

Characterization of spatial Schmidt modes in high-gain SU(1,1) interferometers

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Multimode quantum light has promising applications in many areas of physics, such as quantum communications and quantum computing. However, its multimode nature also makes it challenging to measure its properties. Recently [I. Barakat *et al.*, *Optica Quantum* **3**, 36 (2025)], a technique for the simultaneous measurement of squeezing of multiple broadband modes based on a phase-sensitive amplification approach was experimentally implemented using a setup that effectively corresponds to an SU(1,1) interferometer. Here, we aim to provide a complete theoretical analysis of the modal structure of (generally unbalanced) SU(1,1) interferometers and a detailed theoretical formal derivation of the framework for this technique. Utilizing the joint Schmidt decomposition of the transfer functions, we investigate the shape and phase profiles of the modes of the SU(1,1) interferometer and its components [parametric down-conversion (PDC) sections] for different parametric gain regimes. We discover a complicated interplay between the PDC modes and the modes of the entire interferometer, and analyze it by using their overlap coefficients as a similarity measure. Finally, we develop a rigorous processing method for the aforementioned multimode squeezing measurement technique and discuss necessary approximations to make this method experimentally feasible.

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I. INTRODUCTION

A. Motivation: multimode squeezed light

In the last decade, high-gain parametric down conversion (PDC) has become a topic of great interest in quantum computing, sensing, and metrology [1–7]. Using PDC, it is possible to prepare quantum states with large photon numbers, that is, macroscopic states, which still carry quantum correlations [8,9]. Such states also exhibit quadrature squeezing and therefore have important applications in continuous variable quantum computing [10–12], quantum metrology [13–15], quantum cryptography [16], cluster state generation [17,18], and the realization of Gaussian Boson sampling [19]. In particular, in metrology, squeezed states of light make it possible to implement advanced types of interferometers, called nonlinear interferometers [SU(1,1) interferometers], which consist of two consecutive PDC sections and allow for high-sensitivity measurements that may surpass the shot noise level (supersensitive measurements) [7,20–22].

Multimode squeezed light [5,23–27] consists of multiple squeezed modes in a single pulse, each of which has its own degree of squeezing that can be controlled by proper source engineering [28,29]. Such states of light have important ap-

plications in quantum information processing [30,31] and quantum communication [32] since they are very attractive for scalability aspects in quantum computing and multiparameter estimation in metrology. However, the multimode nature of these states significantly complicates their description and characterization. This is because the conventional measurement techniques for squeezing, such as homodyne detection [5,33,34], require an increasing amount of resources as the number of modes increases.

The use of multimode SU(1,1) interferometers can greatly simplify the problem of measuring and characterizing many modes simultaneously. For example, a recently developed method for performing Wigner function tomography based on multimode SU(1,1) interferometry has demonstrated the advantage of being able to simultaneously tomograph multiple modes, as well as being immune to detection losses [35]. A similar technique was later used to simultaneously measure multimode squeezing of multimode PDC [5], where the first crystal of the interferometer (or the first pass through the PDC section) prepares a squeezed vacuum state, while the second crystal (or the second pass) acts as an analyzer. However, for the correct implementation of such types of interferometers, it is necessary to know their modal structure, as well as the modal structure of the PDC sections that comprise them, and the complete information regarding the modal overlap mismatch in the system.

In this work, we aim to provide a thorough analysis of the modal structure of (generally unbalanced) multimode SU(1,1) interferometers in various gain regimes, taking into account the mismatch of the squeezed light modes generated in the different PDC sections. Based on this analysis, we present

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a rigorous and detailed derivation of the multimode phase-sensitive amplification technique for measuring multimode squeezing using direct intensity measurements. We start with the exact reconstruction of the squeezing as it follows naturally from the presented approach. We then provide a number of simplifications that allow us to significantly reduce the complexity of the presented technique and make it feasible for experimental implementations. This simplified technique has already been successfully realized in an experimental setup in Ref. [5] and demonstrated good agreement between theory and experiment. The discussed approach is quite general and can in principle be applied to arbitrary multimode squeezed states of light and various interferometric configurations where the mode-matching problem is a key bottleneck. In particular, this approach could be useful for systems where a full measurement of the states carrying squeezing is not accessible to standard homodyne detection [26,27].

In order to provide a self-contained description of generally unbalanced SU(1,1) interferometers, in the following section, we will briefly recapitulate important concepts for the general description of multimode SU(1,1) interferometers. The mentioned equations will be later directly used and expanded upon in the following sections.

B. Theoretical framework and Schmidt modes

In order to describe the PDC process in both the low-gain and high-gain regime, we utilize a description of PDC using a system of coupled integro-differential equations that describe the propagation of the plane-wave operators $\hat{a}_s$ and $\hat{a}_i$ through the PDC section, where the labels s and i refer to the signal and idler beam, respectively. The derivation can be found in Ref. [8]. For simplicity, in this work, we will restrict ourselves to the description of frequency-degenerate type-I PDC, meaning that the signal and idler photons are indistinguishable, and the signal and idler labels are redundant. However, to be consistent with the literature, we will keep the s and i labels for now.

The solution of this integro-differential equation approach takes the form of the well-known input-output relations [7,8]:

$$\hat{a}_s^{(\text{out})}(q_s) = \int dq'_s \tilde{\eta}(q_s, q'_s) \hat{a}_s^{(\text{in})}(q'_s) + \int dq'_i \beta(q_s, q'_i) [\hat{a}_i^{(\text{in})}(q'_i)]^\dagger, \quad (1.1a)$$

$$[\hat{a}_i^{(\text{out})}(q_i)]^\dagger = \int dq'_i \tilde{\eta}^*(q_i, q'_i) [\hat{a}_i^{(\text{in})}(q'_i)]^\dagger + \int dq'_s \beta^*(q_i, q'_s) \hat{a}_s^{(\text{in})}(q'_s), \quad (1.1b)$$

where the variables $q_{s/i}$ label the transverse wave-vector components of the plane-wave modes associated with the $\hat{a}_{s/i}$ operators and where the plane-wave operators marked with the superscript $^{(\text{in})}$ are the input operators of the PDC section, which form the boundary condition for the set of integro-differential equations, while the operators marked with $^{(\text{out})}$ are the output operators corresponding to the solutions of the set of integro-differential equations. These two types of operators are connected via the complex-valued transfer functions β and $\tilde{\eta}$ that describe how the input plane-wave modes are

connected to the output plane-wave modes of the PDC section and contain the full information regarding the PDC interaction. Note that this form of the solution already assumes that the signal and idler photons are indistinguishable [7,36].

For a Gaussian pump of the form $E_p(x, z, t) = E_0 e^{-\frac{x^2}{2\sigma^2}} e^{i(k_p z - \omega_p t)}$, the transfer functions can be found as the solution of the integro-differential equations [7]

$$\frac{d\beta(q_s, q'_i, L)}{dL} = \Gamma \int dq_i e^{-\frac{(q_s + q_i)^2 \sigma^2}{2}} \times h(q_s, q_i, L) \tilde{\eta}^*(q_i, q'_i, L), \quad (1.2a)$$

$$\frac{d\tilde{\eta}^*(q_i, q'_i, L)}{dL} = \Gamma \int dq_s e^{-\frac{(q_s + q_i)^2 \sigma^2}{2}} \times h^*(q_s, q_i, L) \beta(q_s, q'_i, L). \quad (1.2b)$$

These equations can be solved using standard methods such as the Runge-Kutta methods for solving ordinary differential equations or more elaborate solvers (see, for example, Appendix B). Note that generally, the equations include a two-dimensional transverse wave vector $\mathbf{q}$. However, below we will consider only systems with the azimuthal symmetry and, therefore, reduce our consideration to a single transverse component of the wave vector. The parameter Γ is connected to the parametric gain of the PDC interaction, which will be elaborated on in Sec. II. Note that the full width at half maximum (FWHM) of the pump intensity distribution is given by $2\sigma\sqrt{\ln 2}$. The function h describes the phase matching inside the PDC section and is symmetric due to the fact that we consider frequency-degenerate type-I PDC; concrete expressions for h will be given and explained in Secs. II and III.

It is well known that the transfer functions for this system admit a *joint Schmidt decomposition* (Bloch-Messiah reduction) of the form [3,7,37]

$$\beta(q, q') = \sum_n \sqrt{\Lambda_n} u_n(q) \psi_n(q'), \quad (1.3a)$$

$$\tilde{\eta}(q, q') = \sum_n \sqrt{\tilde{\Lambda}_n} u_n(q) \psi_n^*(q'), \quad (1.3b)$$

where the $u_n(q)$ modes are the output Schmidt modes (connected to the output Schmidt operators, see below) and the $\psi_n(q')$ are the input Schmidt modes (connected to the input Schmidt operators), while Λ_n and $\tilde{\Lambda}_n$ are the Schmidt eigenvalues associated with the two transfer functions β and $\tilde{\eta}$, respectively. Note that the eigenvalues are connected via [7,37]

$$\tilde{\Lambda}_n = \Lambda_n + 1. \quad (1.4)$$

More details on the joint Schmidt decomposition can be found in Appendix A.

By plugging both Eqs. (1.3a) and (1.3b) into the form of the solutions of the plane-wave operators, Eqs. (1.1a) and (1.1b), multiplying both sides with $u_l^*(q_s)$ and $u_l(q_i)$, respectively, and integrating over q_s and q_i , respectively, the well-known Bogoliubov transformation for the Schmidt operators is recovered [9,37]:

$$\hat{A}_l^{(\text{out})} = \sqrt{\tilde{\Lambda}_l} \hat{A}_l^{(\text{in})} + \sqrt{\Lambda_l} [\hat{A}_l^{(\text{in})}]^\dagger, \quad (1.5)$$

and analogously for the creation operator, where the input and output Schmidt operators are connected to the mode functions ψ and u , respectively [37],

$$\hat{A}_l^{(\text{in})} = \int dq \psi_l^*(q) a^{(\text{in})}(q), \quad (1.6a)$$

$$\hat{A}_l^{(\text{out})} = \int dq u_l^*(q) a^{(\text{out})}(q). \quad (1.6b)$$

Note that, as mentioned above, the signal and idler photons are indistinguishable, meaning that the transformation of the Schmidt operators is fully described by Eq. (1.5) and its Hermitian conjugate. The Bogoliubov transformation for the Schmidt operators provides a diagonalization of the input-output relations, thus coupling input and output Schmidt modes with only the same mode index l .

II. SINGLE CRYSTAL

Before considering the modal structure of the full SU(1,1) interferometer, in this section, we analyze the modal structure of light generated in its first half, namely, a single PDC section, at low and high parametric gain. For a single crystal, the function h describing the phase matching and appearing in Eqs. (1.2a) and (1.2b) is given by [7,8]

$$h(q_s, q_i, L) = e^{i\Delta k(q_s, q_i)L}, \quad (2.1)$$

where $\Delta k(q_s, q_i) = \sqrt{k_p^2 - (q_s + q_i)^2} - \sqrt{k_s^2 - q_s^2} - \sqrt{k_i^2 - q_i^2}$ is the collinear wave vector mismatch inside the crystal. The integro-differential equations are numerically integrated over the interval $[0, L_1]$ for different values of the parametric gain. The connection between the theoretical parameter Γ and the experimental gain is obtained analogously to the experimental procedure, namely, by performing a fit of the collinear output intensity $\langle \hat{N}_s(q=0) \rangle dq$ with a function of the form $y(\Gamma) = B \sinh^2(A\Gamma)$, where A and B are the fitting parameters. The experimental gain is then defined as $G_{\text{exp}} = A\Gamma$ [7–9]. After obtaining the transfer functions by solving Eqs. (1.2a) and (1.2b), we can compute the joint Schmidt decomposition as written in Eqs. (1.3a) and (1.3b) and obtain the Schmidt-mode functions $u_n(q)$ and $\psi_n(q')$. In order to provide an analysis of the modal similarity of the input and output modes, we define the *same-crystal overlap coefficient* c_{lm} between these two modes at different indices as follows:

$$c_{lm} \stackrel{\text{def.}}{=} \int dq u_l(q) \psi_m^*(q). \quad (2.2)$$

For the numerical simulations, we consider a beta barium borate (BBO) crystal with a length of $L_1 = 3$ mm. The half width of the waist ($1/e^2$ radius) of the intensity distribution of the spatial Gaussian pump envelope is chosen as $70 \mu\text{m}$, meaning that $\sigma = (70/\sqrt{2}) \mu\text{m}$ in Eqs. (1.2a) and (1.2b), which corresponds to an FWHM of the intensity distribution of $70\sqrt{2} \ln 2 \mu\text{m} \approx 82.4 \mu\text{m}$ (similar to Ref. [5]). For these parameters, we found the gain fitting constant as $A = 142.12$. Unlike in Ref. [7], where different fitting constants were used depending on the experimental gain regime, in this work, we take the same fitting constant for both low and high gain. This will not affect the following results qualitatively but simplifies the numerical calculations greatly.

Plots of the moduli and the phase distributions of the first input (ψ_0) and output (u_0) modes are shown in Fig. 1. Both functions are plotted over the external angle $\theta \approx q/k^{\text{vac}}$, where k^{vac} is the modulus of the wave vector of the signal (idler) photons in vacuum. One can notice that for low parametric gain, $G_{\text{exp}} = 0.01$, the first input and output modes ($n = 0$) coincide. Additionally, from the plot of the c_{lm} coefficients, it is clear that in the low-gain regime this also holds for higher-order modes. This can also be seen analytically from the Schmidt-mode theory presented in Ref. [9]: For low parametric gain, the two-photon amplitude (TPA) $F(q_s, q_i)$ and the transfer function $\beta(q_s, q_i)$ coincide [36,38]. If the signal and idler photons are indistinguishable, the TPA is symmetric in its arguments. This means that $\beta(q_s, q_i)$ is also symmetric in its arguments and admits a decomposition of the form [39]

$$\beta(q, q') = \sum_n \sqrt{\Lambda_n} u_n(q) u_n(q'). \quad (2.3)$$

However, the decomposition for β as written in Eq. (2.3) is, on its own, not sufficient to conclude that the input and output modes of the system are identical. To conclude this, the full joint decomposition including $\tilde{\eta}$ must be investigated. However, clearly, for low gain, $\tilde{\eta}(q, q') \approx \delta(q - q')$ and $\sqrt{\Lambda_n} \approx 1$, meaning that $\tilde{\eta}$ may be decomposed as

$$\tilde{\eta}(q, q') \approx \sum_n u_n(q) u_n^*(q'), \quad (2.4)$$

for any orthonormal basis with functions u_n , confirming the validity of Eq. (2.3).

As the parametric gain increases, the mode profiles (moduli of the modes) broaden, which has already been observed in Ref. [8]. Moreover, for any gain, the moduli of the first input ($|\psi_0|$) and output modes ($|u_0|$) coincide to a good approximation, which has also been demonstrated, for example, in Ref. [37] in the frequency domain. However, the phase profiles of the Schmidt modes in the high-gain regime have not previously been studied in detail and are here presented in Fig. 1. Clearly, the phase profiles of the input and output modes differ significantly with increasing gain and the input modes have a much flatter phase profile than the output modes.

For the high-gain regime, the difference in the phases between the input and output modes seems to be a manifestation of time-ordering effects: The photons generated at the beginning of the crystal are distinguished by their phase from those generated at the end. These effects become especially relevant at high parametric gain, where the plots of the overlap coefficient c_{lm} explicitly demonstrate that the input and output modes are no longer identical. However, as the mode index increases, the number of photons in the mode decreases. As such, it may be expected that for large mode indices, the same-crystal overlap coefficient again approaches a diagonal form (as in the low-gain regime). Indeed, this behavior can be observed in Fig. 1: For $G_{\text{exp}} = 4$, $|c_{lm}|^2$ approaches a diagonal form after around $l = m = 6$, while for $G_{\text{exp}} = 8$, it can be seen that the overlap coefficients become more diagonal approaching $l = m = 15$. The behavior of the modes $n > 15$ is investigated in more detail below.

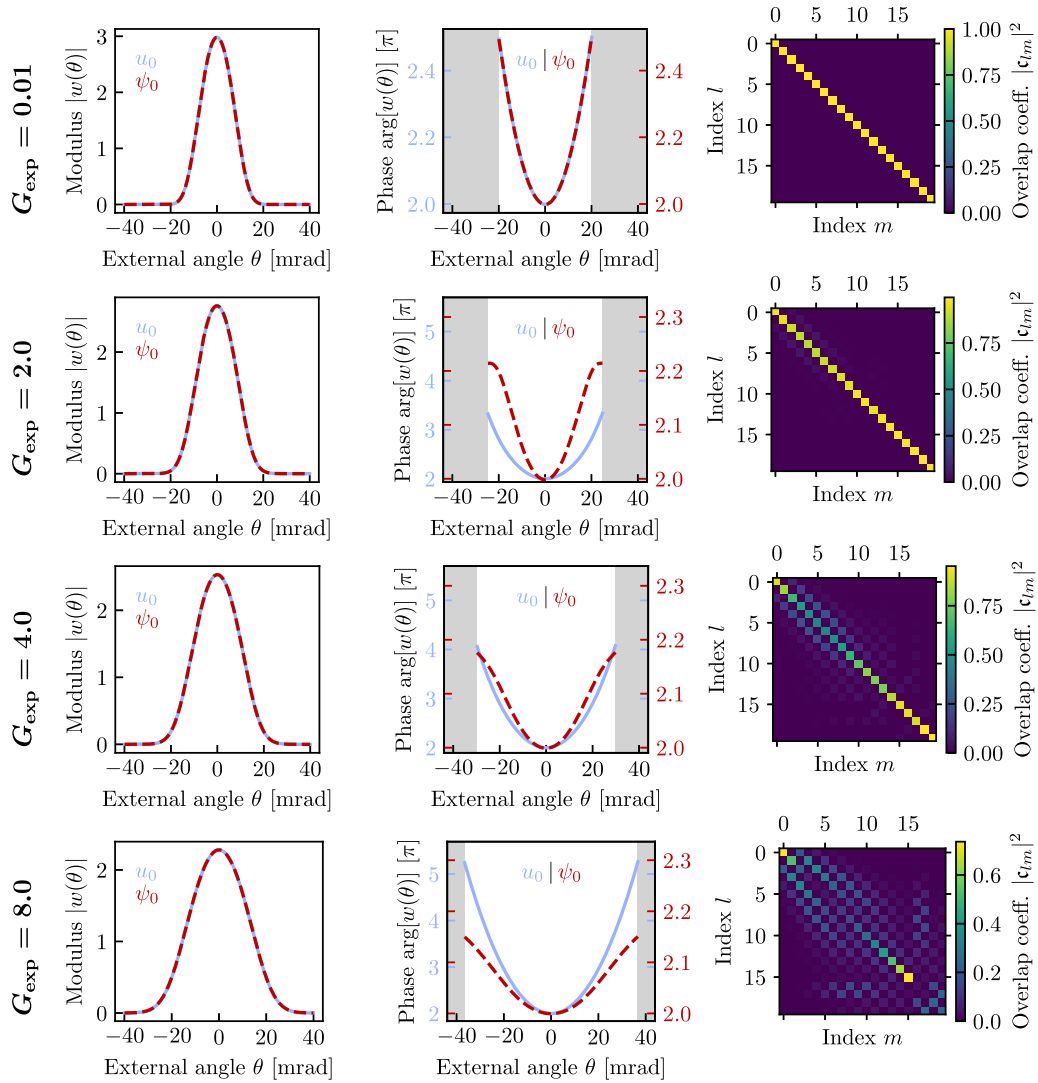


FIG. 1. Modulus (left column) and phase distributions (center column) of the first input ($w = \psi_0$, red dashed curve) and output ($w = u_0$, blue solid curve) Schmidt modes at different experimental gain values as written on the left. The gray margins indicate the range after which the modulus of the modes has decreased by a factor of 10^{-3} compared to the peak value. Since the modulus decreases, inside the margins, the phase remains undefined and is characterized by random π (due to the fact that $w \approx 0$) or 2π (due to the periodicity) phase jumps (not shown). As the gain increases, the width of the modes increases, as already described in Ref. [8]. At low gain, $G_{\text{exp}} = 0.01$, both input and output modes fully coincide, while at high gain, only the modulus of the modes coincides approximately. Starting at $G_{\text{exp}} = 2.0$, the phase distributions of the input and output modes differ significantly. Right column: Modulus squared $|c_{lm}|^2$ of the overlap coefficient c_{lm} as defined in Eq. (2.2). Clearly, for low gain, the modes are identical, as indicated by the fact that $|c_{lm}|^2$ is diagonal, and become increasingly different as the gain increases. Note that similar observations regarding the similarity of the modulus of the input and output modes and their broadening have been made in Ref. [37] for the frequency domain.

It should be noted that generally, at high gain, both the modulus and the phase profile of the input and output modes are different, even though they seem to be identical for lower-order modes. This can be seen explicitly from Fig. 2, which shows the moduli of the input and output modes with indices 14, 15, and 16 at high gain, $G_{\text{exp}} = 8$. Clearly, for the input and output modes ψ_{15} and u_{15} , the modulus does not coincide for broad angles around $\theta = \pm 35$ mrad. The differences in the input and output modes can be explained by inspecting the transfer function β in mode detail (see Appendix B).

Additionally, Figs. 1 and 2 demonstrate that the modes with $n \leq 15$ are mostly localized near the collinear direction and have profiles similar to the Hermite-Gaussian modes: Their

number of peaks increases as the mode index is increased. However, this is no longer the case for $n = 16$, since the PDC intensity distribution has a sinc shape [7,8]. This change in the general shape of the mode explains the behavior of the overlap matrix c_{lm} after $l = m = 15$ (see Fig. 1). Clearly, since the mode with $m = 16$ is no longer localized near the collinear direction, it may share overlap with several low-order modes, which all have a small modulus near these broader angles around -40 to -20 mrad and 20 to 40 mrad. At the same time, there is a strong decrease in the overlap between the input and output modes at $l = m = 16$, which is related to the phase profile of the modes [36]. Indeed, as shown in Fig. 2, the relative phase between the modes is almost flat for

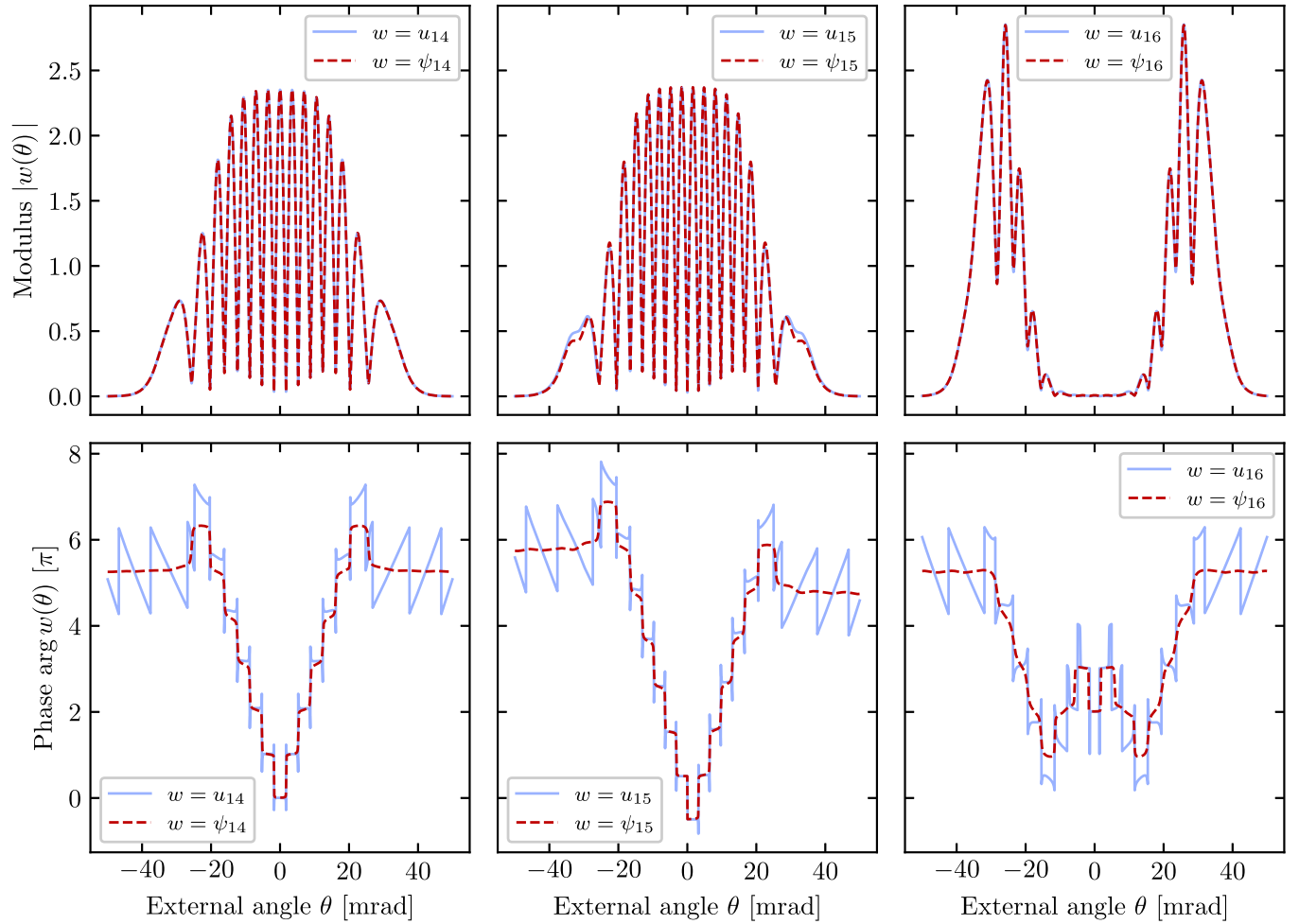


FIG. 2. First row: Plots of the modulus of several higher-order input and output modes of a single BBO crystal at high gain $G_{\text{exp}} = 8$. Second row: Phase distributions of the corresponding mode functions. Evidently, the moduli of the input and output modes ψ_{15} and u_{15} do not coincide at around $\theta = \pm 35$ mrad. Between modes $n = 15$ and $n = 16$, a structural change in the mode shapes occurs: For $n \leq 15$, the moduli of the modes are similar to Hermite-Gaussian modes and are mostly localized near the collinear direction ($\theta = 0$), while for $n = 16$, the modulus of the modes vanishes near the collinear direction and is mostly nonzero between -40 and -20 mrad and from 20 to 40 mrad. See also Fig. 1.

$l = m = 15$ in the region where the intensity of the modes is nonzero, which is not the case for $l = m = 16$. Here, instead, the relative phase has an approximately linear dependence on the angle for the region where the intensity distribution is nonzero, which causes the overlap integral between the modes to drop to almost zero. Note that the index of the mode at which the flip from the Hermite-Gaussian profiles to the double-peak profiles occurs depends on the chosen system parameters.

III. MODAL STRUCTURE OF SU(1,1) INTERFEROMETERS

In the following, we turn to SU(1,1) interferometers consisting of two PDC sections. We will use the superscript (1) for quantities referring to the first crystal and, analogously, (2) for quantities related to the second crystal and (SU) for quantities describing the interferometer as a whole. A general setup of an SU(1,1) interferometer is shown in Fig. 3. Unless specified otherwise, the parameters will be the same as for the single-crystal case discussed in Sec. II.

A. Balanced fully compensated interferometer

In this section, we revisit the properties of the compensated SU(1,1) interferometer based on Ref. [7] and extend its description by accounting for the modal structure of the interferometer and its two PDC sections. For a balanced, lossless, and fully compensated SU(1,1) interferometer, the functions h describing the phase matching are given by [7]

$$h^{(1)}(q_s, q_i, L) = e^{i\Delta k L}, \quad (3.1a)$$

$$h^{(2)}(q_s, q_i, L) = e^{-i\Delta k(q_s, q_i)[L-2L_1]} e^{i\phi}, \quad (3.1b)$$

for the first crystal and second crystal, respectively, where ϕ is the relative phase difference introduced to the pump, signal, and idler radiation. Due to the perfect diffraction compensation, this phase ϕ does not depend on the transverse wave vectors [36]. It should be noted that in Eqs. (3.1a) and (3.1b), the phase-matching functions are written in a form that implies that the integration domain is given by the interval $[0, L_1]$ for both crystals, where L_1 is the crystal length of each crystal.

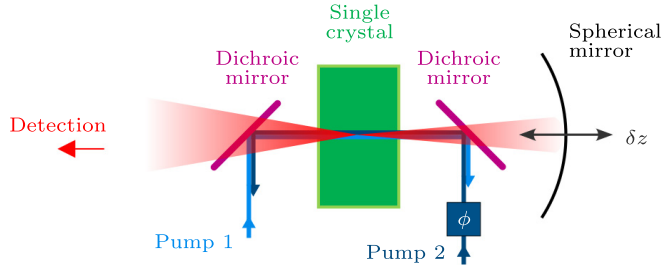


FIG. 3. Sketch of the SU(1,1) interferometer setup in *reflection geometry* as considered in this work. A single crystal is pumped with Pump 1, corresponding to a parametric gain $G^{(1)}$ (first pass). The generated PDC radiation is then reflected back onto the crystal by a spherical mirror. Pump 1 is removed from the setup and Pump 2 is fed into the crystal using two dichroic mirrors, so that the parametric gain $G^{(2)}$ of the second pass may differ from the first pass. Additionally, the interferometric phase ϕ is varied by manipulating the phase of Pump 2. If $G^{(1)} \neq G^{(2)}$, the spherical mirror may be shifted a distance δz toward or away from the crystal in order to improve the interferometric visibility (see the discussion in Sec. III B and Fig. 6). Note that the diffraction of the PDC radiation is shown exaggerated with respect to the rest of the figure elements. See also Refs. [5,7] for similar figures.

As was already described in Ref. [7], the focusing element used for the diffraction compensation introduces a symmetry in the system, which means that the entire interferometer is fully described by the properties of the first crystal and the contribution of the flat phase ϕ , which does not depend on the transverse wave vectors. By expressing the transfer functions of the second crystal in terms of those of the first crystal [7], meaning

$$\beta^{(2)}(q, q') = e^{i\phi} \beta^{(1)}(q', q), \quad (3.2a)$$

$$\tilde{\eta}^{(2)}(q, q') = [\tilde{\eta}^{(1)}(q', q)]^*, \quad (3.2b)$$

and comparing these expressions with the joint Schmidt decomposition for the first crystal as written in Eqs. (1.3a) and (1.3b), it is immediately obvious that the joint decomposition for the transfer functions of the second crystal is given by

$$\beta^{(2)}(q, q') = \sum_n \sqrt{\Lambda_n^{(1)}} [e^{\frac{i}{2}\phi} \psi_n^{(1)}(q)] [e^{\frac{i}{2}\phi} u_n^{(1)}(q')], \quad (3.3a)$$

$$\tilde{\eta}^{(2)}(q, q') = \sum_n \sqrt{\tilde{\Lambda}_n^{(1)}} [e^{\frac{i}{2}\phi} \psi_n^{(1)}(q)] [e^{\frac{i}{2}\phi} u_n^{(1)}(q')]^*. \quad (3.3b)$$

Furthermore, it follows that the eigenvalues of both crystals are connected via

$$\Lambda_n^{(2)} = \Lambda_n^{(1)}, \quad (3.4a)$$

$$\tilde{\Lambda}_n^{(2)} = \tilde{\Lambda}_n^{(1)}, \quad (3.4b)$$

while the Schmidt modes are connected as follows:

$$u_n^{(2)}(q) = e^{\frac{i}{2}\phi} \psi_n^{(1)}(q), \quad (3.4c)$$

$$\psi_n^{(2)}(q') = e^{\frac{i}{2}\phi} u_n^{(1)}(q'). \quad (3.4d)$$

Using the connection relations [7]

$$\beta^{(\text{SU})}(q, q') = \int d\bar{q} \tilde{\eta}^{(2)}(q, \bar{q}) \beta^{(1)}(\bar{q}, q') + \int d\bar{q} \beta^{(2)}(q, \bar{q}) [\tilde{\eta}^{(1)}(\bar{q}, q')]^*, \quad (3.5a)$$

$$\tilde{\eta}^{(\text{SU})}(q, q') = \int d\bar{q} \tilde{\eta}^{(2)}(q, \bar{q}) \tilde{\eta}^{(1)}(\bar{q}, q') + \int d\bar{q} \beta^{(2)}(q, \bar{q}) [\beta^{(1)}(\bar{q}, q')]^*, \quad (3.5b)$$

one can connect the transfer functions of the first and the second crystals to obtain the transfer functions $\tilde{\eta}^{(\text{SU})}$ and $\beta^{(\text{SU})}$ of the whole SU(1,1) interferometer.

Plugging the joint Schmidt decompositions for the transfer functions of the first and second crystals [see Eqs. (1.3a) and (1.3b)] into Eqs. (3.5a) and (3.5b), it becomes clear that the transfer functions of the full SU(1,1) interferometer read

$$\beta^{(\text{SU})}(q, q') = \sum_n \sqrt{\Lambda_n^{(\text{SU})}} u_n^{(\text{SU})}(q) \psi_n^{(\text{SU})}(q'), \quad (3.6a)$$

$$\tilde{\eta}^{(\text{SU})}(q, q') = \sum_n \sqrt{\tilde{\Lambda}_n^{(\text{SU})}} u_n^{(\text{SU})}(q) [\psi_n^{(\text{SU})}(q')]^*, \quad (3.6b)$$

where

$$\Lambda_n^{(\text{SU})} = 4 \cos^2\left(\frac{\phi}{2}\right) \Lambda_n^{(1)} \tilde{\Lambda}_n^{(1)}, \quad (3.6c)$$

$$\tilde{\Lambda}_n^{(\text{SU})} = 1 + \Lambda_n^{(\text{SU})}, \quad (3.6d)$$

$$u_n^{(\text{SU})}(q) = \exp\left[\frac{i}{2}(\mu + \zeta_n)\right] \psi_n^{(1)}(q), \quad (3.6e)$$

$$\psi_n^{(\text{SU})}(q') = \exp\left[\frac{i}{2}(\mu - \zeta_n)\right] \psi_n^{(1)}(q'), \quad (3.6f)$$

with

$$\mu = \arg(1 + e^{i\phi}), \quad (3.6g)$$

$$\zeta_n = \arg[1 + \Lambda_n^{(1)}(1 + e^{i\phi})]. \quad (3.6h)$$

Equations (3.6a)–(3.6h) provide a joint Schmidt decomposition for the transfer functions of the entire interferometer in terms of the input Schmidt modes of the first crystal and supplement the discussion of the Schmidt modes presented in Ref. [7]. Furthermore, Eqs. (3.6e) and (3.6f) suggest that the input and output modes of the SU(1,1) interferometer are equal up to some constant phase. This is similar to the setup described in Ref. [41] for the frequency domain, where the input and output modes of the system are also equal due to a certain symmetry in the system.

B. Unbalanced imperfectly compensated interferometer

For an unbalanced interferometer, the properties of the PDC section on the first and second pass are no longer identical, meaning that the symmetry properties connecting the transfer functions of the first and second crystals described by Eqs. (3.2a) and (3.2b) no longer hold, and it is generally no longer possible to describe the connection between the transfer functions analytically. Thus, the transfer functions for

both crystals have to be obtained by integrating the integro-differential equations numerically.

For this kind of interferometer, we consider phase-matching functions of the form

$$h^{(1)}(q_s, q_i, L) = e^{i\Delta k L}, \quad (3.7a)$$

$$h^{(2)}(q_s, q_i, L) = e^{-i\Delta k(q_s, q_i)[L-2L_1]} \times e^{-i\Delta k^{\text{air}}(q_s, q_i)\delta z} e^{i\phi}. \quad (3.7b)$$

Compared to the balanced interferometer with full diffraction compensation described by Eq. (3.1b), the newly introduced phase term $e^{-i\Delta k^{\text{air}}(q_s, q_i)\delta z}$ in Eq. (3.7b) describes an offset δz of the focusing element in the air gap of the SU(1,1) interferometer from its optimal position in the case of balanced gains. In experiments, this optimal position is normally defined by the position that maximizes the interferometric visibility

$$v = \frac{\langle \hat{N}_{\text{tot}} \rangle_{\text{bf}} - \langle \hat{N}_{\text{tot}} \rangle_{\text{df}}}{\langle \hat{N}_{\text{tot}} \rangle_{\text{bf}} + \langle \hat{N}_{\text{tot}} \rangle_{\text{df}}} \times 100\%, \quad (3.8)$$

when the phase ϕ is varied between the bright and dark fringes with integral intensities $\langle \hat{N}_{\text{tot}} \rangle_{\text{bf}}$ and $\langle \hat{N}_{\text{tot}} \rangle_{\text{df}}$, respectively. Note that the phase ϕ is usually varied by changing the phase of the pump laser and is therefore independent of δz (see also Fig. 3). Clearly, for the perfectly compensated balanced interferometer, $v = 100\%$ (see Appendix C and Ref. [7]). However, for an unbalanced (or imperfectly compensated) interferometer, the optimal visibility will generally be less than 100%. Generally, for a given setup, the optimal value for δz can be found by varying δz and maximizing the visibility [see Eq. (C5)], for which an additional optimization over ϕ is not required. It should be noted that in general, as also described in Appendix C, the bright and dark fringes do not occur at $\phi = 0$ and $\phi = \pi$, respectively, and are instead shifted by some phase Υ [see Eq. (C4)].

Before analyzing the behavior of the Schmidt modes in this kind of interferometer, we first illustrate the modal layouts of two crystals in comparison to the entire interferometer in Fig. 4. One can notice that to describe the full SU(1,1) system, two sets of overlap coefficients [42] g_{km} and h_{km} are required, which will be introduced more formally below.

The first overlap coefficient corresponds to the overlap between the output modes of the first crystal and the input modes of the second crystal and is defined as [5]

$$g_{km} \stackrel{\text{def.}}{=} \int dq [u_k^{(1)}(q)]^* \psi_m^{(2)}(q). \quad (3.9)$$

In general, this overlap coefficient is not diagonal ($g_{km} \not\propto \delta_{km}$), even for an interferometer consisting only of two consecutive crystals with no phase object in between, so that any output Schmidt mode of the first crystal may couple to all input Schmidt modes of the second crystal.

In experimental setups, the modes that are usually measured are the *output modes of the interferometer* and *not* the output modes of the second crystal. These originate from the decomposition of the transfer functions of the entire interferometer and the transfer functions of the second crystal, respectively, and are, in general, not identical since they belong to different (sub)systems. This leads to the definition

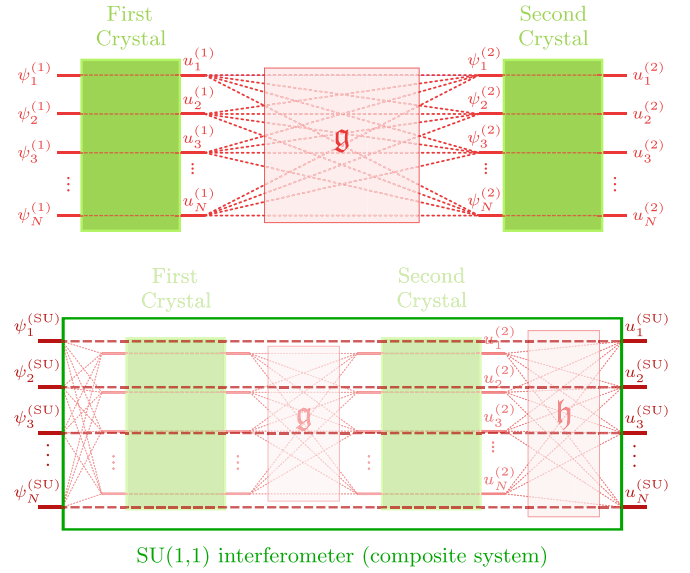


FIG. 4. Upper row: Modal layout of the two crystals comprising the interferometers understood as separate PDC systems. The dashed lines drawn over the crystals illustrate the Bogoliubov transformations connecting the input Schmidt modes $\psi_n^{(k)}$ to the output modes $u_n^{(k)}$ with the same index n (for $k = 1, 2$). The overlap coefficient g as defined in Eq. (3.9) describes the coupling between the output modes of the first crystal and the input modes of the second crystal. Lower row: Modal layout of the SU(1,1) interferometer, with the modal layout of the two separate crystals shown in the background. This illustrates the main difference between the modes of the separate crystals and those of the interferometer: The modes associated with each crystal diagonalize their associated crystal, while the modes of the interferometer diagonalize the input-output relation of the entire interferometer. Notably, since g is in general not diagonal, an input-output relation connecting the $\psi_n^{(1)}$ to the $u_l^{(2)}$ will in general also not be diagonal, and thus the $\psi_n^{(1)}$ and $u_l^{(2)}$ may then not correspond to the input and output modes, respectively, of the entire interferometer. This explains the necessity to introduce the overlap coefficient h as defined in Eq. (3.10) in order to quantify the mismatch between the output modes of the second crystal $u_n^{(1)}$ and those of the interferometer $u_n^{(\text{SU})}$. Figure adapted from Ref. [36].

of the overlap coefficients between the output modes of the second crystal and those of the entire interferometer [5]:

$$h_{nl} \stackrel{\text{def.}}{=} \int dq [u_n^{(2)}(q)]^* u_l^{(\text{SU})}(q). \quad (3.10)$$

Generally, both g_{km} and h_{nl} will depend on the interferometric phase since they depend on the Schmidt modes of the second crystal and the entire interferometer.

Next, in Fig. 5, we provide examples for the overlap matrices g and h defined above, where the first 15×15 entries of the overlap matrices are shown. For this, based on Ref. [5], we consider an unbalanced SU(1,1) interferometer with gains $G_{\text{exp}}^{(1)} = 1$ and $G_{\text{exp}}^{(2)} = 4$ for the first and second crystals, respectively. The remaining parameters are the same as in the case of the single crystal discussed above (see Sec. II). For this setup, the dependence of interferometric visibility v as defined in Eq. (3.8) on δz can be seen in Fig. 6 and takes its maximum

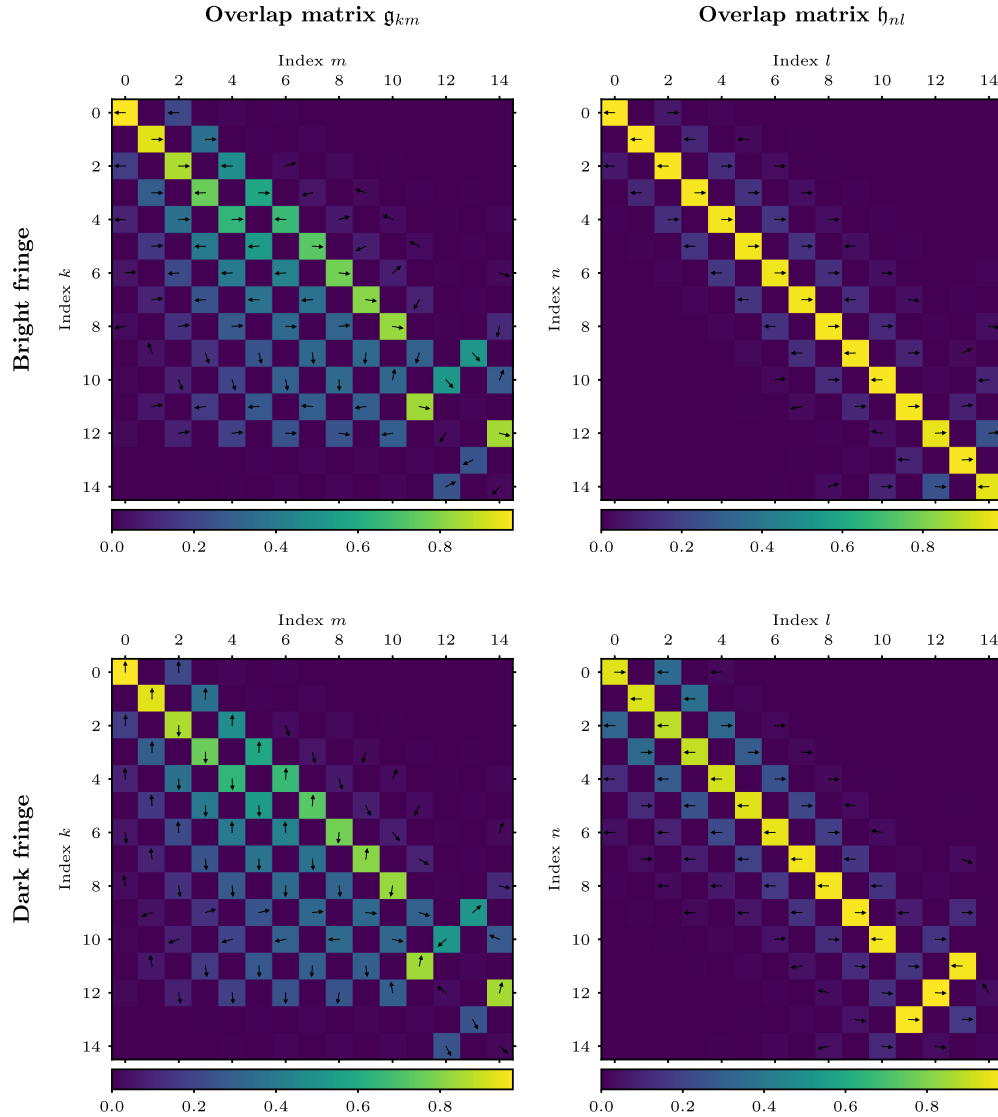


FIG. 5. First 15×15 entries of the overlap matrices g_{km} [left column, defined in Eq. (3.9)] and h_{nl} [right column, defined in Eq. (3.10)] for the bright fringe (first row, $\phi = 0.0337\pi$) and dark fringe (second row, $\phi = 1.03\pi$). The color indicates the modulus of the complex-valued entry of the overlap matrix and the handle pointing outward from the center of each cell indicates the location of the value in the complex plane. That is, if the handle points to the right, the number is purely real and positive, and if it points up, the number is purely imaginary with positive sign. Handles are not shown for entries with modulus smaller than 0.02. It should be noted that the phase of the overlap coefficients is only defined up to adding or subtracting π since the sign of the modes is not well defined from the joint Schmidt decomposition alone (see Appendix A 1). See also Appendix C, which explains why the bright and dark fringes do not occur exactly at $\phi = 0$ and $\phi = \pi$, respectively. Furthermore, Appendix D discusses the alignment of the phases of the first few modes for varying δz .

value if the focusing element inside the interferometer has an offset of $\delta z = 515.3 \mu\text{m}$ from its optimal position in the case of balanced interferometer. Note that, generally, the optimal value of δz will depend on the system parameters. From the plots of g in Fig. 5, it is clear that the output modes of the first crystal and the input modes of the second crystal are substantially different, leading to the checkerboard pattern visible in the lower-left half. It should be noted that when comparing g for the bright and dark fringes, only the phases of the matrix entries are affected. This can be seen as follows: As was shown in Ref. [7], for a PDC section described by the phase-matching functions of the form as in Eqs. (3.1b) and (3.7b), it is possible to analytically account for the flat phase contribution $e^{i\phi}$ by using the fact that the transfer function β

can be written in the form [7]

$$\beta(q_s, q_i) = e^{i\phi} \check{\beta}(q_s, q_i), \quad (3.11)$$

where $\check{\beta}(q_s, q_i)$ is the transfer function resulting from the integro-differential equations by integrating them without the phase contribution $e^{i\phi}$ in the phase-matching function, or, equivalently, for $\phi = 0$. Generally, in the following, the notation

$$\check{x} \quad (3.12)$$

will be used to denote a quantity x obtained by solving the integro-differential equations with the replacement $e^{i\phi} \rightarrow 1$. The transfer function $\check{\eta}$ does not change when removing the

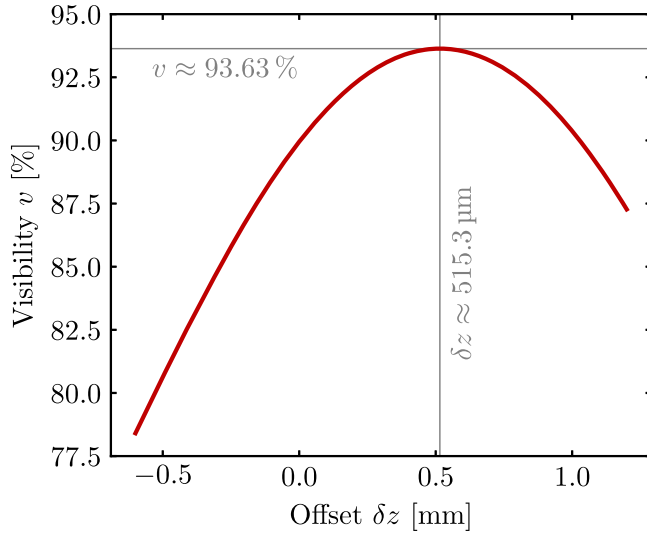


FIG. 6. Visibility v of an unbalanced SU(1,1) interferometer with parametric gains $G^{(1)} = 1$ and $G^{(2)} = 4$ for the first and second crystals, respectively, and the remaining parameters as in Sec. II, as a function of the offset δz . The visibility v reaches its maximum value of 93.63 % at an offset of $\delta z \approx 515.3 \mu\text{m}$, relative to the optimal position of a perfectly compensated interferometer with equal gains in both crystals ($\delta z = 0$).

phase contribution [7], but the notation introduced above may still be applied here, since β and $\tilde{\eta}$ are obtained as a pair when solving the integro-differential equations. The joint decomposition of the transfer functions for the second crystal can then be written in a form analogous to Eqs. (3.3a) and (3.3b):

$$\beta^{(2)}(q, q') = \sum_n \sqrt{\Lambda_n^{(2)}} [e^{\frac{i}{2}\phi} \check{\psi}_n^{(2)}(q)] [e^{\frac{i}{2}\phi} \check{u}_n^{(2)}(q')], \quad (3.13a)$$

$$\tilde{\eta}^{(2)}(q, q') = \sum_n \sqrt{\tilde{\Lambda}_n^{(2)}} [e^{\frac{i}{2}\phi} \check{\psi}_n^{(2)}(q)] [e^{\frac{i}{2}\phi} \check{u}_n^{(2)}(q')]^*, \quad (3.13b)$$

where the full dependence on the phase is given by the terms $e^{\frac{i}{2}\phi}$ and the functions $\check{u}_n$ are phase independent, as described for $\check{\beta}(q_s, q_i)$ above. Thus, to explicitly account for the phase, the defining expression for $\mathbf{g}$ as given in Eq. (3.9) may be rewritten as

$$\mathbf{g}_{km} = e^{i\frac{\phi}{2}} \int dq [u_k^{(1)}(q)]^* \check{\psi}_m^{(2)}(q), \quad (3.14)$$

where $\check{\psi}_m^{(2)}(q)$ now refers to the phase-independent input mode of the second crystal. Importantly, however, note that in this section, Eqs. (3.2a) and (3.2b) no longer hold due to the fact that both crystals have different gain values [43], and thus Eqs. (3.13a) and (3.13b) cannot be trivially expressed in terms of the singular values and mode functions of the first crystal, as was the case in Eqs. (3.3a) and (3.3b).

In Eq. (3.14), the integral does not depend on the interferometer phase ϕ . Hence, the phase dependence is fully described by the term $e^{i\frac{\phi}{2}}$, which explains why the modulus of $\mathbf{g}$ does not change as the phase varies from the dark fringe to the bright fringe. Furthermore, this also implies that the phase of $\mathbf{g}$ differs by exactly π when comparing the bright

and dark fringes, apart from additional π -phase jumps due to the fact that the sign of the Schmidt modes obtained from the numerical decomposition is arbitrary (see Appendix A 1).

Note that for the balanced case with perfect compensation, the overlap coefficients are trivially diagonal:

$$\mathbf{g}_{km} = e^{i\frac{\phi}{2}} \delta_{km} \quad (3.15a)$$

[compare Eqs. (3.9) and (3.4d)] and

$$\mathbf{h}_{ln} = e^{i\frac{\phi}{2}(\mu + \zeta_n - \frac{\phi}{2})} \delta_{ln} \quad (3.15b)$$

[see Eqs. (3.10), (3.6e), and (3.4c)]. For numerical calculations, as mentioned above, it must be taken into account that random phase jumps may lead to sign changes of the overlap coefficients.

The observations made above raise questions regarding the physical interpretations of the Schmidt modes. Care must be taken not to confuse the Schmidt modes of the two isolated crystals of the interferometers (that diagonalize the solution describing each separate crystal) and the Schmidt modes of the composite system (that diagonalize the full two-crystal system), as depicted in Fig. 4.

In this context, it is also important to address a common misconception: While the term “input” Schmidt mode implies that it is connected to the input plane-wave operators, which correspond to the vacuum operators for the *first* PDC section, their shapes ψ_n , as well as the shapes of the output modes u_n , depend on the physical parameters of the PDC process (the length of the PDC section, its refractive indices, the pump profile, the parametric gain etc.). In more direct terms, it is not possible to obtain these modes without having computed the transfer functions $\tilde{\eta}$ and β for the entire interaction, since only after computing them it is possible to find the set of modes, which diagonalizes the plane-wave input/output relations as written in Eqs. (1.1a) and (1.1b). More precisely, the input modes are *not* known “before” the interaction and are *not* a property of the fields as they appear before the interaction at the input faces of the crystals, but rather a property of the PDC interaction [36].

Note that the description of the gain-unbalanced interferometer as given in this section can also be generalized to SU(1,1) interferometers with different crystal lengths, refractive indices, and other differing properties of the two crystals.

IV. MULTIMODE SQUEEZING MEASUREMENT

The modal structure of the SU(1,1) interferometer described above can be used for measuring the level of squeezing in the n th output mode $u_n^{(1)}(q)$ of the first crystal based on measurements of properties of the second crystal and the full SU(1,1) interferometer (phase-sensitive parametric amplification). The simplified version of this technique was experimentally realized in Ref. [5]. In this section, we present a rigorous mathematical derivation of this technique and discuss simplifications that should be provided to make this technique experimentally feasible.

First, it should be highlighted that since the ultimate goal is to measure the squeezing in the modes associated with the mode functions $u_n^{(1)}$ of the first crystal, it is necessary to split the entire interferometer into this crystal and the following crystal. Otherwise, it is not possible to access these modes

$u_n^{(1)}$, as can also be seen by careful inspection of Fig. 4. For the unbalanced interferometer, due to the nondiagonal overlap matrices $\mathbf{g}$ and $\mathbf{h}$, the information regarding this mode $u_n^{(1)}(q)$ is distributed over all output modes $u_k^{(\text{SU})}(q)$ of the entire interferometer, which in turn contain a mixture of the information of all modes $u_n^{(1)}(q)$. However, provided that the overlap coefficients and the eigenvalues of the second crystal and the interferometer are known, it is possible to reconstruct all information regarding the $u_n^{(1)}(q)$. Similarly, it is possible to connect the output Schmidt operators $\hat{A}_l^{(1,\text{out})}$ of the first crystal to the output operators $\hat{A}_m^{(\text{SU},\text{out})}$ of the entire interferometer, as will be seen in the following.

A. Exact reconstruction procedure

When connecting the first crystal of the interferometer to the second crystal, the underlying assumption is that the output plane-wave operators of the first crystal coincide with the input plane-wave operators of the second crystal:

$$\hat{a}^{(1,\text{out})}(q) = \hat{a}^{(2,\text{in})}(q). \quad (4.1)$$

This assumption also leads to the connection relations connecting the transfer functions of the two crystals to those of the entire interferometer as written in Eqs. (3.5a) and (3.5b) [7]. The relation for the corresponding connection between the Schmidt-mode operators is not as simple as Eq. (4.1). Instead, using Eqs. (3.9) and (4.1), one finds

$$\hat{A}_l^{(1,\text{out})} = \sum_m \mathbf{g}_{lm} \hat{A}_m^{(2,\text{in})}. \quad (4.2)$$

The input Schmidt operators of the second crystal are transformed through the second crystal using the Bogoliubov transformations. The inverse transformation for Eq. (1.5) reads [44]

$$\hat{A}_m^{(\text{in})} = \sqrt{\tilde{\Lambda}_m} \hat{A}_m^{(\text{out})} - \sqrt{\Lambda_m} [\hat{A}_m^{(\text{out})}]^\dagger. \quad (4.3)$$

Plugging this into Eq. (4.2) yields an expression connecting the output operators of the first crystal with the output operators of the second crystal.

At the output of the second crystal, the output plane-wave operators of the interferometer coincide with the output plane-wave operators of the second crystal: $\hat{a}^{(2,\text{out})}(q) = \hat{a}^{(\text{SU},\text{out})}(q)$. However, this does, again, not hold analogously for the Schmidt-mode operators. Instead, analogously to Eq. (4.1), one finds

$$\hat{A}_m^{(2,\text{out})} = \sum_k \mathbf{h}_{mk} \hat{A}_k^{(\text{SU},\text{out})}. \quad (4.4)$$

Finally, combining Eqs. (4.2), (4.3), and (4.4) as detailed above, we obtain the following expression connecting the output Schmidt operators of the first crystal with those of the entire interferometer:

$$\begin{aligned} \hat{A}_l^{(1,\text{out})} = & \sum_{m,k} \mathbf{g}_{lm} [\sqrt{\tilde{\Lambda}_m^{(2)}} \mathbf{h}_{mk} \hat{A}_k^{(\text{SU},\text{out})} \\ & - \sqrt{\Lambda_m^{(2)}} \mathbf{h}_{mk}^* (\hat{A}_k^{(\text{SU},\text{out})})^\dagger]. \end{aligned} \quad (4.5)$$

The *generic quadrature* operator for the l th Schmidt mode can be defined analogously to the plane-wave quadrature op-

erators [34] as

$$\hat{P}_{\vartheta,l} \stackrel{\text{def.}}{=} \hat{A}_l e^{-i\frac{\vartheta}{2}} + \hat{A}_l^\dagger e^{i\frac{\vartheta}{2}}, \quad (4.6)$$

where ϑ is the phase of the generic quadrature. For $\vartheta = \pi$ and $\vartheta = 0$, the following special quadrature operators can be defined:

$$\hat{X}_l \stackrel{\text{def.}}{=} \hat{P}_{\pi,l} = -i(\hat{A}_l - \hat{A}_l^\dagger), \quad (4.7a)$$

$$\hat{Y}_l \stackrel{\text{def.}}{=} \hat{P}_{0,l} = \hat{A}_l + \hat{A}_l^\dagger, \quad (4.7b)$$

which are analogous to the quadrature operators defined during the quantization procedure (see, for example, Refs. [34,45]). Generally, the generic quadrature operators fulfill the following commutation relation:

$$[\hat{P}_{\vartheta+\pi,l}, \hat{P}_{\vartheta,l}^\dagger] = \frac{2}{i}. \quad (4.8)$$

For squeezed vacuum, $\langle \hat{P}_{\vartheta,l} \rangle = 0$, and the variance of the generic quadrature operators is given by [46]

$$\Delta^2 \hat{P}_{\vartheta,l} = 1 + 2\langle \hat{A}_l^\dagger \hat{A}_l \rangle + 2\text{Re}[\langle \hat{A}_l \hat{A}_l \rangle e^{-i\vartheta}], \quad (4.9)$$

which reduces to

$$\Delta^2 \hat{X}_l = 1 + 2\langle \hat{A}_l^\dagger \hat{A}_l \rangle - 2\text{Re}[\langle \hat{A}_l \hat{A}_l \rangle], \quad (4.10a)$$

$$\Delta^2 \hat{Y}_l = 1 + 2\langle \hat{A}_l^\dagger \hat{A}_l \rangle + 2\text{Re}[\langle \hat{A}_l \hat{A}_l \rangle], \quad (4.10b)$$

for the special case written in Eqs. (4.7a) and (4.7b).

It should be noted that Eq. (4.9) implies that the concrete value for ϑ to reach the maximally squeezed and antisqueezed quadrature may be shifted away from $\vartheta = \pi$ and $\vartheta = 0$, respectively, if $\langle \hat{A}_l \hat{A}_l \rangle$ is complex valued. Generally, the maximal levels of squeezing and antisqueezing are defined as the minimum and maximum of $\Delta^2 \hat{P}_{\vartheta,l}$, respectively. These extremal values for the quadrature variance may be written as

$$\min_{\vartheta} \Delta^2 \hat{P}_{\vartheta,l} = \Delta^2 \hat{P}_{\vartheta_{\min,l},l} = 1 + 2\langle \hat{A}_l^\dagger \hat{A}_l \rangle - 2|\langle \hat{A}_l \hat{A}_l \rangle|, \quad (4.11a)$$

$$\max_{\vartheta} \Delta^2 \hat{P}_{\vartheta,l} = \Delta^2 \hat{P}_{\vartheta_{\max,l},l} = 1 + 2\langle \hat{A}_l^\dagger \hat{A}_l \rangle + 2|\langle \hat{A}_l \hat{A}_l \rangle|, \quad (4.11b)$$

where we have introduced $\vartheta_{\min,l}$ and $\vartheta_{\max,l}$ as the values for ϑ that minimize and maximize the quadrature variance, respectively. With this, the levels of squeezing (S) and antisqueezing (AS) in mode l are given by

$$S_l = 10 \log_{10} \left[\frac{\min_{\vartheta} \Delta^2 \hat{P}_{\vartheta,l}}{\Delta^2 \hat{P}^{(\text{vac})}} \right] \text{dB}, \quad (4.12a)$$

$$AS_l = 10 \log_{10} \left[\frac{\max_{\vartheta} \Delta^2 \hat{P}_{\vartheta,l}}{\Delta^2 \hat{P}^{(\text{vac})}} \right] \text{dB}, \quad (4.12b)$$

where $\hat{P}^{(\text{vac})}$ is the corresponding vacuum quadrature operator, which has a quadrature variance independent of ϑ , which is why the corresponding index was dropped. To be more precise, from Eq. (4.9) it is immediately clear that

$$\Delta^2 \hat{P}^{(\text{vac})} = 1. \quad (4.13)$$

According to the definitions written in Eqs. (4.12a) and (4.12b), the levels of squeezing and antisqueezing are

measures of the relative change in quadrature variance compared to the vacuum quadrature variance.

Equations (4.9), (4.12a), and (4.12b) show that the levels of squeezing and antisqueezing can be obtained from the expectation values $\langle \hat{A}_l^\dagger \hat{A}_l \rangle$ and $\langle \hat{A}_l \hat{A}_l^\dagger \rangle$. By forming these expectation values for the output Schmidt-mode operators of

the first crystal as it is written in Eq. (4.5), it is possible to express the quadrature variance and thereby also the levels of squeezing and antisqueezing via the overlap matrices $\mathbf{g}$ and $\mathbf{h}$ and the eigenvalues of the second crystal and the entire interferometer. The concrete expressions for the expectation values read

$$\begin{aligned} \langle \hat{A}_l^{(1,\text{out})} \hat{A}_l^{(1,\text{out})} \rangle &= \sum_{n,n',k} \mathbf{g}_{ln} \mathbf{g}_{ln'} [\mathbf{h}_{nk} \mathbf{h}_{n'k} \sqrt{(\Lambda_n^{(2)} + 1)(\Lambda_{n'}^{(2)} + 1)} \sqrt{(\Lambda_k^{(\text{SU})} + 1)\Lambda_k^{(\text{SU})}} + \mathbf{h}_{nk}^* \mathbf{h}_{n'k}^* \sqrt{\Lambda_n^{(2)} \Lambda_{n'}^{(2)}} \sqrt{\Lambda_k^{(\text{SU})}(\Lambda_k^{(\text{SU})} + 1)} \\ &\quad - \mathbf{h}_{nk} \mathbf{h}_{n'k}^* \sqrt{(\Lambda_n^{(2)} + 1)\Lambda_{n'}^{(2)}} \sqrt{(\Lambda_k^{(\text{SU})} + 1)} - \mathbf{h}_{nk}^* \mathbf{h}_{n'k} \sqrt{\Lambda_n^{(2)}(\Lambda_{n'}^{(2)} + 1)} \Lambda_k^{(\text{SU})}] \end{aligned} \quad (4.14a)$$

and

$$\begin{aligned} \langle (\hat{A}_l^{(1,\text{out})})^\dagger \hat{A}_l^{(1,\text{out})} \rangle &= \sum_{n,n',k} \mathbf{g}_{ln}^* \mathbf{g}_{ln'} [\mathbf{h}_{nk}^* \mathbf{h}_{n'k} \sqrt{(\Lambda_n^{(2)} + 1)(\Lambda_{n'}^{(2)} + 1)} \Lambda_k^{(\text{SU})} + \mathbf{h}_{nk} \mathbf{h}_{n'k}^* \sqrt{\Lambda_n^{(2)} \Lambda_{n'}^{(2)}} (\Lambda_k^{(\text{SU})} + 1) \\ &\quad - \mathbf{h}_{nk}^* \mathbf{h}_{n'k} \sqrt{(\Lambda_n^{(2)} + 1)\Lambda_{n'}^{(2)}} \sqrt{\Lambda_k^{(\text{SU})}(\Lambda_k^{(\text{SU})} + 1)} - \mathbf{h}_{nk} \mathbf{h}_{n'k} \sqrt{\Lambda_n^{(2)}(\Lambda_{n'}^{(2)} + 1)} \sqrt{\Lambda_k^{(\text{SU})}(\Lambda_k^{(\text{SU})} + 1)}]. \end{aligned} \quad (4.14b)$$

In order to evaluate these expectation values, we have used the Bogoliubov transformations of the Schmidt operators from Eq. (1.5) written in terms of the entire interferometer to connect its output operators, appearing in Eq. (4.5), with its input operators. The contributing expectation values read

$$\begin{aligned} \langle \hat{A}_l^{(\text{out},\text{SU})} \hat{A}_l^{(\text{out},\text{SU})} \rangle &= \langle (\hat{A}_l^{(\text{out},\text{SU})})^\dagger (\hat{A}_l^{(\text{out},\text{SU})})^\dagger \rangle \\ &= \sqrt{(1 + \Lambda_l^{(\text{SU})}) \Lambda_l^{(\text{SU})}}, \end{aligned} \quad (4.15a)$$

$$\begin{aligned} \langle (\hat{A}_l^{(\text{out},\text{SU})})^\dagger \hat{A}_l^{(\text{out},\text{SU})} \rangle &= \langle \hat{A}_l^{(\text{out},\text{SU})} (\hat{A}_l^{(\text{out},\text{SU})})^\dagger \rangle - 1 \\ &= \Lambda_l^{(\text{SU})}. \end{aligned} \quad (4.15b)$$

Thus, given Eqs. (4.9) and (4.11a)–(4.15b) and complete knowledge regarding the quantities $\mathbf{g}_{ln}$, $\mathbf{h}_{nk}$, $\Lambda_n^{(2)}$, and $\Lambda_k^{(\text{SU})}$, it is possible to compute the levels of squeezing and antisqueezing in each output mode of the first crystal.

B. High-gain amplifier approximation

The exact expressions for the quadrature variance resulting from Eqs. (4.14a) and (4.14b) are rather complex. Furthermore, it requires knowledge of the eigenvalues Λ_n , referring to absolute photon numbers in the modes, which are quite tricky to obtain in experiments. Instead, the relative intensities between modes are free from setup-dependent prefactors and are therefore much easier to extract. To address this, in the following, we will derive an approximated method for obtaining the squeezing in each mode, which ultimately only relies on the *relative* values of the eigenvalues.

If the gain of the second crystal (amplifier) is large enough for the inequalities $\Lambda_n^{(2)} \gg 1$, $\Lambda_n^{(\text{SU})} \gg 1$ to be satisfied (see Fig. 7 for plots of the eigenvalues), it is possible to apply the following approximations for the eigenvalues of the $\tilde{\eta}$ transfer function of the second crystal and the entire interferometer

[47]:

$$\sqrt{\tilde{\Lambda}_n^{(2)}} = \sqrt{\Lambda_n^{(2)} + 1} \approx \sqrt{\Lambda_n^{(2)}} \left(1 + \frac{1}{2\Lambda_n^{(2)}} \right), \quad (4.16a)$$

$$\sqrt{\tilde{\Lambda}_k^{(\text{SU})}} = \sqrt{\Lambda_k^{(\text{SU})} + 1} \approx \sqrt{\Lambda_k^{(\text{SU})}} \left(1 + \frac{1}{2\Lambda_k^{(\text{SU})}} \right). \quad (4.16b)$$

It should be noted that $\Lambda_n^{(2)} \gg 1$ is satisfied by the assumption that the amplifier operates at high gain. However, it is not immediately obvious that $\Lambda_n^{(\text{SU})} \gg 1$ holds in general, since the output intensity of the interferometer should usually be small at the dark fringe (relative to the average intensity as the phase varies). Nevertheless, if the gain of the amplifier is much larger than the gain of the squeezer (the first crystal), meaning $G^{(2)} \gg G^{(1)}$, the eigenvalues of the interferometer are reasonably large even at the dark fringe (see Fig. 7).

Applying the approximations made in Eqs. (4.16a) and (4.16b) to Eq. (4.5), the output Schmidt operator of the first crystal can be written as

$$\begin{aligned} \hat{A}_l^{(1,\text{out})} &= 2i \sum_{n,k} \mathbf{g}_{ln} \sqrt{\Lambda_n^{(2)}} \sqrt{\Lambda_k^{(\text{SU})}} \text{Im}(\mathbf{h}_{nk}) \\ &\quad \times [\hat{A}_k^{(\text{SU},\text{in})} + (\hat{A}_k^{(\text{SU},\text{in})})^\dagger] \\ &\quad + \frac{1}{2} \sum_{n,k} \mathbf{g}_{ln} \sqrt{\frac{\Lambda_n^{(2)}}{\Lambda_k^{(\text{SU})}}} [\mathbf{h}_{nk} \hat{A}_k^{(\text{SU},\text{in})} - \mathbf{h}_{nk}^* (\hat{A}_k^{(\text{SU},\text{in})})^\dagger] \\ &\quad + \frac{1}{2} \sum_{n,k} \mathbf{g}_{ln} \sqrt{\frac{\Lambda_k^{(\text{SU})}}{\Lambda_n^{(2)}}} \mathbf{h}_{nk} [\hat{A}_k^{(\text{SU},\text{in})} + (\hat{A}_k^{(\text{SU},\text{in})})^\dagger] \\ &\quad + \frac{1}{4} \sum_{n,k} \mathbf{g}_{ln} \mathbf{h}_{nk} \frac{1}{\sqrt{\Lambda_n^{(2)}}} \frac{1}{\sqrt{\Lambda_k^{(\text{SU})}}} \hat{A}_k^{(\text{SU},\text{in})}. \end{aligned} \quad (4.17)$$

Note that in the expression above, we have applied the Bogoliubov transformations for the output operators of the entire interferometer [compare Eq. (1.5)] in order to express the output Schmidt operators of the first crystal $\hat{A}_l^{(1,\text{out})}$ in terms

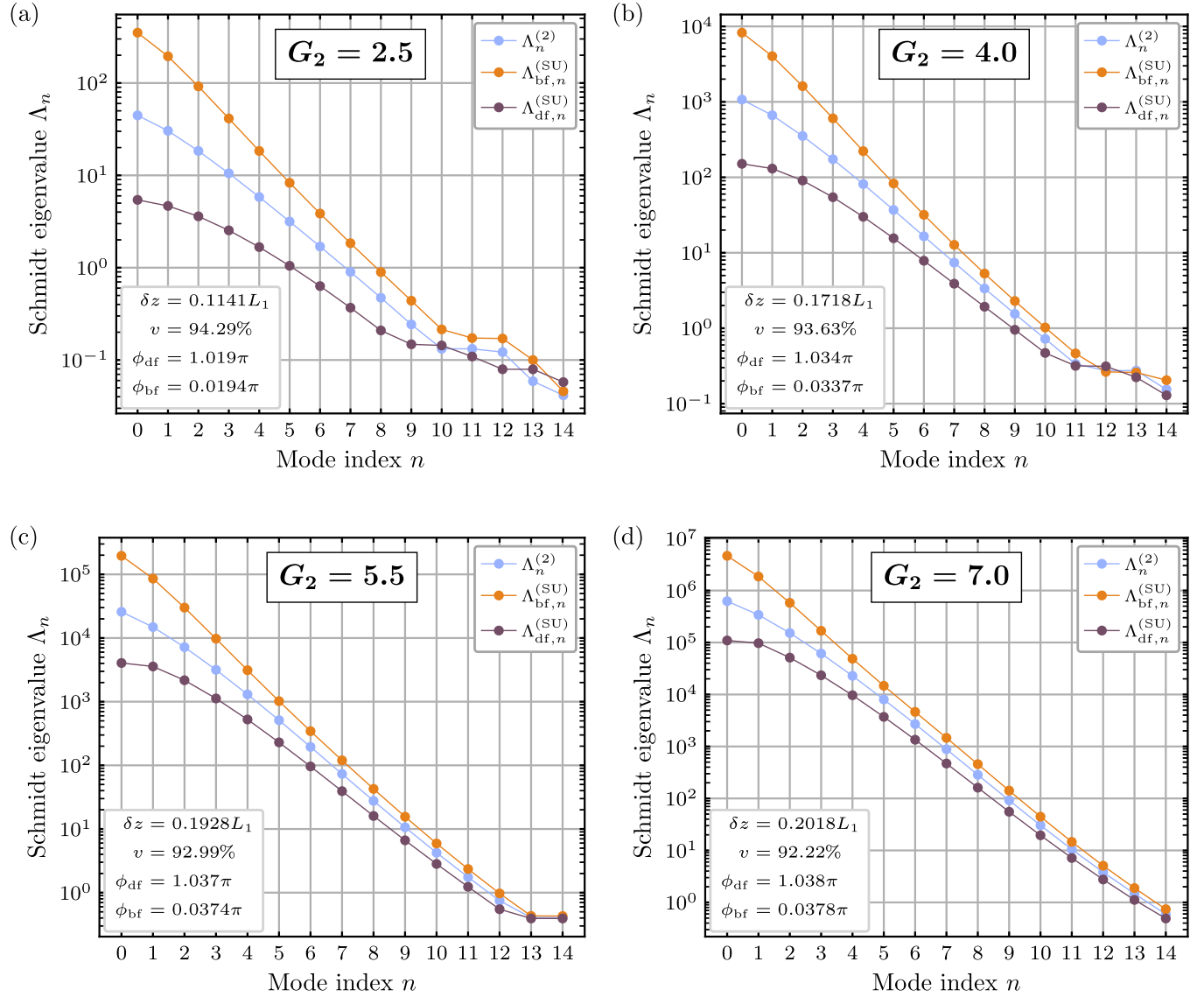


FIG. 7. Eigenvalues Λ_n for the first 15 Schmidt modes of the amplifier (second crystal) and the entire SU(1,1) interferometer at the dark and bright fringes for different gain values as written in the plots. The corresponding values for δz and the visibility v , as well as the phases ϕ_{df} and ϕ_{bf} for the dark and bright fringes, respectively, are also provided in the plots. The lines connecting the data points serve as a visual guide. Clearly, $\Lambda_{df,n}^{(SU)} < \Lambda_n^{(2)} < \Lambda_{bf,n}^{(SU)}$ for the first few modes $n < k$, where k increases as the parametric gain is increased. The condition $\Lambda_{df,n}^{(SU)} \gg 1$ holds true approximately up to $n = 5$ in panel (a), for $n = 9$ in panel (b), for $n = 11$ in panel (c), and for $n = 13$ in panel (d), meaning that as the gain increases, the high-gain amplifier approximation holds up to higher mode indices n .

of the input Schmidt operators of the entire interferometer $\hat{A}_k^{(SU,in)}$.

The conditions of the large number of photons in the mode (large eigenvalues) $\Lambda_n^{(2)} \gg 1$, $\Lambda_n^{(SU)} \gg 1$ imply that the last term in Eq. (4.17) is small and can be neglected. The first term of the Eq. (4.17) is proportional to the imaginary part

of overlap coefficients $\text{Im}(\eta_{nk})$. If focusing is done carefully (see Appendix D), this imaginary part approaches zero [or, equivalently, the phase $\arg(\eta_{nk})$ approaches 0 or π], so that the first term can be omitted (compare also Fig. 5). After neglecting these two terms, the remaining expression for the Schmidt-mode operator reads

$$\hat{A}_l^{(1,out)} = \frac{1}{2} \sum_{n,k} g_{ln} \left\{ \sqrt{\frac{\Lambda_n^{(2)}}{\Lambda_k^{(SU)}}} [\eta_{nk} \hat{A}_k^{(SU,in)} - \eta_{nk}^* (\hat{A}_k^{(SU,in)})^\dagger] + \sqrt{\frac{\Lambda_k^{(SU)}}{\Lambda_n^{(2)}}} \eta_{nk} [\hat{A}_k^{(SU,in)} + (\hat{A}_k^{(SU,in)})^\dagger] \right\}. \quad (4.18)$$

These two remaining terms only contain the *relative* mean photon numbers of the modes of the amplifier and the interferometer. This means that in order to apply the discussed theoretical approach experimentally, it is sufficient to know the intensity in each

Schmidt mode of the amplifier $I_n^{(2)}$ and the interferometer $I_k^{(\text{SU})}$ since their ratio follows the photon number (eigenvalue) ratio: $I_n^{(2)}/I_k^{(\text{SU})} = \Lambda_n^{(2)}/\Lambda_k^{(\text{SU})}$.

Furthermore, in the case of the diagonal $\mathfrak{h}$ matrix,

$$\mathfrak{h}_{nk} = \pm \delta_{nk}, \quad (4.19)$$

where δ_{nk} is the Kronecker delta, Eq. (4.18) simplifies to

$$\hat{A}_l^{(1,\text{out})} = \frac{1}{2} \sum_n \pm \mathfrak{g}_{ln} \left\{ \sqrt{\frac{\Lambda_n^{(2)}}{\Lambda_n^{(\text{SU})}}} [\hat{A}_n^{(\text{SU},\text{in})} - (\hat{A}_n^{(\text{SU},\text{in})})^\dagger] + \sqrt{\frac{\Lambda_n^{(\text{SU})}}{\Lambda_n^{(2)}}} [\hat{A}_n^{(\text{SU},\text{in})} + (\hat{A}_n^{(\text{SU},\text{in})})^\dagger] \right\}. \quad (4.20)$$

Note that the $\pm$ in Eq. (4.19) accounts for the fact that the sign of the modes is not well defined (see Appendix A 1). This leads to random signs for the overlap coefficients $\mathfrak{h}_{nk}$. In Eq. (4.18), this randomness does not pose an issue since the overlap coefficients are multiplied with the corresponding Schmidt-mode operators that contain the mode functions from which the random sign originates. This sign must, however, be taken into account for the approximation expressed by Eq. (4.19). Regardless, as will be seen below, the arbitrary sign does not affect the results for the quadrature variance. Physically, the situation under which the approximation as written in Eq. (4.19) holds occurs when the output Schmidt modes of the second crystal and of the interferometer are sufficiently similar to each other.

The generic quadrature operator $\hat{P}_{\vartheta,l}$ can be obtained by plugging Eq. (4.20) into Eq. (4.6) and simplifying the result:

$$\hat{P}_{\vartheta,l} = \sum_n \left[\text{Re}(\mathfrak{g}_{ln} e^{-i\frac{\vartheta}{2}}) \sqrt{\frac{\Lambda_n^{(\text{SU})}}{\Lambda_n^{(2)}}} \hat{Y}_n^{(\text{SU},\text{in})} - \text{Im}(\mathfrak{g}_{ln} e^{-i\frac{\vartheta}{2}}) \sqrt{\frac{\Lambda_n^{(2)}}{\Lambda_n^{(\text{SU})}}} \hat{X}_n^{(\text{SU},\text{in})} \right], \quad (4.21)$$

with $\hat{X}_n^{(\text{SU},\text{in})}$ and $\hat{Y}_n^{(\text{SU},\text{in})}$ as defined in Eqs. (4.7a) and (4.7b) in terms of the input Schmidt operators of the entire interferometer. Clearly, $\langle \hat{P}_{\vartheta,l} \rangle = 0$ and

$$\langle \hat{P}_{\vartheta,l}^2 \rangle = \sum_n |\check{\mathfrak{g}}_{ln}|^2 \left[\cos^2 \left(\arg(\check{\mathfrak{g}}_{ln}) + \frac{\phi - \vartheta}{2} \right) \frac{\Lambda_n^{(\text{SU})}}{\Lambda_n^{(2)}} + \sin^2 \left(\arg(\check{\mathfrak{g}}_{ln}) + \frac{\phi - \vartheta}{2} \right) \frac{\Lambda_n^{(2)}}{\Lambda_n^{(\text{SU})}} \right], \quad (4.22)$$

where the known phase dependence of the overlap matrix elements $\mathfrak{g}_{ln}$ as written in Eq. (3.14) has been applied so that $\check{\mathfrak{g}}_{ln}$ no longer depends on ϕ :

$$\mathfrak{g}_{ln} = \check{\mathfrak{g}}_{ln} e^{i\frac{\phi}{2}} \quad (4.23)$$

[compare Eqs. (3.14) and (3.12)]. It should also be noted that the $\Lambda_n^{(\text{SU})}$ in Eq. (4.22) also depend on ϕ . In principle, Eq. (4.22) allows for the reconstruction of the quadrature variance $\Delta^2 \hat{P}_{\vartheta,l} = \langle \hat{P}_{\vartheta,l}^2 \rangle$ for any quadrature angle ϑ , given the overlap matrix elements $\check{\mathfrak{g}}_{ln}$, the eigenvalues of the amplifier $\Lambda_n^{(2)}$ and the eigenvalues of the interferometer $\Lambda_n^{(\text{SU})}$ at any known phase ϕ . Then, from $\Delta^2 \hat{P}_{\vartheta,l}$, the levels of squeezing and antisqueezing can be immediately obtained [see Eqs. (4.12a) and (4.12b)].

In general, for the unbalanced interferometer, $\check{\mathfrak{g}}_{ln} = \int dq [u_k^{(1)}(q)]^* \check{\psi}_m^{(2)}(q)$ is still a complex number, even after the ϕ dependence has been split off. However, if the focusing element is moved carefully by some distance δz away from the optimal distance in the gain-balanced case (see Sec. III B), it is possible to reach $\check{\mathfrak{g}}_{ln} \approx \pm 1$ for several low-order modes for the gain-unbalanced interferometer. For the present example, this can be seen directly from Fig. 5, where for the bright fringe ($\phi \approx 0$), $\mathfrak{g}_{ln} = \check{\mathfrak{g}}_{ln} e^{i\phi/2} \approx \pm 1$, while for the dark fringe ($\phi \approx \pi$), $\check{\mathfrak{g}}_{ln} e^{i\phi/2} \approx \pm i$, in both cases for low-order modes. In both cases, this implies $\check{\mathfrak{g}}_{ln} \approx \pm 1$ and therefore $\arg(\check{\mathfrak{g}}_{ln}) \approx m\pi$, for $m \in \mathbb{Z}$. Since both $\cos^2$ and $\sin^2$ are π periodic, $\arg(\check{\mathfrak{g}}_{ln})$ may therefore be neglected in Eq. (4.22), if only the low-order modes have significant contributions to $\langle \hat{P}_{\vartheta,l}^2 \rangle$. For a more

detailed analysis of the behavior of the phases of the overlap coefficients in dependence on δz , see Appendix D.

Finally, neglecting the term $\arg(\check{\mathfrak{g}}_{ln})$ and considering the dark fringe with the phase $\phi_{\text{df}} \approx \pi$, the second contribution in Eq. (4.22) vanishes for $\vartheta = \phi_{\text{df}}/2 + \pi n$, where $n \in \mathbb{Z}$, while the first contribution vanishes for $\vartheta = \phi_{\text{df}}/2 + \pi(n + 1/2)$. At the dark fringe, it would be expected that $\Lambda_n^{(2)} > \Lambda_n^{(\text{SU})}$ [compare Fig. 7], which means that the maximal levels of squeezing and antisqueezing can be obtained as [compare Eqs. (4.12a) and (4.12b)]

$$S_{\text{df},l} = 10 \log_{10} \left[\sum_n |\mathfrak{g}_{ln}|^2 \frac{\Lambda_n^{(\text{SU},\text{df})}}{\Lambda_n^{(2)}} \right] \text{dB}, \quad (4.24a)$$

$$AS_{\text{df},l} = 10 \log_{10} \left[\sum_n |\mathfrak{g}_{ln}|^2 \frac{\Lambda_n^{(2)}}{\Lambda_n^{(\text{SU},\text{df})}} \right] \text{dB}. \quad (4.24b)$$

Similarly, at the bright fringe with phase $\phi_{\text{bf}} \approx 0$, the roles of the eigenvalues are reversed and it should be expected that $\Lambda_n^{(2)} < \Lambda_n^{(\text{SU})}$. Hence, the maximal levels of squeezing and antisqueezing are given by

$$S_{\text{bf},l} = 10 \log_{10} \left[\sum_n |\mathfrak{g}_{ln}|^2 \frac{\Lambda_n^{(2)}}{\Lambda_n^{(\text{SU},\text{bf})}} \right] \text{dB}, \quad (4.25a)$$

$$AS_{\text{bf},l} = 10 \log_{10} \left[\sum_n |\mathfrak{g}_{ln}|^2 \frac{\Lambda_n^{(\text{SU},\text{bf})}}{\Lambda_n^{(2)}} \right] \text{dB}. \quad (4.25b)$$

It should be noted that Eqs. (4.24a)–(4.25b) are estimations based on the assumption that the respective orderings $\Lambda_n^{(2)} > \Lambda_n^{(\text{SU})}$ and $\Lambda_n^{(2)} < \Lambda_n^{(\text{SU})}$ of the eigenvalues in the bright and

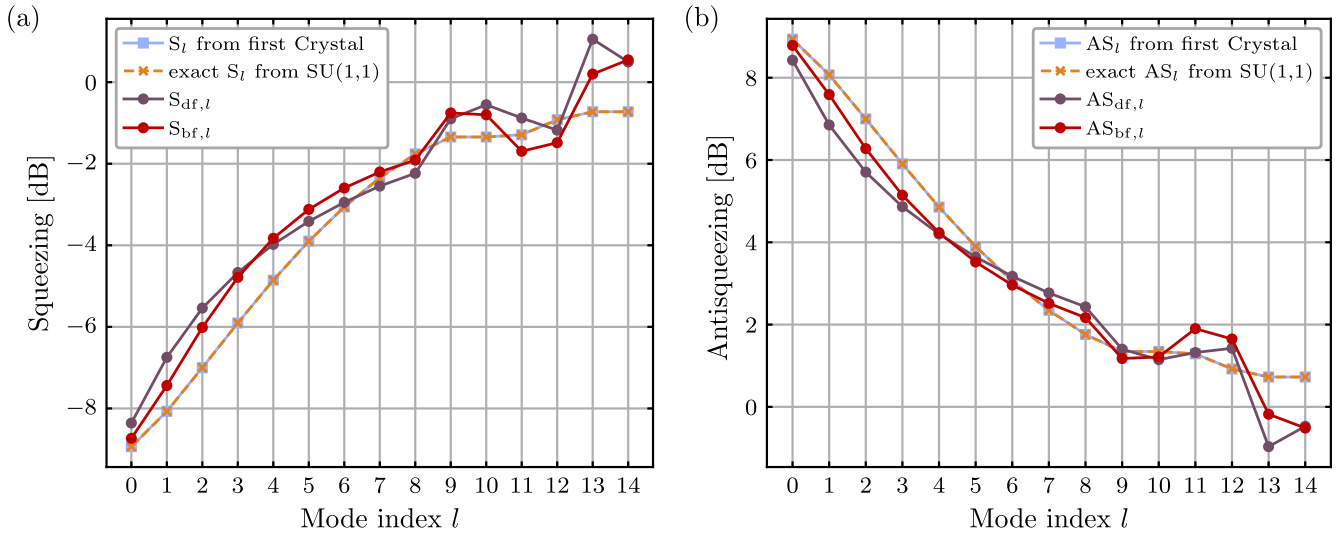


FIG. 8. (a) Comparison of the level of squeezing in each Schmidt mode l of the first crystal (squeezer) obtained directly at the output of the first crystal using Eq. (4.28a), extracted from the output of the SU(1,1) interferometer using the exact approach described in Sec. IV A, and using the high-gain approximation using Eqs. (4.24a) and (4.25a). As expected, the exact procedure coincides with the squeezing obtained directly using the first crystal eigenvalues. The high-gain approximation results in underestimated values for the squeezing. (b) Comparison of the level of antisqueezing obtained directly at the output of the first crystal using Eq. (4.28b), extracted from the output of the SU(1,1) interferometer using the exact approach described in Sec. IV A and using the high-gain approximation using Eqs. (4.24b) and (4.25b). As was the case for the level of squeezing, the exact procedure coincides with the antisqueezing obtained directly using the first crystal eigenvalues, while the high-gain approximation results in underestimated values.

dark fringes, respectively, hold for sufficiently large n , while the $\mathbf{h}_{nk}$ matrix is diagonal. In general, it is otherwise not possible to obtain simple expressions for the levels of squeezing and antisqueezing, and instead Eqs. (4.11a), (4.11b), (4.12a), and (4.12b), together with Eqs. (4.14a) and (4.14b), have to be used directly to obtain expressions for S_l and AS_l .

C. Comparison of the different approaches

Analytically, the levels of squeezing and antisqueezing of the output quadratures of the first crystal can be directly obtained from the Schmidt decomposition and the resulting Bogoliubov transformations of the first crystal. Plugging Eq. (1.5) into Eq. (4.9) yields

$$\Delta^2 \hat{P}_{\vartheta,l} = 1 + 2\Lambda_l^{(1)} + 2\sqrt{\tilde{\Lambda}_l^{(1)}\Lambda_l^{(1)}} \cos(\vartheta), \quad (4.26)$$

while the vacuum quadrature variances are again $\Delta^2 \hat{P}_{\vartheta,l}^{(\text{vac})} = 1$. Using the fact that the eigenvalues can be parametrized as $\Lambda_l = \sinh^2(r_l)$ and $\tilde{\Lambda}_l = \cosh^2(r_l)$ [7,37], where r_l is associated with the gain in mode l according to Schmidt-mode theory [9,38], Eq. (4.26) may be rewritten as

$$\Delta^2 \hat{P}_{\vartheta,l} = \cosh(2r_l) + \sinh(2r_l) \cos(\theta). \quad (4.27)$$

The levels of squeezing and antisqueezing are therefore

$$S_l = -\frac{20r_l}{\ln(10)} \text{dB}, \quad (4.28a)$$

$$AS_l = \frac{20r_l}{\ln(10)} \text{dB}. \quad (4.28b)$$

Note that $20/\ln(10) \approx 8.686$.

In Fig. 8, the levels of squeezing and antisqueezing obtained from the methods described in Secs. IV A and

IV B are compared against the exact values evaluated using Eqs. (4.28a) and (4.28b). As expected, the levels of squeezing and antisqueezing obtained using the exact method presented in Sec. IV A coincide with the values obtained from Eqs. (4.28a) and (4.28b). Contrary to that, the levels of squeezing and antisqueezing obtained using the high-gain approximation as presented in Sec. IV B underestimate the expected values for almost all shown mode indices. For the level of squeezing, this means that the high-gain approximation results in larger (“less negative”) values for S_n compared to the theoretically exact values, while for the antisqueezing, the obtained AS_n values are smaller. A brief discussion of the extension of the presented formalism to include losses is presented in Appendix E.

V. CONCLUSION

We have presented a rigorous theoretical analysis of the spatial Schmidt-mode structure of an unbalanced multimode SU(1,1) interferometer. We analyzed the Schmidt mode shapes for each crystal separately and for the entire interferometer and demonstrated a strong dissimilarity in the phase profile of the input and output Schmidt modes with increasing gain. Considering the complex interplay between the input and output modes of a cascaded crystal setup, we extended the phase-sensitive amplification approach to the multimode scenario and presented a generalized theoretical framework for measuring multimode squeezing based on the direct intensity measurement. To make our framework experimentally feasible, we considered the high-gain amplifier approximation, which has been experimentally realized in Ref. [5] and found to be in good agreement with theoretical predictions. In addition, for the unbalanced interferometer, we discussed

experimental optimizations and showed how shifting the focal point of the focusing element affects the overlap coefficients and helps to improve the visibility (see also Appendix D).

Our work provides a theoretical background for multimode PDC setups and the presented theory can be extended to a wide range of experiments using multimode PDC light. For example, in addition to multimode squeezing measurements, the presented framework can be used to improve the phase sensitivity in quantum-enhanced metrology by eliminating mode mismatch (which acts as internal losses and reduces phase sensitivity). In addition, the lack of experiments on the detailed study of the phases of the Schmidt modes and the overlap coefficients can be filled and motivated by the current work. Indeed, the moduli of the output Schmidt modes can be found by decomposing the covariance matrix [48,49]. However, the measurement of the Schmidt-mode phases requires additional investigations. Moreover, since the input and output Schmidt modes are generally different, their detailed experimental characterization is an important and still open question.

Finally, a different set of modes may be more beneficial and simplify the measurement procedure dramatically: For example, using the full two-dimensional setup gives access to the azimuthal degree of freedom so that the Schmidt modes will be given by modes of the form $u_{mn}(q)e^{in\phi}$ [9,36,38], where q now refers to the radial component of the transverse wave vector, while ϕ is the azimuthal angle. Clearly, these modes carry an orbital angular momentum of n , and therefore, if the measurement basis is extended to orbital angular momentum modes, the cross coupling between the modes inside the interferometer is reduced to being between modes with different radial indices m , which then however have the same orbital angular momentum n . However, it should be noted that this treatment would strongly increase the numerical complexity compared to the case where only one transverse dimension is considered.

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DATA AVAILABILITY

The data used to generate the figures in this article are available in Ref. [50].

APPENDIX A: THE JOINT SCHMIDT DECOMPOSITION

1. Uniqueness of the Schmidt modes

Compared to a simple Schmidt decomposition of only one of the transfer functions, for example, only of β , as written in Eq. (1.3a) on its own, the joint decomposition as written

in Eqs. (1.3a) and (1.3b) imposes restrictions on the sets $\{\psi_n\}$ and $\{u_n\}$ of possible modes. To investigate this, it is helpful to draw analogies between the equations written in the main text for the continuous case over q and their discretized versions. In the discretized case, the Schmidt decomposition corresponds to the singular value decomposition (SVD). SVD algorithms also form the base for the numerical computation of the joint decomposition (see Appendix A 2). The transfer functions $\tilde{\eta}$ and β as defined in Eqs. (1.1a) and (1.1b) are then represented by complex-valued matrices. More precisely, if we label the matrices corresponding to $\tilde{\eta}$ and β by their corresponding uppercase letters, $\tilde{H}$ and B , respectively, and if n discretization points $q_1, \dots, q_n$ are used for the underlying q lattices, one finds that $\tilde{H}, B \in \mathbb{C}^{n \times n}$ and that the joint Schmidt decomposition reads [23,51,52]

$$B = U \Sigma \Psi^T, \quad (A1a)$$

$$\tilde{H} = U \tilde{\Sigma} \Psi^H, \quad (A1b)$$

where U and Ψ are unitary matrices and we denote the transpose and Hermitian conjugates of matrices with T and H , respectively. The columns $\mathbf{u}$ of U and the columns $\boldsymbol{\psi}$ of Ψ correspond to the discretized versions of the mode functions $u(q)$ and $\psi(q')$, respectively,

$$U = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \dots \quad \mathbf{u}_n], \quad (A2a)$$

$$\Psi = [\boldsymbol{\psi}_1 \quad \boldsymbol{\psi}_2 \quad \dots \quad \boldsymbol{\psi}_n], \quad (A2b)$$

so that

$$u_j(q) \leftrightarrow \mathbf{u}_j = \begin{bmatrix} u_j(q_1) \\ u_j(q_2) \\ \vdots \\ u_j(q_n) \end{bmatrix}, \quad (A3a)$$

$$\psi_j(q) \leftrightarrow \boldsymbol{\psi}_j = \begin{bmatrix} \psi_j(q_1) \\ \psi_j(q_2) \\ \vdots \\ \psi_j(q_n) \end{bmatrix}. \quad (A3b)$$

Furthermore,

$$\Sigma = \text{diag}(\sqrt{\Lambda_1}, \sqrt{\Lambda_2}, \dots, \sqrt{\Lambda_n}), \quad (A4a)$$

$$\tilde{\Sigma} = \text{diag}(\sqrt{\tilde{\Lambda}_1}, \sqrt{\tilde{\Lambda}_2}, \dots, \sqrt{\tilde{\Lambda}_n}) \quad (A4b)$$

are diagonal matrices with positive entries containing the singular values of B and $\tilde{H}$, respectively. Note that we neglect additional scaling factors, such as the lattice spacing dq , which appears due to the integrals when making these analogies for simplicity.

For nondegenerate eigenvalues Λ_n and $\tilde{\Lambda}_n$, it is well known that the vectors $\mathbf{u}_n$ and $\boldsymbol{\psi}_n$ resulting from the singular value decomposition are uniquely defined up to a phase factor [53]. In the case of degenerate eigenvalues, the modes have a higher degree of freedom with respect to the phase rotation and it is possible to perform unitary rotations in the corresponding subspace of the degenerate modes. For the rest of this section, we will focus only on the discussion of the case of nondegenerate eigenvalues since for the typical parameters discussed in the main text of this work, the low-order eigenvalues that have the highest contributions are usually nondegenerate (compare

also Fig. 7). See Ref. [36] for a more general description. Furthermore, note that due to the connection between the eigenvalues written in Eq. (1.4), if Λ_n are all nondegenerate, so are all $\tilde{\Lambda}_n$ as well, and vice versa [54]. Similarly, if Λ_j is an m -fold degenerate eigenvalue, $\tilde{\Lambda}_j$ must also be m -fold degenerate.

Transferring this to the case of continuous q variables, we will assume in the following that the modes u_n and ψ_n are also defined up to a constant phase. The joint Schmidt decomposition of the two transfer functions results in a pair P of two sets of modes $\{u_n\}$ and $\{\psi_n\}$: $P = (\{u_n\}, \{\psi_n\})$. Then, given another independent pair of sets of input and output modes, $P' = (\{u'_n\}, \{\psi'_n\})$, the joint Schmidt decomposition for the former sets P takes the form as written in Eqs. (1.3a) and (1.3b), while for P' , the mode functions therein would be replaced with u'_n and ψ'_n . However, due to the fact that the mode functions are only defined up to some phase, the modes in P' may be expressed in terms of the modes of P , and they must be connected via $u'_n(q) = u_n(q)e^{i\xi_{u_n}}$ and $\psi'_n(q) = \psi_n(q)e^{i\xi_{\psi_n}}$. Thus, the decomposition for P' may be written as

$$\beta(q, q') = \sum_n \sqrt{\Lambda_n} u_n(q) \psi_n(q') e^{i\xi_{u_n} + i\xi_{\psi_n}}, \quad (\text{A5a})$$

$$\tilde{\eta}(q, q') = \sum_n \sqrt{\tilde{\Lambda}_n} u_n(q) \psi_n^*(q') e^{i\xi_{u_n} - i\xi_{\psi_n}}. \quad (\text{A5b})$$

Multiplying both transfer functions with $u_m(q)$ and $\psi_m(q')$, integrating over the q variables, and then comparing the result obtained for Eqs. (1.3a) and (1.3b) with that obtained for Eqs. (A5a) and (A5b) yields a system of coupled equations for the possible values of ξ_{ψ_m} and ξ_{ψ_m} , which can be written as

$$\begin{aligned} e^{i\xi_{u_m} + i\xi_{\psi_m}} &= 1, \\ e^{i\xi_{u_m} - i\xi_{\psi_m}} &= 1. \end{aligned} \quad (\text{A6})$$

The solutions of this set of equations are of the form

$$\begin{aligned} \xi_{\psi_m} &= m\pi, \\ \xi_{u_m} &= \xi_{\psi_m} + 2l\pi, \end{aligned} \quad (\text{A7})$$

for $l, m \in \mathbb{Z}$. Thus, in conclusion, the modes of P and P' only differ in the sense that u_n and ψ_n may simultaneously (but for each n independently) be replaced by $-u_n$ and $-\psi_n$, respectively. The modes are therefore actually only unique up to their sign, with the additional restriction that the signs of both the input modes and the output modes must be changed together to preserve the decomposition. In more direct terms, for each n , a numerical procedure may result either in u_n and ψ_n or in the same pair with the opposite signs: $-u_n$ and $-\psi_n$.

2. Remarks on the numerical computation

Equations (A1a) and (A1b) describe a special joint SVD of the matrices B and $\tilde{H}$. Methods for finding such a decomposition have, for example, been described in Refs. [23,40,51]. Note that, generally, it is not possible to simply perform two independent singular value decompositions of B and $\tilde{H}$, since these are usually not unique and the resulting matrices U and Ψ will generally not fulfill Eqs. (A1a) and (A1b) above [52]. Additionally, care must be taken when applying numerical methods due to degeneracies that may appear in the

singular value matrices Σ and $\tilde{\Sigma}$ [see Eqs. (A1a) and (A1b)] and because numerical methods may not necessarily preserve symmetries present in their input matrices.

One way of obtaining the joint decomposition as written in Eqs. (A1a) and (A1b) is as follows. First, two independent SVDs of B and $\tilde{H}$ are performed [55,56]:

$$B = \mathcal{U} \Sigma V_0^H, \quad (\text{A8a})$$

$$\tilde{H} = \tilde{\mathcal{U}} \tilde{\Sigma} \tilde{V}^H, \quad (\text{A8b})$$

yielding the unitary matrices $\mathcal{U}, \tilde{\mathcal{U}}$ (with, in general, $\mathcal{U} \neq \tilde{\mathcal{U}}$), V_0 , and $\tilde{V}$, as well the diagonal matrices Σ and $\tilde{\Sigma}$, which contain the singular values [see Eqs. (A4a) and (A4b)]. Then, the matrix

$$K = \mathcal{U}^H \tilde{\mathcal{U}} \quad (\text{A9})$$

and its Hermitian conjugate K^H are unitary, block-diagonal, and commute with Σ and $\tilde{\Sigma}$. Expressing the $n \times n$ identity matrix I_n as $I_n = K K^H$ and inserting it into Eq. (A8a) yields $B = \mathcal{U} I_n \Sigma V_0^H = (\mathcal{U} K) \Sigma [K^H V_0^H]$, which brings the SVDs in Eqs. (A8a) and (A8b) to the form

$$B = \tilde{\mathcal{U}} \Sigma V^H, \quad (\text{A10a})$$

$$\tilde{H} = \tilde{\mathcal{U}} \tilde{\Sigma} \tilde{V}^H, \quad (\text{A10b})$$

where $V = V_0 K$. Equations (A10a) and (1.2) describe a special SVD of the matrices B and $\tilde{H}$, where the left-hand side unitary $\tilde{\mathcal{U}}$ is identical for both matrices. This SVD can serve as the starting point for the algorithm described in Ref. [51] to bring Eqs. (A10a) and (1.2) into the form of Eqs. (A1a) and (A1b). Alternatively, K may be defined as $K = \tilde{\mathcal{U}}^H \mathcal{U}$ [instead of as written in Eq. (A9)] and inserted into Eq. (A8b) so that retracing the steps above, but for $\tilde{H}$, yields a decomposition of the same form as in Eqs. (A10a) and (1.2).

As another alternative, such a decomposition can be obtained as suggested in Refs. [23,51], that is, by performing an eigendecomposition of BB^H or $\tilde{H}\tilde{H}^H$. According to Ref. [51], this yields $BB^H = \mathcal{U}_1 \Sigma^2 \mathcal{U}_1^H$ or $\tilde{H}\tilde{H}^H = \mathcal{U}_2 \tilde{\Sigma}^2 \mathcal{U}_2^H$, where both $\mathcal{U}_1$ and $\mathcal{U}_2$ are candidates for the left-hand side unitary $\mathcal{U}$ in Eqs. (A10a) and (1.2). This is because $\tilde{H}\tilde{H}^H$ and BB^H are connected via [23,37,51]

$$BB^H + I_n = \tilde{H}\tilde{H}^H, \quad (\text{A11})$$

meaning that any choice for $\mathcal{U}_1$ or $\mathcal{U}_2$ that diagonalizes the matrix BB^H or $\tilde{H}\tilde{H}^H$ will also diagonalize the other matrix and is therefore a candidate for the left-hand side unitary of both SVDs.

The next step is to compute the matrix [51]

$$G = \tilde{V}^H V^*, \quad (\text{A12})$$

where $*$ denotes the elementwise complex conjugate of a matrix. This matrix G is unitary, symmetric, and block diagonal, and commutes with Σ and $\tilde{\Sigma}$ [51]. Such a matrix allows for a special SVD, the *Takagi factorization*, which here is of the form [57]

$$G = D D^T, \quad (\text{A13})$$

with a unitary matrix D , which here must be chosen to have the same block sizes as G . As suggested in Ref. [51], this

can be done by computing the factorization $G_i = D_i D_i^T$ for each block G_i of G separately and building D from the blocks D_i . For the numerical computation, this also ensures that D commutes with Σ and $\tilde{\Sigma}$.

It was also suggested in Ref. [51] to perform the computation of D as

$$D = (V^H \tilde{V}^*)^{\frac{1}{2}}, \quad (\text{A14})$$

where $\frac{1}{2}$ denotes the matrix square root. Note that here the matrix square root also has to be applied for each block of the matrix $V^H \tilde{V}^*$ separately, to preserve the block-diagonal structure of the matrix, as suggested in Ref. [51]. It should, however, be noted that if the matrix square root is computed *numerically*, it is not automatically guaranteed that the resulting square-root matrix is symmetric (meaning $D = D^T$); but this is required for the proof of the decomposition [see Eqs. (A6)–(A8) of Ref. [51]]. Finally, a numerical computation may also result in a matrix D that is not unitary, meaning that the resulting matrices U and Ψ will also ultimately not be unitary [see Eqs. (A15a) and (A15b)] [58].

Note that the computation of the Takagi factorization also requires the computation of a matrix square root, for example, in the algorithm as described in Refs. [52,59]. With this, the requirement that the symmetry of the input matrix is preserved when computing the matrix square root is lifted. The computation of the matrix square root can be, for example, performed using the eigendecomposition or using algorithms for the Schur decomposition (both decompositions coincide for normal matrices).

As noted in Ref. [59], the computation should also be performed for each block of the input matrix separately, as already described above. Let Z be the block-diagonal unitary matrix whose matrix square root $Z^{\frac{1}{2}}$ should be computed and let the blocks of Z be labeled Z_j . The eigendecomposition (or Schur decomposition) of Z_j is of the form $Z_j = K_j S_j K_j^H$, with the unitary matrix K_j and the diagonal eigenvalue matrix S_j , whose entries all have modulus 1, meaning S_j is itself unitary. Then, $Z^{\frac{1}{2}}$ can be built from the unitary blocks $Z_j^{\frac{1}{2}} = K_j S_j^{\frac{1}{2}} K_j^H$ and is itself unitary. Here, $S_j^{\frac{1}{2}}$ can be computed as an element-wise square root since S_j is diagonal. Thus, ultimately, $Z^{\frac{1}{2}}$ is unitary and block diagonal, and the block sizes correspond to those of Z .

Finally, the matrices defined by [23,51]

$$U = \tilde{U}D, \quad (\text{A15a})$$

$$\Psi = V^* D^* \quad (\text{A15b})$$

fulfill the condition of the joint Schmidt decomposition as written in Eqs. (A1a) and (A1b).

APPENDIX B: ASYMMETRY OF THE TRANSFER FUNCTION β

As was discussed in Sec. II, in a general case, the moduli of the higher-order input and output modes of a single crystal are no longer necessarily equal at high gain. This asymmetry can be explained as originating from an asymmetry in the modulus $|\beta(q, q')|$ of the transfer function β . Note that as discussed

in Appendix A, for nondegenerate singular values, the mode functions are defined up to a constant phase. Therefore, if only the moduli of the modes or the relative behavior of their phases over q are of interest, it is sufficient to consider a single Schmidt decomposition of the transfer function β instead of the joint Schmidt decomposition.

Figure 9 shows the modulus and the phase of the transfer function β , as well as the asymmetry in its modulus, which is around a factor of 500 smaller than the maximum value of the modulus of the transfer function itself. In order to verify that this effect is not a result of a numerical error, we have repeated the numerical calculation of the results shown in Fig. 9 using a *Lie group integrator*. All numerical results shown in this work were computed using an adaptive-stepsize Runge-Kutta method of order 5(4) as implemented in *Scipy* [60]. However, the regular Runge-Kutta integration methods are not explicitly constructed to preserve the symplectic structure of the complex-valued $2n \times 2n$ matrix (compare Appendix A)

$$S(L) = \begin{bmatrix} \tilde{H}(L) & B(L) \\ B^*(L) & \tilde{H}^*(L) \end{bmatrix} \quad (\text{B1})$$

during the evolution of the transfer functions over L . Symplectic matrices S are defined by the fact that fulfill [40,61]

$$S^T J S = J, \quad (\text{B2})$$

where

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \quad (\text{B3})$$

is the *canonical symplectic matrix*. If S is a symplectic matrix, its inverse is given by $S^{-1} = J^{-1} S^T J$ and is itself a symplectic matrix. Explicitly evaluating the left-hand side of Eq. (B2) for both S^{-1} and S yields the four well-known [23,37,40,51] conditions on the matrices representing the transfer function:

$$\tilde{H} \tilde{H}^H - B B^H = I_n, \quad (\text{B4a})$$

$$\tilde{H} B^T - B \tilde{H}^T = 0, \quad (\text{B4b})$$

and

$$\tilde{H}^H \tilde{H} - B^T B^* = I_n, \quad (\text{B5a})$$

$$\tilde{H}^H B - B^T \tilde{H}^* = 0, \quad (\text{B5b})$$

respectively [compare also Eq. (A11)]. These are the discretized versions of Eqs. (A3a), (A3b), (D3a), and (D3b) of Ref. [7], respectively. Thus, the symplectic structure of S ensures that the bosonic [51] commutation relations of the output plane-wave operators are preserved.

To be more precise, for simplicity, we have used the *Lie-Euler method* [62,63], which is a variation of the Euler method and ensures that the Lie group structure [here, the structure of the symplectic group $\text{Sp}(2n, \mathbb{C})$] of the solution of the integro-differential equation is preserved [meaning $S(L) \in \text{Sp}(2n, \mathbb{C})$ for all L]. We have found that there are no qualitative differences in the results obtained using the aforementioned Runge-Kutta method (presented in this section and the rest of this work) and those obtained using the Lie-Euler method, meaning that, in other words, all plots shown in this work would remain unchanged when produced using the Lie-Euler method.

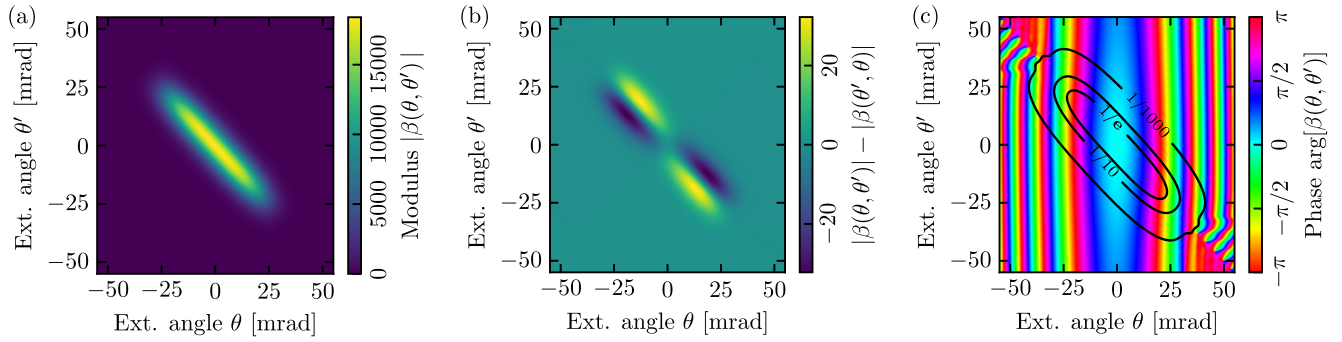


FIG. 9. (a) Modulus $|\beta(\theta, \theta')|$ of the transfer function β at high gain $G_{\text{exp}} = 8$. (b) Plot of the expression $|\beta(\theta, \theta')| - |\beta(\theta', \theta)|$, which reveals an asymmetry in the modulus of the transfer function, whose modulus is around a factor of 500 smaller than the peak value of the transfer function itself. (c) Phase $\arg \beta(\theta, \theta')$ of the transfer function. The contour lines indicate where the modulus has decayed by $1/e$, $1/10$, and $1/1000$ compared to its maximum value, respectively. As the modulus approaches zero, numerical artifacts in the upper left and lower right corner become more visible. However, these values do not contribute as much to the function overall.

The fact that the moduli of the lower-order modes coincide (see Figs. 1 and 2) can be explained as follows: Clearly, Figs. 9(a) and 9(b) indicate that the modulus of the transfer function β is approximately symmetric: $|\beta(q, q')| \approx |\beta(q', q)|$. The Schmidt decomposition of the modulus of β can be written as

$$|\beta(q, q')| = \sum_n \sqrt{\Lambda_n^{\text{abs}}} u_n^{\text{abs}}(q) \psi_n^{\text{abs}}(q'), \quad (\text{B6})$$

one can calculate the same-crystal overlap matrix $\mathbf{c}$ [see Eq. (2.2); here, $c_{lm}^{\text{abs}} = \int dq u_l^{\text{abs}}(q) [\psi_m^{\text{abs}}(q)]^*$], see Fig. 11 (left column), where the first 20×20 overlap matrix entries are presented. This matrix is almost diagonal, $|c_{lm}^{\text{abs}}|^2 \approx \delta_{lm}$, which means that $|u_n^{\text{abs}}(q)| \approx |\psi_n^{\text{abs}}(q')|$ and the functions $u_n^{\text{abs}}(q)$ and $\psi_n^{\text{abs}}(q')$ differ only by a constant phase.

Furthermore, as illustrated by Fig. 10, the phase distribution $\arg[\beta(q, q')]$ is, to a good approximation, factorizable.

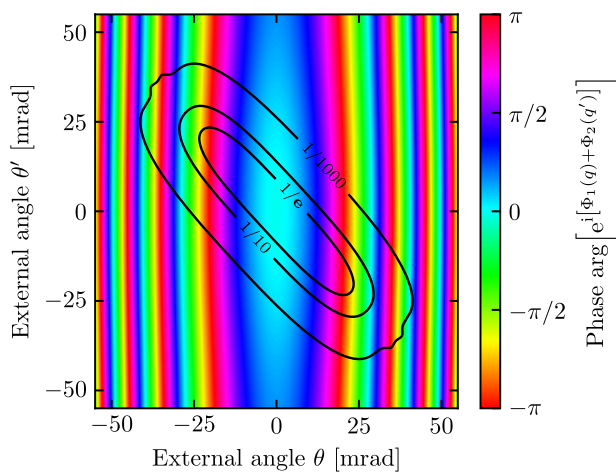


FIG. 10. Plot of the fitted phase distribution $\arg[e^{i(\Phi_1(q) + \Phi_2(q'))}]$, where $\Phi_j(q) = a_{j,1} + a_{j,2}q^2 + a_{j,3}q^4 + a_{j,4}q^6$, with $a_{j,k}$ being the fit parameters. The fit suggests that $\arg[\beta(q, q')]$ is approximately factorizable in the form $\arg[\beta(q, q')] = \Phi_1(q) + \Phi_2(q')$. The contour lines as shown in Fig. 9(c) are added as a visual guide regarding the modulus of β . The values of the fitting parameters are listed in Table I.

This means that an approximation to the Schmidt decomposition of β can be written using the fitted phase distribution (see Fig. 10 for details) as

$$\begin{aligned} \beta(q, q') &\approx \beta^{\text{fit}}(q, q') = |\beta(q, q')| e^{i(\Phi_1(q) + \Phi_2(q'))} \\ &= \sum_n \sqrt{\Lambda_n^{\text{fit}}} u_n^{\text{fit}}(q) \psi_n^{\text{fit}}(q'), \end{aligned} \quad (\text{B7a})$$

where the mode functions are given by

$$u_n^{\text{fit}}(q) = u_n^{\text{abs}}(q) e^{i\Phi_1(q)}, \quad (\text{B7b})$$

$$\psi_n^{\text{fit}}(q') = \psi_n^{\text{abs}}(q') e^{i\Phi_2(q')}, \quad (\text{B7c})$$

and the eigenvalues are the same as for $|\beta(q, q')|$:

$$\Lambda_n^{\text{fit}} = \Lambda_n^{\text{abs}} \quad (\text{B7d})$$

These mode functions are clearly orthonormal, as required by the Schmidt decomposition. Furthermore, they have the same modulus as the modes defined by Eq. (B6):

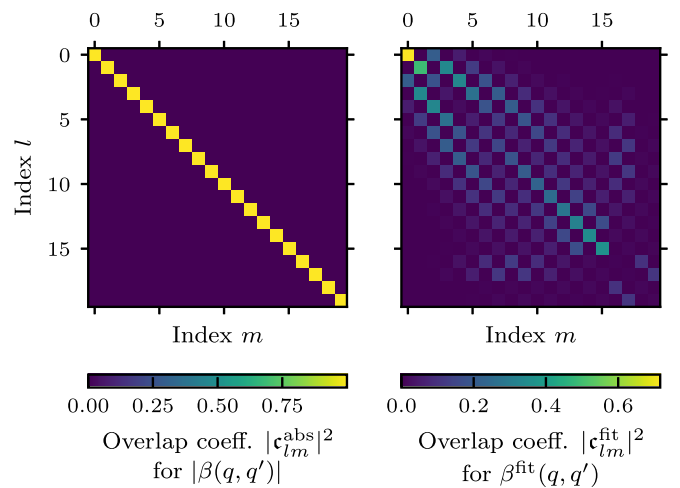


FIG. 11. Left: Modulus squared of the first 20×20 overlap matrix elements $|c_{lm}^{\text{abs}}|^2$ for the decomposition of $|\beta(q, q')|$ as written in Eq. (B6). Right: Modulus squared of the first 20×20 overlap matrix elements $|c_{lm}^{\text{fit}}|^2$ for the transfer function β composed of its modulus and the fit of its phase function as described by Eqs. (B7a)–(B7c).

TABLE I. Fitting parameters for the phase distribution of β as shown in Fig. 10, in units of π rad $\mu\text{m}^{2(k-1)}$.

k	$a_{1,k}$	$a_{2,k}$
1	-1.000	0.995
2	31.125	1.208
3	2.758	-2.671
4	-2.119	2.100

$|u_n^{\text{fit}}(q)| = |u_n^{\text{abs}}(q)|$ and $|\psi_n^{\text{fit}}(q')| = |\psi_n^{\text{abs}}(q')|$. As such, of these mode functions $u_n^{\text{fit}}(q)$ and $\psi_n^{\text{fit}}(q')$, the lower-order ones for the same index will have an approximately equal modulus: $|u_n^{\text{fit}}(q)| \approx |\psi_n^{\text{fit}}(q')|$. However, this will not hold for higher-order modes: Due to asymmetry of β and the fact that the phase is only approximately separable, the matrix $|c_{lm}^{\text{abs}}|$ is not fully diagonal, which explains the difference in the moduli of u_{15} and ψ_{15} as visible in Fig. 2.

Additionally, it immediately follows that the complex behavior of the overlap matrix shown in Fig. 1 for $G_{\text{exp}} = 8$ is partially due to the additional phases $e^{i\Phi_1(q)}$ and $e^{i\Phi_2(q')}$ added onto the mode functions. It can be shown that the mode functions $u_n^{\text{fit}}(q)$ and $\psi_n^{\text{fit}}(q')$ are very close to the mode functions of the exact decomposition: Figure 11 (right column) shows the overlap matrix coefficients $c_{lm}^{\text{fit}} = \int dq u_l^{\text{fit}}(q) [\psi_m^{\text{fit}}(q')]^*$ for the modes as constructed in Eqs. (B7b) and (B7c). Clearly, the behavior of the overlap matrix is strikingly similar to that shown in Fig. 1.

APPENDIX C: COMPUTATION OF THE VISIBILITY

For fixed δz , the full phase dependence of the transfer functions of the entire interferometer can be expressed via the transfer functions of the two crystals as [7]

$$\begin{aligned} \beta^{(\text{SU})}(q, q') &= \int d\bar{q} \check{\eta}^{(2)}(q, \bar{q}) \beta^{(1)}(\bar{q}, q') \\ &+ e^{i\phi} \int d\bar{q} \check{\beta}^{(2)}(q, \bar{q}) [\check{\eta}^{(1)}(\bar{q}, q')]^*, \end{aligned} \quad (\text{C1a})$$

$$\begin{aligned} \tilde{\eta}^{(\text{SU})}(q, q') &= \int d\bar{q} \check{\eta}^{(2)}(q, \bar{q}) \tilde{\eta}^{(1)}(\bar{q}, q') \\ &+ e^{i\phi} \int d\bar{q} \check{\beta}^{(2)}(q, \bar{q}) [\beta^{(1)}(\bar{q}, q')]^*, \end{aligned} \quad (\text{C1b})$$

where we explicitly take the phase out of the transfer functions, considering that the function h for the integration over the second crystal does not contain the phase term $e^{i\phi}$ [compare Eq. (3.7b) and see the notation introduced in Eq. (3.12)]. With the shorthand

$$\check{\mathcal{X}}(q, q') = \int d\bar{q} \check{\eta}^{(2)}(q, \bar{q}) \beta^{(1)}(\bar{q}, q'), \quad (\text{C2a})$$

$$\check{\mathcal{Y}}(q, q') = \int d\bar{q} \check{\beta}^{(2)}(q, \bar{q}) [\tilde{\eta}^{(1)}(\bar{q}, q')]^*, \quad (\text{C2b})$$

the integral intensity for the interferometer can be written as

$$\begin{aligned} \langle \hat{N}_{s,\text{tot}}^{(\text{SU})} \rangle &= \iint dq dq' (|\check{\mathcal{X}}(q, q')|^2 + |\check{\mathcal{Y}}(q, q')|^2) \\ &+ 2 \text{Re}(\mathcal{C} e^{i\phi}), \end{aligned} \quad (\text{C3a})$$

where

$$\mathcal{C} = \iint dq dq' \check{\mathcal{X}}^*(q, q') \check{\mathcal{Y}}(q, q'). \quad (\text{C3b})$$

Since, in general, $\mathcal{C}$ is a complex constant, the positions of the bright and dark fringes, where $\langle \hat{N}_{s,\text{tot}}^{(\text{SU})} \rangle$ is maximized and minimized, respectively, do not, in general, occur at $\phi = 0$ and $\phi = \pi$, respectively. Instead, the positions are shifted by some phase

$$\Upsilon = \arg(\mathcal{C}). \quad (\text{C4})$$

Usually, this phase offset is relatively small; as an example, for the interferometer as described in Sec. III B, $\Upsilon \approx -0.0337\pi$. The visibility as defined in Eq. (3.8) can ultimately be written in the form

$$v = \frac{2|\iint dq dq' \check{\mathcal{X}}^*(q, q') \check{\mathcal{Y}}(q, q')|}{\iint dq dq' (|\check{\mathcal{X}}(q, q')|^2 + |\check{\mathcal{Y}}(q, q')|^2)} \times 100\%. \quad (\text{C5})$$

For a perfectly compensated SU(1,1) interferometer as discussed in Sec. III A, we find

$$|\mathcal{C}| = \iint dq dq' |\check{\mathcal{X}}(q, q')|^2 = \iint dq dq' |\check{\mathcal{Y}}(q, q')|^2 = \mathcal{A}, \quad (\text{C6})$$

where [7]

$$\mathcal{A} = \sum_n \Lambda_n^{(1)} (1 + \Lambda_n^{(1)}), \quad (\text{C7})$$

and therefore it follows from Eq. (1.5) that $v = 100\%$, regardless of any other parameter. This is also evident from the fact that at the dark fringe, which here occurs exactly at $\phi = \pi$, the integral intensity is always identically zero (see the discussion in Ref. [7]).

APPENDIX D: BEHAVIOR OF THE PHASES OF THE OVERLAP COEFFICIENTS NEAR THE OPTIMAL VISIBILITY

As discussed in Sec. III B, for an unbalanced SU(1,1) interferometer, the focusing element must be moved some distance δz away from the optimal position of a balanced SU(1,1) interferometer in order to maximize the interferometric visibility. Furthermore, Fig. 5 shows the behavior of the phases of the overlap matrices for the bright and dark fringes for the value of δz , which maximizes the visibility.

In Fig. 12, this observation is analyzed in more detail in terms of the phase of the g-overlap matrix for varying δz . Clearly, as the visibility is maximized, the phases of the first few overlap coefficients assume approximately the same value. This can also be seen from Fig. 5. Since the full dependence on the phase is described by Eq. (3.14), it is clear that these phases take approximately the same value for all ϕ , if they assume the same phase for *any* specific ϕ . This observation was also required in Sec. IV B.

Similarly, Fig. 13 shows the behavior of the phases of the h-overlap matrix entries at the bright and dark fringes for varying δz . Compared to the g-overlap matrix, the phase values of the h-overlap coefficients are already very close to zero on the entire shown region (note that the y axis here is in mrad). Furthermore, it is visually clear that for the dark

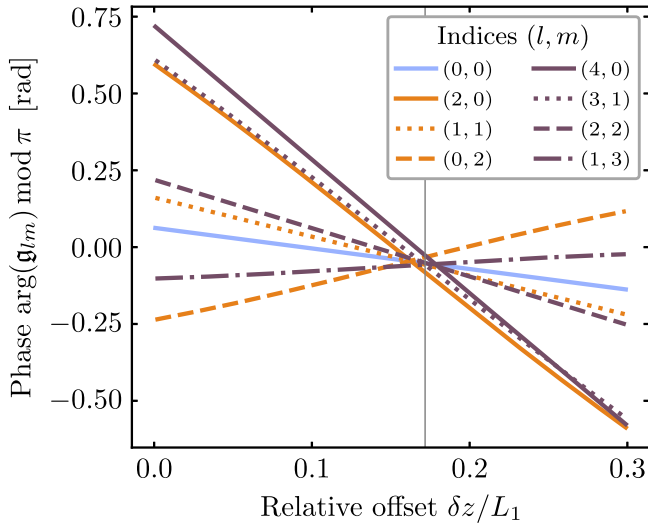


FIG. 12. Plot of the phase $\arg(g_{lm})$ of the first significant entries of the g -overlap coefficient matrix for varying offset δz of the focusing element inside the unbalanced $SU(1,1)$ interferometer. The vertical thin gray line indicates the offset at which the visibility is maximized (compare Fig. 6). As discussed in Appendix A1, the modes and therefore also the g -overlap coefficients are only defined up to their phase. Therefore, the plotted values are corrected by shifting them to the interval $[-\pi/2, \pi/2]$ to achieve a smooth plot. Clearly, at the point of the optimal visibility, the phase values of the shown overlap matrix entries almost coincide. See also Fig. 13, which discusses the phases of the h -overlap matrix elements for the same situation.

fringe, the point at which the phase values are most similar is slightly toward larger values of δz compared to the point at which the visibility is maximized. For the bright fringe, both points coincide.

APPENDIX E: ACCOUNTING FOR INTERNAL LOSSES

It is well known that internal losses, which the signal and idler fields suffer between the two crystals of the $SU(1,1)$ interferometer, degrade the interferometer performance [64,65]. One common way of modeling these losses is by introducing a *virtual* beam splitter inside the interferometer whose two input ports receive the output fields of the first crystal and the vacuum state [65]. This beam splitter is virtual in the sense that it only serves as a theoretical way of describing the losses and would not be included in an experimental implementation of the setup. One of the output ports of the beam splitter is then sent to the input of the second crystal, while the output field of the second output port models the lost radiation and is essentially discarded. Note that this description bears similarity to the description of lossy PDC provided in Ref. [23].

Using this description to model the $SU(1,1)$ interferometer with losses, it is clear that the transfer functions $\beta^{(1)}$ and $\tilde{\eta}^{(1)}$ describing the first crystal remain unchanged, when internal losses are included. However, assuming that the beam splitter applies a certain amount of losses, described by the loss factor ι , to the output field of the first crystal, the relationship connecting the plane-wave input operators for the second crystal to its output operators are, instead of Eqs. (1.1a) and (1.1b), given by

$$\hat{a}_s^{(2,\text{out})}(q_s) = \sqrt{1-\iota} \left\{ \int dq'_s \tilde{\eta}(q_s, q'_s) \hat{a}_s^{(2,\text{in})}(q'_s) + \int dq'_i \beta(q_s, q'_i) [\hat{a}_i^{(2,\text{in})}(q'_i)]^\dagger \right\} + \sqrt{\iota} \left\{ \int dq'_s \tilde{\eta}_0(q_s, q'_s) \hat{b}_s^{(2,\text{in})}(q'_s) + \int dq'_i \beta_0(q_s, q'_i) [\hat{b}_i^{(2,\text{in})}(q'_i)]^\dagger \right\}, \quad (\text{E1a})$$

$$[\hat{a}_i^{(2,\text{out})}(q_i)]^\dagger = \sqrt{1-\iota} \left\{ \int dq'_i \tilde{\eta}^*(q_i, q'_i) [\hat{a}_i^{(2,\text{in})}(q'_i)]^\dagger + \int dq'_s \beta^*(q_i, q'_s) \hat{a}_s^{(2,\text{in})}(q'_s) \right\} + \sqrt{\iota} \left\{ \int dq'_i \tilde{\eta}_0^*(q_i, q'_i) [\hat{b}_i^{(2,\text{in})}(q'_i)]^\dagger + \int dq'_s \beta_0^*(q_i, q'_s) \hat{b}_s^{(2,\text{in})}(q'_s) \right\}, \quad (\text{E1b})$$

where the newly appearing $\hat{b}_{s/i}$ operators originate from the vacuum input port of the beam splitter. These new operators unconditionally commute with the $\hat{a}_{s/i}$ operators:

$$[\hat{b}_{s/i}(q_{s/i}), \hat{a}_{s/i}(q'_{s/i})] = 0, \quad (\text{E2a})$$

$$[\hat{b}_{s/i}^\dagger(q_{s/i}), \hat{a}_{s/i}(q'_{s/i})] = 0, \quad (\text{E2b})$$

$$[\hat{b}_{s/i}^\dagger(q_{s/i}), \hat{a}_{s/i}^\dagger(q'_{s/i})] = 0, \quad (\text{E2c})$$

which is to say that the state belonging to these operators is orthogonal to that belonging to the $\hat{a}_{s/i}^{(2,\text{in})}$ operators [64]. While the $\hat{a}_{s/i}^{(2,\text{in})}$ operators connect the second crystal to the first crystal, the $\hat{b}_{s/i}^{(2,\text{in})}$ operators introduce a new vacuum component to the input of the second crystal. Importantly, these

operators are therefore not connected to the output operators via the same transfer functions as the $\hat{a}_{s/i}^{(2,\text{in})}$ operators, but rather by the newly introduced transfer functions that have the subscript 0 added.

Plugging Eqs. (E1a) and (1.2) into the operator-value integro-differential equations as derived in Ref. [8] (compare also Ref. [7]) now yields two sets of coupled integro-differential equation, in one of which $\tilde{\eta}$ and β , and in the other of which $\tilde{\eta}_0$ and β_0 are each coupled via the transverse wave vectors q_s and q_i . The form of each set is the same as written in Eqs. (1.2a) and (1.2b). Additionally, both sets require different phase-matching functions h : For the newly introduced $\hat{b}_{s/i}$ operators, the phase-matching function is that of a single (first) crystal [Eqs. (3.1a) and

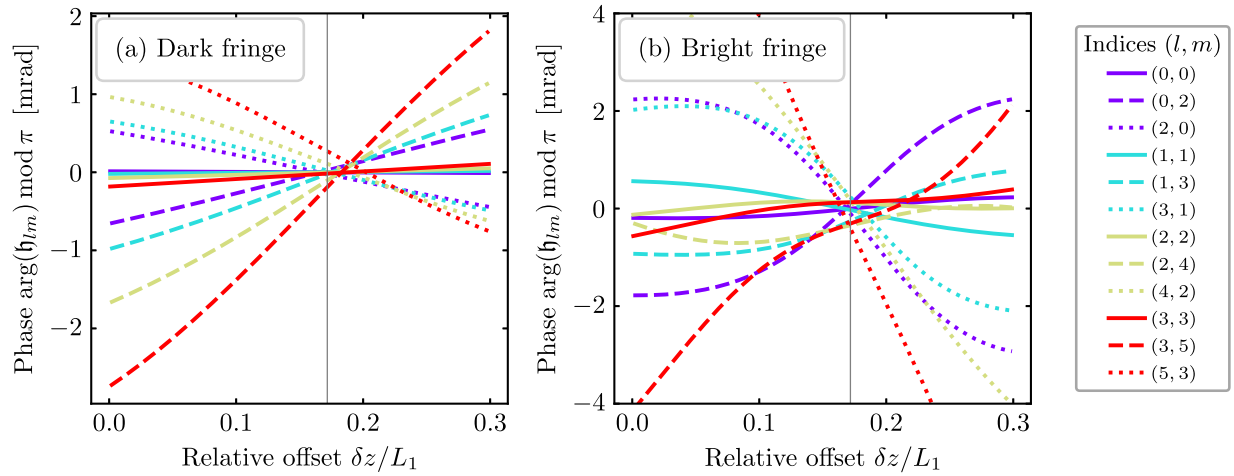


FIG. 13. Plot of the phase $\arg(h_{lm})$ of the first significant entries of the h -overlap coefficient matrix for varying offset δz of the focusing element inside the unbalanced $SU(1,1)$ interferometer for the (a) dark fringe and (b) bright fringe. The vertical thin gray line indicates the offset at which the visibility is maximized (compare Fig. 6). As discussed in Appendix A 1, the modes and therefore also the h -overlap coefficients are only defined up to their phase. Therefore, the plotted values are corrected by shifting them to the interval $[-\pi/2, \pi/2]$ to achieve a smooth plot. Clearly, near the point of the optimal visibility, the phase values of the shown overlap matrix entries almost coincide. Note that unlike Fig. 12, the y axis here has its units in mrad. Thus, the phase in the entire shown region is already very close to zero compared to the phase $\arg(g_{lm})$.

(3.7a)] instead of that of the second crystal of an $SU(1,1)$ interferometer.

As a consequence, it is not longer possible to diagonalize the input-output relation in the same way as it was done with the help of the Schmidt operators in Eq. (1.5). This can be seen from the fact that the Schmidt modes resulting from a decomposition of $\tilde{\eta}$ and β and of $\tilde{\eta}_0$ and β_0 will necessarily be different and it is no longer possible to bring the plane-wave input-output relations to the form of the Bogoliubov transformations. A similar observation was also made in Ref. [66] where input-output relations of a similar structure were obtained. Ultimately, this means that the mathematical formalisms of Secs. III and IV are no longer directly applicable in the case of nonvanishing internal losses.

An alternative method, which allows for the partial inclusion of the internal losses, is as follows: For the construction of the overlap matrices, internal losses are neglected, which allows for the utilization of the formalisms as developed in Secs. III and IV. For the eigenvalues $\Lambda_n^{(2)}$ and $\Lambda_n^{(SU)}$, however,

internal losses can be taken into account by computing them via the covariance function. Similar to what was described in Ref. [7], the covariance function for the output photons

$$\text{cov}(q, q') = \langle \hat{N}(q) \hat{N}(q') \rangle - \langle \hat{N}(q) \rangle \langle \hat{N}(q') \rangle \quad (\text{E3})$$

can be obtained from the output plane-wave operators and expressed in terms of $\tilde{\eta}$, β , $\tilde{\eta}_0$, and β_0 . If the signal and idler photons are treated as distinguishable in some degree of freedom, for example, their polarization or when they are measured at slightly different frequencies but in such a way that the corresponding refractive indices can still be treated as being equal, the eigenvalues can be obtained by decomposing the square root of the covariance function $\text{cov}(q, q')$ (see Refs. [5,49] for more details). These eigenvalues may then be used instead of the eigenvalues obtained from the Schmidt decomposition since they take into account the internal losses. It should be noted that this approximation is only valid for high gain and requires $\Lambda_n \gg 1$. See also Eq. (20) of Ref. [8].

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- [43] Even if both crystals have the same parametric gain, the q_s - and q_t -dependent phase term $e^{-i\Delta k^{\text{eff}}(q_s, q_t)\delta z}$ in Eq. (3.7b) still prevents this.
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- [56] Note that numerical implementations of the SVD may return either V_0^H and $\tilde{V}^H$ or V_0 and $\tilde{V}$ as the right-hand side unitary.
- [57] Generally, the Takagi factorization of a symmetric matrix G is defined as $G = DSD^T$ [40,51,59], where D is unitary and S is the diagonal matrix containing the singular values. However, for a unitary matrix, all singular values are equal to 1, meaning $D = I_n$, and therefore the Takagi factorization takes the form as in Eq. (1.4).
- [58] As an example for the abovementioned issues that may arise, consider the 2×2 identity matrix $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, which is a unitary, symmetric, and block-diagonal matrix consisting of two blocks of size 1. The matrix $S = \begin{bmatrix} 1 & i \\ 0 & -1 \end{bmatrix}$ is one of the matrix square roots of I_2 : $S^2 = I_2$. However, S is clearly neither symmetric nor unitary; see, for example, $S^H S = \begin{bmatrix} 1 & i \\ -i & 2 \end{bmatrix} \neq I_2$. Furthermore, S consists of a single block of size 2. Generally, a numerical algorithm may therefore not necessarily preserve the symmetry, unitarity, and/or the block-diagonal structure of the input matrix.
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