

Enhanced superresolution by entanglement of the unified quantum wave function

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When two point sources are separated by less than the Abbe-Rayleigh limit, conventional resolution methods fail to distinguish whether there is one source or two. Consequently, it becomes nearly impossible to resolve important relative properties of the two sources, such as mutual coherence, relative phase, and brightness contrast. In this regime, the incoming composite quantum signal must be treated as a single entity described by an overall quantum wave function. In this study, we explore the superresolution of such two-point sources by examining a special wave property, i.e., entanglement between spatial degree of freedom (DoF) and remaining DoFs (both internal and external). Remarkably, we find that increasing this inter-DoF entanglement enhances the precision of superresolution, as quantified by the quantum Fisher information (QFI) in single-parameter estimation. Moreover, we show that this entanglement reliably ensures a nonvanishing QFI in two-parameter estimation, highlighting its role as a valuable resource for quantum superresolution.

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I. INTRODUCTION

The Abbe-Rayleigh diffraction limit has long posed a significant challenge to achieving superfine resolution and imaging [1–4]. Over the past few decades, various techniques have been successfully demonstrated to surpass this diffraction limit for actively controllable sources [5–8]. For passive incoherent and balanced point sources, an alternative method of quantum spatial mode demultiplexing has proven to be an effective way to achieve subwavelength resolution (or superresolution) [9–24]. In such a state-based spatial mode decomposition method, the fundamental quantum wave properties play a critical role. In most previous such studies, the quantum state is often analyzed from the perspective of two individual sources. As a result, two-source relative wave properties such as the coherence between the sources [11,14,16,18,24–26], relative phases [19], brightness contrast [27], and inter-source entanglement [28–30] have been shown to significantly influence the capability and quality of superresolution.

However, at the detection stage, the signal from two closely spaced sources is typically treated as a single physical entity, without distinguishing between the relative properties of the two sources. Therefore, this composite two-source entity should be analyzed from the perspective of a unified quantum state. A quantum wave function encompasses multiple degrees of freedom (DoFs), which can exhibit special

correlations such as entanglement and coherence. Understanding how these inter-DoF wave properties impact the superresolution process is thus crucial.

In this work, we aim to fill this gap by focusing on the inter-DoF entanglement based on a unified two-source wave function. Specifically, we investigate the effect of entanglement between the spatial DoF and the remaining DoFs (regardless of being internal or external). This approach represents a significant departure from previous studies, which have primarily investigated entanglement between the two individual sources [28–30]. Additionally, we extend the analysis to the quantitative level by examining how the degree of entanglement, quantified by concurrence [31], influences superresolution.

Interestingly, it is found that in single-parameter estimation, where only the two-source separation s is unknown, increasing inter-DoF entanglement significantly enhances the precision of superresolution, as evidenced by an increase in the quantum Fisher information (QFI) [32–36]. In the case of two-parameter estimation, where an additional parameter (i.e., the degree of entanglement) is also unknown, our results demonstrate that nonzero entanglement consistently guarantees a nonvanishing QFI, thereby enabling reliable superresolution. Such entanglement is directly related to the purity (opposite of mixedness) of the reduced state of the spatial domain. These findings underscore the critical role of inter-DoF entanglement in superresolution and highlight the importance of maximizing such entanglement to achieve superior resolution performance.

II. TWO-POINT SOURCE

We start by considering an incoming field of two equal-intensity point sources separated by a distance s . At the shift-invariant imaging plane, the spatial profiles of the two

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sources, denoted as $h_{\pm}(x) = h(x \pm s/2)$, respectively, can be viewed as positive and negative displacements of $s/2$ from the amplitude function $h(x)$. The point-spread function (PSF) is taken to be in the Gaussian form $h^2(x) = (1/\sqrt{2\pi\sigma^2})\exp[-x^2/2\sigma^2]$, with σ being the width. The general spatial state can be described by a non-normalized statistical mixture of photon spatial states, i.e.,

$$\rho_s = |h_+\rangle\langle h_+| + |h_-\rangle\langle h_-| + \gamma^*|h_+\rangle\langle h_-| + \gamma|h_-\rangle\langle h_+|, \quad (1)$$

where $|h_{\pm}\rangle$ represent the spatial wave functions with $\langle x|h_{\pm}\rangle = h_{\pm}(x) \equiv h(x \pm s/2)$. Here, $\gamma = \cos\theta e^{i\varphi}$ is the coherence parameter with $|\gamma| \leq 1$ due to the Cauchy-Schwarz inequality and φ being the relative phase.

Despite the nonorthogonality of the source states $\langle h_+|h_- \rangle = \alpha \neq 0$ (due to the fact that the two PSFs always have nonzero overlaps), the density matrix ρ of the spatial domain in Eq. (1) remains fully general for all equal-strength two-state mixture. It reduces to the incoherent scenario considered in Ref. [11] by Tsang *et al.* and in Ref. [16] by Hradil *et al.*, when one takes $|\gamma| = 0$.

For such a reduced density matrix ρ in the spatial domain, it can always be purified by an auxiliary party to become a larger unified two-source joint pure state:

$$|\Psi\rangle = |h_+\rangle|\phi_1\rangle + |h_-\rangle|\phi_2\rangle, \quad (2)$$

where $|\phi_1\rangle$ and $|\phi_2\rangle$ are two normalized auxiliary states that represent the nonspatial DoF (e.g., temporal domain, polarization, etc.) as well as any external systems that may be involved. The partial coherence parameter γ is related directly to the overlap of the two auxiliary states, i.e., $\gamma = \langle\phi_1|\phi_2\rangle$.

For the convenience of the following analysis, we re-express the remaining DoFs in an orthogonal basis, i.e., $\{|\phi_1\rangle, |\phi_1^\perp\rangle\}$, where $|\phi_2\rangle = e^{i\varphi}\cos\theta|\phi_1\rangle + \sin\theta|\phi_1^\perp\rangle$, $\langle\phi_1|\phi_1^\perp\rangle = 0$, $|\gamma| = \cos\theta$, and $\theta \in [-\pi/2, \pi/2]$. Then, the optical field (2) can be rewritten as

$$|\Psi\rangle = |\Phi_1\rangle|\phi_1\rangle + |\Phi_2\rangle|\phi_1^\perp\rangle, \quad (3)$$

where $|\Phi_1\rangle = (|h_+\rangle + e^{i\varphi}\cos\theta|h_-\rangle)$ and $|\Phi_2\rangle = \sin\theta|h_-\rangle$ are two non-normalized spatial amplitudes. Again, it is noted that in general $|\Phi_1\rangle$ and $|\Phi_2\rangle$ are nonorthogonal states with $\langle\Phi_1|\Phi_2\rangle \neq 0$.

The degree of entanglement of the two-source field (3), equivalent to Eq. (2), between the spatial domain and the remaining DoFs can also be straightforwardly computed, in terms of concurrence [31], to be

$$C = |\sin\theta|\sqrt{1-d^2}, \quad (4)$$

with $d = \exp[-s^2/8\sigma^2]$. This inter-DoF entanglement C controlled by two parameters, θ and s (through the quantity d). We remark that the concurrence C is obtained after the normalization of the state (3) with the factor $\sqrt{\langle\Phi_1|\Phi_1\rangle + \langle\Phi_2|\Phi_2\rangle}$.

Since Eq. (2) is the purified state of the spatial mixed state ρ in Eq. (1), the measurement of entanglement between $\{|h_+\rangle, |h_-\rangle\}$ and $\{|\phi_1\rangle, |\phi_2\rangle\}$ can be just achieved by measuring either of the two reduced density matrices, i.e., the spatial density matrix ρ_s or the reduced density matrix of the $\{|\phi_1\rangle, |\phi_2\rangle\}$ DoF after tracing off the spatial domain. According to Schmidt analysis of quantum states [37], the two

subparty reduced density matrices share the same eigenvalues, which determine the value of entanglement (e.g., in terms of concurrence [31], von Neumann entropy [38], etc.). These eigenvalues can be obtained by performing a tomography measurement of the matrix ρ_s of Eq. (1), e.g., in the Hermite-Gaussian (HG) spatial mode basis or any other convenient basis.

III. ONE-PARAMETER ESTIMATION

The measurement of the target parameter, i.e., the two-source separation s , is always compensated with an indefinite variance. Here, we adopt the usual quantum estimation theory quantity, QFI [32,33], to characterize measurement precision. Throughout this article, we assume the measurements being single event repetition, for example, detecting a single photon repeatedly, measuring a group of identical photons within the coherence time frame, or a singular light intensity measurement. Our analysis does not cover environment-induced loss or vacuum contribution cases where the photon numbers are unknown [23].

We first consider the case when separation s is the only unknown parameter. From the Cramér-Rao theory [39,40], quantum Fisher information F corresponds to the lower bound of the variance of the estimator s , i.e., $\text{Var}(s) \geq 1/F$. Generally, the QFI for a pure state $|\Phi\rangle/\sqrt{N}$ can be specifically defined as [16,33–35,41,42]

$$F_\Phi(s) = \frac{4}{N} \langle\partial_s\Phi|\partial_s\Phi\rangle - \frac{4}{N^2} |\langle\Phi|\partial_s\Phi\rangle|^2, \quad (5)$$

where $N = \langle\Phi|\Phi\rangle$ is the normalization factor of $|\Phi\rangle$. It is important to note that the Fisher information expression takes exactly the same form (5) regardless of whether N is dependent on s or not.

To achieve the QFI of the two-source state (3), different approaches have been taken in the literature to characterize appropriately the measurement precision in different situations [11,14,16]. Considering the normalization requirement of the state (3), here we adopt the weighted QFI proposed in Ref. [16], based on the probabilities of the two components $p_1 = \langle\Phi_1|\Phi_1\rangle/(\langle\Phi_1|\Phi_1\rangle + \langle\Phi_2|\Phi_2\rangle)$ and $p_2 = \langle\Phi_2|\Phi_2\rangle/(\langle\Phi_1|\Phi_1\rangle + \langle\Phi_2|\Phi_2\rangle)$, i.e.,

$$F_{\text{tot}} = p_1 F_{\Phi_1}(s) + p_2 F_{\Phi_2}(s), \quad (6)$$

where $F_{\Phi_1}(s)$ and $F_{\Phi_2}(s)$ correspond to the quantum Fisher information of the pure states $|\Phi_{1,2}\rangle$, respectively, and can be achieved through Eq. (5). Straightforward calculation leads to the weighted QFI to be

$$F_{\text{tot}} = \frac{1}{4\sigma^2} - \frac{d(4\sigma^2 - s^2)\sqrt{1-d^2} - C^2}{16\sqrt{1-d^2}\sigma^4} - \frac{d^2 s^2 (1-d^2 - C^2)/8}{2 - 2d^2 - C^2 + 2d\sqrt{1-d^2}\sqrt{1-d^2 - C^2}}, \quad (7)$$

which is a function of separation s and entanglement C . It is worth to note that when the two sources are completely incoherent, which corresponds exactly to the well-known Tsang-Nair-Lu scenario [11], it will require $\langle\phi_1|\phi_2\rangle = e^{i\varphi}\cos\theta = 0$. This means the entanglement reduces directly to $C = \sqrt{1-d^2}$. Straightforward evaluation of the above

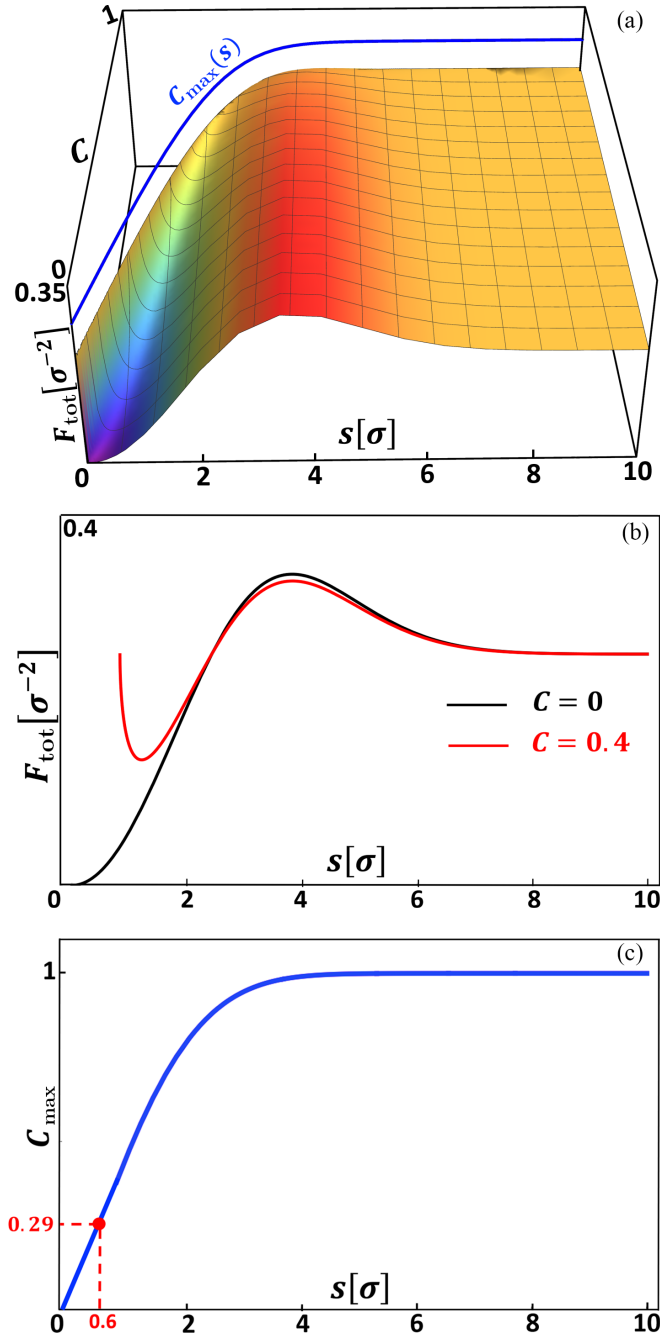


FIG. 1. Entanglement effects on QFI under one-parameter estimation. (a) Illustration of F_{tot} with respect to separation s and entanglement C . The maximum possible entanglement value $C_{\text{max}}(s)$ is shown on the top plane by the blue curve to indicate the range $[0, C_{\text{max}}(s)]$ it can take for any given s , see also in plot (c). (b) An example behavior of F_{tot} with respect to separation s for fixed entanglement $C = 0, 0.4$. (c) Behavior of maximum entanglement C_{max} with respect to separation s .

result (7) directly leads to a constant Fisher information $F_{\text{tot}} = 1/4\sigma^2$, exactly consistent with the result in Ref. [11].

Figure 1(a) illustrates the comprehensive behavior of F_{tot} with respect to both s and C . For any fixed nonzero amount of entanglement, the QFI starts with a finite value and decreases to a minimum, then increases to a maximum, and finally approaches asymptotically to a finite value as the separation

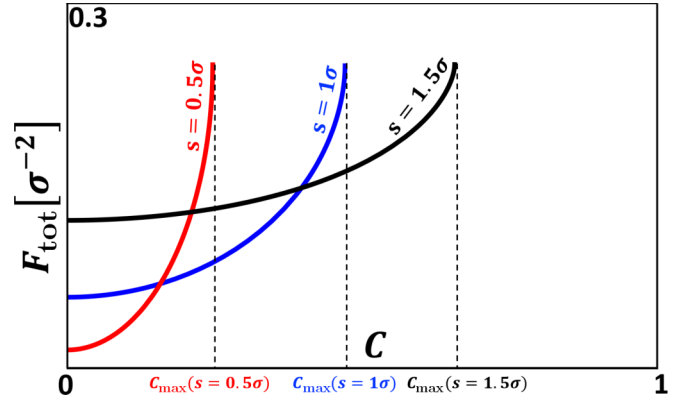


FIG. 2. Entanglement effects on QFI for the fixed separations $s = 0.5\sigma$, $s = 1\sigma$, and $s = 1.5\sigma$. Note that for a given s , entanglement C is restricted by a maximum value C_{max} . Therefore, the corresponding F_{tot} curves stop at C_{max} and will not go up without bound.

s increases. Such a behavior guarantees high measurement precision at the small separation regime.

A crucial feature of F_{tot} , illustrated in Fig. 1(a), is that it increases as the amount of entanglement C increases for any fixed separation s in the separation region $0 \leq s \lesssim 3\sigma$, see details in Fig. 2. This is a region of most interest as it includes the domain when the two-source separation is beyond the diffraction limit, i.e., $s \in [0, \sim 1.22\sigma)$. Therefore, one is led to the conclusion that entanglement is a beneficial feature that can enhance superresolution precision.

Figure 1(b) also illustrates the example behavior of F_{tot} with respect to s when the entanglement is fixed at $C = 0.4$ (red curve) and $C = 0$ (black curve), respectively. Apparently, it is noted that higher amount of entanglement will significantly enhance the Fisher information, particularly at the small separation regime. To have a systematic illustration, Fig. 2 shows the monotonic increasing behavior of $F_{\text{tot}}(C)$ in terms of C in the fixed separations $s = 0.5\sigma$, $s = 1\sigma$, and $s = 1.5\sigma$. It is worth to mention that the degree of entanglement C is also partially dependent on separation s , thus restricting the freedom of C . That is, for a given s , C can vary freely only within a range $[0, C_{\text{max}}(s)]$, where $C_{\text{max}}(s) = \sqrt{1 - d^2}$. The behavior of the maximum entanglement value $C_{\text{max}}(s)$ is plotted on top of Fig. 1(a) as a function of s and is also shown explicitly in Fig. 1(c). It is also shown for each given s in Fig. 2. The singularity of $F_{\text{tot}}(C)$ exists only in the nonentangled case. As a result, one can conclude that the existence of this inter-DoF entanglement is crucial for subdiffraction-limit resolution. When the two-source separation is the only unknown parameter, the increase of entanglement enhances the quality of superresolution with higher measurement precision (i.e., larger Fisher information).

It is important to note that the two-source coherence $|\gamma| = |\cos \theta| = |\langle \phi_1 | \phi_2 \rangle| \in [0, 1]$ is quantitatively connected to the entanglement measure C . Crucially, C also depends on the parameter $d = \exp[-s^2/8\sigma^2]$, which reflects the spatial separation s of the sources. Consequently, the behavior of entanglement C cannot be used to unambiguously infer that of coherence $|\gamma|$. Similarly, the role of entanglement in determining the Fisher information should not be conflated

with the direct contributions of coherence, as these quantities are governed by distinct physical mechanisms.

IV. TWO-PARAMETER ESTIMATION

While single-parameter estimation is relatively straightforward to handle, there are often more parameters unknown and need to be experimentally determined in practice. Therefore, a multiparameter estimation analysis needs to be applied [34,35]. In this section, we consider the two-parameter estimation case when the amount of entanglement is also unknown besides the two-source separation s . As shown in Eq. (4), entanglement C depends both on the separation s and the parameter θ . Therefore, for convenience, we analyze the quantum Fisher information by estimating both s and θ .

Similar to the single-parameter estimation case, the measurement precision quantities (i.e., the quantum Fisher information) H_s and H_θ for the two-parameter case also correspond to the Cramér-Rao lower bound, i.e., $\text{Var}(s) \geq H_s^{-1}$ and $\text{Var}(\theta) \geq H_\theta^{-1}$, for the respective parameters s and θ . The difference is that H_s and H_θ are now related to a 2×2 quantum Fisher information matrix (QFIM) [13,14,34,35] $\hat{F}$ with

$$H_s = F_{ss} - \frac{F_{s\theta}^2}{F_{\theta\theta}} \quad \text{and} \quad H_\theta = F_{\theta\theta} - \frac{F_{s\theta}^2}{F_{ss}}, \quad (8)$$

where F_{ij} with $i, j = s, \theta$ are the matrix elements that can be achieved as

$$F_{ij} = \frac{1}{2} \text{Tr}[(L_i L_j + L_j L_i) \rho]. \quad (9)$$

Here, $L_{i(j)}$ are the symmetric logarithmic derivative (SLD) operators with respect to the parameter $i(j)$ and are defined as

$$L_i = \sum_{k,l=1}^4 \frac{2}{\lambda_k + \lambda_l} \left(\langle e_k | \frac{\partial \rho}{\partial i} | e_l \rangle \right) | e_k \rangle \langle e_l |. \quad (10)$$

Note that the commutator of operators L_i satisfy the zero-expectation value relation $\text{Tr}[\rho[L_i, L_j]] = 0$, indicating that a single measurement can be optimal and simultaneously obtained for both s and θ . The vectors $|e_k\rangle$ with $k = 1, 2$ are the eigenvectors of ρ given as $|e_1\rangle = (|h_-\rangle - |h_+\rangle)/\sqrt{2(1-d)}$ and $|e_2\rangle = (|h_-\rangle + |h_+\rangle)/\sqrt{2(1+d)}$ with corresponding eigenvalues $\lambda_1 = (d-1)(\cos\theta-1)/2(1+d\cos\theta)$ and $\lambda_2 = (d+1)(\cos\theta+1)/2(1+d\cos\theta)$. The vectors $|e_k\rangle$ with $k = 3, 4$ are transformed from $|e_{1,2}\rangle$, i.e., $|e_3\rangle = a_3^{-1} \partial |e_1\rangle / \partial s$ and $|e_4\rangle = a_4^{-1} \partial |e_2\rangle / \partial s$, where the coefficients are given as

$$a_3 = \sqrt{\frac{1-A}{16\sigma^4(1-d)} - \frac{B^2}{4(1-d)^2}},$$

$$a_4 = \sqrt{\frac{1+A}{(1+d)} - \frac{B^2}{4(1+d)^2}},$$

with $A = d(s^2/4 - 1)$ and $B = -ds/4\sigma^2$. Then, the derivative of the density matrix ρ can be expressed as

$$\begin{aligned} \frac{\partial \rho}{\partial s} = & \frac{B(1 - \cos^2 \theta)}{2(1 + d \cos \theta)^2} [-|e_1\rangle \langle e_1| + |e_2\rangle \langle e_2|] \\ & + \lambda_1 a_3 [|e_3\rangle \langle e_1| + |e_1\rangle \langle e_3|] \\ & + \lambda_2 a_4 [|e_4\rangle \langle e_2| + |e_2\rangle \langle e_4|], \end{aligned} \quad (11)$$

$$\frac{\partial \rho}{\partial \theta} = \frac{(1-d^2) \sin \theta}{2(1+d \cos \theta)^2} [|e_1\rangle \langle e_1| - |e_2\rangle \langle e_2|]. \quad (12)$$

As the four vectors $|e_k\rangle$, $k = 1, 2, 3, 4$, form a new orthonormal basis [14], the SLD operators represent various 4×4 matrices. Then, the QFIM elements F_{ij} can be specifically achieved through SLD operator matrix elements and expressed as

$$F_{ss} = \lambda_1 ([L_s]_{11}^2 + [L_s]_{13}^2) + \lambda_2 ([L_s]_{22}^2 + [L_s]_{24}^2), \quad (13)$$

$$F_{\theta\theta} = \lambda_1 [L_\theta]_{11}^2 + \lambda_2 [L_\theta]_{22}^2, \quad (14)$$

$$F_{s\theta} = \lambda_1 [L_s]_{11} [L_\theta]_{11} + \lambda_2 [L_s]_{22} [L_\theta]_{22}, \quad (15)$$

where the relevant L_i operator matrix elements are computed as

$$[L_s]_{11} = \frac{-B(1 - \cos^2 \theta)}{2\lambda_1(1 + d \cos \theta)^2}, \quad [L_s]_{13} = [L_s]_{31} = 2a_3, \quad (16)$$

$$[L_s]_{22} = \frac{-B(1 - \cos^2 \theta)}{2\lambda_2(1 + d \cos \theta)^2}, \quad [L_s]_{24} = [L_s]_{42} = 2a_4, \quad (17)$$

$$[L_\theta]_{11} = \frac{(1-d^2) \sin \theta}{2\lambda_1(1 + d \cos \theta)^2}, \quad [L_\theta]_{22} = \frac{-[L_\theta]_{11} \lambda_1}{\lambda_2}. \quad (18)$$

Then, the quantum Fisher information quantities H_s and H_θ are fully determined, through Eq. (8), as functions of s and θ (entanglement C). We will now label the two-parameter Fisher information as $H_s \equiv H_s(s, C)$ with respect to the estimation parameters, separation s and inter-DoF entanglement C . Its behavior is illustrated in Fig. 3(a). It is noted that the quantum Fisher information reduces to zero, i.e., $H_s(s, C) = 0$, only in the zero-entanglement $C = 0$ case when the two-source separation is also at zero $s = 0$. This simply means that any nonzero entanglement $C \neq 0$ will ensure a nonvanishing quantum Fisher information $H_s(s, C) > 0$.

Also, it is noted that for a given entanglement C , the Fisher information $H_s(s, C)$ has a minimum value $H_s^{\min}(C)$. To show this feature specifically, a 2D illustration is provided in Fig. 3(b) for fixed entanglement $C = 0.4$ (red curve) and $C = 0$ (black curve), respectively. Apparently, higher entanglement ($C = 0.4$) will ensure a higher minimum Fisher information, i.e., $H_s^{\min}(C = 0.4) > H_s^{\min}(C = 0)$. This echoes with the conclusion from the single-parameter estimation case that entanglement is indeed a favorable feature for superresolution.

V. EXPERIMENTAL PROPOSAL

Generally, the states $|\phi_1\rangle$ and $|\phi_2\rangle$ in Eq. (2) cannot be fully accessed. However, since they fully purify the spatial mixed state ρ , entanglement between the spatial domain $\{|h_+\rangle, |h_-\rangle\}$ and the remaining DoFs $\{|\phi_1\rangle, |\phi_2\rangle\}$ can be achieved solely

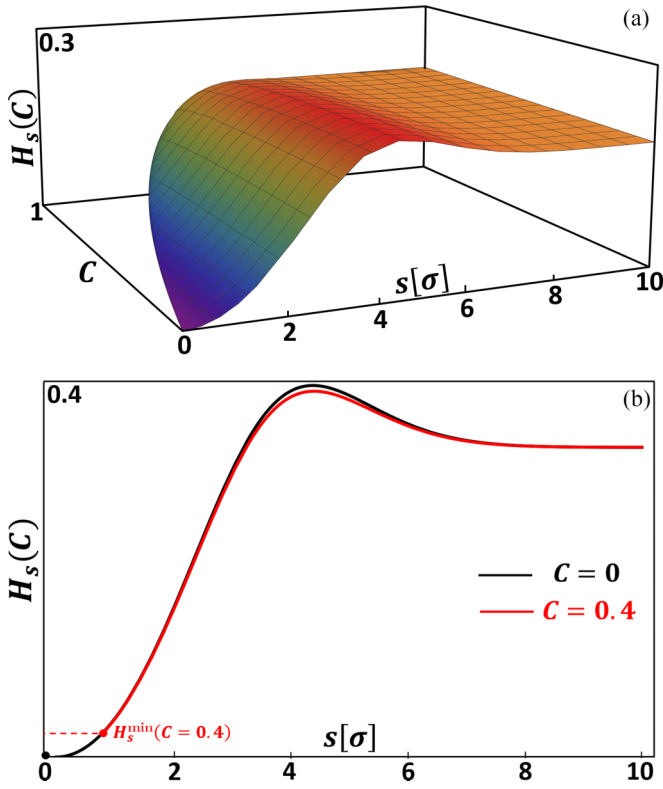


FIG. 3. Entanglement effects on QFI precision under two-parameter estimation. (a) Behavior of precision $H_s(s, C)$ with respect to separation s and concurrence C . (b) A 2D example behavior of H_s with respect to separation s for fixed entanglement $C = 0.4$ (red curve) and $C = 0$ (black curve), respectively.

by measuring ρ via tomography [37,43]. To facilitate experimental verification of our theoretical results, we propose a controllable setup where both degrees of freedom are easily accessible: a two-point source entangled in its spatial and polarization degrees of freedom.

As shown in Fig. 4, a Mach-Zehnder interferometer (MZI) generates an entangled state between the spatial states $|h_+\rangle, |h_-\rangle$ and polarization states $|p_1\rangle, |p_2\rangle$. The brightness balance of the two sources is controlled by half waveplate

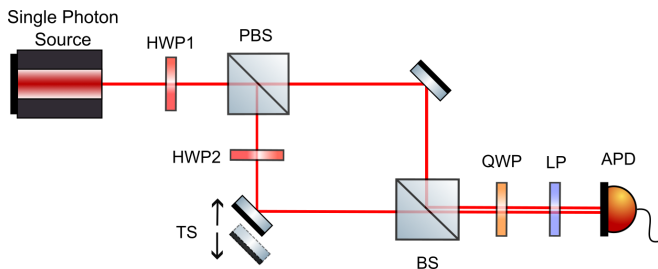


FIG. 4. Schematic illustration of setup for generating a polarization-spatial entangled two-point source and measuring its entanglement. The source is created using a modified Mach-Zehnder interferometer with a translation stage in one arm. Entanglement is characterized via polarization Stokes tomography using wave plates, a linear polarizer (LP), and a single photon avalanche photodiode (APD).

HWP1 combined with the polarizing beamsplitter (PBS). The polarization-state overlap $\langle p_1 | p_2 \rangle = \cos \theta$, and thus the amount of polarization-spatial entanglement, is adjusted by HWP2 setting at angle $\theta/2$. Partially overlapping spatial states are achieved by parallel alignment at the output beamsplitter (BS), with their separation s controlled by the translation stage (TS) in one of the MZI arms. The output of the BS then produces the two-source state given in Eq. (2).

To measure entanglement, we propose to perform polarization Stokes tomography [44,45] using wave plates and a linear polarizer. This involves measuring projection probabilities on key polarization bases: P_H and P_V for horizontal $|H\rangle$ and vertical $|V\rangle$ states; P_D and P_A for diagonal $|D\rangle = (|H\rangle + |V\rangle)/\sqrt{2}$ and antidiagonal $|A\rangle = (|H\rangle - |V\rangle)/\sqrt{2}$ states; P_L and P_R for left circular $|L\rangle = (|H\rangle + i|V\rangle)/\sqrt{2}$ and right circular $|R\rangle = (|H\rangle - i|V\rangle)/\sqrt{2}$ states. The diagonal and antidiagonal states are realized with an HWP and the left and right circular polarization states are realized with a combination of HWP and QWP. The Stokes parameters are then calculated as

$$\begin{aligned} S_0 &= P_H + P_V, & S_1 &= P_H - P_V, \\ S_2 &= P_D - P_A, & S_3 &= P_L - P_R. \end{aligned} \quad (19)$$

Following Ref. [45], entanglement between the polarization and spatial DoFs is directly connected to the Stokes parameters and can be achieved as

$$C = \sqrt{1 - \frac{S_1^2 + S_2^2 + S_3^2}{S_0^2}}. \quad (20)$$

This allows full entanglement characterization through single-photon polarization measurements.

With C obtained, the quantum Fisher information F_{tot} in Eq. (7), expressed in terms of source separation s and concurrence C , can be computed straightforwardly. One then can verify its theoretical monotonic increasing behavior with respect to C at fixed s by (1) fixing the separation using the translation stage and (2) continuously varying C through HWP2 rotation.

VI. CONCLUSION

In summary, we present a unified quantum treatment of two-source superresolution by analyzing their combined wave function across multiple degrees of freedom. Our work demonstrates that entanglement between spatial and the remaining DoFs (regardless being of internal or external) fundamentally enhances measurement precision as measured by quantum Fisher information. When source separation (s) is the only unknown parameter, increasing inter-DoF entanglement boosts QFI. When entanglement magnitude is also unknown, any nonzero entanglement guarantees finite precision.

To verify these predictions, we propose an experimental implementation using polarization states $\{|p_1\rangle, |p_2\rangle\}$ as the accessible remaining DoFs $\{|\phi_1\rangle, |\phi_2\rangle\}$. Spatial-polarization entanglement is controllably generated via a modified MZI. Stokes tomography measures concurrence directly from polarization projections.

These results highlight that inter-DoF entanglement benefits superresolution in two important ways. First, it suggests to measure the amount of entanglement between spatial

and the remaining DoFs to optimize superresolution performance. Prior entanglement measurement (via spatial mixed states) is achievable even when nonspatial DoFs are inaccessible. This corresponds to the single-parameter estimation case, and the measured entanglement knowledge will guide to gain higher precision of superresolution measures. Second, in the two-parameter estimation case, where the exact amount of entanglement cannot be determined, implementing entanglement-enhancing techniques (e.g., basis projection, postselection [33]) ensures finite precision, as vanishing entanglement represents the only failure case.

Our analysis establishes a general framework for quantum-enhanced metrology where the signal is treated as a single

entity with correlated DoFs, moving beyond conventional source-separation paradigms.

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DATA AVAILABILITY

No data were created or analyzed in this study.

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