

Markov inequality as a tool for linear-scaling estimation of local observables

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(Received 20 November 2025; accepted 1 June 2026; published 6 July 2026)

We introduce a linear-scaling stochastic method to compute real-space maps of any positive local spectral operator in a tight-binding model. By employing positive semidefinite estimators, the sampling error at each site can be rigorously bounded relative to the mean via the Markov inequality, overcoming the lack of self-averaging and enabling accurate estimates even under strong spatial fluctuations. The approach extends to nondiagonal observables, such as local currents, through local unitary transformations and its effectiveness is showcased by benchmark calculations in the disordered two-dimensional π -flux model, where the local density of states and steady-state current maps are computed. This method will enable simulations of disorder-driven mesoscopic physics in realistically large lattices and accelerate real-space self-consistent mean-field calculations.

DOI: [10.1103/qb1w-44r1](https://doi.org/10.1103/qb1w-44r1)

I. INTRODUCTION

Revealing spatially resolved quantum observables in large lattice systems remains a central challenge in condensed-matter theory. Quantities such as the local density of states (LDoS), local currents, magnetization textures, and topological markers encode rich information about underlying quantum phenomena, including localization, Mott transitions, and quantum Hall effects. Understanding these observables is crucial not only for characterizing exotic phases of matter, but also for predicting how disorder, interactions, and topology interplay in realistic materials. Moreover, advancement of experimental techniques over the last 20 years—e.g., scanning tunneling microscopy [1–3], scanning superconducting quantum interference device (SQUID) [4], and nitrogen-vacancy center magnetometry [5]—has uncovered intricate spatial patterns in quantum materials, sparking the development of simulation methods capable of comparable resolution.

Spatial inhomogeneities are core to the physics of disordered systems. Anderson localization, for example, appears as an exponential suppression of wavefunction amplitudes, which directly reflects in the LDoS [6], while their multifractal scaling near a metal-to-insulator (MI) transition is a clear signature of quantum criticality [7–13]. Likewise, observing local currents in a system can unveil robust edge channels for particle transport [14,15], signal presence or absence of backscattering [16–18], or even demonstrate the existence of

vortices in steady-state transport regimes, which had previously been connected to MI transition [19,20]. In addition, real-space topological markers, such as local Chern and $\mathbb{Z}_2$ invariants, have been used to detect topological order even in the absence of translational symmetry [21,22].

Linear-scaling algorithms based on Chebyshev expansions are renowned as very efficient ways to estimate volume-averaged spectral functions (i.e., operator traces) of sparse real-space Hamiltonians [23–28]. While these can be adapted to compute local quantities, the generation of full-sample maps inevitably increases the computational effort to $\mathcal{O}(N^2)$ (N being the system size), as the use of stochastic evaluation becomes limited by the lack of self-averaging. This limitation is mainly significant for observables that fluctuate by orders of magnitude from point to point or if repeated sampling is required [29,30].

In this work, we introduce a stochastic vector-based method that efficiently computes maps of positive local operators in large lattices, all at once and with controlled accuracy. Crucial to this approach is the use of positive semidefinite stochastic estimators, for which the celebrated *Markov inequality* enables a rigorous bound of sitewise stochastic errors to a multiple of the local mean value, thus ensuring homogeneous convergence across the sample. Local unitary transformations naturally extend the approach to off-diagonal matrix elements.

The effectiveness of the method is showcased by benchmark calculations done on large samples of the disordered two-dimensional (2D) π -flux model in the square lattice. First, vacancies are considered as the disorder source and full-sample maps of the LDoS are computed. For a single vacancy at the center, the zero-energy LDoS is shown to decay as r^{-2} away from the vacancy. This is an expected behavior caused by critical zero-energy states [24,31], but its numerical verification over several orders of magnitude is only accessible by the convergence properties of our method.

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For multiple vacancies, converged LDoS maps were obtained around zero energy, demonstrating the method's ability to accurately capture details of the particle density field, even when it strongly fluctuates in space. Finally, steady-state current maps were computed in the presence of long-range potential scatterers. Their high spatial resolution reveals vortexlike patterns similar to those previously identified as hallmarks of the localization transition in graphene [19], belonging to the Kosterlitz-Thouless universality class.

II. LOCAL LATTICE OBSERVABLES

For independent quantum particles on a disordered lattice, much of the underlying physics is captured by real-space matrix elements of functions involving the single-particle Hamiltonian, $\mathcal{H}$. A central example is the spectral function,

$$\mathcal{A}(\sigma', \mathbf{r}'; \sigma, \mathbf{r} | \varepsilon) = \langle \mathbf{r}', \sigma' | \delta(\varepsilon - \mathcal{H}) | \mathbf{r}, \sigma \rangle, \quad (1)$$

where ε is the energy, $\{\mathbf{r}, \mathbf{r}'\}$ are lattice positions, and $\{\sigma, \sigma'\}$ label orbitals within the unit cell. This function reveals the spatial structure of the eigenstates of $\mathcal{H}$: It captures their static real-space properties around a given energy, and its diagonal elements yield the LDoS. Off-diagonal elements, in turn, may also describe local dynamics—such as particle currents induced by an applied electric field. Specifically, the current from site $(\mathbf{r}, \sigma)$ to $(\mathbf{r} + \mathbf{\Delta}, \sigma')$ at time t (in units of $2e/\hbar$) is

$$I_{\mathbf{r}, \mathbf{r} + \mathbf{\Delta}}^{\sigma \sigma'}(t) = \text{Im} \{ t_{\mathbf{\Delta}}^{\sigma \sigma'} \langle \mathbf{r}, \sigma | \rho(t) | \mathbf{r} + \mathbf{\Delta}, \sigma' \rangle \}, \quad (2)$$

where $\rho(t) = \mathcal{U}_t \rho_0 \mathcal{U}_t^\dagger$ is the time-evolved single-particle density matrix and $t_{\mathbf{\Delta}}^{\sigma \sigma'}$ is the hopping parameter [32]. For fermions in equilibrium at inverse temperature T , $\beta = (k_B T)^{-1}$, and chemical potential μ , the density matrix ρ_0 is $\{1 + \exp[\beta(\mathcal{H} - \mu)]\}^{-1}$.

As the method is not tailored to any specific observable, the presentation is kept general by considering real-space matrix elements of a Hermitian operator X . The diagonal element at site $\mathbf{r}$, denoted as $x_{\mathbf{r}} = \langle \mathbf{r} | X | \mathbf{r} \rangle$, is considered with the orbital index σ omitted for brevity. In general, the full matrix representation of X in the real-space basis is unavailable, so each $x_{\mathbf{r}}$ must be computed via $\mathcal{O}(N)$ matrix-vector multiplication, where N is the Hilbert-space dimension. This way, obtaining the full spatial map of $x_{\mathbf{r}}$ would take $\mathcal{O}(N^2)$ operations [25,26]. In contrast, the trace of large matrices (e.g., the sum over all $x_{\mathbf{r}}$) can be computed much more efficiently using stochastic sampling methods [23,29,33–35]. Stochastic trace estimation starts by generating R square-normalized vectors, $\{|\xi_n\rangle\}_{n=1,\dots,R}$, each being a linear combination of basis states with coefficients $\xi_{n,\mathbf{r}}$. These coefficients are real or complex random variables such that $\overline{\xi_{n,\mathbf{r}}} = 0$ and $\overline{\xi_{n,\mathbf{r}}^* \xi_{n',\mathbf{r}'}} = \delta_{n,n'} \delta_{\mathbf{r},\mathbf{r}'}$ (complemented by $\overline{\xi_{n,\mathbf{r}} \xi_{n',\mathbf{r}'}} = 0$ in the complex case) where $(\dots)$ is an average over independent random vectors. Following Ref. [23], the trace of X may be approximated as the average $\text{Tr}(X) \approx \frac{1}{R} \sum_{n=1}^R \langle \xi_n | X | \xi_n \rangle$, which is expected to be self-averaging, that is, with a sampling error $\propto N^{-1}$.

Inspired by this algorithm, it is possible to recast the calculation of the full spatial map $x_{\mathbf{r}}$ as the average of a global stochastic estimator that samples all positions at once. A

simple *naive* unbiased estimator can be written as

$$\chi_{\mathbf{r}} = \frac{1}{R} \sum_{n=1}^R \xi_{n,\mathbf{r}}^* \sum_{\mathbf{r}'} \langle \mathbf{r} | X | \mathbf{r}' \rangle \xi_{n,\mathbf{r}'} = \frac{1}{R} \sum_n \langle \xi_n | \circ X | \xi_n \rangle, \quad (3)$$

with the notation $\langle \psi_1 | \circ | \psi_2 \rangle = \psi_1^*(\mathbf{r}) \psi_2(\mathbf{r})$ representing the (element wise) Hadamard product [36]. Although Eq. (3) represents a legitimate estimator [30] ($\overline{\chi_{\mathbf{r}}} = x_{\mathbf{r}}$), a subtle but important issue remains regarding its convergence properties. The sample-to-sample fluctuations of this estimator are quantified by its variance,

$$\sigma_{\mathbf{r}}^2 = [X^2]_{\mathbf{r},\mathbf{r}} + \Xi[\xi] X_{\mathbf{r},\mathbf{r}}^2, \quad \Xi[\xi] = \overline{|\xi|^4} - 2, \quad (4)$$

where $\overline{|\xi|^4}$ is the fourth moment of the random-vector component distribution. Specifically, in the eigenbasis of X , the relative sampling error takes the form of

$$\frac{\sigma_{\mathbf{r}}}{X_{\mathbf{r},\mathbf{r}}} = \frac{\sqrt{\sum_{\alpha} [X^2]_{\alpha,\alpha} |\psi_{\alpha}(\mathbf{r})|^2}}{\sum_{\alpha} X_{\alpha,\alpha} |\psi_{\alpha}(\mathbf{r})|^2}, \quad (5)$$

revealing its dependence on the spatial structure of the wavefunctions. If X [as in Eq. (1)] selects a narrow spectral window that effectively isolates a single eigenstate $\psi_{\alpha}(\mathbf{r})$, the relative local error scales as $|\psi_{\alpha}(\mathbf{r})|^{-1}$. This results in a nonuniform spatial convergence, preventing an accurate LDoS estimation with feasible computational time in localized systems, where wavefunction amplitudes can vary by orders of magnitude in space.

III. POSITIVE SEMIDEFINITE ESTIMATOR

The spatially uneven convergence can be eliminated by using an estimator whose sampling noise scales with the local mean. For positive semidefinite operators ($X_{\mathbf{r}} \geq 0$), this can be achieved by forcing the samples to be non-negative, in contrast to the sign-changing ones produced by Eq. (3). If a random field $\eta_{\mathbf{r}}$ is non-negative, the *Markov inequality* bounds

$$P(\eta_{\mathbf{r}} \geq C) \leq \overline{\eta_{\mathbf{r}}} C^{-1} \quad (6)$$

for any $C > 0$. Consequently, the α th quantile is at most a multiple of its mean ($q_{\alpha} \leq (1 - \alpha)^{-1} \overline{\eta_{\mathbf{r}}}$) implying that the field's relative fluctuations remain constant in space. To exploit this property, a positive semidefinite unbiased estimator is constructed for the diagonal elements of the operator $X_{\mathbf{r}}$. A new random field

$$\phi_{n,\mathbf{r}} = \langle \xi_n | \circ \sqrt{X} | \xi_n \rangle \quad (7)$$

is introduced so that

$$X_{\mathbf{r}} = \frac{1}{R} \sum_n |\phi_{n,\mathbf{r}}|^2 \quad (8a)$$

for complex random vectors and

$$X_{\mathbf{r}} = \frac{1}{R-1} \left[\sum_n \phi_{n,\mathbf{r}}^2 - \frac{1}{R} \left(\sum_n \phi_{n,\mathbf{r}} \right)^2 \right] \quad (8b)$$

for real random vectors satisfying $\overline{|\xi|^4} = 2$, as follows directly from Eq. (4). In both cases, the field $X_{\mathbf{r}}$ is estimated by non-negative sampling, which leverages Markov inequality and ensures controlled relative errors across the system.

A. Rigorous upper bound for the sampling error

If the local map of a positive semidefinite operator is estimated from Eq. (8a), it is possible to show that the corresponding sampling standard deviation is bounded by its expected value, x_r . For that, we rewrite Eq. (7) as

$$\phi_{n,r} = \sum_{r_1} \xi_{n,r}^* \mathcal{X}_{r,r_1} \xi_{n,r_1}, \quad (9)$$

where $\mathcal{X}_{r,r_1} = \langle \mathbf{r} | \sqrt{X} | \mathbf{r}_1 \rangle$. The mean value of this estimator approximates $x_r = \langle \mathbf{r} | X | \mathbf{r} \rangle$ with R random vectors. The corresponding variance of the estimates is given as follows:

$$\sigma_r^2 = \frac{1}{R} \sum_n (|\phi_{n,r}|^2 - x_r)^2, \quad (10)$$

where

$$|\phi_{n,r}|^2 = x_r + \sum_{r_1 \neq r_2} \xi_{n,r_1} \mathcal{X}_{r,r_1} \mathcal{X}_{r,r_2} \xi_{n,r_2}^*. \quad (11)$$

Thus, the expression in Eq. (10) is equivalent to

$$\sigma_r^2 = \frac{1}{R} \sum_n \sum_{\substack{r_1 \neq r_2 \\ r_3 \neq r_4}} X_{r,r_1} \mathcal{X}_{r,r_2} \mathcal{X}_{r,r_3} \mathcal{X}_{r,r_4} \Upsilon_{n,\{r_i\}}, \quad (12)$$

with

$$\Upsilon_{n,\{r_i\}} = \xi_{n,r_2}^* \xi_{n,r_1} \xi_{n,r_4}^* \xi_{n,r_3}. \quad (13)$$

Since $\overline{\xi_{n,r}^* \xi_{n,r'}} = \delta_{r,r'}$ and $\overline{\xi_{n,r} \xi_{n,r'}} = 0$, one concludes that

$$\overline{\Upsilon_{n,\{r_i\}}} = \delta_{r_1,r_4} \delta_{r_2,r_3}, \quad (14)$$

which simplifies Eq. (12) to

$$\sigma_r^2 = \frac{1}{R} \left(x_r^2 - \sum_{r_1} \mathcal{X}_{r,r_1}^4 \right). \quad (15)$$

Because the subtracted term is strictly non-negative, the standard deviation associated with the stochastic sampling is upper bounded by x_r as $\sigma_r \leq x_r / \sqrt{R}$.

B. Off-diagonal generalization

The estimators are basis independent; i.e., if the basis $\{|\mathbf{r}\rangle\}$ is replaced by a unitarily transformed basis $\{|\mathbf{u}\rangle\} = \{U|\mathbf{r}\rangle\}$, the corresponding random field $\{\phi_{\mathbf{u}}\}$ can be likewise used to estimate diagonal elements in that basis. The reasoning also applies to the unitary time evolution of Eq. (2). This means that even though the method had initially been formulated for the estimation of diagonal matrix elements, it can be naturally extended to evaluate off-diagonal elements [as in Eq. (2)]. The key idea is that diagonal elements in a suitably rotated basis encode information about the corresponding off-diagonal terms. Specifically, if one applies a local unitary transformation to a pair of sites $\{\mathbf{r}, \mathbf{r}'\}$ of the form

$$\begin{aligned} |\alpha_\gamma\rangle &= \frac{1}{\sqrt{1+|\gamma|^2}} (|\mathbf{r}\rangle + \gamma|\mathbf{r}'\rangle), \\ |\beta_\gamma\rangle &= \frac{1}{\sqrt{1+|\gamma|^2}} (-\gamma^*|\mathbf{r}\rangle + |\mathbf{r}'\rangle), \end{aligned} \quad (16)$$

$$\begin{pmatrix} \text{---} \end{pmatrix} = \frac{1}{\sqrt{1+|\gamma|^2}} \begin{pmatrix} 1 & \gamma \\ -\gamma^* & 1 \end{pmatrix}$$

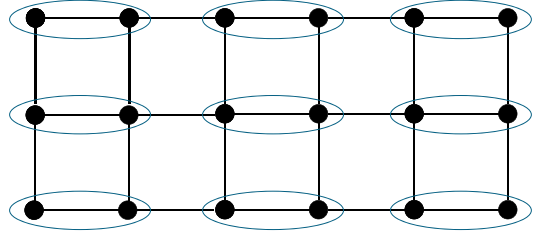


FIG. 1. Schematic representation of a bipartite squared lattice. The blue ellipses represent connected pairs. These form a complete set, and for each pair we apply the change of basis matrix shown in the red box.

where $\gamma \in \mathbb{C}$ is a parameter controlling the rotation, the diagonal elements in the rotated basis will be related to the original off-diagonal matrix elements as

$$2\langle \mathbf{r} | X | \mathbf{r}' \rangle = \Upsilon_1 - i\Upsilon_i, \quad (17)$$

where $\Upsilon_\gamma = \langle \alpha_\gamma | X | \alpha_\gamma \rangle - \langle \beta_\gamma | X | \beta_\gamma \rangle$. Importantly, because each bond rotation acts only on disjoint site pairs and the different components of the random vector are uncorrelated, the procedure can be applied to all such bonds in parallel without inducing additional sampling correlations (illustrated in Fig. 1). Hence, by iterating through the different groupings needed to cover all intersublattice bonds, this method yields a complete map of the off-diagonal structure of X . For observables requiring off-diagonal elements from multiple hopping ranges (nearest neighbor, next-nearest-neighbor,...), we treat each range as a separate bond class and measure them in independent simulations.

IV. DEMONSTRATIONS

A. Electron density fields

The 2D π -flux model on a square lattice is used as a test bed for the method. Its initially pristine periodic Hamiltonian

$$\mathcal{H}_{\mathbf{k}} = -2w(\cos k_x \sigma_x + \cos k_y \sigma_z), \quad (18)$$

where w is the hopping parameter, loses its translation symmetry due to the presence of point defects (vacancies). As a first demonstration, we test our method by unveiling this zero-energy mode for a single central vacancy in a 4096×4096 supercell under twisted boundary conditions. The LDoS, $\rho_r(\varepsilon)$, was computed along a longitudinal cut starting at the vacancy for a set of energies approaching $\varepsilon = 0$, using kernel polynomial method (KPM) with a spectral resolution of 1 meV. Random sampling was performed with both the proposed positive semidefinite estimator and the conventional mean estimator using 32 random vectors. Figures 2(a)–2(c) show that the positive semidefinite estimator faithfully captures the exact algebraic decay of the zero-energy LDoS, $\rho_r(0) \propto |\mathbf{r}|^{-2}$, over several orders of magnitude, whereas the naive estimator does not. The

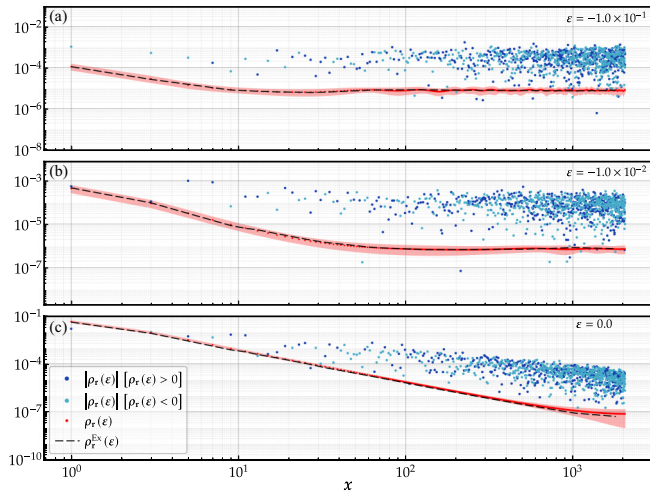


FIG. 2. (a)–(c) Longitudinal cuts through the LDoS of a single central vacancy in a 4096×4096 supercell at the energies shown in the lower panels. Dark blue denotes the naive estimator, light blue the negative values from the conventional estimator, red the positive semidefinite estimator, and the black dashed curve the exact LDoS. The shaded region indicates a $\pm 2\sigma$ interval, with σ being the standard deviation from averages over 32 random vectors.

latter fails in regions of very low LDoS, where sampling noise generates nonphysical negative values and large relative errors—an instance of a random-sampling *sign problem* that the positive semidefinite formulation inherently avoids.

Figure 3 shows feature-rich real-space LDoS maps at a finite vacancy concentration (0.5% on a 2048×2048 supercell). Computing them with a fixed $100 \mu\text{eV}$ spectral resolution, averaging over 1000 random vectors and twist-angle configurations, we demonstrate the controlled accuracy of the positive semidefinite estimator. Despite the LDoS maps spanning several orders of magnitude, the 95th percentile of the local relative fluctuations is upper bounded at 3.5%. The final panel highlights the overlap of the zero-energy critical states, with local amplitudes at least two orders of magnitude larger than the noncritical regions.

Next, we analyze a 512×512 sample with 1% randomly placed vacancies at $\varepsilon = 0$ with a $200 \mu\text{eV}$ spectral resolution.

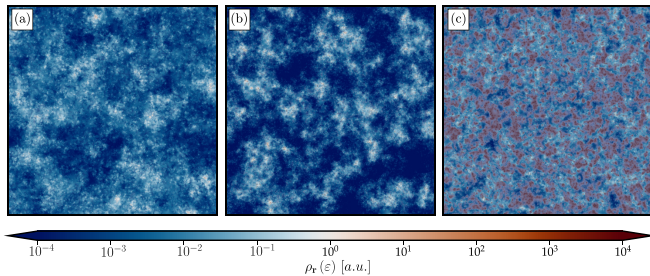


FIG. 3. LDoS maps for a 2048×2048 supercell with 0.5% vacancy concentration. In panel (a), we set $\varepsilon_F = -0.1 \text{ eV}$, while in panel (b) $\varepsilon_F = -0.01 \text{ eV}$ and in panel (c) we probe this observable at zero energy.

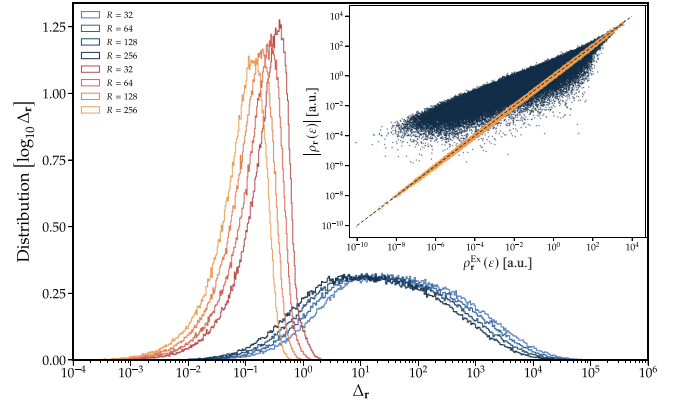


FIG. 4. Probability density functions of the local relative error Δ_r for the conventional (shades of blue) and positive semidefinite (shades of red) stochastic estimators, as a function of the number of random vectors, R . The latter reduces the mode of Δ_r by approximately two orders of magnitude, while scaling as $R^{-0.5}$. The inset is a scatter plot of stochastic and exact LDoS for 256 random vectors.

To quantify the accuracy of both estimators, we measure the local relative error

$$\Delta_r = \frac{|\rho_r^{\text{St}}(\varepsilon) - \rho_r^{\text{Ex}}(\varepsilon)|}{\rho_r^{\text{Ex}}(\varepsilon)}, \quad (19)$$

where ρ_r^{Ex} is the exact LDoS and ρ_r^{St} is the estimate.

Figure 4 demonstrates that the positive semidefinite estimator (shades of red) dramatically reduces relative errors. The corresponding statistical distribution mode lies about two orders of magnitude below that of the naive estimator (shades of blue), which also exhibits a much broader spread. The first moment of the error distribution follows the expected $R^{-1/2}$ scaling with the number of random vectors R . The inset shows the local observable scatter plot comparing stochastic estimates with the exact LDoS maps (256 random vectors). The naive estimator (blue) fails to reproduce values below approximately 10^2 . Furthermore, it produces nonphysical negative LDoS (here plotted as absolute values). Contrastingly, the positive semidefinite estimator (yellow) exhibits a controlled dispersion for over ten orders of magnitude. Moreover, for the highest R , each stochastic estimation took ≈ 4 CPU hours, whereas the exact result took ≈ 171 CPU days.

B. Local currents fields

To illustrate the versatility of this method, we studied the local current fields that develop when translation symmetry is broken by strong long-range impurities. The disordered potential is given by $V_r = \sum_{m=1}^M w_m \exp(-0.5|\mathbf{r} - \mathbf{r}_m|^2/\kappa^2)$, where the scatterer amplitudes w_m are drawn uniformly from $\frac{W}{2}[-1, 1]$. We analyze a 512×512 sample with 0.5% randomly placed impurities with $W = 1.1 \text{ eV}$ and $\kappa = 9.0$. This sample is connected to two finite pristine leads (each 512×512) and periodic boundary conditions are imposed in the transverse direction. Following the linear-scaling time-evolution methodology described in Refs. [32,37], we compute the real-space distribution of the steady-state local

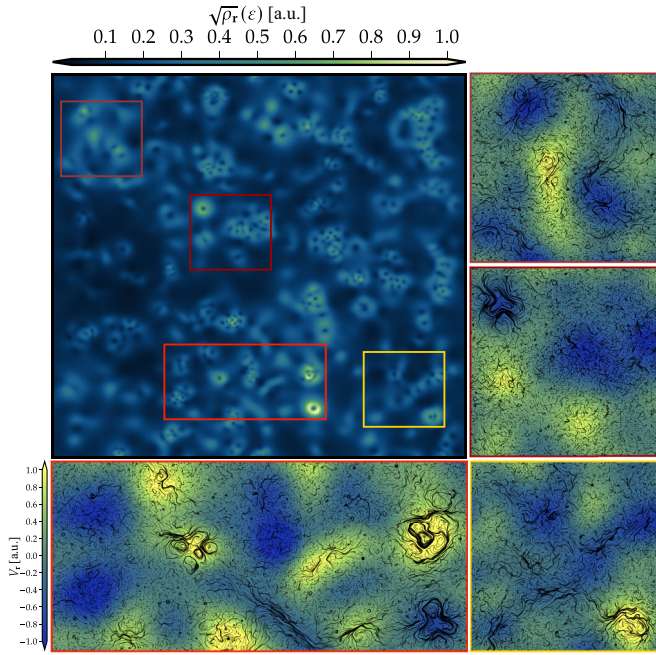


FIG. 5. Square-root LDoS and steady-state local currents for a 512×512 sample in the two-terminal setup with 0.5% concentration of long-range impurities ($W = 1.1$ eV, $\kappa = 9.0$). Top: Square-root LDoS map (500 random vectors and spectral resolution $100 \mu\text{eV}$). Magnified regions demonstrate the local currents' steady-state fields. Intricate patterns in the real-space distributions of currents are evident with vortex-antivortex pairs being visible.

current fields. The LDoS shown in Fig. 5 was obtained using 500 random vectors. The spectral probe is centered on the Fermi energy $\varepsilon_F = 0.1$ eV and has a width of $100 \mu\text{eV}$. In terms of vacancy calculations, the LDoS exhibits controlled accuracy. The 95th percentile of local relative fluctuations is $\lesssim 5\%$. The statistical properties of the diagonal matrix elements carry over to the matrix elements in the rotated basis. However, the relation between the magnitudes of the individual matrix elements and their differences is model dependent, so it is challenging to draw general conclusions for off-diagonal estimates. The simulation of the linear response steady-state current fields is performed by exploiting statistical correlations. Both the equilibrium and steady-state fields are computed with the same random vector. Subtraction of both estimates with 2.4×10^4 random vectors reveals intricate structures, with tightly bound vortex-antivortex pairs and complex flow patterns that appear only when local currents are resolved at this scale.

V. SUMMARY AND DISCUSSION

To summarize, we introduced a new linear-scaling stochastic method to compute full-sample maps of positive local spectral operators in tight-binding systems. By employing positive semidefinite estimators, the sampling error at each site is bounded relative to its mean value via the Markov inequality, ensuring homogeneous sampling convergence across the sample even when the observable fluctuates strongly

from point to point. This precise control of the sitewise error is crucial, since unlike stochastic trace evaluations, random sampling of local quantities lacks self-averaging. The method was further extended to evaluate nondiagonal observables (e.g., local currents) by means of local unitary transformations.

The method was illustrated using three representative cases in the 2D π -flux model. For a single vacancy, we computed the real-space LDoS and recovered the expected zero-energy r^{-2} decay [24,31], demonstrating that our approach can accurately capture the converged tail over several orders of magnitude—something not attainable with a naive mean estimator. Moreover, full-sample LDoS maps near the nodal energy were obtained, maintaining a controlled accuracy even when vacancy-induced critical states dominate and drive fast spatial variations spanning several orders of magnitude. Finally, for the π -flux model with Gaussian potential scatterers, we computed steady-state current maps in a two-terminal setup using the time-evolution approach introduced in Refs. [32,37]. The resulting current patterns reveal subtle vortex structures akin to those previously associated with the metal-to-insulator transition in graphene [19], belonging to the Kosterlitz-Thouless universality class.

VI. OUTLOOK

Our proposal opens up possibilities for large-scale simulations of mesoscopic phenomena aided by a reduction in computational complexity, from *quadratic* to *linear* in the lattice size. This enables the measurement of local quantities across entire samples, which can be applied, for example, to characterize multifractality of disordered phases [7–9,38,39] or to study disorder-driven transitions [6,10–13,40–43]. In addition, its efficiency and precision make it suitable for use in self-consistent mean-field approaches to interactions in real space [25,26]. The positive-estimator principle can further improve evaluation of other global quantities without significant changes on current KPM implementations.

ACKNOWLEDGMENTS

This work was supported by Fundação para a Ciência e a Tecnologia (FCT, Portugal) in the framework of the Strategic Funding UIDB/04650—Centro de Física das Universidades do Minho e do Porto. Further support from FCT through Project No. POCI-01-0145-FEDER028887 (J.M.V.P.L.) and Ph.D. Grant No. 2024.00560.BD (H.P.V.) is acknowledged. H.P.V. acknowledges the University of York High-Performance Computing service, Viking. The authors acknowledge fruitful discussions with Aires Ferreira, Bruno Amorim, Caio H. Lewenkopf, Vitor M. Pereira, J. M. Alendouro Pinho, J. M. B. Lopes dos Santos, and Simão M. João.

DATA AVAILABILITY

The data that support the findings of this article are openly available [44].

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