau}), \quad (\text{B5})$$

where x_{10} and x_{20} are the initial positions, and c_1 and c_2 are integration constants that must be determined by minimizing the mean work, as given in Eq. (18). To determine the optimal values of c_1 and c_2 , we substitute the solutions (B4) and (B5) into Eq. (18) and perform the optimization of the resulting expression with respect to c_1 and c_2 . This leads to the following conditions:

$$\left. \frac{\partial \omega(t_f)}{\partial c_1} \right|_{c_1=c_1^*} = 0, \quad \left. \frac{\partial \omega(t_f)}{\partial c_2} \right|_{c_2=c_2^*} = 0. \quad (\text{B6})$$

Solving these equations yields the optimal coefficients c_1^* and c_2^* ,

$$c_1^* = k_0 T_{13} L_1 - \gamma k_0^2 t_f T_3^{-1} L_2 - k_0 T_{23} x_{10} - 2\gamma k_0 T_3^{-1} x_{20} - \frac{k_0 \tau}{2} \rho(t) T_{23} f_{10} - k_0 \gamma \tau \rho(t) T_3^{-1} f_{20} \quad (\text{B7})$$

and

$$c_2^* = -\gamma k_0^2 t_f T_3^{-1} L_1 + k_0 T_{13} L_2 - 2\gamma k_0 T_3^{-1} x_{10} - k_0 T_{23} x_{20} - k_0 \gamma \tau \rho(t) T_3^{-1} f_{10} - \frac{k_0 \tau}{2} \rho(t) T_{23} f_{20}, \quad (\text{B8})$$

where

$$\begin{aligned} \rho(t) &= (1 - e^{-t/\tau}), \\ T_1 &= 2 + k_0 t_f, \quad T_2 = 2 + k_0 t_f (1 - \gamma^2), \\ T_3 &= 4 + 4k_0 t_f + k_0^2 t_f^2 (1 - \gamma^2), \\ T_{13} &= T_1/T_3, \quad T_{23} = T_2/T_3. \end{aligned} \quad (\text{B9})$$

For $\gamma = 0$, one obtains $T_{13} = T_{23} = 1/(2 + k_0 t_f)$, and the expressions for the parameters simplify to

$$c_1^* = k_0 T_{13} \left[L_1 - x_{10} - \frac{\tau}{2} \rho(t) f_{10} \right], \quad (\text{B10})$$

$$c_2^* = k_0 T_{13} \left[L_2 - x_{20} - \frac{\tau}{2} \rho(t) f_{20} \right]. \quad (\text{B11})$$

APPENDIX C: OPTIMAL PROTOCOLS AND AVERAGE WORK

Using the optimal coefficients obtained from Eqs. (B7) and (B8) and substituting them into Eqs. (16) and (17), the optimal protocols $\lambda_1^*(t)$ and $\lambda_2^*(t)$ can be explicitly constructed as

$$\begin{aligned} \lambda_1^*(t) &= \frac{c_1^*}{k_0} (1 + k_0 t) - c_2^* \frac{\kappa}{k_0} + x_{10} + f_{10} \frac{\tau}{2} \rho(t) \\ &\quad - \left(\frac{f_{10}}{2k_0} + \frac{f_{20}\kappa}{2k_0} \right) e^{-t/\tau} + \gamma (x_{20} + c_2^* t) + \gamma f_{20} \frac{\tau}{2} \rho(t) \end{aligned} \quad (\text{C1})$$

and

$$\begin{aligned} \lambda_2^*(t) &= c_1^* \frac{\kappa}{k_0} + \frac{c_2^*}{k_0} (1 + k_0 t) + x_{20} + f_{20} \frac{\tau}{2} \rho(t) \\ &\quad + \left(\frac{f_{10}\kappa}{2k_0} - \frac{f_{20}}{2k_0} \right) e^{-t/\tau} + \gamma (x_{10} + c_1^* t) + \gamma f_{10} \frac{\tau}{2} \rho(t). \end{aligned} \quad (\text{C2})$$

(1) *Open-loop control.* Let us first consider the open-loop protocol, for which the initial conditions are set as $\langle f_{10} \rangle = \langle f_{20} \rangle = 0$ and $x_{10} = x_{20} = 0$. Under these assumptions, the optimal coefficients are obtained, by virtue of Eqs. (B7) and (B8), as

$$c_1^o = (c_1^*)_{ss} = k_0 T_{13} L_1 - \gamma k_0^2 t_f T_3^{-1} L_2, \quad (\text{C3})$$

$$c_2^o = (c_2^*)_{ss} = -\gamma k_0^2 t_f T_3^{-1} L_1 + k_0 T_{13} L_2. \quad (\text{C4})$$

As a sanity check, setting $\gamma = 0$ recovers the well-known result for an overdamped Brownian particle in a thermal bath [37]:

$$c_1^o|_{\gamma=0} = k_0 \frac{L_1}{2 + k_0 t_f}, \quad c_2^o|_{\gamma=0} = k_0 \frac{L_2}{2 + k_0 t_f}. \quad (\text{C5})$$

For the open-loop protocol with finite γ , the corresponding optimal control protocols are

$$\lambda_1^*(t) = \frac{c_1^o}{k_0} (1 + k_0 t) + c_2^o \left(\gamma t - \frac{\kappa}{k_0} \right), \quad (\text{C6})$$

$$\lambda_2^*(t) = c_1^o \left(\gamma t + \frac{\kappa}{k_0} \right) + \frac{c_2^o}{k_0} (1 + k_0 t). \quad (\text{C7})$$

Thus, the initial discontinuities (jumps) in the protocols at $t = 0$ are given by

$$\begin{aligned} \Delta \lambda_1^*(0) &= \lambda_1^*(0^+) - \lambda_1^*(0) = \frac{c_1^o}{k_0} - \frac{c_2^o \kappa}{k_0}, \\ \Delta \lambda_2^*(0) &= \lambda_2^*(0^+) - \lambda_2^*(0) = \frac{c_1^o \kappa}{k_0} + \frac{c_2^o}{k_0}. \end{aligned} \quad (\text{C8})$$

Similarly, the discontinuities at the final time $t = t_f$ in the optimal protocols can be computed as

$$\begin{aligned}\Delta\lambda_1^*(t_f) &= \lambda_1^*(t_f) - \lambda_1^*(t_f^-) \\ &= L_1 - \frac{c_1^o}{k_0}(1 + k_0 t_f) - c_2^o \left(\gamma t_f - \frac{\kappa}{k_0} \right), \\ \Delta\lambda_2^*(t_f) &= \lambda_2^*(t_f) - \lambda_2^*(t_f^-) \\ &= L_2 - c_1^o \left(\gamma t_f + \frac{\kappa}{k_0} \right) - \frac{c_2^o}{k_0}(1 + k_0 t_f). \quad (\text{C9})\end{aligned}$$

To examine the time-reversal symmetry of the optimal protocols, one can compute

$$\lambda_1^*(t) + \lambda_1^*(t_f - t) = \lambda_1^*(t_f^-) + \lambda_1^*(0^+). \quad (\text{C10})$$

This relation can be equivalently rewritten as

$$\lambda_1^*(t) + \lambda_1^*(t_f - t) + (\Delta\lambda_1^*(t_f^-) - \Delta\lambda_1^*(0)) = L_1, \quad (\text{C11})$$

and similarly for the second protocol:

$$\lambda_2^*(t) + \lambda_2^*(t_f - t) + (\Delta\lambda_2^*(t_f^-) - \Delta\lambda_2^*(0)) = L_2. \quad (\text{C12})$$

These relations show that time-reversal symmetry of the protocols is preserved when the discontinuities at the initial and final times are explicitly accounted for.

(2) *Closed-loop control.* For closed-loop control, the initial active force $\mathbf{f}_0 = (f_{10}, f_{20})$ is known. By substituting the

expressions from Eq. (A14) into Eqs. (A18) and (A19), the optimal coefficients from Eqs. (B7) and (B8) can be rewritten as

$$c_1^* = c_1^o - c_{11}f_{10} - c_{12}f_{20}, \quad (\text{C13})$$

$$c_2^* = c_2^o - c_{21}f_{10} - c_{22}f_{20}, \quad (\text{C14})$$

where

$$c_{11} = k_0 T_{23} \left[\tilde{x}_{11} + \frac{\tau}{2} \rho(t) \right] + 2\gamma k_0 T_3^{-1} \tilde{x}_{21}, \quad (\text{C15})$$

$$c_{12} = 2k_0 \gamma T_3^{-1} \left[\frac{\rho(t)\tau}{2} + \tilde{x}_{22} \right] + k_0 T_{23} \tilde{x}_{12}, \quad (\text{C16})$$

$$c_{21} = 2k_0 \gamma T_3^{-1} \left[\frac{\rho(t)\tau}{2} + \tilde{x}_{11} \right] + k_0 T_{23} \tilde{x}_{21}, \quad (\text{C17})$$

and

$$c_{22} = k_0 T_{23} \left[\tilde{x}_{22} + \frac{\tau}{2} \rho(t) \right] + 2\gamma k_0 T_3^{-1} \tilde{x}_{12}, \quad (\text{C18})$$

with $\rho(t)$, T_3 , and T_{23} given explicitly in Eq. (B9). Using Eqs. (C13) and (C14) and substituting the expressions from Eq. (A14), the optimal protocols given in Eqs. (C1) and (C2) can be written explicitly as

$$\begin{aligned}\lambda_1^*(t) &= \frac{c_1^o - c_{11}f_{10} - c_{12}f_{20}}{k_0}(1 + k_0 t) - (c_2^o - c_{21}f_{10} - c_{22}f_{20}) \frac{\kappa}{k_0} + \tilde{x}_{11}f_{10} + \tilde{x}_{12}f_{20} + f_{10} \frac{\tau}{2} \rho(t) \\ &\quad - \left(\frac{f_{10}\kappa}{2k_0} + \frac{f_{20}\kappa}{2k_0} \right) e^{-\frac{t}{\tau}} + \gamma (\tilde{x}_{21}f_{10} + \tilde{x}_{22}f_{20} + (c_2^o - c_{21}f_{10} - c_{22}f_{20})t) + \gamma f_{20} \frac{\tau}{2} \rho(t) \\ &= \lambda_1^o(t) + f_{10}\lambda_{11}^*(t) + f_{20}\lambda_{12}^*(t)\end{aligned} \quad (\text{C19})$$

and

$$\begin{aligned}\lambda_2^*(t) &= (c_1^o - c_{11}f_{10} - c_{12}f_{20}) \frac{\kappa}{k_0} + \frac{(c_2^o - c_{21}f_{10} - c_{22}f_{20})}{k_0}(1 + k_0 t) + \tilde{x}_{21}f_{10} + \tilde{x}_{22}f_{20} + f_{20} \frac{\tau}{2} \rho(t) \\ &\quad + \left(\frac{f_{10}\kappa}{2k_0} - \frac{f_{20}\kappa}{2k_0} \right) e^{-\frac{t}{\tau}} + \gamma (\tilde{x}_{11}f_{10} + \tilde{x}_{12}f_{20} + (c_1^o - c_{11}f_{10} - c_{12}f_{20})t) + \gamma f_{10} \frac{\tau}{2} \rho(t) \\ &= \lambda_2^o(t) + f_{10}\lambda_{21}^*(t) + f_{20}\lambda_{22}^*(t).\end{aligned} \quad (\text{C20})$$

The protocols in the open-loop control setting, given in Eqs. (C6) and (C7), are denoted here as

$$\lambda_1^o(t) = \frac{c_1^o}{k_0}(1 + k_0 t) - c_2^o \frac{\kappa}{k_0} + \gamma c_2^o t, \quad \lambda_2^o(t) = \frac{c_2^o}{k_0}(1 + k_0 t) + c_1^o \frac{\kappa}{k_0} + \gamma c_1^o t. \quad (\text{C21})$$

In addition, we define the following auxiliary terms associated with the activity-dependent components of the optimal protocols:

$$\begin{aligned}\lambda_{11}^*(t) &= c_{21} \frac{\kappa}{k_0} - \frac{c_{11}}{k_0}(1 + k_0 t) + \tilde{x}_{11} + \frac{\tau}{2} \rho(t) - \frac{e^{-t/\tau}}{2k_0} + \gamma \tilde{x}_{21} - \gamma c_{21}t, \\ \lambda_{12}^*(t) &= c_{22} \frac{\kappa}{k_0} - \frac{c_{12}}{k_0}(1 + k_0 t) + \tilde{x}_{12} - \frac{\kappa e^{-t/\tau}}{2k_0} + \gamma \tilde{x}_{22} - \gamma c_{22}t + \gamma \frac{\tau}{2} \rho(t), \\ \lambda_{21}^*(t) &= -c_{11} \frac{\kappa}{k_0} - \frac{c_{21}}{k_0}(1 + k_0 t) + \tilde{x}_{21} + \gamma \frac{\tau}{2} \rho(t) + \frac{\kappa e^{-t/\tau}}{2k_0} + \gamma \tilde{x}_{11} - \gamma c_{11}t, \\ \lambda_{22}^*(t) &= -c_{12} \frac{\kappa}{k_0} - \frac{c_{22}}{k_0}(1 + k_0 t) + \tilde{x}_{22} - \frac{e^{-t/\tau}}{2k_0} + \gamma \tilde{x}_{12} - \gamma c_{12}t + \gamma \frac{\tau}{2} \rho(t).\end{aligned} \quad (\text{C22})$$

The average work for a given initial condition a is computed using Eq. (18), where the average optimal trajectories are obtained from Eqs. (B4) and (B5), and the average active force is given by Eq. (A1). Upon substituting these into the expression for work, we obtain

$$\omega^*(t_f) = \Delta\tilde{V} + c_1^2 t_f + c_2^2 t_f - \frac{f_{10}^2 \tau}{8} (1 - e^{-\frac{2t_f}{\tau}}) - \frac{f_{20}^2 \tau}{8} (1 - e^{-\frac{2t_f}{\tau}}), \quad (\text{C23})$$

where c_1 and c_2 are given by Eqs. (C13) and (C14), and the superscript “*” on c_1 and c_2 has been omitted for notational simplicity. The expression in Eq. (41) is valid as it stands for the average work in the closed-loop protocol, where the initial active force is fixed at a definite value. For the open-loop protocol, however, the initial forces are fixed to $\langle f_{10} \rangle = \langle f_{20} \rangle = 0$. This modifies the optimal trajectories from Eqs. (B4) and (B5) to $x_1^a(t) = c_1 t$, $x_2^a(t) = c_2 t$, and correspondingly, the average work simplifies to

$$\omega^*(t_f) = \Delta\tilde{V} + c_1^2 t_f + c_2^2 t_f. \quad (\text{C24})$$

-
- [1] F. L. Curzon and B. Ahlborn, Efficiency of a carnot engine at maximum power output, *Am. J. Phys.* **43**, 22 (1975).
 - [2] L. Chen, C. Wu, and F. Sun, Finite time thermodynamic optimization or entropy generation minimization of energy systems, *J. Non-Equilib. Thermodyn.* **24**, 327 (1999).
 - [3] M. Esposito, R. Kawai, K. Lindenberg, and C. Van den Broeck, Efficiency at maximum power of low-dissipation carnot engines, *Phys. Rev. Lett.* **105**, 150603 (2010).
 - [4] H. Wang and M. Pumera, Micro/nanomachines and living biosystems: From simple interactions to microcyborgs, *Adv. Funct. Mater.* **28**, 1705421 (2018).
 - [5] B. Wang, Y. Zhang, and L. Zhang, Recent progress on micro- and nano-robots: Towards *in vivo* tracking and localization, *Quant. Imaging Med. Surg.* **8**, 461 (2018).
 - [6] U. Seifert, Stochastic thermodynamics, fluctuation theorems and molecular machines, *Rep. Prog. Phys.* **75**, 126001 (2012).
 - [7] K. Goswami, Heat fluctuation of a harmonically trapped particle in an active bath, *Phys. Rev. E* **99**, 012112 (2019).
 - [8] V. Holubec and R. Marathe, Underdamped active Brownian heat engine, *Phys. Rev. E* **102**, 060101(R) (2020).
 - [9] K. Goswami, Work fluctuations in a generalized Gaussian active bath, *Physica A* **566**, 125609 (2021).
 - [10] L. Cocconi and L. Chen, Efficiency of an autonomous, dynamic information engine operating on a single active particle, *Phys. Rev. E* **110**, 014602 (2024).
 - [11] I. Derényi and R. D. Astumian, Efficiency of Brownian heat engines, *Phys. Rev. E* **59**, R6219 (1999).
 - [12] I. Derényi, M. Bier, and R. D. Astumian, Generalized efficiency and its application to microscopic engines, *Phys. Rev. Lett.* **83**, 903 (1999).
 - [13] G. E. Crooks, Measuring thermodynamic length, *Phys. Rev. Lett.* **99**, 100602 (2007).
 - [14] S. Toyabe, T. Watanabe-Nakayama, T. Okamoto, S. Kudo, and E. Muneyuki, Thermodynamic efficiency and mechanochemical coupling of fl-atpase, *Proc. Natl. Acad. Sci. USA* **108**, 17951 (2011).
 - [15] Y. Jun and P.-Y. Lai, Minimal dissipation protocols of an instantaneous equilibrium Brownian particle under time-dependent temperature and potential variations, *Phys. Rev. Res.* **4**, 023157 (2022).
 - [16] T. Schmiedl and U. Seifert, Efficiency at maximum power: An analytically solvable model for stochastic heat engines, *Europhys. Lett.* **81**, 20003 (2008).
 - [17] V. Blickle and C. Bechinger, Realization of a micrometre-sized stochastic heat engine, *Nat. Phys.* **8**, 143 (2012).
 - [18] I. A. Martínez, É. Roldán, L. Dinis, D. Petrov, J. M. R. Parrondo, and R. A. Rica, Brownian carnot engine, *Nat. Phys.* **12**, 67 (2016).
 - [19] V. Holubec and A. Ryabov, Cycling tames power fluctuations near optimum efficiency, *Phys. Rev. Lett.* **121**, 120601 (2018).
 - [20] S. A. M. Loos, S. Monter, F. Ginot, and C. Bechinger, Universal symmetry of optimal control at the microscale, *Phys. Rev. X* **14**, 021032 (2024).
 - [21] J. A. C. Albay, S. R. Wulaningrum, C. Kwon, P.-Y. Lai, and Y. Jun, Thermodynamic cost of a shortcuts-to-isothermal transport of a Brownian particle, *Phys. Rev. Res.* **1**, 033122 (2019).
 - [22] J. A. C. Albay, C. Kwon, P.-Y. Lai, and Y. Jun, Work relation in instantaneous-equilibrium transition of forward and reverse processes, *New J. Phys.* **22**, 123049 (2020).
 - [23] J. A. C. Albay, P.-Y. Lai, and Y. Jun, Realization of finite-rate isothermal compression and expansion using optical feedback trap, *Appl. Phys. Lett.* **116**, 103706 (2020).
 - [24] Y. Jun and P.-Y. Lai, Instantaneous equilibrium transition for Brownian systems under time-dependent temperature and potential variations: Reversibility, heat and work relations, and fast isentropic process, *Phys. Rev. Res.* **3**, 033130 (2021).
 - [25] A. Gomez-Marín, T. Schmiedl, and U. Seifert, Optimal protocols for minimal work processes in underdamped stochastic thermodynamics, *J. Chem. Phys.* **129**, 024114 (2008).
 - [26] E. Aurell, C. Mejía-Monasterio, and P. Muratore-Ginanneschi, Optimal protocols and optimal transport in stochastic thermodynamics, *Phys. Rev. Lett.* **106**, 250601 (2011).
 - [27] S. Blaber, M. D. Louwerse, and D. A. Sivak, Steps minimize dissipation in rapidly driven stochastic systems, *Phys. Rev. E* **104**, L022101 (2021).
 - [28] D. A. Sivak and G. E. Crooks, Thermodynamic metrics and optimal paths, *Phys. Rev. Lett.* **108**, 190602 (2012).
 - [29] S. Blaber and D. A. Sivak, Optimal control in stochastic thermodynamics, *J. Phys. Commun.* **7**, 033001 (2023).
 - [30] M. C. Engel, J. A. Smith, and M. P. Brenner, Optimal control of nonequilibrium systems through automatic differentiation, *Phys. Rev. X* **13**, 041032 (2023).
 - [31] C. A. Plata, D. Guéry-Odelin, E. Trizac, and A. Prados, Optimal work in a harmonic trap with bounded stiffness, *Phys. Rev. E* **99**, 012140 (2019).
 - [32] S. Blaber and D. A. Sivak, Optimal control with a strong harmonic trap, *Phys. Rev. E* **106**, L022103 (2022).
 - [33] A. Zhong and M. R. DeWeese, Limited-control optimal protocols arbitrarily far from equilibrium, *Phys. Rev. E* **106**, 044135 (2022).

- [34] P. Abiuso, V. Holubec, J. Anders, Z. Ye, F. Cerisola, and M. Perarnau-Llobet, Thermodynamics and optimal protocols of multidimensional quadratic Brownian systems, *J. Phys. Commun.* **6**, 063001 (2022).
- [35] L. Cocconi, J. Knight, and C. Roberts, Optimal power extraction from active particles with hidden states, *Phys. Rev. Lett.* **131**, 188301 (2023).
- [36] L. K. Davis, K. Proesmans, and É. Fodor, Active matter under control: Insights from response theory, *Phys. Rev. X* **14**, 011012 (2024).
- [37] T. Schmiedl and U. Seifert, Optimal finite-time processes in stochastic thermodynamics, *Phys. Rev. Lett.* **98**, 108301 (2007).
- [38] J. A. C. Albay, Z.-Y. Zhou, C.-H. Chang, H.-Y. Chen, J. S. Lee, C.-H. Chang, and Y. Jun, Colloidal heat engine driven by engineered active noise, *Phys. Rev. Res.* **5**, 033208 (2023).
- [39] Y. Wen, Z. Li, H. Wang, J. Zheng, J. Tang, P.-Y. Lai, X. Xu, and P. Tong, Activity-assisted barrier crossing of self-propelled colloids over parallel microgrooves, *Phys. Rev. E* **107**, L032601 (2023).
- [40] Y. Wen, Z. Li, H. Wang, J. Zheng, J. Tang, X. Xu, P.-Y. Lai, and P. Tong, Ratchet effect of self-propelled colloids in an asymmetric periodic potential, *J. Chem. Phys.* **162**, 204903 (2025).
- [41] D. Gupta, S. H. L. Klapp, and D. A. Sivak, Efficient control protocols for an active ornstein-uhlenbeck particle, *Phys. Rev. E* **108**, 024117 (2023).
- [42] R. Garcia-Millan, J. Schüttler, M. E. Cates, and S. A. M. Loos, Optimal closed-loop control of active particles and a minimal information engine, *Phys. Rev. Lett.* **135**, 088301 (2025).
- [43] C. Casert and S. Whitelam, Learning protocols for the fast and efficient control of active matter, *Nat. Commun.* **15**, 9128 (2024).
- [44] J. Schüttler, R. Garcia-Millan, M. E. Cates, and S. A. M. Loos, Active particles in moving traps: Minimum work protocols and information efficiency of work extraction, *Phys. Rev. E* **112**, 024119 (2025).
- [45] S. C. Takatori, T. Quah, and J. B. Rawlings, Feedback control of active matter, *Annu. Rev. Condens. Matter Phys.* **16**, 319 (2025).
- [46] J. Toner, Y. Tu, and S. Ramaswamy, Hydrodynamics and phases of flocks, *Ann. Phys. (NY)* **318**, 170 (2005).
- [47] N. C. Darnton, L. Turner, S. Rojevsky, and H. C. Berg, Dynamics of bacterial swarming, *Biophys. J.* **98**, 2082 (2010).
- [48] A. Zöttl and H. Stark, Emergent behavior in active colloids, *J. Phys.: Condens. Matter* **28**, 253001 (2016).
- [49] H. R. Vutukuri, M. Lisicki, E. Lauga, and J. Vermant, Light-switchable propulsion of active particles with reversible interactions, *Nat. Commun.* **11**, 2628 (2020).
- [50] R. Bebon, J. F. Robinson, and T. Speck, Thermodynamics of active matter: Tracking dissipation across scales, *Phys. Rev. X* **15**, 021050 (2025).
- [51] S. A. M. Loos and S. H. L. Klapp, Irreversibility, heat and information flows induced by non-reciprocal interactions, *New J. Phys.* **22**, 123051 (2020).
- [52] S. M. J. Khadem, R. Klages, and S. H. L. Klapp, Stochastic thermodynamics of fractional Brownian motion, *Phys. Rev. Res.* **4**, 043186 (2022).
- [53] H.-M. Chun, X. Durang, and J. D. Noh, Emergence of nonwhite noise in Langevin dynamics with magnetic Lorentz force, *Phys. Rev. E* **97**, 032117 (2018).
- [54] H. D. Vuijk, J. M. Brader, and A. Sharma, Anomalous fluxes in overdamped Brownian dynamics with lorentz force, *J. Stat. Mech.* (2019) 063203.
- [55] J.-M. Park and H. Park, Thermodynamic uncertainty relation in the overdamped limit with a magnetic Lorentz force, *Phys. Rev. Res.* **3**, 043005 (2021).
- [56] E. Kalz, H. D. Vuijk, I. Abdoli, J.-U. Sommer, H. Löwen, and A. Sharma, Collisions enhance self-diffusion in odd-diffusive systems, *Phys. Rev. Lett.* **129**, 090601 (2022).
- [57] M. Fruchart, C. Scheibner, and V. Vitelli, Odd viscosity and odd elasticity, *Annu. Rev. Condens. Matter Phys.* **14**, 471 (2023).
- [58] P. L. Muzzeddu, E. Kalz, A. Gambassi, A. Sharma, and R. Metzler, Self-diffusion anomalies of an odd tracer in soft-core media, *New J. Phys.* **27**, 033025 (2025).
- [59] F. Effenberger, H. Fichtner, K. Scherer, S. Barra, J. Kleimann, and R. D. Strauss, A generalized diffusion tensor for fully anisotropic diffusion of energetic particles in the heliospheric magnetic field, *Astrophys. J.* **750**, 108 (2012).
- [60] D. Schick, M. Weißenhofer, L. Rózsa, J. Rothörl, P. Virnau, and U. Nowak, Two levels of topology in skyrmion lattice dynamics, *Phys. Rev. Res.* **6**, 013097 (2024).
- [61] A. P. Petroff, X.-L. Wu, and A. Libchaber, Fast-moving bacteria self-organize into active two-dimensional crystals of rotating cells, *Phys. Rev. Lett.* **114**, 158102 (2015).
- [62] C. W. Chan, D. Wu, K. Qiao, K. L. Fong, Z. Yang, Y. Han, and R. Zhang, Chiral active particles are sensitive reporters to environmental geometry, *Nat. Commun.* **15**, 1406 (2024).
- [63] J.-L. Auriault, C. Moyne, and H. P. A. Souto, On the asymmetry of the dispersion tensor in porous media, *Transp. Porous Media* **85**, 771 (2010).
- [64] C. J. O. Reichhardt and C. Reichhardt, Active rheology in odd-viscosity systems, *Europhys. Lett.* **137**, 66004 (2022).
- [65] X. Cao, D. Das, N. Windbacher, F. Ginot, M. Krüger, and C. Bechinger, Memory-induced magnus effect, *Nat. Phys.* **19**, 1904 (2023).
- [66] A. Shalchi, Applicability of the Taylor-Green-Kubo formula in particle diffusion theory, *Phys. Rev. E* **83**, 046402 (2011).
- [67] K. Goswami and R. Metzler, Effects of active noise on transition-path dynamics, *J. Phys. Complexity* **4**, 025005 (2023).
- [68] K. Goswami and K. L. Sebastian, Diffusion caused by two noises—active and thermal, *J. Stat. Mech.* (2019) 083501.
- [69] C. Maggi, M. Paoluzzi, N. Pellicciotta, A. Lepore, L. Angelani, and R. Di Leonardo, Generalized energy equipartition in harmonic oscillators driven by active baths, *Phys. Rev. Lett.* **113**, 238303 (2014).
- [70] C. Bechinger, R. Di Leonardo, H. Löwen, C. Reichhardt, G. Volpe, and G. Volpe, Active particles in complex and crowded environments, *Rev. Mod. Phys.* **88**, 045006 (2016).
- [71] K. Sekimoto, Langevin equation and thermodynamics, *Prog. Theor. Phys. Suppl.* **130**, 17 (1998).
- [72] L. Caprini, U. M. B. Marconi, A. Puglisi, and A. Vulpiani, The entropy production of ornstein-uhlenbeck active particles: A path integral method for correlations, *J. Stat. Mech.* (2019) 053203.
- [73] D. Martin, J. O'Byrne, M. E. Cates, É. Fodor, C. Nardini, J. Tailleur, and F. van Wijland, Statistical mechanics of active ornstein-uhlenbeck particles, *Phys. Rev. E* **103**, 032607 (2021).

- [74] K. Goswami, Inertial particle under active fluctuations: Diffusion and work distributions, *Phys. Rev. E* **105**, 044123 (2022).
- [75] K. Goswami, A. G. Cherstvy, A. Godec, and R. Metzler, Anomalous diffusion of active Brownian particles in responsive elastic gels: Nonergodicity, non-Gaussianity, and distributions of trapping times, *Phys. Rev. E* **110**, 044609 (2024).
- [76] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/qrvi-mng6> for additional figures.
- [77] L. Caprini, H. Löwen, and U. Marini Bettolo Marconi, Chiral active matter in external potentials, *Soft Matter* **19**, 6234 (2023).
- [78] W. R. DiLuzio, L. Turner, M. Mayer, P. Garstecki, D. B. Weibel, H. C. Berg, and G. M. Whitesides, *Escherichia coli* swim on the right-hand side, *Nature (London)* **435**, 1271 (2005).
- [79] F. V. Reyes, M. A. López-Castaño, and Á. Rodríguez-Rivas, Diffusive regimes in a two-dimensional chiral fluid, *Commun. Phys.* **5**, 256 (2022).
- [80] C. Hargus, F. Ghimentì, J. Tailleur, and F. van Wijland, Odd dynamics of passive objects in a chiral active bath, *Phys. Rev. Lett.* **135**, 167102 (2025).
- [81] K. Goswami, Dataset for “Optimal work control for an odd-diffusive tracer under active fluctuations”, Zenodo (2025), <http://doi.org/10.5281/zenodo.16808964>.
- [82] C. W. Gardiner, *Handbook of Stochastic Methods* (Springer, Berlin, 2004), Vol. 3.
- [83] I. Gelfand and S. Fomin, *Calculus of Variations* (Prentice-Hall, Inc., Englewood Cliffs, NJ, 1963).