

Uncertainty-disturbance relations and applications

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Uncertainty and intrinsic measurement disturbance, two fundamental concepts in quantum measurement, have conventionally been viewed as distinct and studied separately. In this work, we establish a fundamental connection between them, proving that uncertainty not only serves as a prerequisite for intrinsic disturbance but also bounds it from above. We formalize this connection via uncertainty-disturbance relations (UDRs) with direct applications in quantum information science. We show that for rank-one projective measurements, these UDRs effectively function as uncertainty relations by bounding the uncertainties of incompatible measurements. They also enable the experimental estimation of key quantum resources—including von Neumann entropy, purity, coherence, and genuine randomness. Our findings thus unify the understanding of uncertainty and disturbance and provide a versatile framework for quantum resource detection.

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I. INTRODUCTION

Unlike classical measurements—which passively record deterministic observable values—quantum measurements are inherently probabilistic and inevitably perturb the measured quantum state. These two defining features, namely, uncertainty and disturbance, lie at the heart of quantum measurement theory and underpin nearly all quantum phenomena, including nonlocality [1–3], contextuality [4,5], and coherence [6,7]. Moreover, they give rise to fundamental distinctions between classical and quantum information science, serving as key ingredients for quantum advantages in a wide range of information-processing tasks. Given their theoretical and practical significance, a central challenge remains: to refine and expand our understanding of these phenomena.

To date, uncertainty and disturbance are primarily associated with the uncertainty principle [8–20] and complementary principles [21–42], respectively. The uncertainty principle states that a quantum system cannot be prepared in a state such that incompatible quantities possess exact values simultaneously. This principle has been articulated through various uncertainty relations that utilize different measures, including standard deviation [8–10], the α -Rényi entropy [15], and general Schur-concave functions [19]. Moreover, uncertainty relations have been extended to encompass scenarios involving memory [16,43], multiple observables [44,45], and quantum dynamical processes [46]. These advancements have proven crucial in establishing the security of quantum communications [47,48], verifying quantum randomness [49], constraining the strength of quantum nonlocality [3,50], and delving into the foundations of quantum thermodynamics [51].

The complementary principle asserts that incompatible quantities cannot be precisely measured in sequential measurements, as expressed by the inherent trade-off between the error of the first measurement and the disturbance imposed on the subsequent one. This principle fundamentally limits the information extractable from a single particle through incompatible measurements [52], thereby underpinning the security of quantum cryptographic protocols [53–56]. To date, numerous error-disturbance relations have been investigated

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both theoretically and experimentally [21–42]. Nevertheless, a universally accepted definition of intrinsic disturbance in the context of general quantum measurement processes remains elusive [57].

The above two kinds of relations treat uncertainty and disturbance as independent properties. However, in a quantum measurement process, these two effects manifest simultaneously in a “random jump” of the measured state, known as the reduction of the wave function, raising the question of whether they are fundamentally interconnected. Winter first observed that, when a measurement is implemented using a Lüders instrument, the state change induced by the measurement can be upper bounded by the probability of outcomes—a result now known as the gentle measurement lemma. Although this lemma is specific to Lüders instruments, it has proven remarkably useful, playing key roles in diverse areas such as channel coding theorem [58], state tomography [59–61], self-testing [62], differential privacy [63], quantum OR problem [64], quantum process learning [65,66], quantum correlation [67,68], etc. In our previous papers, by analyzing quantum measurement within the framework of Heisenberg’s picture [69], we show that the uncertainty associated with one measurement can serve as an upper bound for its intrinsic disturbance effect. This finding allowed us to establish an uncertainty-disturbance relation (UDR) and a generalized gentle measurement. Furthermore, we showed that these uncertainty-disturbance relations can be employed to gain insights into the strength of nonlocality in the framework of generalized probabilistic theories [3].

In this paper, we present a systematic approach to derive uncertainty-disturbance relations, followed by illustrating their applications in quantum information science. We first review basic notions of a general measurement. We then introduce the concept of distance-based intrinsic disturbance and show that, for commonly used state-distance measures, the defined intrinsic disturbance induced by a measurement is always upper bounded by its uncertainty and provides many UDRs. When restricted to rank-one projection measurements, these UDRs imply constraints on the uncertainties of incompatible measurements, leading to uncertainty relations. We discuss the tightness of these uncertainty relations in the contexts of qubit and qutrit systems. Additionally, UDRs are utilized to design protocols for experimentally estimating key quantities in quantum information science, including the von Neumann entropy, purity, coherence, and quantum Fisher information.

II. UNCERTAINTY-DISTURBANCE RELATION

We begin with a brief review of the notions of quantum measurement. A general quantum measurement is described with a set of positive operator-valued measures (POVMs) $\{M_i\}_i$. To realize a nonprojective POVM, one has to extend it, using Naimark’s dilation theorem, to be a projective measurement at the cost of introducing a higher-dimensional space. This extension is often described as follows [70]: coupling the measured state ρ_s with an ancillary state $\rho_a = |0\rangle\langle 0|_a$ via unitary operation U and performing projective measurement $\{P_i = |i\rangle\langle i|_a \otimes \mathbb{1}_s\}$ on the resulted joint state $U\rho_s \otimes \rho_a U^\dagger$, where $\{|i\rangle\langle i|_a\}$ is a set of orthogonal basis acting on the

ancillary system. Then, upon outcome i the measured state is updated (up to a normalization) according to the rule $\mathcal{J}_i^{\mathcal{D}} : \rho_s \rightarrow U_i \sqrt{M_i} \rho_s \sqrt{M_i} U_i^\dagger$, where U_i is determined by U and can be arbitrary. This channel does not well characterize the intrinsic disturbance in the measurement process, as the channels $\{\mathcal{J}_i^{\mathcal{D}}\}_i$ include additional and reversible state changes induced by U_i . In contrast, intrinsic disturbance stems from solely the irreversible wave function reduction. To rigorously characterize intrinsic disturbance, we adopt the Heisenberg picture. The POVM is equivalently realized by performing $\{Q_i = U^\dagger P_i U\}$ directly on $\rho_s \otimes \rho_a$ as $\text{Tr}(\rho_a \otimes \rho_s Q_i) = \text{Tr}(\rho_a \otimes \rho_s U^\dagger P_i U) = \text{Tr}(U \rho_a \otimes \rho_s U^\dagger P_i)$. Then, one has a different state updating rule

$$\mathcal{J}_i^{\mathcal{N}} : \rho_s \rightarrow \text{Tr}_a(Q_i \rho_s \otimes \rho_a Q_i), \quad (1)$$

where the partial trace is taken over the ancillary system. This channel actually characterizes the intrinsic disturbance in a general measurement process. Note that an arbitrary measurement channel $\mathcal{J}_i(\cdot)$ can be formulated with some projective measurement $\mathbf{Q}$, a dilation of the given measurement, on the composite system $\rho_s \otimes \rho_a$ followed by unitary operations $\{U_i\}$ [71] that possibly depend on the outcome i : $\mathcal{J}_i(\rho_s) = \text{Tr}_a(U_i Q_i \rho_s \otimes \rho_a Q_i U_i^\dagger)$, where the intrinsic disturbance in the joint system comes from $\mathbf{Q}$ (U_i introduces a reversible state change). We refer Ref. [69] for a detailed discussion of this channel. Intrinsic disturbance relevant to $\mathbf{Q}$ can be uncontroversially defined in terms of the distance $\mathcal{D}$ between the initial state $\rho_{sa} \equiv \rho_s \otimes \rho_a$ and the postmeasurement state $\rho'_{sa} \equiv \sum_i Q_i \rho_s \otimes \rho_a Q_i$ as

$$\mathcal{D}_D(\rho_{sa}) \equiv D(\rho_{sa}, \rho'_{sa}). \quad (2)$$

where state-distance measure $D(\cdot, \cdot)$ assumes a few nice properties, namely, positivity, unitary invariance, and monotonicity under physical operations. By tracing out the ancillary system, we have the disturbance in the measured system $\mathcal{D}_D(\rho_s)$:

$$\mathcal{D}_D(\rho_s) \equiv D(\rho_s, \rho'_s), \quad (3)$$

where $\rho'_s \equiv \sum_i \mathcal{J}_i^{\mathcal{N}}(\rho_s)$ and our first result is the following theorem.

Theorem 1 (Uncertainty upper-bound disturbance). For a general state-distance-based disturbance $\mathcal{D}_D(\rho_s)$ or $\mathcal{D}_D(\rho_{sa})$, there exists at least one distance-dependent uncertainty measure $\delta_D(\mathbf{p})$, as listed in Table I, serving as an upper bound

$$\delta_D(\mathbf{p}) \geq \mathcal{D}_D(\rho_{sa}) \geq \mathcal{D}_D(\rho_s), \quad (4)$$

where $\mathbf{p} \equiv \{p_i = \text{Tr}(\rho_s M_i)\}$ specifies the distribution of the measurement of interest.

In Table I, we give the uncertainty relevant to almost all the frequently used state-distance measures (see the appendix for the proof), including the infidelity (IF) measure, the trace distance (Tr) [70], Tsallis relative entropy (Ts) [72], the Rényi divergence (Rd) measure [73], the relative entropy measure (Re), and the Hilbert-Schmidt (HS) distance measure [74,75]. Concerning the results provided in Table I, we note that some state distances are not symmetric, namely, $D(\rho_1, \rho_2) \neq D(\rho_2, \rho_1)$, which may induce different UDRs, and an example is the Rényi divergence (see the Appendix). In the Appendix, we also establish Winter’s gentle measurement lemma for $\mathcal{J}_i^{\mathcal{N}}$, a result originally developed for Lüders instrument.

TABLE I. The uncertainty measures induced by distance-based disturbance are specifically listed in the third column of the table below. A list of uncertainty measures induced by distance-based disturbance, where $\text{IF}(\rho_1, \rho_2) = \sqrt{1 - F^2(\rho_1, \rho_2)}$ with $F(\rho_1, \rho_2) = [\text{tr}(\sqrt{\sqrt{\rho_2}\rho_1\sqrt{\rho_2}})]^2$, $\delta_{\text{Tr}}(\mathbf{p}) \equiv \sqrt{1 - \|\mathbf{p}\|_2^2}$, whose square is the so-called operationally invariant information [76], the notions $\bar{\alpha} \equiv 1 - \alpha$, $H_\alpha(\mathbf{p}) \equiv \frac{\alpha}{\bar{\alpha}} \log_2 \|\mathbf{p}\|_\alpha$ for Rényi entropy, the von Neumann $S(\rho) \equiv -\text{Tr}(\rho \log_2 \rho)$ entropy, and the Shannon entropy reads $H(\mathbf{p}) \equiv -\sum_i p_i \log_2 p_i$ and consider the regions $0 \leq \alpha < 1$ for Rényi divergence and $0 \leq \alpha < 1$ for Tsallis relative entropy.

State distance	$\mathcal{D}(\rho_1, \rho_2)$	Uncertainty
IF	$\sqrt{1 - F^2(\rho_1, \rho_2)}$	$\delta_{\text{Tr}}(\mathbf{p})$
D_{Tr}	$\frac{1}{2} \text{Tr} \rho_1 - \rho_2 $	$\delta_{\text{Tr}}(\mathbf{p})$
D_{Ts}^α	$\frac{1}{\bar{\alpha}} [1 - \text{tr}(\rho_1^\alpha \rho_2^{\bar{\alpha}})]$	$\frac{1 - \ \mathbf{p}\ _{2-\alpha}^{2-\alpha}}{\bar{\alpha}}$
D_{Rd}^α	$\frac{1}{\bar{\alpha}} \log_2 \text{tr}[(\rho_2^{\frac{\bar{\alpha}}{\alpha}} \rho_1 \rho_2^{\frac{\bar{\alpha}}{\alpha}})^\alpha]$	$H_{\frac{1}{\alpha}}(\mathbf{p})$
$S(\cdot \parallel \cdot)$	$\text{Tr}[\rho_1(\log_2 \rho_1 - \log_2 \rho_2)]$	$H(\mathbf{p}) - S(\rho)$
D_{HS}	$\sqrt{\text{Tr}((\rho_1 - \rho_2)^2)}$	$\sqrt{\text{Tr}(\rho^2) - \ \mathbf{p}\ _2^2}$

Next, we consider the disturbance in a sequential measurement scheme where another following measurement $B \equiv \{M_{j|B}\}$ is performed on the postmeasurement state ρ'_s after measurement A , giving a disturbed statistics $\mathbf{q}' \equiv \{q'_j = \text{Tr}(M_{j|B} \rho'_s)\}$. Compared to probability distribution $\mathbf{q} \equiv \{q_j = \text{Tr}(\rho_s M_{j|B})\}$ from measurement performed on the original state, a disturbance in measurement B can be defined with the distance between $\mathbf{q}'$ and $\mathbf{q}$. According to the data processing inequality, $D(\rho_s, \rho'_s) \geq \tilde{D}(\mathbf{q}, \mathbf{q}')$, where $\tilde{D}(\mathbf{q}, \mathbf{q}')$ is the classical counterpart of state distance D .

$$\delta_D(\mathbf{p}) \geq D(\mathbf{q}, \mathbf{q}'). \quad (5)$$

This relation is also called UDR as it is an immediate consequence of Theorem 1. We also give this kind of UDR for each disturbance in the Appendix. In what follows, we demonstrate that UDRs can provide a systematic framework for deriving uncertainty relations when measurements A and B are rank-one projective measurements.

Here, we consider the tightness of these UDRs. In the derivation of $\delta_D(\mathbf{p}) \geq D(\rho_s, \rho'_s)$, we have purified the joint state $\rho_s \otimes \rho_a$ by introducing environment and then tracing out both the ancilla and the environment. This generally relaxes the inequality. However, when measurement A is projective measurement and the measured state is pure, this relaxation can be avoided, and the inequality for IF, D_{Rd}^α , S , D_{Ts}^α , D_{HS} , and D_{Tr} can be saturated. Furthermore, in relation (5) we have relaxed $D(\rho, \rho_s)$ to $D(\mathbf{q}, \mathbf{q}')$. For IF and D_{HS} , the relation $D(\rho, \rho_s) \geq D(\mathbf{q}, \mathbf{q}')$ can be saturated when B is an eigenvector of $\rho' - \rho_s$. To gain further insights into the tightness, when restricted to rank-one projective measurements, these UDRs can be viewed as standard uncertainty relations. We compare their tightness with the Maassen-Uffink (MU) uncertainty relation and the relation associated with Winter's gentle measurement lemma (after summing all the outcomes). A summary of these UDRs is provided in Table II.

TABLE II. Computed *volumes* according to different UDRs and their duals.

Relations	Volume ($d = 2$)	Volume ($d = 3$)
U_{Tr}	0.930	0.946 75
$U_{\text{Rd}}^{0.5}$	0.787	0.917
U_{Re}	0.770	0.905
$U_{\text{Ts}}^{0.5}$	0.814	0.937
U_{HS}	0.705	0.887
MU	0.974	0.999

III. UDRs AS PREPARATION UNCERTAINTY RELATIONS

Consider sequential rank-one projective measurements $A = \{|A_i\rangle\langle A_i|\}$ and $B = \{|B_j\rangle\langle B_j|\}$ performed in the order $A \rightarrow B$ on a quantum state ρ (here ρ_s is written as ρ for brevity). After measuring A , the postmeasurement state becomes $\rho' = \sum_i p_i |A_i\rangle\langle A_i|$, where $p_i = \text{Tr}(\rho |A_i\rangle\langle A_i|)$. A subsequent measurement of B on ρ' yields the statistics $\mathbf{q}' \equiv \{q'_j = \sum_i p_i c_{ij}\}$, where the matrix $\mathcal{C} \equiv \{c_{mn} = |\langle A_m | B_n \rangle|^2\}$ encodes the squared overlaps between the eigenvectors of A and B . This matrix $\mathcal{C}$ is a unistochastic matrix [77,78]. We observe that the disturbed statistics $\mathbf{q}'$ are simply a linear transformation of the original probabilities $\mathbf{p}$, meaning that all UDRs listed in Table I are expressed in terms of $\mathbf{p}$, $\mathbf{q}$, and $\mathcal{C}$, and UDRs impose constraints on the allowed probability distributions for a given overlap matrix $\mathcal{C}$, thus functioning as uncertainty relations.

Theorem 2 (UDRs as uncertainty relations). When considering rank-one projective measurements, UDRs constrain the probability distributions from incompatible and rank-one measurements on identically prepared ensembles, thereby serving as uncertainty relations in this context.

As uncertainty relations, UDRs are specified with a subscript notation. For example, U_{Re} represents the relation

$$H(\mathbf{p}) \geq H(\mathbf{q} \parallel \mathbf{q}'),$$

where $H(\mathbf{p}) = -\sum_i p_i \log_2 p_i$ is the Shannon entropy and $H(\mathbf{q} \parallel \mathbf{q}') = \sum_i q_i \log_2 \frac{q_i}{q'_i}$ is the classical relative entropy. This corresponds to an uncertainty relation based on the relative entropy distance measure. From U_{Re} , we can derive the MU uncertainty relation [15]:

$$H(\mathbf{p}) + H(\mathbf{q}) \geq -\log_2 c,$$

where $c \equiv \max_{ij} c_{ij}$. The derivation follows from $H(\mathbf{q} \parallel \mathbf{q}') \geq \sum_i q_i \log_2 \frac{q_i}{q'_i} \geq -H(\mathbf{q}) - \sum_i q_i \log_2 q'_i \geq -H(\mathbf{q}) - \log_2 c$, where we have used $q'_i = \sum_j c_{ij} p_j \leq c$. Combining this with $H(\mathbf{p}) \geq H(\mathbf{q} \parallel \mathbf{q}')$ yields the MU relation. Similarly, from U_{Ts} we can derive an independent entropic uncertainty relation:

$$\frac{1}{2-\alpha} H_{2-\alpha}(\mathbf{q}) + H_\alpha(\mathbf{p}) \geq -\log_2 c, \quad 0 \leq \alpha < 1, \quad (6)$$

which represents a distinct family of relations beyond the Maassen-Uffink framework [15].

Furthermore, in previous uncertainty relations, e.g., the MU entropic uncertainty relation, the incompatibility of measurements is characterized by a single number c , namely, the

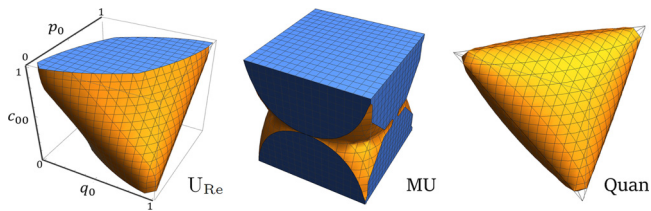


FIG. 1. Geometrical visualization of the constraints of uncertainty relations. In the qubit case, the distributions from measuring A and B can be specified, respectively, by p_0 and q_0 , and their incompatibility is characterized by c_{00} . They are subject to uncertainty relations. Plots of three typical uncertainty relations correspond to U_{Re} , U_{HS} , and exact quantum uncertainty relation.

maximum element of the matrix $\mathcal{C}$. In contrast, UDRs involve the whole matrix of $\mathcal{C}$. Hence, these uncertainty relations arising from UDRs may provide tighter constraints than the MU entropic uncertainty relations. Below we present two case studies of the *uncertainty-disturbance*-type uncertainty relation regarding the tightness in the case of qubit ($d = 2$) and qutrit ($d = 3$).

For the qubit case, $\mathbf{p}$, $\mathbf{q}$, and $\mathcal{C}$ are determined by three independent parameters, $\{p_0, q_0, c_{00}\} \in [0, 1]$. We shall investigate constraints placed by various uncertainty relations to p_0 , q_0 , and c_{00} . Geometrically one may visualize that each one of the inequalities in U_{Tr} , U_{Ts}^α , U_{Rd}^α , U_{Re} and the MU relation for measurements $A \rightarrow B$ together with their dual inequalities for measurements $B \rightarrow A$ enclose a region of parameter space shown in Fig. 1.

We also compute their *volumes* of the enclosed region in Table II. To this end, we randomly select 10^7 triples $\{p_0, q_0, c_{00}\}$ and count the number n of them that satisfy the uncertainty relation. The *volume* is then estimated as $n \times 10^{-7}$. This value provides a direct measure for comparing the tightness of different relations: A smaller *volume* indicates a tighter constraint. The volume is normalized to 1 if without any other constraints. Under constraint of UDRs, the volumes are in the range $[0.705, 0.930]$, which are significantly smaller than the *volume* (0.974) for the MU relation. Here, we note that tight uncertainty relation is $\langle A \rangle^2 + \langle B \rangle^2 + C^2 - 2C\langle A \rangle\langle B \rangle \leq 1$, where $\langle A \rangle = 2p_0 - 1$, $\langle B \rangle = 2q_0 - 1$, and $C = c_{00} - c_{01} = 2c_{00} - 1$, for which the volume under exact quantum constraint [79] is 0.612. For the qutrit case, the UDRs enclose regions with volumes in the range $[0.887, 0.947]$, which are also significantly smaller than the *volume* (0.999) for the MU relation (see Table II).

Uncertainty and disturbance, as two fundamental concepts in quantum measurement, have been investigated through a variety of relations. Among these, the uncertainty relations [8–20] and the error-disturbance relations [21–42] are the most studied, while Winter’s gentle measurement lemma offers a distinct perspective by directly linking a measurement’s uncertainty and its disturbance. The lemma is specific to Lüders instruments and disturbance quantified by fidelity and trace distance. Here, we develop the lemma and present a method applicable to general measurement processes and nearly all state-distance measures, with which many variants of UDRs are given. These UDRs, while differing from uncertainty relations and error-disturbance relations, can be

related to them in specific cases. On one hand, they can be interpreted as uncertainty relations when the measurements are rank one and projective. However, when measurement A is not rank one, the disturbed statistics $\mathbf{q}'$ cannot be expressed solely in terms of the statistics relevant to measurement A and the overlaps between the measurement elements of A and B ; consequently, the corresponding UDRs are not uncertainty relations. On the other hand, certain uncertainty measures employed in UDRs—such as Shannon entropy—allow for an informational interpretation; thus, UDRs reveal that disturbance lower bounds information gain. This differs conceptually from error-disturbance relations, which typically provide some *lower* bounds on the disturbance in terms of the error of the first measurement. From a technical standpoint, our approach makes full use of the results from quantum information science, including state-distance measures and the data processing inequality, in comparison to commonly used tools in prior uncertainty relations, such as the Riesz theorem in entropic uncertainty relations [15], the Schwarz inequality in Heisenberg-type variance-based relations [8–10], and the methods underlying majorization-based relations [17,19].

IV. APPLICATIONS OF UDRs IN QUANTUM INFORMATION

In quantum information science, fundamental quantum properties [80]—including purity, randomness, and coherence—play vital roles in enabling key information-processing tasks such as quantum cryptography, quantum algorithms, and quantum metrology. Experimentally estimating these resources is essential for assessing a quantum system’s capability to perform such tasks. In this section, we demonstrate how UDRs provide an effective framework for addressing this challenging problem.

The estimation of von Neumann entropy and purity with UDRs. The first quantity we consider is von Neumann entropy $S(\rho) = -\rho \log_2 \rho$. It is generally estimated via the well-known theory of majorization [81], where a projective measurement specified by A performed on ρ gives a distribution $\mathbf{p}$. One has $S(\rho) \leq H(\mathbf{p})$. However, by the UDR corresponding to the relative entropy, we have

$$S(\rho) \leq H(\mathbf{p}) - H(\mathbf{q}||\mathbf{q}'),$$

where the upper bound is optimized by the disturbance $H(\mathbf{q}||\mathbf{q}')$ measured by the relative entropy between the disturbed (by A) and undisturbed statistics of B .

Second, we consider the purity of quantum state, which can be quantified via Rényi 2-purity [82] defined as $\log_2(d\text{Tr}(\rho^2))$ with d specifying the dimension of the system of interest, i.e., which is a simple function of linear purity $\text{Tr}(\rho^2)$. According to our UDR $U_{\text{HS}} : \sqrt{\text{Tr}(\rho^2)} - \|\mathbf{p}\|_2 \geq \sqrt{\|\mathbf{q} - \mathbf{q}'\|_2^2}$, we have

$$\text{Tr}(\rho^2) \geq \|\mathbf{p}\|_2^2 + \|\mathbf{q} - \mathbf{q}'\|_2^2. \quad (7)$$

One can achieve such an estimation with only $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{q}'$, where measurement B needs to be projective.

Estimation of coherence and randomness. In this section, we estimate coherence and randomness measures defined via convex-roof construction [83–85]. That is, provided that for a pure state $|\phi\rangle$ the given resource is quantified by some measure, say, $\mathcal{R}(\phi)$, the measure for a mixed state ρ is defined

via a convex-roof construction as

$$\mathcal{R}(\rho) = \min_{\{r_n, |\phi_n\rangle\}} \sum_n \mathcal{R}(\phi_n),$$

where the optimization is taken over all the possible pure-state ensembles of $\rho = \sum_n r_n \cdot |\phi_n\rangle\langle\phi_n|$. This kind of measure is notoriously difficult to compute and estimate in experiments as there is infinite possible decomposition of ρ . Our UDRs serve well the issue of estimating convex-roof-based resource for coherence and quantum randomness. This is because, on one hand, the measures are often defined as uncertainty or a related function for a pure state. On the other hand, UDRs have favorable properties: Uncertainty is convex, while disturbance is concave. This allows one to convert a convex uncertainty into a concave one. Then, we have the following theorem.

Theorem 3 (Estimation of convex-roof-based single partite resource). If a measure for pure-state resource $\mathcal{R}(\phi)$ admits a lower bound in terms of an increasing convex function $f(\cdot)$ of uncertainty $\delta(\mathbf{p})$, i.e., $\mathcal{R}(\phi) \geq f(\delta(\mathbf{p}))$, then we have

$$\mathcal{R}(\rho) \geq f[D(\rho, \rho')] \geq f[\tilde{D}(\mathbf{q}, \mathbf{q}')], \quad (8)$$

where $D(\rho, \rho')$ is state-distance measure corresponding to $\delta(\mathbf{p})$ in an uncertainty-disturbance relation and $\tilde{D}(\mathbf{q}, \mathbf{q}')$ can be measured in experiment.

Proof. Specifying the optimal decomposition of ρ achieving the convex roof is $\{r_n, |\phi_n\rangle\}$:

$$\begin{aligned} \mathcal{R}(\rho) &= \sum_n r_n \mathcal{R}(\phi_n) \geq \sum_n r_n f(\delta(\mathbf{p}_n)) \geq f\left(\sum_n r_n \delta(\mathbf{p}_n)\right) \\ &\geq f\left[\sum_n r_n D(\phi_n, \rho'_n)\right] \geq f[D(\rho, \rho')] \geq f[\tilde{D}(\mathbf{q}, \mathbf{q}')], \end{aligned} \quad (9)$$

where ρ'_n specify the postmeasurement state relevant to $|\phi_n\rangle$ and $\rho' = \sum_n r_n \rho'_n$ and the inequality follows the defining property of f and UDRs. ■

Case 1: Convex roof of infidelity. Coherence is a property characterized and quantified with respect to a computation basis, say, $\{|A_i\rangle\}$. Consider a convex-roof-based measure [86] $C_{\text{if}}(\rho) := \min_{\{r_n, |\phi_n\rangle\}} \sum_n r_n \cdot C(\phi_n)$, where the measure for a pure state $|\phi\rangle$ is $C(\phi) = \sqrt{1-p}$ with $p \equiv \max_i |\langle\phi|A_i\rangle|^2$. We immediately have a lower bound $C_{\text{if}}(\rho) \geq \frac{\sqrt{2}}{2} D_{\text{Tr}}(\rho, \rho') \geq D(\mathbf{q}, \mathbf{q}')$ with $\rho' = \sum_i p_i |A_i\rangle\langle A_i|$ (see the Appendix).

Case 2: Quantum Fisher information. Quantum Fisher information quantifies the information of one parameter, say θ ,

carried in a quantum state $\rho(\theta)$ evolving according to unitary operation $e^{-iA\theta}$. It determines the ultimate precision of the parameter estimation via the quantum Cramér-Rao bound [87]. Quantum Fisher information is given as $F(\rho, A) = \sum_{kl} 2 \frac{(r_k - r_l)^2}{r_k + r_l} |A_{kl}|^2$, where $A_{kl} \equiv \langle\psi_l|A|\psi_k\rangle$ and r_i and $|\psi_i\rangle$ are the eigenvalues and eigenvectors of density matrix ρ , respectively. By UDRs, we finally have (see the Appendix)

$$F(\rho, A) \geq 4D_{\text{Tr}}^2(\mathbf{q}, \mathbf{q}'); \quad (10)$$

one can estimate $F(\rho, A)$ via computing $D_{\text{Tr}}(\rho, \rho')$ and detect it via experimentally estimating $D_{\text{Tr}}(\mathbf{q}, \mathbf{q}')$.

V. DISCUSSION AND CONCLUSIONS

Uncertainty and disturbance are two fundamental notions of quantum measurements, classically explored through uncertainty relations and error-disturbance trade-offs. Winter's gentle measurement lemma bridges these concepts in an elegant light, revealing a direct and insightful connection. In this work, we introduce a systematic framework for deriving such uncertainty-disturbance relations. We demonstrate that the intrinsic disturbance of a general quantum measurement can be bounded by its uncertainty—with each choice of state distance yielding a distinct variant of the relation. When the measurement is rank one, these UDRs naturally simplify into familiar uncertainty relations. To assess their tightness, we develop a volume-based approach, showing that these UDRs imply stronger constraints than the Maassen-Uffink relation for incompatible measurements on qubit and qutrit systems. Furthermore, we apply these UDRs across quantum information science, offering practical protocols for estimating von Neumann entropy, quantum purity, coherence, and randomness extraction. Thus, our uncertainty-disturbance relations not only deepen the conceptual foundations of quantum mechanics but also open promising pathways for quantum information applications—bringing theory closer to experiment.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: PROOF OF THEOREM 1

Here, we shall give the proofs for the results in Table I in the main text. Let the auxiliary system be in a pure state ρ_a (as a mixed state can be purified by including an environment) and specify ρ_{sae} the purification of $\rho_s \otimes \rho_a$ and ρ'_{sae} the postmeasurement state $\sum_i Q_i \rho_{sae} Q_i$, where $\{Q_i\}$ is the extended measurement realizing the measurement of interest.

a. Infidelity

Uncertainty-disturbance relation Provided the measure of $\text{IF}(\rho_1, \rho_2) = \sqrt{1 - F^2(\rho_1, \rho_2)}$ with $F(\rho_1, \rho_2) = [\text{tr}(\sqrt{\sqrt{\rho_2}\rho_1\sqrt{\rho_2}})]^2$.

$$\begin{aligned} F(\rho_s, \rho'_s) &\geq F(\rho_{sae}, \rho'_{sae}) = [\text{Tr}(\sqrt{\sqrt{\rho_{sae}}\rho'_{sae}\sqrt{\rho_{sae}}})]^2 \\ &= [\text{Tr}(\sqrt{\rho_{sae}\rho'_{sae}\rho_{sae}})]^2 \\ &= [\text{Tr}(\sqrt{\rho_{sae} \sum_i Q_i \rho_{sae} Q_i \rho_{sae}})]^2 \\ &= [\sqrt{\sum_i \text{Tr}(\rho_{sae}(Q_i \rho_{sae} Q_i))}]^2 \\ &= \|\mathbf{p}\|_2^2. \end{aligned} \quad (\text{A1})$$

where the first inequality is due to the monotonic property of fidelity under partial trace and the second and the fourth quality are due to that the state ρ_{sae} is pure. It then follows that $\delta_{\text{IF}}(\mathbf{p}) \equiv \sqrt{1 - \|\mathbf{p}\|_2^2}$ and $\text{IF}(\rho_s, \rho'_s) = \sqrt{1 - F^2(\rho_s, \rho'_s)}$, then we have

$$\delta_{\text{IF}} \geq \text{IF}(\rho_s, \rho'_s) \geq \text{IF}(\mathbf{q}, \mathbf{q}').$$

with $\text{IF}(\mathbf{q}, \mathbf{q}') := \sum_i \sqrt{q_i q'_i}$ being the infidelity between the disturbed and undisturbed statistics.

b. Trace-distance

For trace-distance $D_{\text{Tr}}(\rho_1, \rho_2) = \frac{1}{2} \text{Tr}|\rho_1 - \rho_2|$, we have proved the corresponding Theorem 1 in the main text. Then we have

$$\begin{aligned} \sqrt{1 - \|\mathbf{p}\|_2^2} &= \sqrt{1 - \sum_i p_i^2} = \sqrt{1 - \text{Tr}(\rho_{sae}\rho'_{sae})} \\ &\geq \sqrt{1 - F(\rho_{sae}, \rho'_{sae})} \geq \frac{1}{2} \text{tr}|\rho_{sae} - \rho'_{sae}| \\ &\geq \frac{1}{2} \text{Tr}|\rho_s - \rho'_s| \end{aligned}$$

where in the second equality we have used the fact that $\text{Tr}(\rho_{sae}\rho'_{sae}) = \text{Tr}(\rho_{sae}^2) = \sum_i p_i^2$ and in the first inequality we have used the inequality $\text{tr}(\rho_1 \rho_2) \leq F(\rho_1, \rho_2)$ and the Fuchs–van de Graaf inequality $\sqrt{1 - F(\rho_1, \rho_2)} \geq \frac{1}{2} \text{Tr}|\rho_1 - \rho_2|$ in the second inequality. The last inequality is due to the data processing inequality.

c. α -Rényi divergence

Provided state distance α -Rényi divergence $D_{\text{Rd}}^\alpha(\rho_1, \rho_2) = \frac{1}{\alpha} \log_2 \text{tr}(\rho_1^{\frac{\alpha}{2\alpha}} \rho_2^{\frac{\alpha}{2\alpha}})^{\frac{2\alpha}{\alpha}}$ with $0 < \alpha < 1$, we present below the variants of theorem. As α -Rényi divergence is not symmetric, i.e., $D_{\text{Rd}}^\alpha(\rho'_s, \rho_s)$ maybe not equal to $D_{\text{Rd}}^\alpha(\rho_s, \rho'_s)$, we consider both two cases.

Adhere to the notions $\bar{\alpha} \equiv 1 - \alpha$ and $\kappa \equiv \frac{\bar{\alpha}}{2\alpha}$, we have

$$\begin{aligned} D_{\text{Rd}}^\alpha(\rho_{sae}, \rho'_{sae}) &= \frac{1}{\alpha} \log_2 \text{Tr}(\rho_{sae}^{\kappa} \rho_{sae}^{\kappa} \rho_{sae}^{\kappa})^\alpha \\ &\leq \frac{1}{\alpha} \log_2 [\text{Tr}(\rho_{sae}^{\kappa} \rho_{sae}^{\kappa} \rho_{sae}^{\kappa})]^\alpha \\ &= \frac{1}{\alpha} \log_2 [\text{Tr}(\rho_{sae} \rho_{sae}^{2\kappa})]^\alpha \\ &= \frac{1}{\alpha} \log_2 [\sum_i p_i^{2\kappa} \text{Tr}(\rho_{sae} \cdot \rho'_{i|sae})]^\alpha \\ &= \frac{1}{\alpha} \log_2 [\sum_i p_i^{2\kappa+1}]^\alpha \\ &= \frac{1}{\alpha} \log_2 (\sum_i p_i^{\frac{1}{\alpha}})^\alpha = H_{\frac{1}{\alpha}}(\mathbf{p}), \end{aligned}$$

where we have used that, for a positive Hermitian operator O , $\text{Tr} O^\alpha \geq (\text{Tr} O)^\alpha$ for $0 < \alpha < 1$ in the second inequality and we have used notation $\rho'_{i|sae} = Q_i \rho_{sae} Q_i / p_i$ to specify the post-measurement state relevant to outcome i that is a pure state and classical α -Rényi entropy $H_\alpha(\mathbf{p}) = \frac{\alpha}{\alpha} \log_2 \|\mathbf{p}\|_\alpha$. Using data processing inequality we have

$$H_{\frac{1}{\alpha}}(\mathbf{p}) \geq D_{\text{Rd}}^\alpha(\rho_{sae}, \rho'_{sae}) \geq D_{\text{Rd}}^\alpha(\rho_s, \rho'_s) \geq \tilde{D}_{\text{Rd}}^\alpha(\mathbf{q} \|\mathbf{q}'),$$

where the classical Rényi relative entropy $\tilde{D}_{\text{Rd}}^\alpha(\mathbf{q} \|\mathbf{q}') \equiv \frac{1}{\alpha-1} \log_2 \sum_i q_i^\alpha q'^{1-\alpha}$.

Consider the distance $D_{\text{Rd}}^\alpha(\rho'_{sae}, \rho_{sae})$, we have

$$\begin{aligned} D_{\text{Rd}}^\alpha(\rho'_{sae}, \rho_{sae}) &= \frac{1}{\alpha} \log_2 \text{Tr}(\rho_{sae}^{\kappa} \rho'_{sae} \rho_{sae}^{\kappa})^\alpha \\ &= \frac{1}{\alpha} \log_2 \text{Tr}(\rho_{sae} \rho'_{sae} \rho_{sae})^\alpha \\ &= \frac{1}{\alpha} \log_2 (\sum_i p_i^2)^\alpha \\ &= \frac{\alpha}{\alpha} H_2(\mathbf{p}), \end{aligned} \quad (\text{A2})$$

where the second inequality is due to that ρ_{sae} is a pure state. Using the data processing inequality, we have

$$\frac{\alpha}{\alpha} H_2(\mathbf{p}) \geq D_{\text{Rd}}^\alpha(\rho'_s, \rho_s) \geq \tilde{D}_{\text{Rd}}^\alpha(\mathbf{q}' \|\mathbf{q}). \quad (\text{A3})$$

d. Relative entropy

Proof of the variant of Theorem 1 corresponding to Relative entropy $S(\rho_1 \|\rho_2) = \text{Tr}[\rho_1(\log_2 \rho_1 - \log_2 \rho_2)]$. In this case, we only assume ρ_a is pure while ρ_s is not necessarily a pure state. We have

$$\begin{aligned} S(\rho_{sa} \|\rho'_{sa}) &= \text{Tr}(\rho_{sa}(\log_2 \rho_{sa} - \log_2 \rho'_{sa})) \\ &= -S(\rho_{sa}) - \sum_i p_i \log_2 p_i + \sum_i p_i S(\rho_{i|sa}) \\ &= -S(\rho_{sa}) - \sum_i p_i \log_2 p_i \\ &\leq -S(\rho_s \otimes \rho_a) + H(\mathbf{p}) \\ &= -S(\rho_s) - S(\rho_a) + H(\mathbf{p}) \\ &= -S(\rho_s) + H(\mathbf{p}). \end{aligned}$$

where $\rho'_{sa} = \sum_i Q_i \rho_{sa} Q_i$, $\rho_{i|sa} = Q_i \rho_{sa} Q_i / p_i$, and $\rho_{sa} \equiv \sum_i p_i \rho_{i|sa}$, and $-\text{Tr}(\rho_{sa} \log_2 \rho'_{sa}) = -\sum_i p_i \log_2 p_i + S(\rho_{i|sa}) \leq H(\mathbf{p})$. Using the data processing inequality, we have

$$-S(\rho_s) + H(\mathbf{p}) = S(\rho_{sa} \|\rho'_{sa}) \geq S(\rho_s \|\rho'_s) \geq H(\mathbf{q} \|\mathbf{q}').$$

e. Tsallis relative entropy

Proof of the variant of Theorem 1 corresponding to D_{Ts}^α . Tsallis relative entropy (Ts) $D_{\text{Ts}}^\alpha(\rho_1, \rho_2) = \frac{1}{\alpha} [1 - \text{tr}(\rho_1^\alpha \rho_2^{\bar{\alpha}})]$ with $0 \leq \alpha < 1$ and $\bar{\alpha} = 1 - \alpha$. The distance in terms of Tsallis relative entropy ($0 < \alpha < 1$) is given by

$$\begin{aligned} D_{\text{Ts}}^\alpha(\rho_{sae}, \rho'_{sae}) &= \frac{1}{\alpha} [1 - \text{Tr}(\rho_{sae}^\alpha (\rho'_{sae})^{\bar{\alpha}})] \\ &= \frac{1}{\alpha} [1 - \sum_i p_i^{\bar{\alpha}} \text{Tr}(\rho_{sae} \cdot \rho'_{i|sae})] \\ &= \frac{1 - \sum_i p_i^{2-\alpha}}{\alpha} = \frac{1 - \|\mathbf{p}\|_{2-\alpha}^{2-\alpha}}{\alpha}. \end{aligned} \quad (\text{A4})$$

Considering the disturbance in the following measurement B ,

$$\frac{1 - \|\mathbf{p}\|_{2-\alpha}^{2-\alpha}}{\bar{\alpha}} = D_{Ts}^{\alpha}(\rho_{sae}, \rho'_{sae}) \geq D_{Ts}^{\alpha}(\rho_s, \rho_s^{(r)}) \geq \tilde{D}_{Ts}^{\alpha}(\mathbf{q}, \mathbf{q}'), \quad (\text{A5})$$

which leads to a neat expression of UDR as

$$H_{2-\alpha}(\mathbf{p}) \geq \tilde{D}_{Rd}^{\alpha}(\mathbf{q}|\mathbf{q}'). \quad (\text{A6})$$

f. Hilbert-Schmidt distance

Let the state distance be defined as $D_{HS}(\rho_1 - \rho_2) = \sqrt{\text{Tr}((\rho_1 - \rho_2)^2)}$. This measure is not monotonic under general physical operations but is monotonic under projective measurements and partial-trace

$$\begin{aligned} D_{HS}(\rho_{sae}, \rho'_{sae}) &= \sqrt{\text{Tr}(\rho_{sae} - \rho'_{sae})^2} \\ &= \sqrt{\text{Tr}(\rho_{sae}^2) + \text{Tr}(\rho'_{sae})^2 - 2\text{Tr}(\rho_{sae} \cdot \rho'_{sae})} \\ &= \sqrt{\text{tr}\rho_s^2 - \|\mathbf{p}\|_2^2} \\ &\geq D_{HS}(\rho_s, \rho'_s) \geq \tilde{D}_{HS}(\mathbf{q}, \mathbf{q}'). \end{aligned}$$

where $\mathbf{q}$ and $\mathbf{q}'$ are statistics relevant to a projection measurement respectively performed on ρ_s and ρ'_s .

Recovering gentle measurement Lemma

GML considers Lüders' instrument that updates a measured state with respect to outcome i into $\rho_{i|\mathcal{L}} = \sqrt{M_i}\rho_s\sqrt{M_i}/p_i$ and states

$$F(\rho, \rho_{i|\mathcal{L}}) \geq p_i. \quad (\text{A7})$$

Consider the fidelity between initial state ρ_s and the i th postmeasurement state, we have

$$F(\rho_s, \mathcal{J}_i(\rho)) \geq F(\rho_{sae}, \rho_{i|sae}) = \text{Tr}(\rho_{sae}\rho_{i|sae}) = p_i, \quad (\text{A8})$$

where $\rho_{i|sae} = Q_i\rho_{sae}Q_i$.

APPENDIX B: THE ESTIMATE OF QUANTUM COHERENCE AND QUANTUM FISHER INFORMATION

Consider a convex-roof based coherence measure $C_{\text{if}}(\rho) := \min_{\{r_n, |\phi_n\rangle\}} \sum_n r_n \cdot C(\phi_n)$, where the measure for a pure state $|\phi\rangle$ is $C(\phi) = \sqrt{1-p}$ with $p \equiv \max_i |\langle\phi|A_i\rangle|^2$. As $\sqrt{2}C(\phi) \geq \sqrt{(1-p)(1+p)} \geq \sqrt{1 - \|\mathbf{p}\|_2^2} = \delta_{\text{Tr}}(\mathbf{p})$. From Table I, we have different choices of UDRs. For an instance, if we take the UDR for trace distance $\sqrt{1 - \|\mathbf{p}\|_2^2} \geq \frac{1}{2} \text{tr}|\rho_s - \rho'_s|$ with $\rho'_s = \sum_i p_i |A_i\rangle\langle A_i|$, we immediately have a lower bound $C_{\text{if}}(\rho) \geq \frac{\sqrt{2}}{4} \text{tr}|\rho_s - \rho'_s|$, where the lower bound does not involve any minimization process thus is easy to calculate.

Case 2. Quantum fisher information. Quantum fisher information quantifies the information of one parameter, say θ , carried in a quantum state $\rho(\theta)$ evolving according to unitary operation $e^{-iA\theta}$. It determines the ultimate precision of the parameter estimation via the quantum Cramér-Rao bound. Quantum fisher information is given as

$$F(\rho, A) = \sum_{kl} 2 \frac{(r_k - r_l)^2}{r_k + r_l} |A_{kl}|^2, \quad (\text{B1})$$

where $A_{kl} \equiv \langle\psi_l|A|\psi_k\rangle$ and r_i and $|\psi_i\rangle$ are the eigenvalues and eigenvectors of density matrix ρ , respectively. It is found that $F(\rho, A)$ actually is a convex-roof of variance, namely,

$$F(\rho, A) = 4 \min_{\{r_i, |\psi_i\rangle\}} \sum_i r_i V(|\psi_i\rangle, A), \quad (\text{B2})$$

where $V(\rho, A) = \text{tr}(\rho \cdot A^2) - \text{tr}(\rho \cdot A)^2$. The minimization is taken over any possible decomposition of $\rho = \sum_i r_i \cdot |\psi_i\rangle\langle\psi_i|$. In the case that A is binary observable with outcome ± 1 , we have $V(\rho, A) = 2(1 - \|\mathbf{p}\|_2^2)$. By UDRs, we finally have

$$F(\rho, A) \geq 4\delta_{\text{Tr}}^2(\mathbf{p}) \geq 4D_{\text{Tr}}^2(\rho, \rho') \geq 4D_{\text{Tr}}^2(\mathbf{q}, \mathbf{q}'), \quad (\text{B3})$$

one can estimate $F(\rho, A)$ via computing $D_{\text{Tr}}(\rho, \rho')$ and detect it via experimentally estimating $D_{\text{Tr}}(\mathbf{q}, \mathbf{q}')$.

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