

Symmetry-driven zero-frequency-pulling lasing via superradiant sidebands for active optical clocks

Mingyu Jeon,¹ Jinuk Kim,^{2,*} and Kyungwon An^{1,†}

¹*Department of Physics and Astronomy & Institute of Applied Physics, Seoul National University, Seoul 08826, South Korea*

²*Korea Research Institute of Standards and Science, Daejeon 34113, South Korea*



(Received 14 December 2025; accepted 1 April 2026; published 20 April 2026)

Active optical clocks based on collective atomic emission promise narrow lasing linewidths as well as reduced sensitivity to cavity fluctuations, yet residual frequency pulling limits their ultimate precision. Here, we propose a laser scheme that simultaneously achieves vanishing frequency pulling and ultranarrow linewidths by exploiting symmetry in collective superradiant emission. Atoms in a dense beam traversing a bad cavity are prepared in controlled superposition states using a pump laser with a sinusoidally modulated Rabi frequency Ω_m . The resulting emission spectrum consists of a central lasing peak at the atomic resonance and two superradiant sidebands at symmetric detunings $\pm\Omega_m$. Owing to their symmetric interaction, the central peak is protected against cavity-induced frequency shifts over detunings comparable to the cavity linewidth. As the atom number increases, its linewidth narrows far beyond the conventional gain-narrowing limit, reaching values as small as one millionth of the cavity linewidth. Small pump detunings break the symmetry of the sideband intensities, providing a robust error signal to stabilize the pump frequency without an external reference cavity. These results establish a symmetry-driven route toward ultrastable active optical clocks and collective-radiation sources with exceptional frequency stability.

DOI: [10.1103/q6lc-f9b8](https://doi.org/10.1103/q6lc-f9b8)

I. INTRODUCTION

The precise measurement of time has been one of the most important endeavors in the development of science. Many modern technologies including communications, computer networks, and Global Navigation Satellite System (GNSS) rely on precise measurements of time in order to operate properly [1]. Accurate and stable clocks also help advancing fundamental sciences. A test of Lorentz symmetry has been performed using two ytterbium optical clocks with precision on the order of 10^{-18} [2]. More precise time measurement would lead to discoveries as well as technologies and enable many fundamental sciences that might be impossible otherwise [3].

A great deal of efforts are made to create better clocks [4], such as clocks using quantum logic technology [5,6], single ions [7], multi-ions [8,9], rare-earth atoms cooled and localized in an optical lattice [10–12], etc. An idea of an active optical clock based on superradiance has also been pursued [13] and its initial tests demonstrated superradiant emission on a highly forbidden transition, highlighting the

promise of active optical clock architectures [14]. In an active optical clock, the atomic system itself acts as an oscillator and generates radiation of a narrow linewidth without relying on an external cavity [15].

In a typical superradiant laser operating as an active optical clock [13], the atoms with an ultranarrow linewidth are trapped in a bad cavity and excited by a pulsed pump laser repeatedly. Because the atomic linewidth γ_a is much narrower than the cavity linewidth γ_c , the so-called frequency pulling of the resulting superradiance lasing of frequency ω_{sr} is given by $\delta\omega \equiv \omega_{sr} - \omega_c \simeq \frac{\gamma_a}{\gamma_c}(\omega_c - \omega_a)$, thereby achieving a frequency-pulling coefficient on the order of $\frac{\gamma_a}{\gamma_c} \ll 1$, where $\omega_{a(c)}$ is the atomic (cavity) resonance frequency. This is an advantage of a superradiant active optical clock over the passive optical clocks that need to be stabilized to an external cavity, which is prone to thermal noise, and thus limiting the clock's ultimate precision [16–20].

The superradiant laser in its present form, however, has a shortcoming, namely, that its pulsed operation fundamentally limits the short-term stability. In order to overcome this shortcoming, a quasisuperradiant laser employing a dense beam of two-level atoms traversing a bad cavity has been proposed for a continuous-wave operation [21]. In addition, a slow atomic beam traversing a ring cavity [22] as well as a chip-scale atomic beam source [23] has been developed toward this and other applications. The bad-cavity condition in the above proposal [21] enforces the total atomic polarization to be proportional to the field amplitude, analogously to the rate equation approximation, and as a result, the system exhibits superradiancelike behavior in the steady state, because

*Contact author: jinuk.kim@kriss.re.kr

†Contact author: kwon@phya.snu.ac.kr

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI.

interatomic coherence is established via cavity-mediated collective interactions as the number of atoms in the cavity grows beyond a critical number, even though the atoms are initially injected in a fully inverted and incoherent state. The resulting cavity field shows both the linewidth and the frequency-pulling coefficient decreasing as the mean number of atoms, and consequently the number of collective emission events, in the cavity is increased.

The remaining frequency-pulling coefficient in the work of Ref. [21], although very small, would eventually limit the performance of the active optical clock based on this principle. Note that a coherent atomic superposition can transfer its phase to the cavity field through collective superradiant emission. As a result, the cavity field inherits the atomic phase reference and its oscillation frequency is not determined by the cavity resonance [24]. One may therefore wonder what happens if the atoms are initially prepared with both the population inversion and the atomic coherence by using a superposition state close to the north pole of the Bloch sphere.

The present study is motivated by this question. Through numerical simulations, we have found that a proper initial superposition state results in simultaneous operation of an ordinary lasing at the atomic resonance and superradiance at symmetrically shifted frequencies at $\pm\Omega_m$ when the pump laser for preparing the initial state was resonant with the atom while its amplitude sinusoidally modulated at Ω_m . Moreover, the ordinary lasing exhibits both linewidth narrowing and a vanishing frequency-pulling coefficient, as symmetric coupling to the two superradiant sidebands leads to a symmetry-induced cancellation of cavity-induced frequency shifts. The zero frequency pulling took place over the cavity-atom detuning comparable to the cavity linewidth, which is assumed to be much larger than that of the atoms in the bad-cavity limit. For a nonzero pump-atom detuning, the superradiant lasing components showed asymmetric peak heights, the difference between which was proportional to the pump-atom detuning. By using this asymmetry, a feedback loop can be implemented to nullify the remaining pump-atom detuning. We found the expected signal-to-noise ratio of the lasing signal would be enough to eliminate the pump frequency fluctuations through the feedback, and as a result, the ultimate linewidth of the central peak can be as narrow as 10^{-6} of the cavity linewidth. Our simulation parameters such as a 50-MHz cavity decay rate and a 0.5-MHz atom-cavity coupling constant are comparable with high-density high-velocity rare-earth ion beams [25] such as Ca^+ with about 400 nm transition wavelengths. Our results can be tailored for a specific atom system and can be used as a guideline for building an actual active clock system.

This paper is organized as follows. In Sec. II, we briefly recapitulate the quantum Langevin equations formulated by Liu *et al.* [21] and then introduce three different configurations in the superposition-state pumping that are considered in our simulations and provide the expressions for the corresponding initial states for the atoms. We also explain the tilted atomic beam geometry and its physical consequence as well as how the pump laser linewidth is included and how the power spectral density is calculated. Simulation results are discussed in Sec. III, where we present the results with a nonmodulated pump laser with a nonzero pump-atom detuning exciting an

initial superposition state close to the north pole of the Bloch sphere for each atom. The results motivate us to perform simulations with a modulated pump laser with a zero pump-atom detuning. The results with an ordinary lasing component occurring at the atomic resonance with two sidebands of superradiant lasing are presented. The linewidth narrowing of the ordinary lasing component as a function of the mean number of atoms in the cavity is discussed. In Sec. IV, we consider the effect of a small nonzero pump-atom detuning and discuss a feedback scheme to nullify the overall shift of the three-peak spectrum using the asymmetric heights of the superradiant peaks. In Sec. V, we summarize our work and remark on possible extension of the present work to actual experimental studies.

II. THEORY

A. Equations of motion

We consider a laser made of a beam of two-level atoms and a cavity. The atoms are prepared, immediately before entering the cavity mode, in a common coherent superposition of the ground and excited states of the same transition that subsequently participates in the lasing process. We assume an atomic beam crossing the cavity mode at a small tilt angle so as to ensure traveling-wave atom-cavity interaction [26,27] with the atom-cavity coupling constant uniform inside the cavity. We employ the same set of differential equations reported by Liu *et al.* [21]. Their formalism is recapitulated below.

Our system is described by the Tavis-Cummings Hamiltonian [28] in the rotating frame of the atom as

$$\hat{H}/\hbar = \Delta \hat{a}^\dagger \hat{a} + g \sum_j \eta(\mathbf{x}_j) (\hat{\sigma}_j^+ \hat{a} + \hat{a}^\dagger \hat{\sigma}_j^-), \quad (1)$$

where $\Delta = \omega_c - \omega_a$ is the cavity-atom detuning, $\hat{a}$ ($\hat{a}^\dagger$) is the photon annihilation (creation) operator, $\hat{\sigma}_j^-$ ($\hat{\sigma}_j^+$) is the lowering (raising) operator for the j th atom, g is the atom-cavity coupling constant (half width), also known as the vacuum Rabi frequency, and $\eta(\mathbf{x}_j)$ is the normalized cavity mode function evaluated at the position $\mathbf{x}_j$ of the j th atom ($|\eta(\mathbf{x}_j)| \leq 1$).

A set of differential equations in terms of complex-valued variables derived from the Hamiltonian equations with noise operators phenomenologically added are [21,29]

$$\begin{aligned} \frac{da}{dt} &= -(\kappa/2 + i\Delta)a - ig(J^x - iJ^y) \\ &\quad + i\frac{\sqrt{\kappa}}{2}(\xi^p + i\xi^q), \end{aligned} \quad (2)$$

$$\begin{aligned} \frac{ds_j^x}{dt} &= \frac{\Gamma_c}{2}\eta_j J^x s_j^z - \frac{\Gamma_\Delta}{2}\eta_j J^y s_j^z \\ &\quad - \frac{\Gamma_c}{\sqrt{\Gamma_0}}\eta_j s_j^z \xi^p + \frac{\Gamma_\Delta}{\sqrt{\Gamma_0}}\eta_j s_j^z \xi^q, \end{aligned} \quad (3)$$

$$\begin{aligned} \frac{ds_j^y}{dt} &= \frac{\Gamma_c}{2}\eta_j J^y s_j^z + \frac{\Gamma_\Delta}{2}\eta_j J^x s_j^z \\ &\quad + \frac{\Gamma_c}{\sqrt{\Gamma_0}}\eta_j s_j^z \xi^q - \frac{\Gamma_\Delta}{\sqrt{\Gamma_0}}\eta_j s_j^z \xi^p, \end{aligned} \quad (4)$$

$$\begin{aligned} \frac{ds_j^z}{dt} &= -\frac{\Gamma_c}{2}\eta_j (J^x s_j^x + J^y s_j^y) + \frac{\Gamma_\Delta}{2}\eta_j (J^y s_j^x - J^x s_j^y) \\ &\quad + \frac{\Gamma_c}{\sqrt{\Gamma_0}}\eta_j [s_j^x (\xi^p + \xi^q) + s_j^y (\xi^p - \xi^q)], \end{aligned} \quad (5)$$

where $a = \langle \hat{a} \rangle$, κ is the cavity decay rate (full width), and ξ^p and ξ^q are complex-valued-variable or c -number representation of noise operators corresponding to $\hat{\xi}^p = -i(\hat{\xi}^\dagger - \hat{\xi})$ and $\hat{\xi}^q = \hat{\xi}^\dagger + \hat{\xi}$ with $\hat{\xi}$ and $\hat{\xi}^\dagger$ associated with $\hat{a}$ and $\hat{a}^\dagger$, respectively. The noise terms arise from the coupling of $\hat{a}$ and $\hat{a}^\dagger$ to the vacuum outside through the cavity mirrors. In addition, $J^{x(y)} \equiv \sum_{k=1}^N \eta(\mathbf{x}_k) s_k^{x(y)}$, where s_j^x , s_j^y , and s_j^z are the c -number representations of the operators $\hat{\sigma}_j^x \equiv \hat{\sigma}^+ + \hat{\sigma}^-$, $\hat{\sigma}_j^y \equiv -i(\hat{\sigma}^+ - \hat{\sigma}^-)$, and $\hat{\sigma}_j^z$ under the semiclassical approximation, respectively. In writing Eqs. (3)–(5), the interaction terms, except the noise terms, not proportional to J^x or J^y such as $s_j^{x,y}(s_j^z + 1)$ are neglected since these terms are on the order of unity and thus much smaller than the other terms proportional to J^x and J^y since $N \gg 1$ in all of our simulations to be presented below. Moreover, the atomic decay time or the inverse of the atomic decay rate Γ_{sp} is assumed to be much larger than the atom-cavity interaction time τ , so the atomic decay is neglected in Eqs. (3)–(5). The variables Γ_Δ , Γ_c , and Γ_0 are defined as

$$\Gamma_\Delta \equiv \frac{2g^2\Delta}{\Delta^2 + (\kappa/2)^2}, \quad \Gamma_c \equiv \frac{g^2\kappa}{\Delta^2 + (\kappa/2)^2}, \quad \Gamma_0 = \frac{4g^2}{\kappa}. \quad (6)$$

We need to calculate a in order to obtain the first-order correlation or the power spectral density as to be explained in Sec. II C. We assume a bad cavity, for which $\kappa \gg \sqrt{N}g$ is satisfied with $\bar{N}$ the mean number of atoms in the cavity. Under this condition, Eq. (2) quickly reaches a steady state, so we can approximately set $\frac{da}{dt} = 0$. The bad-cavity approximation, similar to the rate equation approximation in quantum optics, is expected to give a correct steady-state solution, which we are interested in. It allows us to express a as

$$a = -\frac{1}{4g}(\Gamma_\Delta + i\Gamma_c) \left[J^x - iJ^y - \frac{2}{\sqrt{\Gamma_0}}(\xi^p + i\xi^q) \right]. \quad (7)$$

B. Superposition-state pumping

In this study, we are interested in what happens if atoms are initially prepared in a common superposition state for all atoms with a uniform atom-cavity coupling constant. We consider three different pumping cases for the initial superposition states.

1. Superposition state with a nonzero pump-atom detuning

The pump laser for preparing the superposition state is detuned from the atomic resonance by $\delta \neq 0$. The pumping time is assumed to be much shorter than the atomic damping time, so we can neglect the atomic damping during the pumping process. The initial atomic state can be calculated with the optical Bloch equation given by

$$\begin{aligned} \dot{s}^x &= \delta \cdot s^y, \\ \dot{s}^y &= -\delta \cdot s^x + \Omega s^z, \\ \dot{s}^z &= -\Omega s^y, \end{aligned} \quad (8)$$

in the rotating frame of the pump field for $t_0 \leq t \leq t_0 + \tau_p$. The atom is assumed to enter the pump field at $t = t_0$ and leave the pump field at $t = t_0 + \tau_p$. With the initial condition $s_0^x \equiv s^x(t_0) = 0$, $s_0^y \equiv s^y(t_0) = 0$, $s_0^z \equiv s^z(t_0) = -1$ and

$\Omega \gg |\delta|$ with a top-hat distribution for the pump Rabi frequency Ω in time, we obtain

$$\begin{aligned} s_0^x &\simeq -\sin(\Omega\tau_p)\sin(\delta\tau_p), \\ s_0^y &\simeq -\sin(\Omega\tau_p)\cos(\delta\tau_p), \\ s_0^z &\simeq -\cos(\Omega\tau_p). \end{aligned} \quad (9)$$

Once the atom leaves the pump field, it enters the cavity and interacts with the cavity field there, generating frequency components in the spectrum of the field according to Eqs. (3)–(5), which are expressed in the rotating frame of the atom. Therefore, we need to transform the above result back to the rotating frame of the atom at $t = t_0 + \tau_p$:

$$\begin{aligned} s_0^x &\rightarrow s_0^x \cos[\delta(t_0 + \tau_p)] - s_0^y \sin[\delta(t_0 + \tau_p)] \\ &= \sin(\Omega\tau_p) \sin(\delta t_0), \\ s_0^y &\rightarrow s_0^x \sin[\delta(t_0 + \tau_p)] + s_0^y \cos[\delta(t_0 + \tau_p)] \\ &= -\sin(\Omega\tau_p) \cos(\delta t_0), \\ s_0^z &\rightarrow s_0^z = -\cos(\Omega\tau_p). \end{aligned} \quad (10)$$

Note that the pump phase at time t_0 is encoded in s_0^x and s_0^y . Using the relation $s_0^z = \rho_{ee} - \rho_{gg} = 2\rho_{ee} - 1$ with $\rho_{ee}(\rho_{gg})$ the initial excited (ground)-state population, Eq. (10) can be rewritten as

$$\begin{aligned} s_0^x &\simeq 2\sqrt{\rho_{ee}(1 - \rho_{ee})} \sin(\delta t_0), \\ s_0^y &\simeq -2\sqrt{\rho_{ee}(1 - \rho_{ee})} \cos(\delta t_0), \\ s_0^z &\simeq 2\rho_{ee} - 1. \end{aligned} \quad (11)$$

We have reproduced the results of Liu *et al.* [21] with $\rho_{ee} = 1$, $\delta = 0$ and random η_j 's.

2. The case with two opposite pump-atom detunings

In this case, the atoms are pumped with two opposite detunings, $\pm\Omega_m$. The pump field can be expressed as $\frac{1}{2}E_0[\exp[-i(\omega_a + \Omega_m)t] + \exp[-i(\omega_a - \Omega_m)t]] = E_0 \cos(\Omega_m t) \exp(-i\omega_a t)$. Under the rotating wave approximation in the frame of the atom, it is as if the pump is resonant with the atoms with its Rabi frequency modulated as $\Omega \cos(\Omega_m t)$ with $\Omega \gg \Omega_m$. So, the Bloch equation can be written as

$$\begin{aligned} \dot{s}^x &= 0, \\ \dot{s}^y &= \Omega \cos(\Omega_m t) s^z, \\ \dot{s}^z &= -\Omega \cos(\Omega_m t) s^y. \end{aligned} \quad (12)$$

Differentiating s^z equation and substituting s^y and $\dot{s}^y$ expressions there, we obtain

$$\ddot{s}^z + \Omega_m \tan(\Omega_m t) \dot{s}^z + \Omega^2 \cos^2(\Omega_m t) s^z = 0. \quad (13)$$

We have solved Eq. (13) with the initial condition $s^z(t_0) = -1$, $s^x(t_0) = s^y(t_0) = 0$, for which the atom is assumed to enter the pump field at $t = t_0$ and leave the pump field at

$t = t_0 + \tau_p$. The atomic state after the pump is obtained as

$$\begin{aligned} s_0^x &= 0, \\ s_0^y &= -\sin \left\{ \frac{\Omega}{\Omega_m} [\sin \Omega_m(t_0 + \tau_p) - \sin(\Omega_m t_0)] \right\}, \\ s_0^z &= -\cos \left\{ \frac{\Omega}{\Omega_m} [\sin \Omega_m(t_0 + \tau_p) - \sin(\Omega_m t_0)] \right\}. \end{aligned} \quad (14)$$

With $\Omega_m \tau_p \ll 1$, Eq. (14) can be approximated as

$$\begin{aligned} s_0^x &= 0, \\ s_0^y &\simeq -\sin[\Omega \cos(\Omega_m t_0) \tau_p], \\ s_0^z &\simeq -\cos[\Omega \cos(\Omega_m t_0) \tau_p]. \end{aligned} \quad (15)$$

Note that the pump phase at time t_0 is encoded in s_0^y and s_0^z .

3. The case with two opposite pump-atom detunings with a common offset

Here, we are interested in the effect of overall pump detunings. The pump detunings are given by $\pm\Omega_m + \delta$, respectively, where $|\delta| \ll \Omega_m$ is a common offset or the average of two detunings. The Bloch equation can be written in the rotating frame of the pump frequency ($\omega_a + \delta$) as

$$\begin{aligned} \dot{s}^x &= \delta \cdot s^y, \\ \dot{s}^y &= -\delta \cdot s^x + \Omega \cos(\Omega_m t) s^z, \\ \dot{s}^z &= -\Omega \cos(\Omega_m t) s^y. \end{aligned} \quad (16)$$

Assuming $\Omega \gg |\delta|$, we can apply the same approximation as the one used in Eq. (9) to obtain

$$\begin{aligned} s_0^x &\simeq -\sin \left\{ \frac{\Omega}{\Omega_m} [\sin \Omega_m(t_0 + \tau_p) - \sin(\Omega_m t_0)] \right\} \sin(\delta \tau_p), \\ s_0^y &\simeq -\sin \left\{ \frac{\Omega}{\Omega_m} [\sin \Omega_m(t_0 + \tau_p) - \sin(\Omega_m t_0)] \right\} \cos(\delta \tau_p), \\ s_0^z &\simeq -\cos \left\{ \frac{\Omega}{\Omega_m} [\sin \Omega_m(t_0 + \tau_p) - \sin(\Omega_m t_0)] \right\}. \end{aligned} \quad (17)$$

For $\Omega_m \tau_p \ll 1$, Eq. (17) is further simplified as

$$\begin{aligned} s_0^x &\simeq -\sin[\Omega \cos(\Omega_m t_0) \tau_p] \sin(\delta \tau_p), \\ s_0^y &\simeq -\sin[\Omega \cos(\Omega_m t_0) \tau_p] \cos(\delta \tau_p), \\ s_0^z &\simeq -\cos[\Omega \cos(\Omega_m t_0) \tau_p]. \end{aligned} \quad (18)$$

We should transform this result back to the rotating frame of the atom at $t = t_0 + \tau_p$ as we did in Eq. (10). Comparing Eqs. (9) and (18), we find the correspondence $\Omega \leftrightarrow \Omega \cos(\Omega_m t_0)$. Therefore, the result in the rotating frame of the atom is

$$\begin{aligned} s_0^x &\simeq \sin[\Omega \cos(\Omega_m t_0) \tau_p] \sin(\delta t_0), \\ s_0^y &\simeq -\sin[\Omega \cos(\Omega_m t_0) \tau_p] \cos(\delta t_0), \\ s_0^z &\simeq -\cos[\Omega \cos(\Omega_m t_0) \tau_p]. \end{aligned} \quad (19)$$

We recover Eq. (15) when $\delta = 0$.

C. Simulation methods

In the numerical simulations, we incorporate the projection noise of spin operators in the initial values of the Bloch-vector

components in a way similar to the Monte Carlo simulation. If the flux of atoms incident on the cavity is sufficiently large, it can be approximated that the atoms are injected as a cluster in a superposition state with a common phase over a short time interval—specifically, within the coherence time during which the phase of the pump laser can be regarded as constant. This cluster-based injection is adopted to reduce computational burden. We have explicitly verified that simulations based on single-atom injection and those using small clusters yield identical results in the low-flux regime where single-atom injection is computationally tractable.

We model the collective spin of such a two-level atomic cluster as a classical vector with Gaussian-distributed fluctuations. Each atom is assumed to be prepared in the same superposition state, giving rise to a collective spin $\hat{j}^\mu = \sum_i^{n_{cs}} \hat{\sigma}_i^\mu$ ($\mu = x, y, z$) with the corresponding c -number initial value $\langle j_0^\mu \rangle = n_{cs} s_0^\mu$, where s_0^μ denotes the single-atom Bloch-vector components given by Eq. (11), (15), or (19), and n_{cs} is the number of atoms in a cluster. Since the clusters are uncorrelated with each other in a spin-coherent state, the spin projection j_0^μ follows binomial statistics with a variance of $n_{cs}[1 - (s_0^\mu)^2]$. In the large atomic number limit, this distribution is well approximated by a normal distribution, allowing one to treat the collective spin as a c -number variable with small stochastic noise. Accordingly, in the simulation each spin component j_0^μ is sampled from a Gaussian distribution as $j_0^\mu = \langle j_0^\mu \rangle + f_{NR}(0, \sqrt{n_{cs}[1 - (s_0^\mu)^2]})$, where $f_{NR}(m, \sigma)$ denotes a normal random variable with mean m and standard deviation σ .

The linewidth $\Delta\omega_p$ of the pump laser is included in terms of an additional random-fluctuating phase term in the phase of the pump laser. In Eq. (11), (15), or (19), the pump phase δt_0 is replaced with $\delta t_0 + \phi_{\text{noise}}$, where the value of ϕ_{noise} is given by a Gaussian distribution, the standard deviation of which equals the square root of $\Delta\omega_p \tau_p$. The mean and variance of j_0^μ then evolve in time according to s_0^μ , which includes the stochastic phase evolution of the pump laser. This approach reproduces the correct statistical widths of an atomic coherent state and the pump laser while enabling a semiclassical treatment of collective spin dynamics.

The residual Doppler shift of the atomic beam in the direction of the cavity mode is considered in terms of the atomic transverse velocity v_{tr} , which is given by the Maxwell-Boltzmann velocity distribution centered at $v_{tr} = \bar{v}_{tr}$ with a width $\Delta v_{tr} = \frac{\delta_D}{k}$ with $k = \omega_c/c$. During the interaction time, the position of the atom changes at the speed of v_{tr} , which has no effect in our simulation assuming the traveling-wave atom-cavity interaction [26,27] and thus a uniform coupling constant. But it introduces a Doppler shift in the cavity-atom detuning as $\Delta = \omega_c - \omega_a - kv_{tr}$ in Eq. (2) and Eqs. (3), (4), and (5) through Γ_Δ . Since $k\bar{v}_{tr}$ gives a common offset, we let $\bar{v}_{tr} = 0$ for simplicity in our simulations but still assuming a uniform atom-cavity coupling constant.

The injection rate of atoms in the cavity is adjusted to obtain a desired mean number of atoms in the cavity. Each new atom interacts with the cavity for the atom-cavity interaction time τ and then it is removed from the calculations of Eqs. (3), (4), and (5) and excluded from the summations for J^x and J^y . Integration of Eqs. (3), (4), and (5) are done utilizing

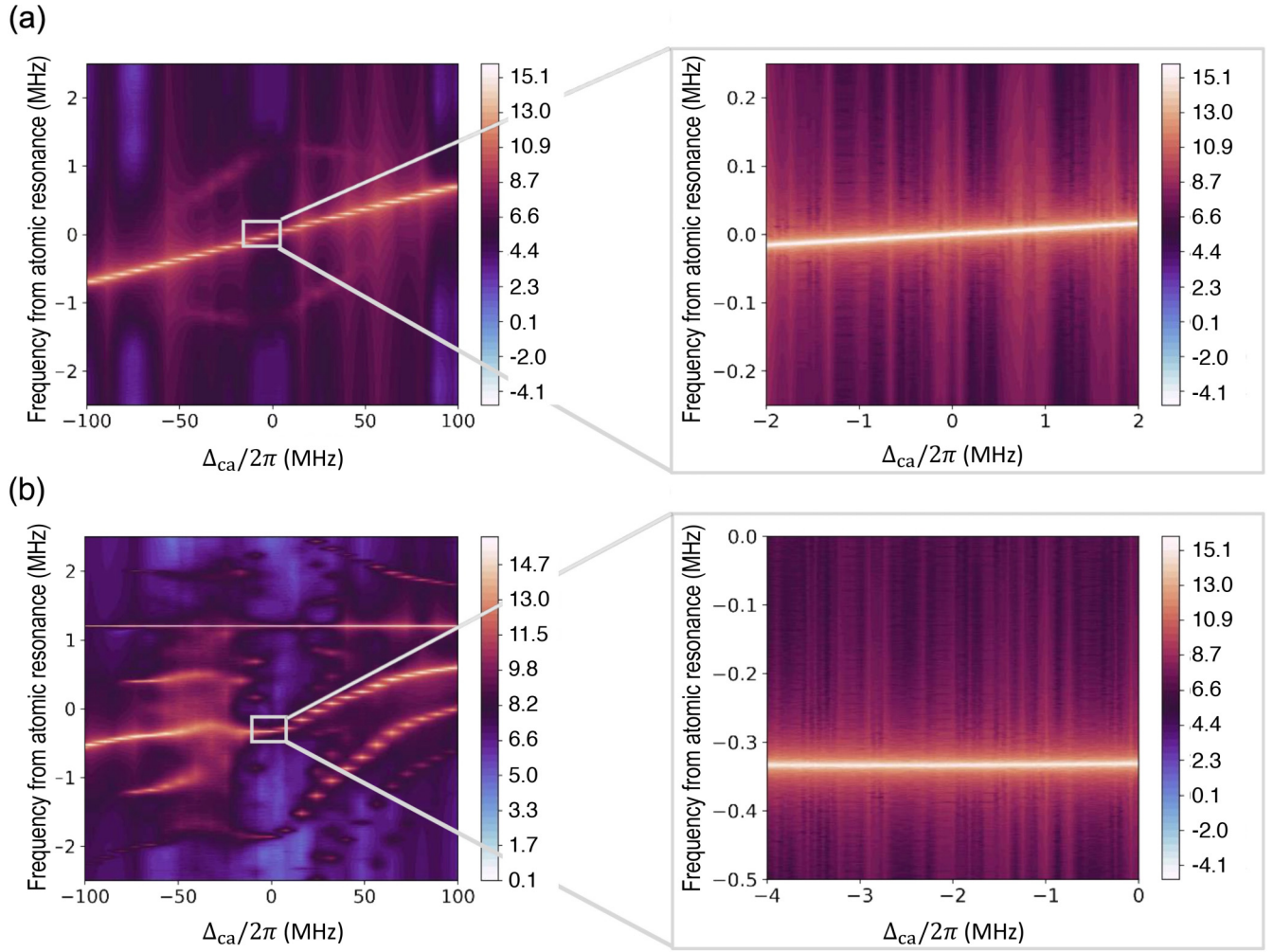


FIG. 1. Comparison of on-resonance population pumping and superposition-state pumping with nonzero pump-atom detuning. (a) Power spectral density of the intracavity field with the initial conditions, $\Omega\tau_p = \pi$ ($\rho_{ee} = 1$) and $\delta = 0$ in Eq. (11). On the right, the magnified view of the region enclosed by a rectangle in panel (a) shows a weak frequency-pulling effect as a function of Δ_{ca} . (b) Power spectral density of the intracavity field with the initial conditions, $\Omega\tau_p = 0.7952\pi$ ($\rho_{ee} = 0.8$) and $\delta/2\pi = 1.4$ MHz in Eq. (11). The magnified view of the rectangular region in panel (b) exhibits the central lasing frequency independent of the cavity-atom detuning Δ_{ca} but with a small offset. It is due to the interaction of the ordinary lasing component with the superradiance component (the horizontal line in the power spectral density plot) occurring at the pump frequency. Common simulation parameters: $\kappa/2\pi = 50$ MHz (full width), $g/2\pi = 0.25$ MHz (half width), $\tau = 1.0$ μ s, $\delta_D/2\pi = 0.1/\tau$, $\Delta\omega_p/2\pi = 10$ kHz, and $N = 2000$. Each power spectral density is plotted on a logarithmic scale in arbitrary units and is color coded according to the scale bar on the right. The spectra are obtained from the Fourier transform of $(J^+(t + \tau)J^-(t))$.

the fourth-order Runge-Kutta method. After each time step, J^x and J^y are evaluated. Under the bad-cavity assumption, a is proportional to $J^- = J^x - iJ^y$ as shown in Eq. (7). This allows us to calculate the first-order correlation $g^{(1)}(\tau)$ of the cavity field and its spectrum in terms of J^- and $J^+ = (J^-)^*$ instead of $\hat{a}$ and $\hat{a}^\dagger$: $g^{(1)}(\tau) \propto \bar{J^+(t + \tau)J^-(t)}$, where the upper bar indicates an average over time t . The spectrum of the cavity field is then obtained by the Fourier transform of $g^{(1)}(\tau)$.

III. SIMULATION RESULT

Let us first consider the simulation with the initial superposition state with nonzero pump-atom detuning corresponding to the first case discussed in Sec. II B 1 with Eqs. (8)–(11). When s_0^x or s_0^y is nonzero in the initial superposition state, J^x and J^y can build up directly from initial nonzero s_j^x and s_j^y in

Eqs. (3)–(5). Since $s_j^x \propto \sin(\delta t_j)$ and $s_j^y \propto \cos(\delta t_j)$ according to Eq. (11) with t_j the arrival time of the j th atom at the pump field (note t_0 is replaced with t_j for the j th atom), the pump phase is encoded in the individual atomic initial states and thus $\langle \hat{a}^\dagger \rangle \propto J^x + iJ^y$ oscillates as $e^{-i\delta t}$ in time, resulting in a superradiance peak at the pump frequency. If $|s_0^x|$ and $|s_0^y|$ are much smaller than $s_0^z \sim 1$, an ordinary lasing due to population inversion can also take place, and we can have both the ordinary lasing and the superradiance as shown in Fig. 1(b), whereas Fig. 1(a) shows the results with initial population inversion only ($s_0^x = s_0^y = 0$, $s_0^z = 1$). If s_0^z significantly deviates from unity, we only get the superradiance peak.

When both the superradiance and the ordinary lasing occur simultaneously, because of the cross terms such as $J^x s_j^x$ and $J^y s_j^y$ in $\dot{s}_j^z$ equation, Eq. (5), the frequency of the ordinary lasing is affected by that of the superradiance, resulting in

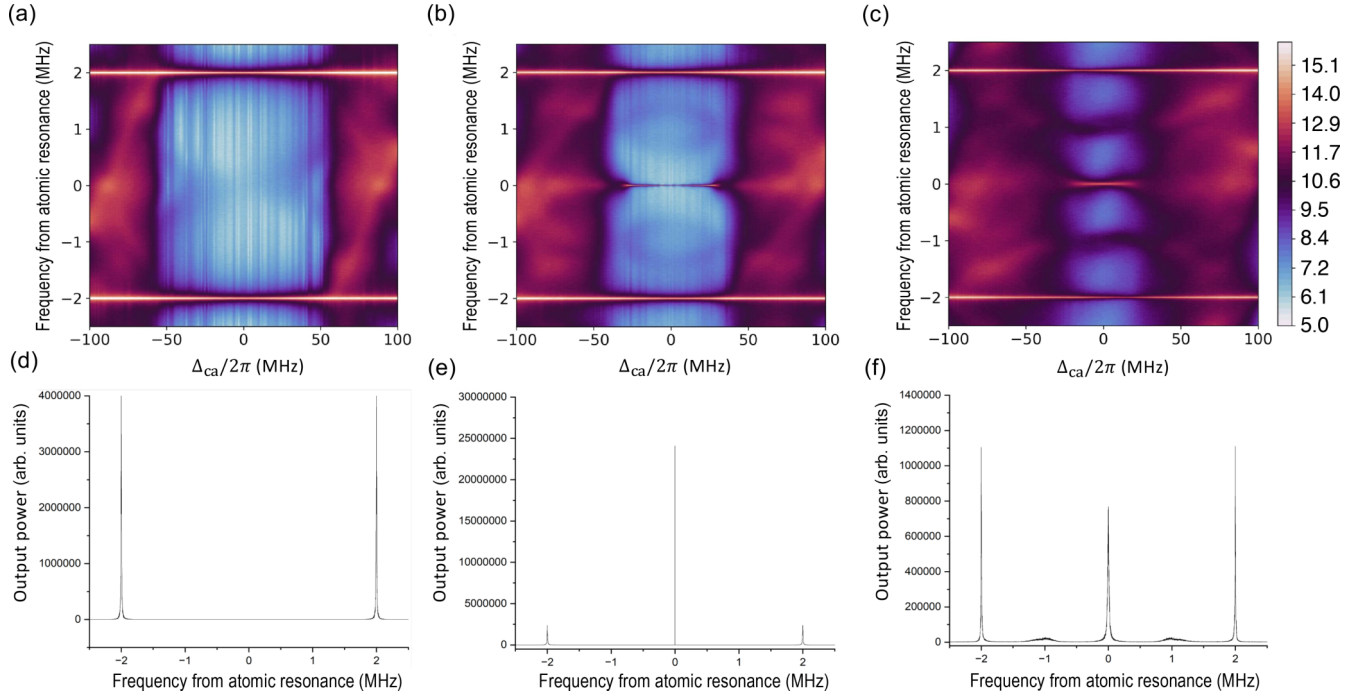


FIG. 2. Power spectral densities under the superposition-state pumping with two opposite pump-atom detunings. Common simulation conditions: $\Omega_m/2\pi = 2.0$ MHz, $\delta = 0$, $\Omega/2\pi = 12$ MHz, $\tau_p = 0.0414$ μ s ($\Omega\tau_p = 0.936\pi$), $\kappa/2\pi = 50$ MHz, $g/2\pi = 0.25$ MHz, $\delta_D/2\pi = 0.1/\tau$, $\Delta\omega_p/2\pi = 10$ kHz, and $N = 10\,000$. Spectral densities of the cavity field as a function of the cavity-atom detuning Δ_{ca} in the steady state are shown in panel (a) with the atom-cavity interaction time $\tau = 0.36$ μ s, in panel (b) with $\tau = 0.40$ μ s, and in panel (c) with $\tau = 0.44$ μ s, respectively. Spectra at $\Delta_{ca} = 0$ in each case are shown in panels (d), (e), and (f), respectively. The horizontal line segment in the center in panel (b) is the ordinary lasing line exhibiting zero frequency-pulling effect. It is fixed at the atom frequency, unaffected by the cavity-atom detuning over a significant range comparable to the cavity linewidth (~ 50 MHz). The horizontal lines at ± 2.0 MHz in the power spectral density plots correspond to the superradiance occurring at the pump frequencies.

its frequency shift. When we compare Fig. 1(a) showing the ordinary lasing spectrum and Fig. 1(b), the spectrum when both the ordinary lasing and the superradiance occur simultaneously, we can notice that the ordinary lasing frequency is pulled to or pushed away from that of the superradiance. We also notice frequency components corresponding to the sum and difference of those two frequencies, originating from the cross terms such as $J^x s_j^z$ and $J^y s_j^z$ in s_j^x and s_j^y equations.

The pulling and pushing of the ordinary lasing frequency by the superradiance component results in a flat region as indicated by a rectangle in the power spectral density plot of Fig. 1(b). In this region, the lasing frequency is independent of the cavity-atom detuning, exhibiting a zero frequency-pulling coefficient. It implies that the lasing frequency would not be affected by the fluctuations in the cavity frequency. Such zero frequency-pulling effect is a desired asset in optical clocks. However, the lasing frequency also exhibits a constant offset, not locked to the atomic resonance frequency. Since the offset depends on the initial atomic states in a complicated way, the zero frequency pulling seen in Fig. 1(b) cannot be used in optical clocks.

It might be possible, however, to overcome this problem by introducing another superradiance component with the opposite frequency detuning. If two superradiance components at $\pm\Omega_m$ symmetrically affect the ordinary lasing component in the middle, we might be able to lock the ordinary lasing frequency at the atomic resonance regardless of the cavity-atom detuning. In order to check the effectiveness of this

idea, we now prepare the initial superposition state of atom according to the second case discussed in Sec. II B 2 with Eqs. (12)–(15) and perform numerical simulations. The results are summarized in Fig. 2. Only difference among three cases there is the atom-cavity interaction times τ , which are 0.36, 0.40, and 0.44 μ s in panels (a), (b), and (c), respectively. The central lasing peak is sandwiched between two superradiance peaks at $\pm\Omega_m$ around the atomic resonance. Because of the symmetric interaction with the two opposite superradiance peaks, the central lasing peak is locked to the atomic resonance regardless of the cavity-atom detuning, i.e., exhibiting zero frequency pulling, as best seen in Fig. 2(b). The zero frequency pulling occurs over a range of the cavity-atom detuning comparable to the cavity decay rate, which is about 50 MHz in our simulation. Clean central peak is obtained when the empirical condition $\Omega_m\tau \simeq 5$ is satisfied with τ the atom-cavity interaction time.

As can be seen in Fig. 3, the central peak corresponds to the ordinary lasing, whereas the two symmetric side peaks correspond to the superradiance with their linewidth the same as the pump laser linewidth. As discussed in Sec. II C, the power spectral density of the cavity field is obtained by Fourier transforming the first-order correlation function of $\langle \hat{a}^\dagger(t+\tau)\hat{a}(t) \rangle \propto \overline{J^+(t+\tau)J^-(t)}$ with time average over t chosen after the steady state is reached. We varied the mean number N of atoms in the cavity and the pump pulse area $\Omega\tau_p$ independently and calculated the power spectral density. The results are summarized in Fig. 3. The vertical axis

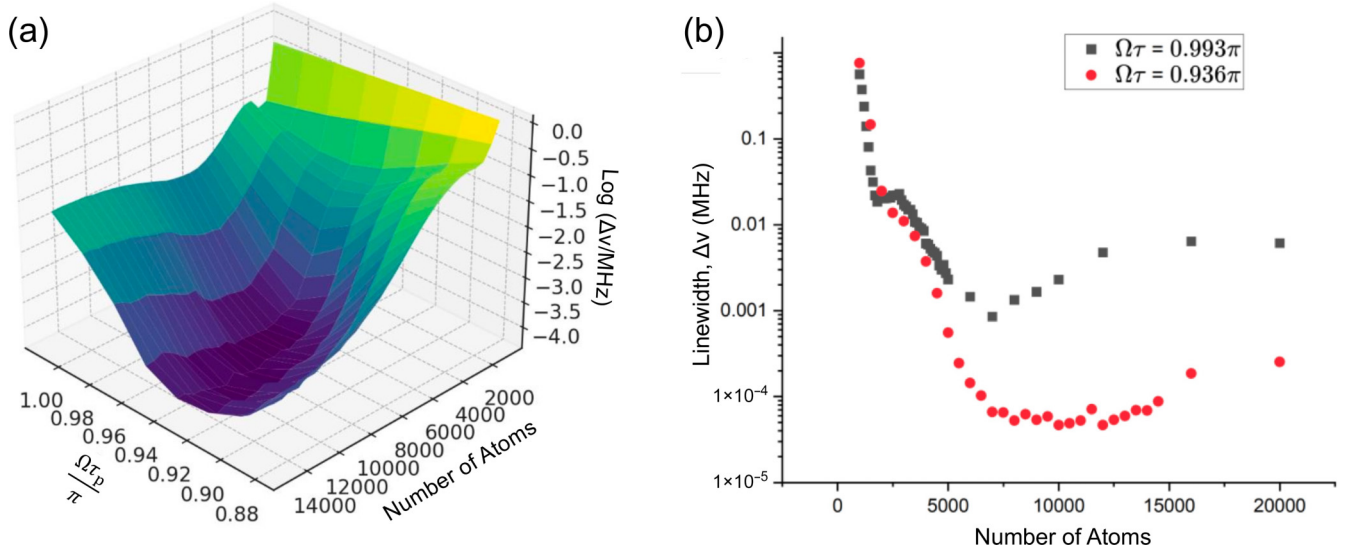


FIG. 3. Linewidth of the central peak in the spectrum under the superposition-state pumping with two opposite pump-atom detunings ($\delta = 0$). (a) Linewidth in MHz in log (for base 10) scale as a function of the pump pulse area $\Omega\tau_p$ and the mean number N of atoms in the cavity. The minimum occurs when $\Omega\tau_p \simeq 0.94\pi$ and $N \simeq 10000$ under our simulation conditions listed below. (b) Cross sections of the surface plot in panel (a) at $\Omega\tau_p = 0.936\pi$ and 0.993π . For the former, the minimum linewidth is about 50 Hz, amounting to 1% of the collective emission rate $\Gamma_c = 4g^2/\kappa$, corresponding to the number of collective emission of $N\Gamma_c\tau \simeq 130$. Simulation conditions are $\kappa/2\pi = 50$ MHz, $g/2\pi = 0.25$ MHz, $\tau = 0.4$ μ s, $\delta_D/2\pi = 0.1/\tau$, $\Omega_m/2\pi = 2$ MHz, $\Omega/2\pi = 12$ MHz, and $\Delta\omega_p/2\pi = 10$ kHz. The pump pulse area is varied from 0.879π to 1.02π by changing the pumping time τ_p from 0.0366 to 0.0422 μ s.

corresponds to the logarithm of the central-peak linewidth in MHz. Over the range of N and $\Omega\tau_p$ varied, the linewidth changes by 4 orders of magnitude, reaching the minimum of about 50 Hz when $N \simeq 10000$ and $\Omega\tau_p \simeq 0.94\pi$. Since the collective emission rate is given by $\Gamma_c = 4g^2/\kappa$, it is natural to compare the linewidth with Γ_c . We note that the minimum linewidth amounts to 1/100 of Γ_c under our simulation conditions. In contrast, the two side peaks at $\pm\Omega_m$ correspond to superradiance and their linewidths are the same as that of the pump field. It is because the superradiant peaks coherently result from initially injected s_0^x and s_0^y in Eq. (15) and the linewidth originates from the frequency uncertainty in Ω_m , i.e., the linewidth of the pump field.

IV. DISCUSSION

A. Evolution of linewidth reduction

When the linewidth of the central peak in the power spectral density of the cavity field is plotted as a function of the mean number of atoms in the cavity as in Fig. 3(b), several distinct features can be noticed. First, the linewidth rapidly drops to about 20 kHz, which is close to the pump-field linewidth in our simulations. This drop corresponds to conventional gain narrowing, where the linewidth is inversely proportional to the mean number of photons in the cavity. When the linewidth is plotted as a function of the mean photon number in the log-log scale as in Fig. 4, we find the initial slope is close to -1 .

Second, the linewidth is further decreased with the mean number of atoms, exhibiting an abrupt slope change around the pump linewidth in its decrease. The reduction of the linewidth beyond the pump linewidth is qualitatively different from the gain narrowing in that the mean photon

number is clamped or even reduced while the linewidth drops, corresponding to near vertical segments in Fig. 4. During this drop, the linewidth gets narrowed while the peak height increased and thus the total number of photons remains about the same.

The linewidth does not drop forever. Eventually, the linewidth starts to grow, so does the mean photon number, as the number of atoms is increased. This linewidth broadening must be related to power broadening or dressed-state interactions under the strong intracavity field. As the mean photon number increases, the atoms experience a large Rabi frequency $2g\sqrt{n}$, and the turnaround of the linewidth in Fig. 4 occurs approximately when $2g\sqrt{n} \sim \kappa$, indicating a departure from the ideal bad-cavity (superradiant) regime.

In our simulations, the significant linewidth reduction, down to 100 Hz or less, takes place in a narrow range of the pump pulse area corresponding to $0.90 < \rho_{ee} < 0.98$, judging from Fig. 4. It is important to have just enough amount of s_0^x and s_0^y . Too much or too little s_0^x and s_0^y will make either the ordinary lasing component or the superradiance components too small to induce sufficient interaction between them. For those pulse areas, the minimum linewidth is obtained with $N_{\min} \sim 8000$ with $\Gamma_c/2\pi = 2g^2/(\pi\kappa) = 5$ kHz, so the number $\mathcal{N}_{\text{coll}}$ of collective emission events is given by $\mathcal{N}_{\text{coll}} = N_{\min}\Gamma_c\tau \sim 100$. One then expects the minimum linewidth to be given by $(\Gamma_c/2\pi)/\mathcal{N}_{\text{coll}} = 1/(2\pi N_{\min}\tau) \sim 50$ Hz, which is exactly what we observe.

From Fig. 3(b), one can also note that the linewidth reduction beyond the gain narrowing starts to occur when the number $\mathcal{N}_{\text{coll}}$ of collective emission events becomes much larger than unity. So, the linewidth reduction of the central conventional lasing peak beyond the gain narrowing can be

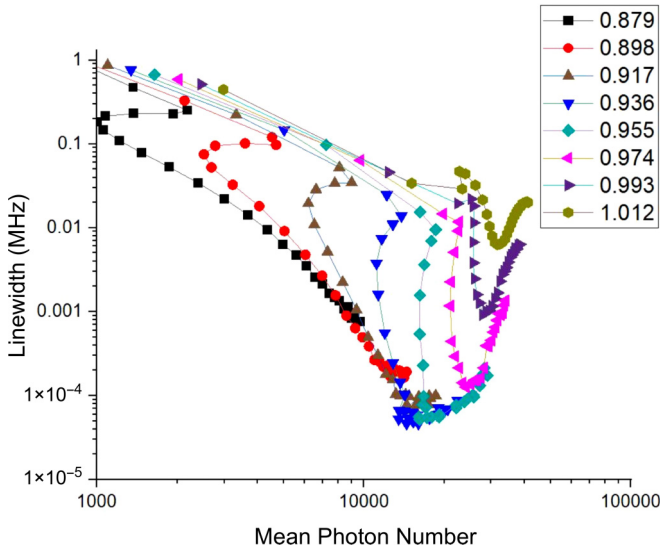


FIG. 4. Linewidth of the central peak vs the intracavity mean photon number associated with the central peak for various $\Omega\tau_p$. For the pump pulse area $0.92\pi < \Omega\tau_p < \pi$, the linewidth of the central peak decreases, inversely proportional to the mean photon number, until it reaches the pump linewidth. This is nothing but the gain narrowing in the conventional lasers. Beyond the pump linewidth, as the gain increases, the mean photon number does not change but the linewidth is further decreased to reach its minimum. The linewidth starts to increase as the gain is further increased as the mean photon number also increases. The smallest linewidth of 50 Hz is obtained when $\Omega\tau_p = 0.936\pi$ and $N \simeq 10\,000$.

thought as being taking place when the conventional lasing evolves to superfluorescence, the collective interaction starting from pure population inversion.

1. Collective interaction to induce the minimum linewidth

Note that $1/(2\pi\tau) \sim 390$ kHz corresponds to the transit-time broadening (within a factor of 2), so the minimum linewidth can be thought as the transit time broadening divided by the number of atoms. This interpretation is analogous to the linewidth in laser mode locking, where each pulse has a broad linewidth but the each spectral peak has a linewidth reduced by the factor equal to the number of pulses in the pulse train. Likewise, the emission from each atom in our case would exhibit a transit-time broadening but the collective emission of $N_{\min}$ such atoms results in a linewidth reduced from the transit-time broadening by a factor equal to the number $N_{\min}$ of atoms.

In order to reduce the minimum linewidth, one tempts to reduce the transit time broadening and increase the number of atoms corresponding to the minimum. But they are not independent from each other, and therefore such adjustment is not possible. We rather pay attention to the empirical fact that the minimum linewidth is obtained when $\mathcal{N}_{\text{coll}} \sim 10^2$, so the minimum linewidth is roughly given by $\Gamma_c/100 = g^2/(25\kappa)$, suggesting that smaller g and larger κ would reduce the minimum linewidth further at the expense of the larger mean number of atoms.

B. Effect of pump carrier detuning and its compensation for an active optical clock

In practice, the frequency of the pump laser itself may change. If the frequency of the central peak changes in accordance to that of the pump, it is necessary to have a nonfrequency indicator with which we can form a feedback loop to correct the pump frequency detuning for optical clock applications. For this purpose, we investigate the power spectral density as a function of the pump-atom detuning δ . The pump laser is detuned from the atomic resonance by δ , while its amplitude is modulated as $\cos(\Omega_m t)$, leading to two pump detuning frequencies at $\delta \pm \Omega_m$. The initial conditions for s_0^μ ($\mu = x, y, z$) are derived in Eq. (19). The simulation results are summarized in Fig. 5, which shows in panel (b) that the three-peak spectrum shifts as a whole by the pump-atom detuning δ . Nonzero δ also induces imbalance in the side-peak heights. The superradiant peak closer to the atomic resonance becomes larger, while the other becomes smaller. It is understandable since the gain gets larger toward the atomic resonance.

In applications, the pump-atom detuning tends to be minimized, so we varied δ in a small range from -0.03 to $+0.03$ MHz, within 1.5% of the modulation frequency $\Omega_m = 2$ MHz. In Fig. 5(a), the contrast ratio of the peak heights, defined as $(h_+ - h_-)/(h_+ + h_-)$ with $h_{>(<)}$ the height of the superradiance peak at the higher (lower) detuning, is plotted as a function of δ . We are interested in the contrast ratio for large number of atoms, for which the linewidth of the central peak is greatly reduced. The slope of the contrast ratio is saturated as the number of atoms is increased, resulting in a slope of about $-5/3 \text{ MHz}^{-1}$. This height difference can be used as a nonfrequency indicator for a feedback loop.

1. Pump stability with the feedback by peak height difference

The pump laser stability achievable by a feedback loop can be estimated as follows. Suppose we have a pump-atom detuning of δ . The peak height difference would then be $-2 \times (5/3)\delta/(1 \text{ MHz})$ times the mean peak height. To resolve this difference, we need a signal-to-noise ratio larger than $(1 \text{ MHz})/(3.3\delta)$. Conversely, if the signal-to-noise ratio is equal to $(1 \text{ MHz})/(3.3\delta)$, the frequency stability of the pump laser by the feedback mechanism would be just 2δ (full width). If we want the pump stability comparable to the minimum linewidth ($2\delta = 50$ Hz) in Fig. 4, the signal-to-noise ratio should be $(1 \text{ MHz})/(83 \text{ Hz}) = 1.2 \times 10^4$. The output photon flux is estimated as $n\kappa \sim 3n \times 10^8$ photons/s according to Fig. 4, with n the number of photons associated with the average of the side-peak heights. For an averaging time of T , we get $3n(T/s) \times 10^8$ photons, and thus the signal-to-noise ratio is about $\sqrt{3n(T/s)}10^4$, and thus an enough signal-to-noise ratio is obtained if $n > 0.48/(T/s)$. For Fig. 5(b), $n \simeq 3.6$, so the condition is easily satisfied in our simulations for $T \sim 0.2$ s, making the pump laser stability comparable to the minimum linewidth of the central peak. As a whole, the pump laser and our system may then form an active optical clock without the need for a separate high-finesse reference cavity, which is typically required for laser stabilization in conventional passive optical clocks.

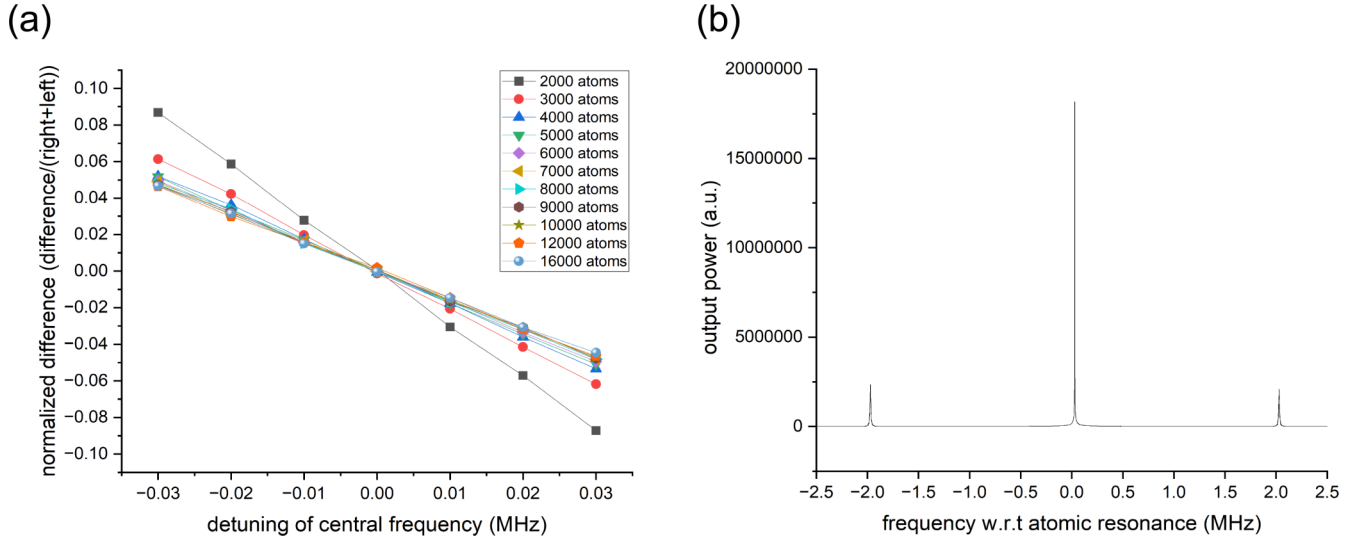


FIG. 5. The effect of the pump-atom detuning δ from the atomic resonance. Nonzero pump-atom detuning introduces an overall shift of the spectrum by δ as well as an imbalance between the superradiant peak heights. (a) Normalized difference or contrast ratio defined as (peak difference)/(peak sum) as a function of δ for various mean number of atoms in the cavity. (b) Spectrum of the cavity field for $\delta/2\pi = 0.03$ MHz and $N = 10\,000$. Simulation conditions are $\kappa/2\pi = 50$ MHz, $g/2\pi = 0.25$ MHz, $\tau = 0.4$ μ s, $\delta_D/2\pi = 0.1/\tau$, $\Omega_m/2\pi = 2$ MHz, $\tau_p = 0.0390$ μ s, $\Omega/2\pi = 12$ MHz ($\Omega\tau_p = 0.936\pi$), and $\Delta\omega_p/2\pi = 10$ kHz.

C. Regime of validity and experimental considerations

The results presented in this work rely on several assumptions that define the regime of validity of our model. First, our scheme works in the bad-cavity limit, such that the cavity field adiabatically follows the collective atomic polarization and the lasing frequency is primarily determined by the atomic ensemble rather than the cavity resonance. Second, a large number of atoms in the cavity is assumed, which justifies the semiclassical c -number treatment of collective spin variables. Third, we neglect the atomic spontaneous decay rate $\Gamma_{sp} (\ll \kappa)$ under the condition $\Gamma_{sp}(\tau + \tau_p) \ll 1$. This implies that the natural lifetime of the chosen transition must be much longer than both the state-preparation time τ_p and the interaction time τ . If free-space spontaneous emission during the transit becomes significant, it would lead not only to additional decoherence and linewidth broadening but also to a loss of excited-state population, thereby reducing the effective gain. This represents a practical constraint on experimental implementation. Therefore, an appropriate optical transition is required, and ion beams with higher velocities than neutral atom beams can broaden the range of viable options. Moreover, since different transitions exhibit different sensitivities to technical noise sources, the ability to explore multiple transitions can be advantageous.

A key advantage of the present scheme is its strong suppression of sensitivity to cavity frequency fluctuations. Instead, the dominant frequency reference is set by the atomic transition and the pump laser used to prepare the atomic superposition. As a result, stabilization of the pump laser phase noise and amplitude noise associated with modulation of the pump field becomes an important experimental consideration. Additional technical noise sources that may degrade the achievable linewidth include fluctuations in atomic flux and velocity, as well as residual Doppler broadening.

V. CONCLUSION

In summary, we considered a beam of two-level atoms traversing a low- Q cavity, where the atoms prepumped to a superposition state of the ground and excited states with the excited-state amplitude dominant prior to entering the cavity. We numerically solved the quantum Langevin equations under conditions where the pump laser, resonant with the atomic transition, was modulated at a frequency Ω_m for various cavity-atom detunings. Our simulations reveal that conventional lasing occurs at the atomic resonance, while superradiance simultaneously emerges at $\pm\Omega_m$, independent of the cavity-atom detuning. The central peak, originating from the initial population inversion, exhibits a zero frequency-pulling coefficient. Unlike previously studied steady-state superradiant lasers, where residual frequency pulling is suppressed but not eliminated, the present scheme achieves exact cancellation of cavity-induced shifts through symmetry in the collective emission spectrum. Moreover, the linewidth of the central lasing peak narrows beyond the conventional gain-narrowing limit while the mean photon number remains stationary as the mean number of atoms increases. This transition takes place once the number of collective emission events greatly exceeds unity, marking the crossover from conventional lasing to superfluorescence. The minimum linewidth achieved is as small as one millionth of the cavity linewidth. When the pump frequency was detuned by a small offset δ from the atomic resonance, all three lasing peaks shifted by δ , with the side peaks exhibiting asymmetric intensities. A signal-to-noise analysis indicates that this asymmetry could be exploited in a simple feedback scheme to lock the pump laser to the atomic resonance, with an uncertainty on the order of the central-peak linewidth. The combination of zero frequency pulling and ultranarrow linewidths highlights attractive features for clock applications. When implemented in suitable atomic systems [30], these results may enable the

realization of a class of ultrastable active optical clocks for future frequency standards.

ACKNOWLEDGMENTS

We thank Myoungseon Hur for helpful discussions on eliminating the pump laser detuning. This work was supported by National Research Foundation of Korea (Grant No. RS-2025-00553232).

DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

-
- [1] S. Diddams, T. Udem, J. Bergquist, A. Curtis, R. Drullinger, L. Hollberg, W. Itano, W. Lee, C. Oates, K. Vogel, and D. Wineland, An optical clock based on a single trapped $^{199}\text{Hg}^+$ ion, *Science* **293**, 825 (2001).
 - [2] C. Sanner, N. Huntemann, R. Lange, C. Tamm, E. Peik, M. S. Safronova, and S. G. Porsev, Optical clock comparison for Lorentz symmetry testing, *Nature (London)* **567**, 204 (2019).
 - [3] A. Aepli, K. Kim, W. Warfield, M. S. Safronova, and J. Ye, Clock with 8×10^{-19} systematic uncertainty, *Phys. Rev. Lett.* **133**, 023401 (2024).
 - [4] S. M. Brewer, J.-S. Chen, A. M. Hankin, E. R. Clements, C. W. Chou, D. J. Wineland, D. B. Hume, and D. R. Leibbrandt, $^{27}\text{Al}^+$ quantum-logic clock with a systematic uncertainty below 10^{-18} , *Phys. Rev. Lett.* **123**, 033201 (2019).
 - [5] P. O. Schmidt, T. Rosenband, C. Langer, W. M. Itano, J. C. Bergquist, and D. J. Wineland, Spectroscopy using quantum logic, *Science* **309**, 749 (2005).
 - [6] T. Rosenband, D. B. Hume, P. O. Schmidt, C. W. Chou, A. Brusch, L. Lorini, W. H. Oskay, R. E. Drullinger, T. M. Fortier, J. E. Stalnaker, S. A. Diddams, W. C. Swann, N. R. Newbury, W. M. Itano, D. J. Wineland, and J. C. Bergquist, Frequency ratio of Al^+ and Hg^+ single-ion optical clocks; metrology at the 17th decimal place, *Science* **319**, 1808 (2008).
 - [7] P. Micke, T. Leopold, S. A. King, E. Benkler, L. J. Spieß, L. Schmöger, M. Schwarz, J. R. C. López-Urrutia, and P. O. Schmidt, Coherent laser spectroscopy of highly charged ions using quantum logic, *Nature (London)* **578**, 60 (2020).
 - [8] M. Steinel, H. Shao, M. Filzinger, B. Lipphardt, M. Brinkmann, A. Didier, T. E. Mehlstäubler, T. Lindvall, E. Peik, and N. Huntemann, Evaluation of a $^{88}\text{Sr}^+$ optical clock with a direct measurement of the blackbody radiation shift and determination of the clock frequency, *Phys. Rev. Lett.* **131**, 083002 (2023).
 - [9] H. N. Hausser, J. Keller, T. Nordmann, N. M. Bhatt, J. Kiethe, H. Liu, I. M. Richter, M. von Boehn, J. Rahm, S. Weyers, E. Benkler, B. Lipphardt, S. Dörscher, K. Stahl, J. Klose, C. Lisdat, M. Filzinger, N. Huntemann, E. Peik, and T. E. Mehlstäubler, $^{115}\text{In}^+ - ^{172}\text{Yb}^+$ coulomb crystal clock with 2.5×10^{-18} systematic uncertainty, *Phys. Rev. Lett.* **134**, 023201 (2025).
 - [10] G. Wilpers, T. Binnewies, C. Degenhardt, U. Sterr, J. Helmcke, and F. Riehle, Optical clock with ultracold neutral atoms, *Phys. Rev. Lett.* **89**, 230801 (2002).
 - [11] H. Katori, M. Takamoto, V. G. Pal'chikov, and V. D. Ovsiannikov, Ultrastable optical clock with neutral atoms in an engineered light shift trap, *Phys. Rev. Lett.* **91**, 173005 (2003).
 - [12] M. Takamoto, F.-L. Hong, R. Higashi, and H. Katori, An optical lattice clock, *Nature (London)* **435**, 321 (2005).
 - [13] J. G. Bohnet, Z. Chen, J. M. Weiner, D. Meiser, M. J. Holland, and J. K. Thompson, A steady-state superradiant laser with less than one intracavity photon, *Nature (London)* **484**, 78 (2012).
 - [14] M. A. Norcia, M. N. Winchester, J. R. K. Cline, and J. K. Thompson, Superradiance on the millihertz linewidth strontium clock transition, *Sci. Adv.* **2**, e1601231 (2016).
 - [15] J. Zhang, T. Shi, J. Miao, D. Yu, and J. Chen, An extremely bad-cavity laser, *npj Quantum Inf.* **10**, 87 (2024).
 - [16] Y. Huang, H. Guan, P. Liu, W. Bian, L. Ma, K. Liang, T. Li, and K. Gao, Frequency comparison of two $^{40}\text{Ca}^+$ optical clocks with an uncertainty at the 10^{-17} level, *Phys. Rev. Lett.* **116**, 013001 (2016).
 - [17] K. Beloy, M. I. Bodine, T. Bothwell, S. M. Brewer, S. L. Bromley, J.-S. Chen, J.-D. Deschênes, S. A. Diddams, R. J. Fasano, T. M. Fortier, *et al.*, Frequency ratio measurements at 18-digit accuracy using an optical clock network, *Nature (London)* **591**, 564 (2021).
 - [18] M. Schioppo, R. C. Brown, W. F. McGrew, N. Hinkley, R. J. Fasano, K. Beloy, T. H. Yoon, G. Milani, D. Nicolodi, J. A. Sherman, N. B. Phillips, C. W. Oates, and A. D. Ludlow, Ultra-stable optical clock with two cold-atom ensembles, *Nat. Photon.* **11**, 48 (2017).
 - [19] J. Grotti, S. Koller, S. Vogt, S. Häfner, U. Sterr, C. Lisdat, H. Denker, C. Voigt, L. Timmen, A. Rolland, *et al.*, Geodesy and metrology with a transportable optical clock, *Nat. Phys.* **14**, 437 (2018).
 - [20] S. Origlia, M. S. Pramod, S. Schiller, Y. Singh, K. Bongs, R. Schwarz, A. Al-Masoudi, S. Dörscher, S. Herbers, S. Häfner, U. Sterr, and C. Lisdat, Towards an optical clock for space: Compact, high-performance optical lattice clock based on bosonic atoms, *Phys. Rev. A* **98**, 053443 (2018).
 - [21] H. Liu, S. B. Jäger, X. Yu, S. Touzard, A. Shankar, M. J. Holland, and T. L. Nicholson, Rugged mHz-linewidth superradiant laser driven by a hot atomic beam, *Phys. Rev. Lett.* **125**, 253602 (2020).
 - [22] J. R. K. Cline, V. M. Schäfer, Z. Niu, D. J. Young, T. H. Yoon, and J. K. Thompson, Continuous collective strong coupling of strontium atoms to a high finesse ring cavity, *Phys. Rev. Lett.* **134**, 013403 (2025).
 - [23] B. J. Larsen, H. Hensley, G. Martinez, A. Staron, W. R. McGehee, J. Kitching, and J. K. Thompson, A chip-scale atomic beam source for non-classical light, *arXiv:2506.00199* [physics.atom-ph].
 - [24] J. Kim, S.-H. Oh, D. Yang, J. Kim, M. Lee, and K. An, A photonic quantum engine driven by superradiance, *Nat. Photon.* **16**, 707 (2022).

- [25] High-current ion beams at tens of keV with centimeter-scale diameters have been experimentally demonstrated. In particular, M. Zhang *et al.* [31] reported extracted proton beams with energies of 50 keV and currents up to 100 mA, with measured beam diameters on the order of 20 mm. These experimentally achieved parameters support the feasibility of realizing high-flux ion beams in the tens-of-keV energy range relevant to the present proposal.
- [26] W. Choi, J.-H. Lee, K. An, C. Fang-Yen, R. R. Dasari, and M. S. Feld, Observation of sub-Poisson photon statistics in the cavity-QED microlaser, *Phys. Rev. Lett.* **96**, 093603 (2006).
- [27] H.-G. Hong, W. Seo, Y. Song, M. Lee, H. Jeong, Y. Shin, W. Choi, R. R. Dasari, and K. An, Spectrum of the cavity-QED microlaser: Strong coupling effects in the frequency pulling at off resonance, *Phys. Rev. Lett.* **109**, 243601 (2012).
- [28] M. Tavis and F. W. Cummings, Exact solution for an N -molecule—Radiation-field Hamiltonian, *Phys. Rev.* **170**, 379 (1968).
- [29] C. W. Gardiner and M. J. Collett, Input and output in damped quantum systems: Quantum stochastic differential equations and the master equation, *Phys. Rev. A* **31**, 3761 (1985).
- [30] A feasible candidate for an initial experimental test is the 397-nm transition of $^{40}\text{Ca}^+$ in a 50-keV, 100-mA ion beam. The relevant parameters are a cavity linewidth $\kappa/2\pi = 6$ GHz, an atomic linewidth $2\gamma_a/2\pi = 21.6$ MHz, a single-ion coupling strength $g/2\pi = 190$ kHz, and an ion transit time $\tau = 1.4$ ns. The number of ions interacting with the cavity mode is estimated to be $N = 1.5 \times 10^8$, assuming a pump-cavity separation of 1.0 mm. The pump Rabi frequency is taken to be $\Omega_m/2\pi = 10$ MHz, with a pulse area $\Omega\tau_p = 0.936\pi$. Based on our numerical simulations, the projected collective decay rate is $\Gamma_c/2\pi \approx 23$ Hz and the number of collective emission events is $\mathcal{N}_{\text{coll}} \sim 31$, leading to an expected linewidth on the order of 1 Hz. The proposed parameters satisfy the bad-cavity condition, $\kappa \gg \sqrt{N}g, \sqrt{n}g$, with an estimated mean intracavity photon number of $n \sim 1.4 \times 10^6$.
- [31] M. Zhang *et al.*, Upgrade of the extraction system of permanent magnet electron cyclotron resonance ion source, *Rev. Sci. Instrum.* **81**, 02B715 (2010).