

Penning trap single-photon counter for axion detection

Jack A. Devlin^{✉,*}, Marko L. Wojtkowiak, Shreyak R. Banhatti, He Zhang[✉], Jiacheng Shi[✉], Toren S. Dofher[✉],
 Jonathan M. H. Gosling[✉], Michael R. Tarbutt[✉], and Richard C. Thompson[✉]
Centre for Cold Matter, Blackett Laboratory, Imperial College London, Prince Consort Road, London SW7 2AZ, United Kingdom



(Received 2 March 2026; accepted 3 June 2026; published 22 October 2026)

Discovering the microscopic composition of dark matter is one of the most important open problems in physics today. Axions are a leading candidate to be dark matter; however, a search of the full range of all likely axion masses is hampered by the standard quantum noise limit. This makes haloscope searches for axions with masses above $\sim 100 \mu\text{eV}$ unfeasible with current technologies. To overcome this limitation, we propose a photon counting technique designed to operate at 30–60 GHz for detecting axions with masses between 124 and 248 μeV , based on a single electron in a Penning trap. The electron cyclotron mode absorbs microwave photons, and, via the continuous Stern-Gerlach effect, this absorption imparts a measurable phase shift onto the axial motion. In this paper, we comprehensively analyze this photon detection method. We introduce a type of fast, phase-sensitive axial detection technique, using axial-magnetron parametric amplification to overcome detector Johnson noise and cancel associated frequency shifts. This method may find other applications in precision Penning trap frequency measurements. We compare the efficiency of the electron single-photon counter with an ideal device and find that our proposed photon counter has sufficient performance to search for high-mass axions.

DOI: [10.1103/pg4b-jpg2](https://doi.org/10.1103/pg4b-jpg2)

I. INTRODUCTION

Dark matter is an unknown substance that affects astrophysical observations on subgalactic to cosmological distances. There is no Standard Model particle that fits the bill to be dark matter, and considerable efforts are underway to experimentally discover particles that could be this missing mass. One plausible particle that could be dark matter is the axion. Axions are pseudoscalar bosons, first proposed to explain why CP symmetry appears to be conserved by the strong force [1–4]. Axions are predicted not to be completely dark, with an axion-to-photon interaction Lagrangian, in natural units, of

$$\mathcal{L}_a = -\frac{1}{4}g_{a\gamma\gamma}aF^{\mu\nu}\tilde{F}_{\mu\nu}, \quad (1)$$

where F is the electromagnetic tensor, $\tilde{F}$ is its dual, a is the axion field, and the strength of the interaction is governed by the axion-to-photon coupling constant, again in natural units

$$g_{a\gamma\gamma} = C_{a\gamma} \frac{\alpha}{2\pi} \frac{m_a}{\sqrt{\chi}}. \quad (2)$$

Here, α is the fine structure constant, m_a is the axion mass, $\chi = [75.44(34)\text{MeV}]^4$ is a QCD parameter called the topological susceptibility [5], and $C_{a\gamma}$ is a dimensionless coupling constant. There are various models that produce axions that solve the strong CP problem, and the choice of model

affects the value of $C_{a\gamma}$, sometimes also written as $g_\gamma = C_{a\gamma}/2$. Two well-established models are the Kim-Shifman-Vainshtein-Zakharov (KSVZ) [6,7] and Dine-Fischler-Srednicki-Zhitnitsky (DFSZ) [8,9] models, for which $C_{a\gamma} = -1.92(4)$ and $C_{a\gamma} = 0.75(4)$, respectively [10]. Thus, although the axion mass is not predicted, there is a well-understood relation between the axion mass and the coupling strength. Experiments have ruled out axions with masses above 1 eV, but sub-eV axions remain a viable and well-motivated dark matter candidate, and there are production mechanisms in the early universe that straightforwardly generate axion abundances consistent with the Λ Cold Dark Matter (ΛCDM) model and observed cold dark matter density [11–14]. In our universe, axions would form a gravitationally bound dark matter halo, which could interact with a terrestrial “haloscope” detector. Most experiments seek to use the axion-photon coupling given by Eq. (1) to convert axions into rf photons. Building a device with sufficient sensitivity to produce a detectable signal from axion-normal matter interactions is challenging. Single-photon counters dramatically improve the search for axions with masses above about $\sim 100 \mu\text{eV}$, potentially boosting the rate at which different axion masses can be investigated by many orders of magnitude. This application requires low dark counts and high duty cycle; currently, there are no single-photon counters with the necessary characteristics in this frequency range. In this work, we propose that an electron in a Penning trap device could fill this niche.

Trapped electrons have been used for a rich series of fundamental physics tests. They can be used for measurements of the electron g -factor [15] and leptonic CPT tests [16] as well as quantum computing in both Paul [17–21] and Penning traps [22–24]. Trapped electrons have also been proposed as

*Contact author: j.devlin11@imperial.ac.uk

Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI.

sensors for directly detecting dark matter-normal matter collisions [25,26] and the conversion of dark photons to normal photons [27], part of a range of recent experiments to detect dark matter with quantum techniques [28]. Electrons can also be used to detect very weak electromagnetic radiation. Individual thermal microwave photons inside a cavity have been detected using electrons in a Penning trap [29]. The microwave photons were observed to drive transitions between the modified cyclotron states of the trapped electron, and these transitions could be detected by measuring the electron axial frequency. In this paper, we propose an extension of this technique to detect incoming microwave photons produced outside of the trap, so that a trapped electron can serve as a single-photon counter. The basic physics behind our photon counting concept is similar to proposals by Cridland *et al.* [30] and Fan *et al.* [31]; however, we choose a different experimental configuration and measurement protocol to achieve photon detection.

The structure of the paper is as follows: In Sec. II, we consider why single-photon counters are vital for certain experiments searching for dark matter axions, and we review the technologies for single microwave photon counting at the relevant microwave frequencies. In Sec. III, we give an overview of the method for counting microwave photons with a trapped electron. Section IV deals with the cavity QED interaction between the electron and the cavity, and how this leads to an additional phase shift on the electron's axial motion, conditional on a photon being absorbed by the electron. In Sec. V, we consider how the axial and magnetron modes evolve during the photon counting sequence and describe the noise sources that enter into a measurement of the electron axial phase, which affect our ability to resolve the signal from a photon absorption. Section VI presents the detection efficiencies expected under various experimental configurations. Finally, in Sec. VII we discuss our results, compare these to other photon counting proposals at these frequencies, and suggest avenues for further improvements in efficiency.

II. SINGLE-PHOTON COUNTERS FOR AXION SEARCH EXPERIMENTS

Our aim is to develop a photon counter that will improve axion search experiments. To understand why such a device is beneficial, we begin by considering a generic axion cavity haloscope experiment. In this experiment, axions are converted into photons in a resonant cavity, placed in a strong magnetic field. The output of a particular mode of the cavity is monitored and if the axion (angular) Compton frequency $\omega_a = m_a(c^2 + v^2/2)/\hbar$, found from the axion mass m_a and speed v , matches the haloscope cavity's resonant frequency, then some axions may be converted into photons inside the cavity at a frequency ω_a . The spread of speeds v causes a spread in the axion energies prior to conversion, with the fractional energy spread full width at half maximum given by $1/Q_a$, where the axion quality factor is $Q_a \sim 10^6$.

The axion-sourced rf power coupled out of a generic cavity haloscope is given by [32,33]

$$P_a = \left(\frac{\alpha^2 \hbar^3 c^3 C_{ay}^2 \rho_{\text{DM}}}{4\pi^2 \mu_0 \chi} \right) \left(\kappa_c \omega_a B^2 V_m \frac{Q_c Q_a}{Q_c + Q_a} \right). \quad (3)$$

The factors inside the first pair of brackets are out of the experimenter's control. In addition to the factors introduced previously, there is also ρ_{DM} , the local dark matter density. In the second bracket are factors within experimental control (apart from Q_a): The coupling constant κ_c , which gives the ratio of power coupled out of the cavity to the total power lost per cycle, the loaded cavity Q factor Q_c , the average applied magnetic field amplitude B into which the cavity is placed, and the volume factor for the m th mode:

$$V_m = \frac{|\int \mathbf{E}_m \cdot \mathbf{B} dV|^2}{B^2 \int \epsilon_r |\mathbf{E}_m|^2 dV}. \quad (4)$$

Here, $\mathbf{E}_m$ is the electric field of the cavity, ϵ_r is (position-dependent) relative permittivity, and $\mathbf{B}$ is the external magnetic field applied. Using $\rho_{\text{DM}} = 0.45 \text{ GeV/cm}^3$ and a critically coupled cavity of $\kappa_c = 1/2$, the rate at which photons are emitted (in photons per second) is $R_a = \frac{P_a}{\hbar \omega_a} \simeq 5 \times 10^{-4} C_{ay}^2 B^2 V_m Q_\mu$, with all quantities in Système International (SI) units and $Q_\mu = \frac{Q_c Q_a}{Q_c + Q_a}$. For $\omega_a = 2\pi \times 30 \text{ GHz}$, we expect R_a to be around 1 count/s to 1 count/h. Our group is developing a high- Q cavity like this for axion-to-photon conversion, but for higher axion masses, there are also other noncavity approaches, including using magnetized mirrors [34,35] or using plasma resonances [36]. In all these cases, the signal is same: an expected increase in the number of photons emitted from the structure, when it is configured to be sensitive to a particular range of axion masses. Rather than focusing on the axion-to-photon conversion step, in this paper we describe a method to detect R_a .

Any experiment that searches for axions by converting them into microwave photons has to somehow detect the power emitted from the axion-to-photon converter. Traditionally, this has involved using a linear amplifier based on a high electron mobility transistor (HEMT), superconducting quantum interference device (SQUID), or Josephson parametric amplifier to increase the low-power microwave signal from axion decay. The amplified signal can then be down-mixed to a convenient frequency, digitized, and Fourier transformed in a spectrum analyzer. The axion signal would lead to excess noise at ω_a , above the amplifier noise floor. To assess the size of a resolvable axion signal, the axion-sourced power needs to be compared to the noise at this particular frequency. For linear amplification, there is a fundamental lower limit to the noise floor that can be reached, even as both the cavity and detector approach zero temperature. This is referred to as the standard quantum limit (SQL) and arises because linear amplification allows both the phase and amplitude of the field to be measured simultaneously, but these quantities obey a Heisenberg uncertainty relation which prevents simultaneous measurement at arbitrary precision [37].

The SQL can be avoided by counting the photons, a process that does not preserve phase information and so is not subject to SQL noise. To compare single-photon counting to linear amplification for axion searches, the best figure of merit is the time required to scan a frequency range $\Delta\omega_a = \omega_a/Q_a$ to the level where both a single-photon counter and a linear amplifier can detect or exclude an axion-sourced signal at the same level of significance. The ratio between the scan rates for single-photon counting and linear amplification is

approximately given by [33]

$$E = \frac{Q_c}{2\pi Q_a} \frac{\eta_{\text{sp}}^2}{\eta_{\text{la}}^2} \left(\frac{\Delta\omega_a}{\eta_{\text{sp}} R_a + \eta_{\text{sp}} R_T + R_{\text{RO}}} \right). \quad (5)$$

Here, η_{sp} and η_{la} are the detection efficiencies for single-photon counting and linear amplification, respectively, R_T is the rate of thermal blackbody photons, and R_{RO} represents other nonthermal dark counts from the device readout. As $\Delta\omega_a = \omega_a/Q_a$, where Q_a is independent of ω_a , this expression suggests that single-photon counting will be more advantageous at higher axion frequencies. We can use this expression to consider the point at which photon counting is beneficial to an axion experiment. At low enough temperatures, $R_T \ll R_a$. We also can typically control R_{RO} to be much smaller than ηR_a . For most current experiments, Q_c/Q_a is around 1/100, but in future cavity experiments, including those under development in our laboratory, it is feasible to imagine $Q_c/Q_a \sim 1$. In this case, assuming $\eta_{\text{la}} = 1$, the enhancement is

$$E = \frac{\omega_a \eta_{\text{sp}}}{2\pi Q_a R_a} = 3 \times 10^4 \left(\frac{\eta_{\text{sp}}}{1} \right) \left(\frac{\omega_a}{2\pi \times 30 \text{ GHz}} \right) \left(\frac{1 \text{ cps}}{R_a} \right). \quad (6)$$

At 30 GHz, a single-photon counter with an efficiency of $\eta_{\text{sp}} > 3 \times 10^{-5} \frac{R_a}{1 \text{ cps}}$ will outperform a linear amplifier. Typical count rates R_a for accessible values of B , V_m , and Q_h are of order 0.001–1 counts/s, suggesting huge boosts for even very low single-photon counter efficiencies. Assuming $R_a = 0.1$ counts/s, a single-photon counter with $\eta_{\text{sp}} = 0.5\%$ will outperform a linear amplifier by a factor of 1500 at 30 GHz, while $\eta_{\text{sp}} = 10\%$ improves the scan rate by a factor 30 000 at the same frequency.

The benefits of single-photon counting for axion experiments operating at frequencies above around 10 GHz have long been recognized [37]. Unfortunately, the technology to count single photons in the window 30–300 GHz at high efficiency and low dark count rates does not currently exist. At frequencies above this band, transition-edge sensors [38] and superconducting nanowires [39] typically operate at terahertz frequencies, whereas superconducting hot electron bolometers operate above 300 GHz. Work is underway to make transition-edge sensors sensitive down to 90 GHz [40] to partially bridge this gap. Below this range of frequencies, circuit quantum electrodynamics (cQED), superconducting qubits interacting with microwave cavities, can be used to count single photons [41]. In the standard cQED configuration, these devices may not have dark count rates low enough for axion searches; however, recent progress [42] using improved measurement protocols has demonstrated counting efficiencies up to 0.409(55) with false positive rates of $4.3(1.1) \times 10^{-4}$ at 6.011 GHz with a duty cycle of 65%. This method seems directly extendable up to 30 GHz or above, and recent progress has been made in this direction [43]. In the future, higher microwave frequencies could potentially be detected using superconducting qubits made from higher T_c superconductors. Another approach is to use aluminum-based tunnel Josephson junctions in a current-biased regime [44]. If the Josephson junction is coupled to a photon field, it switches to a resistive state, and this leads to a voltage across the junction.

Few-photon detection sensitivity was demonstrated at 10 GHz via this method. Single-photon counters are also proposed in the 30–300 GHz range using quantum dots [45] and optomechanical sensors [46]. Rydberg atoms have also been used in single-photon counting experiments with a view to conducting axion searches [47,48] and are also proposed to explore the range 10–50 GHz [33]. Finally, electrons in surface [30] and macroscopic [31] Penning traps have also been proposed to detect microwave photons. In the sections that follow, we present our method for counting single photons with a trapped electron. In Sec. VII, we contrast our approach with other methods and draw some conclusions.

III. OVERVIEW OF THE PHOTON DETECTION METHOD

Our microwave single-photon counter consists of a single electron in a cryogenic Penning trap. The Penning trap, pictured in Fig. 1(a), confines the electron with mass m in a strong static magnetic field $B_0 \hat{z}$, and a quadratic electrostatic potential $\varphi = \frac{V_0 C_2}{2d^2} (z^2 - \rho^2/2)$ generated by applying voltages to a closed set of cylindrical electrodes surrounding the electron. Here, the total effect of applying different voltages to the various electrodes of potentially different geometries is captured by a single effective voltage V_0 , a characteristic length $d^2 = \frac{1}{2}(z_t^2 + \rho_t^2/2)$, determined by the trap radius ρ_t and the distance between the trap center and the endcap electrode z_t , and a dimensionless constant C_2 . In a hyperbolic trap, V_0 would be the real potential difference between the ring and endcaps and $C_2 = 1$. The Hamiltonian of this ideal system can be written as the sum of three quantum harmonic oscillators and a spin in a magnetic field [49]:

$$H_0 = \hbar\omega_+ (a_+^\dagger a_+ + \frac{1}{2}) + \hbar\omega_z (a_z^\dagger a_z + \frac{1}{2}) - \hbar\omega_- (a_-^\dagger a_- + \frac{1}{2}) + \frac{1}{2} \hbar\omega_s \sigma_z.$$

Here, “+” denotes the modified cyclotron mode, “z” the axial mode, “−” the magnetron mode, σ_z is the Pauli z matrix, $a_i^\dagger$ the creation, and a_i the annihilation operators for the respective modes. The mode oscillation frequencies for a trapped electron are given by $\omega_z^2 = \frac{eV_0 C_2}{md^2}$, $\omega_\pm = \frac{1}{2}(\omega_c \pm (\omega_c^2 - 2\omega_z^2)^{1/2})$, and $\omega_s = \frac{g_S}{2}\omega_c$, where $\omega_c = \frac{eB_0}{m}$ is the free cyclotron frequency, g_S is the electron g factor, and ω_s is the Larmor frequency. This Hamiltonian does not include relativistic effects, which can be neglected in this case. The electron motion is illustrated in the top part of Fig. 1(a). By tuning the magnetic field B_0 , the modified cyclotron frequency can be matched to the frequency ω_a of the microwave photon; to match photon frequencies between 30–60 GHz requires fields 1–2 T. In principle, higher frequencies could also be accessible with this method. The axial frequency, set by a choice of V_0 and the size of d , can be chosen to be in the range $\omega_z = 2\pi \times 10$ –100 MHz, with a corresponding magnetron frequency ω_- of a few tens to hundreds of kilohertz. In our case, these frequencies are always positive and real. The trap is held at temperatures $T_e < 100$ mK by thermal connections to the mixing chamber of a dilution fridge. At this temperature, the cyclotron mode cools to its ground state via radiative emission with a time constant of < 1 ms. The axial and magnetron modes can be brought to the same mode occupation number by coupling them using an rf drive at

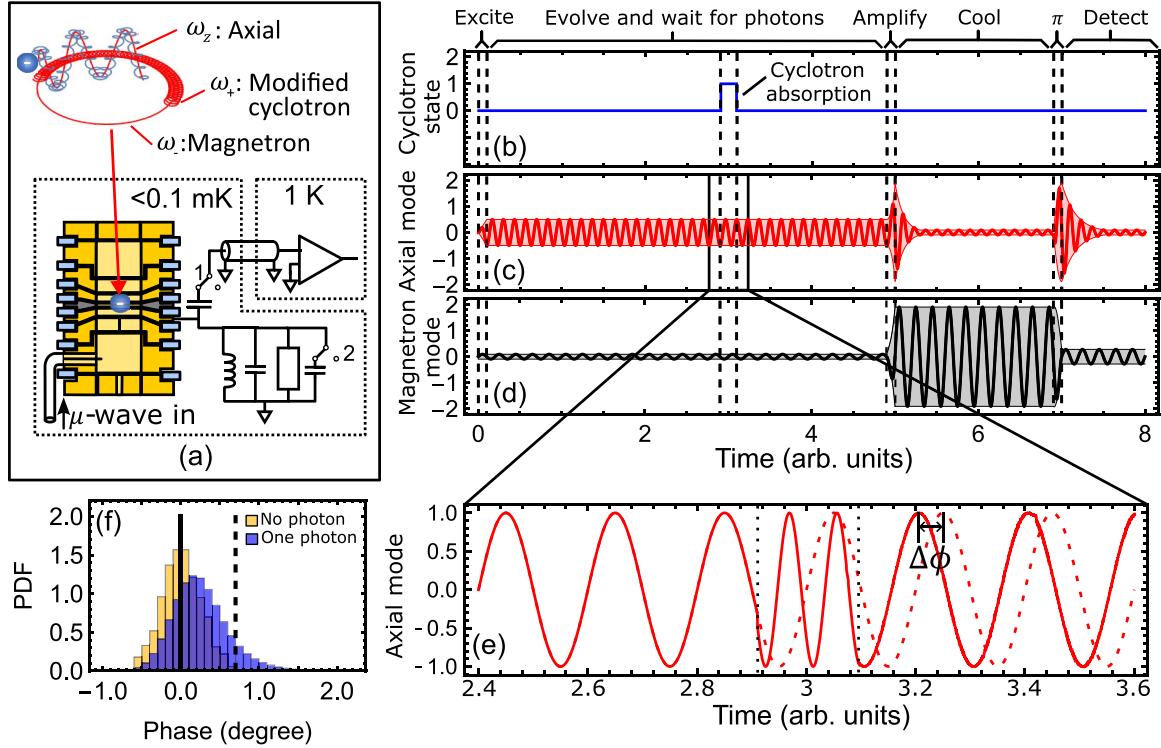


FIG. 1. An illustration of the photon counter and detection sequence. Panel (a) shows the three motions in the Penning trap and a cross section of the trap and electronic detection circuit. Panel (b) shows the cyclotron state during a photon detection sequence, panel (c) illustrates the axial amplitude, and panel (d) shows the magnetron amplitude. (e) An expanded version of the axial oscillation illustrating how a phase shift due to the cyclotron absorption event arises from a transient frequency shift. The size of the frequency shift has been exaggerated to make it more visible in this illustration. (f) An illustration of a phase distribution after repeated photon counting sequences deliberately exaggerated to illustrate the effect: Yellow gives the distribution when no photon was absorbed and blue gives the distribution after an absorption.

$\omega_z + \omega_-$, and then both modes can be cooled to the ground state by alternating this drive with a sideband coupling between the cyclotron and axial modes using microwaves at $\omega_+ - \omega_z$. A resonant LCR circuit can be used to interact with the axial mode, as shown schematically in Fig. 1(a). It has three operating configurations depending on the position of the two switches shown. In the detection mode, the LCR circuit is resonant with the axial mode, and the switch labeled “1,” which connects the LCR circuit via a capacitor to a transmission line and amplifier, is closed, and the switch labeled “2” is open. This enables the amplitude and phase of the axial mode to be read out; however, the electronic temperature of the detection circuit in this configuration is rather high, $T \simeq 1$ K, and the damping time constant is low, limited by losses in the amplifier, which reduces the Q factor of the LCR circuit. The second configuration is the cooling configuration, in which switch 1 between the amplifier and the LCR coil is open. This allows rapid cooling of the axial mode to T_e in situations where sideband cooling causes unwanted phase shifts. Finally, in the decoupled state, switch 1 remains open, and an additional capacitance is connected across the LCR circuit using switch 2 to shift it out of resonance with the axial mode, preventing both heating and damping of this mode. This is the default state of the resonant circuit.

In our Penning trap, the electrodes surrounding the electron are designed to act as a resonant cavity, with a quality factor Q and resonant frequency ω , which is also adjusted to match ω_+ and ω_a , by control of the cavity endcap positions. Photons

enter the cavity via a transmission line shown in the bottom left of Fig. 1(a), and once in the cavity, they can be absorbed by the cyclotron mode of the trapped electron. Adding an extra magnetic field of the form $\Delta\mathbf{B} = B_2((z^2 - \rho^2/2)\hat{z} - z\rho\hat{\rho})$ by means of a ferromagnetic ring electrode, pictured in gray in Fig. 1(a), allows the absorption of the microwave photon to affect the axial frequency [49]. This is because the magnetic field leads to an additional term in the Hamiltonian

$$H_1 = -\boldsymbol{\mu} \cdot \Delta\mathbf{B}. \quad (7)$$

The cyclotron mode has a magnetic moment $\boldsymbol{\mu}$ parallel to $\hat{z}$, and the magnitude of the magnetic moment is proportional to the cyclotron quantum number n_+ . A change in the cyclotron state following a photon absorption leads to a change in the magnetic moment equal to $|\Delta\boldsymbol{\mu}|$. For a particle oscillating along the z axis, the change in potential is therefore $|\Delta\boldsymbol{\mu}|B_2z^2$ imparting a frequency shift

$$\Delta\omega_z = \frac{\Delta n_+ \omega_+ \hbar B_2}{m\omega_z B_0}, \quad (8)$$

to the axial frequency, where Δn_+ is the change in cyclotron quantum number.

Figures 1(b)–1(e) illustrate the basic sequence of how a single-photon counting measurement is performed in our scheme, allowing the frequency shift $\Delta\omega_z$ to be detected. The top graph [panel (b)] shows the cyclotron state of the particle. The particle begins in the cyclotron ground state. At $t = 2.91$, the cyclotron state is briefly excited to $n_+ = 1$

for a time t_{ex} by an incoming photon before returning to the ground state. Figure 1(c) shows the axial motion during the measurement sequence, while graph (d) shows the magnetron motion projected onto one axis. Initially, during the period marked “Excite,” the axial amplitude is excited with an rf pulse at a frequency ω_z . There then follows a period labeled “Evolve and wait for photon,” during which the axial motion freely evolves and the experiment is sensitive to incoming photons. The absorption of the microwave photon causes a brief period where the axial frequency shifts, as described by Eq. (8). This is shown more clearly in the magnified graph [Fig. 1(e)]. Here, the dashed red line shows the axial motion in the absence of the cyclotron excitation, and the solid line shows the actual axial motion. The phase shift is $\Delta\phi = \Delta\omega_z t_{\text{ex}}$. The particle continues to freely evolve until a predetermined time when the phase shift is read out. There are several steps to this process, designed to minimize the contribution of the detector’s Johnson noise to the phase uncertainty without introducing additional phase noise. First, the axial motion and magnetron motions are parametrically amplified using an rf pulse at $\omega_{\text{rf}} = \omega_z - \omega_-$ during the period labeled “Amplify.” The amplitude in the axial mode is then rapidly cooled by bringing the resonant circuit into resonance with the axial mode in the cooling configuration, during the period marked “Cool.” The magnetron mode remains unaffected during this period. The magnetron and axial amplitudes are now exchanged using a π pulse at a frequency $\omega_{\text{rf}} = \omega_z + \omega_-$, during the time marked “ π .” The time between the start of the axial cooling and the start of the π pulse is carefully tuned so that unwanted frequency shifts from the amplification of the axial mode in the presence of the magnetic bottle are canceled by the magnetron evolution, discussed further in Sec. VE2. This maneuver results in a final amplified axial amplitude that retains the phase information of the axial excitation before amplification but has a significantly larger magnitude. Now, during the stage marked “Detect,” switch 1 between the amplifier and the detection circuit is closed, and the axial phase and amplitude are read out in a phase-sensitive measurement [50]. Finally, switch 1 is opened again to disconnect the amplifier from the LCR circuit, and both axial and magnetron modes are cooled using sideband drives to the ground state in preparation for the next cycle.

An indicative histogram from repeated measurements of the axial phase can be seen in Fig. 1(f). In both cases, the mean axial phase when no photons are incident onto the cavity has been subtracted. Here, the yellow histogram corresponds to the situation when no microwave photon is absorbed by the electron. There is a distribution of phases rather than just a constant phase because of the phase noise associated with the axial zero-point fluctuations, even after it is cooled to the ground state. The blue histogram is generated from points where a microwave photon is absorbed by the electron. In this case, the final phase is the convolution of two contributions, one Gaussian distributed noise contribution from the axial zero-point fluctuations and a second from the phase shift due to the transient occupation of the $n_+ = 1$ excited state, which approximately follows an exponential distribution because of the distribution in lifetimes. We can see that in this hypothetical case, if we set a cutoff phase at around 0.75° [dashed line in Fig. 1(f)], we can conclude that phases higher than this must

correspond to a photon absorption event, with a certain false positive rate, while phases less than 0.75° mostly correspond to no photon absorption, with a certain false negative rate. Changing the location of the cutoff reduces the false positive rate, which is a form of readout noise or dark counts, at the expense of detection efficiency. The distributions shown in this figure are purely for illustration, and in the course of the paper, we will derive their true form, so that we can accurately characterize the overall detection probability. In the remainder of the paper, we analyze this detection method in detail to calculate the achievable detection probability. Although we typically analyze the photon counting method in a relatively abstract fashion, without reference to a particular trap geometry or choice of experimental parameters, it is helpful to evaluate the expressions using a particular choice of realizable parameters to make sure the method remains feasible. These reasonable experimental values are listed in Appendix A and are used throughout unless otherwise stated. The reason governing the choice of parameters will be elucidated in the paper. We now proceed to analyze the photon detection sequence in detail.

IV. CAVITY-CYCLOTRON MODE INTERACTION

We now consider the interaction between the trapped electron and the field of the cylindrical cavity formed by the Penning trap electrodes. We note that recently, a closely related system consisting of an electron inside a cylindrical waveguide has also been solved [51]. The quantum mechanical evolution of the cyclotron and cavity states, including the effect of losses from the cavity, can be treated using the Lindblad master equation for the density matrix ρ :

$$\dot{\rho} = -\frac{i}{\hbar}[H, \rho] + \mathcal{L}_D(\rho), \quad (9)$$

where $\mathcal{L}_D(\rho)$ is the dissipative part defined below and H is the Hamiltonian of the system. The cyclotron mode is assumed to only decay via interactions with the cavity electric field; other decays are neglected. We determine the form of the interaction Hamiltonian before solving Eq. (9) to characterize the probability of absorption and the lifetime of an electron in the cyclotron excited state. We then consider how short-lived occupations of the excited cyclotron state can be measured by means of phase-sensitive detection of the axial mode and examine how phase shifts caused by photon absorption can be resolved. Finally, we return to consider the total photon absorption probability of the proposed Penning trap photon counter.

A. Form of the Hamiltonian

Ignoring the axial and magnetron modes, the Hamiltonian $H = H_2 + H_{\text{int}}$ is the sum of the Hamiltonian H_2 associated with the cavity and the cyclotron mode,

$$H_2 = \hbar\omega_+(a_+^\dagger a_+ + \frac{1}{2}) + \hbar\omega_-(a_-^\dagger a_- + \frac{1}{2}), \quad (10)$$

and an interaction Hamiltonian H_{int} between the electromagnetic vector potential A of the cavity mode and the electron’s momentum p :

$$H_{\text{int}} = \frac{e}{m} A \cdot p. \quad (11)$$

Assuming that the cavity is designed so that only a single cavity mode interacts with the electron, we can expand $\mathbf{A}$ in terms of the field creation and annihilation operators $a_\gamma^\dagger$, a_γ for that particular mode, so that in the Heisenberg picture the operator becomes

$$\mathbf{A}(t) = \sqrt{\frac{\hbar}{2\varepsilon_0\omega\tilde{V}}}(\boldsymbol{\epsilon}_p(\mathbf{r})a_\gamma e^{-i\omega t} + \boldsymbol{\epsilon}_p(\mathbf{r})^*a_\gamma^\dagger e^{i\omega t}). \quad (12)$$

Here, $\boldsymbol{\epsilon}_p(\mathbf{r})$ is the normalized vector potential of the mode in question at the electron position, taken as the origin, and $a_\gamma = a_\gamma(0)$, $a_\gamma^\dagger = a_\gamma^\dagger(0)$ are the values of the operators at $t = 0$. For the interaction between the cyclotron mode close to the ground state, where the cyclotron radius is small compared to the wavelength, and the TE_{11q} cavity field, where q is odd and the mode is centered at the electron position, we can ignore the spatial dependence and write $\boldsymbol{\epsilon}_p(\mathbf{r}) = \boldsymbol{\epsilon}_p$, where $\boldsymbol{\epsilon}_p$ is a linear polarization in the xy plane. For more complex modes, $\boldsymbol{\epsilon}_p(\mathbf{r})$ can be evaluated using the standard expressions for the fields in a cavity (see Ref. [52]). The field is normalized so that the total energy in the cavity volume V is equal to that of a single photon $\hbar\omega = \frac{1}{2} \int dV (\varepsilon_0 \mathbf{E} \cdot \mathbf{E} + \frac{1}{\mu_0} \mathbf{B} \cdot \mathbf{B})$, and the effective mode volume is given by

$$\tilde{V} = \frac{\int |\mathbf{E}|^2 dV}{|E_p|^2}, \quad (13)$$

where E_p is measured at the position of the trapped electron.

The cyclotron mode is purely radial, so we can write $\mathbf{p} = m\dot{\mathbf{p}}$ where $\dot{\mathbf{p}}$ is the radial velocity in the x - y plane. This can be written [49] in the form

$$\dot{\mathbf{p}} = \frac{V^+ \omega_+ - V^- \omega_-}{\omega_+ - \omega_-}.$$

The time dependence of the two vectors $V^\pm$ is $e^{i\omega_\pm t}$. As the cavity operates at around ω_+ , we can neglect the V^- term to write

$$H_{\text{int}} = \sqrt{\frac{e^2 \hbar}{2\varepsilon_0 \omega \tilde{V}}} \frac{\omega_+ (a_\gamma e^{-i\omega_+ t} + a_\gamma^\dagger e^{i\omega_+ t})}{\omega_+ - \omega_-} \boldsymbol{\epsilon}_p \cdot \mathbf{V}^+. \quad (14)$$

The velocity vector $\mathbf{V}^+$ can itself be written [49] in terms of the raising and lowering operators $a_+^\dagger$ and a_+ of the modified cyclotron mode. To illustrate a simple case, we consider a linearly polarized field mode along the x axis and use the fact that $\omega_+ \gg \omega_-$ and $\omega_+ \simeq \omega$. Then, neglecting the far-off-resonant terms, this becomes

$$H_{\text{int}} = \frac{e\hbar}{\sqrt{4m\varepsilon_0\tilde{V}}} (a_+^\dagger a_\gamma + a_+ a_\gamma^\dagger). \quad (15)$$

The matrix elements can be written using a basis $|n_+, n_\gamma\rangle$, where n_γ represents the number of photons in the cavity and n_+ the cyclotron state of the electron:

$$\langle n_+ + 1, n_\gamma - 1 | H_{\text{int}} | n_+, n_\gamma \rangle = e\hbar \sqrt{\frac{n_\gamma(n_+ + 1)}{4m\varepsilon_0\tilde{V}}} \quad (16)$$

and

$$\langle n_+ - 1, n_\gamma + 1 | H_{\text{int}} | n_+, n_\gamma \rangle = e\hbar \sqrt{\frac{n_+(n_\gamma + 1)}{4m\varepsilon_0\tilde{V}}}. \quad (17)$$

In Appendix B, we confirm that defining the matrix elements in this way leads to the familiar formula for the emission rate of an electron in free space. In the specific case where there is at most one photon in the cavity and the modified cyclotron mode is either in the ground or first excited state, the matrix elements become

$$\langle 1, 0 | H_{\text{int}} | 0, 1 \rangle = \langle 0, 1 | H_{\text{int}} | 1, 0 \rangle = e\hbar \sqrt{\frac{1}{4m\varepsilon_0\tilde{V}}},$$

and we can define the coupling rate as

$$g = e \sqrt{\frac{1}{4m\varepsilon_0\tilde{V}}}.$$

We can now solve the Linblad equation.

B. Solution of the Linblad equation

We consider a restricted basis set $|0\rangle = |0, 0\rangle$, $|\gamma\rangle = |0, 1\rangle$, and $|+\rangle = |1, 0\rangle$ to investigate either the emission of a single microwave photon into the cavity or the absorption and subsequent emission of a single photon by a particle initially in the cyclotron ground state. We form the density matrix in the usual way, such that its matrix elements are $\varrho_{ij} = \langle i | \varrho | j \rangle$. The dissipative part of the Linblad superoperator in this case is

$$\mathcal{L}_D(\varrho) = \kappa (a_\gamma \varrho a_\gamma^\dagger - \frac{1}{2} (a_\gamma^\dagger a_\gamma \varrho + \varrho a_\gamma^\dagger a_\gamma)), \quad (18)$$

where $\kappa = \omega/Q$. This is one of the most basic examples of the interaction between a harmonic oscillator and a quantized field, and it has been studied extensively. For completeness, we provide the solutions in this simple case. The equation can be solved analytically for this restricted basis set. Consider first the situation where the electron is initially $n_+ = 1$ and all other density matrix elements are zero, and the cavity frequency matches the modified cyclotron frequency $\omega = \omega_+$. The evolution of the cyclotron state is then given by

$$\begin{aligned} \varrho_{++}(t) &= \frac{e^{-\frac{1}{2}\kappa t} (\kappa^2 - 8g^2) \cosh\left(\frac{Wt}{2}\right)}{W^2} \\ &\quad - \frac{e^{-\frac{1}{2}\kappa t} (8g^2 - \kappa W \sinh\left(\frac{Wt}{2}\right))}{W^2}, \end{aligned} \quad (19)$$

where $W = \sqrt{\kappa^2 - 16g^2}$. In the weak field coupling limit when $g \ll \kappa$, this gives the familiar form for the Purcell enhancement of the spontaneous emission rate inside a cavity

$$\varrho_{++}(t) \simeq e^{-\frac{4g^2 t}{\kappa}}. \quad (20)$$

Interestingly, this quantum calculation of the rate of cyclotron emission exactly matches the classical rate that is calculated from a consideration of the image charges induced on the trap electrodes in the weak coupling limit. This equivalence is demonstrated in Appendix C. Comparing this expression to the free-space formula (see Appendix B), we find that in general the spontaneous emission rate can be significantly enhanced by the Purcell factor

$$\frac{4g^2}{\kappa} \left(\frac{1}{4\pi\varepsilon_0} \frac{4e^2\omega^2}{3mc^3} \right)^{-1} = \frac{3\pi c^3 Q}{4V\omega^3} \gg 1. \quad (21)$$

Inserting some typical numbers from Appendix A into the expression for the decay rate $\gamma_q = \frac{4g^2}{\kappa}$, valid when $g \ll \kappa$, we find $\tau = 1/\gamma_q = 1.6$ ms for $\omega = 2\pi \times 30$ GHz for a single electron in a cylindrical cavity of radius 3.7 mm and length 24.55 mm in the TE₁₁₃ mode with $\tilde{V} = 2.4 \times 10^{-7}$ m³ and cavity quality factor $Q = 20\,000$. This lifetime is very short compared to the timescales over which an image current measurement of the electron motion is typically performed. For instance, in Ref. [53], the authors describe a self-excited oscillator method to detect small frequency shifts for a trapped electron. Depending on the precise method used, the bandwidth varies from 150 to 8 Hz, implying response times above 6 ms. It is possible that significantly faster measurements can be performed [31]. Our approach is to use a method that does not require the electron to be measured in the excited state and will be described fully in the next section.

We now consider the situation in which the system begins in the $|0, 1\rangle$ state with one photon in the cavity, which is relevant for photon counting. The solution of the master equation for the diagonal elements of the density matrix corresponding to the probability for the system to be found, respectively, in the ground state, $n_+ = 1$ and $n_\gamma = 1$ states are

$$\varrho_{00}(t) = 1 - \varrho_{++}(t) - \varrho_{\gamma\gamma}(t), \quad (22)$$

$$\varrho_{++}(t) = \frac{16g^2 e^{-\frac{1}{2}\kappa t} \sinh^2\left(\frac{Wt}{4}\right)}{W^2}, \quad (23)$$

$$\begin{aligned} \varrho_{\gamma\gamma}(t) = & \frac{e^{-\frac{1}{2}\kappa t} (\kappa^2 - 8g^2) \cosh\left(\frac{Wt}{2}\right)}{W^2} \\ & - \frac{e^{-\frac{1}{2}\kappa t} (8g^2 + \kappa W \sinh\left(\frac{Wt}{2}\right))}{W^2}. \end{aligned} \quad (24)$$

The resulting probabilities are plotted in Fig. 2, and the parameters used in these evaluations are listed in the caption. Figure 2(a) shows the weak coupling limit with $g \ll \kappa$. In this case, the initial excitation in the cavity, $\varrho_{\gamma\gamma}$, mostly decays quickly away, while a small fraction, around 1%, is excited to $n_+ = 1$, which decays more slowly. Figure 2(b) shows the onset of strong coupling with $\kappa \simeq g$. In this case, nearly 50% of the population is transferred into $n_+ = 1$, and then it subsequently oscillates between $n_+ = 1$ and $n_\gamma = 1$ while slowly decaying. Increasing g further with respect to κ further increases the height of the initial peak in ϱ_{++} so that it approaches 100%. The first amplitude peak, where the probability for the photon to transfer to the excited state is maximized, is found from the first maximum of Eq. (23) and depends only on the ratio $S = \frac{\kappa}{4g}$:

$$P_e = (S + \sqrt{S^2 - 1})^{-\frac{2S}{\sqrt{S^2 - 1}}}, \quad (25)$$

which occurs at a time

$$t = \frac{4}{W} \ln \left[\frac{\kappa + W}{2g} \right]. \quad (26)$$

Equation (25) suggests that the cavity will need to have as high a quality factor as possible, ideally as close to the maximum set by the frequency width of the axion signal, $Q = Q_a = 10^6$, to make g as large as possible with respect to κ . This is much larger than the Q factors achieved in Penning traps that double as microwave cavities, which typically

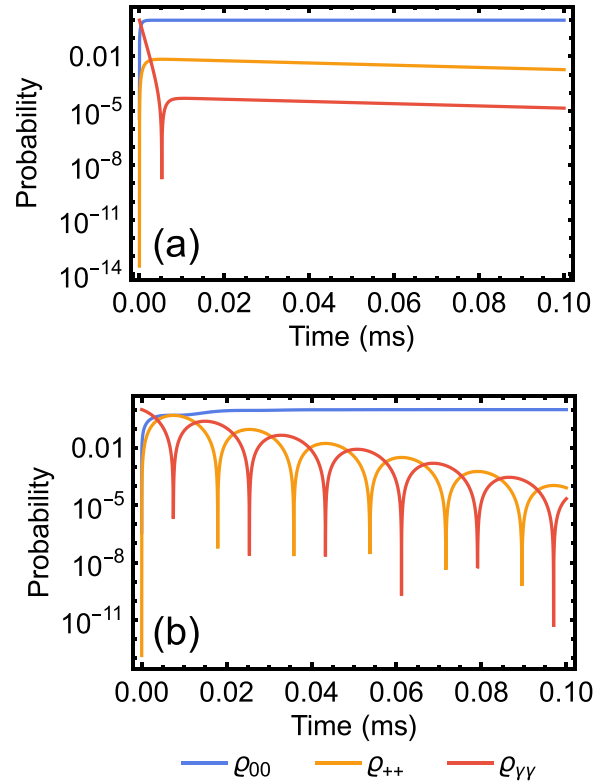


FIG. 2. Evaluation of the populations in the absolute ground state, excited cavity state, and excited cyclotron state given by Eqs. (22)–(24). Parameters are (a) $\omega = 2\pi \times 30$ GHz, $\tilde{V} = 1.2 \times 10^{-7}$ m³, and $Q = 10^5$. (b) $\omega = 2\pi \times 30$ GHz, $\tilde{V} = 2.4 \times 10^{-8}$ m³, and $Q = 10^6$.

have around $Q = 5000$ for the TE modes. Increasing the Q factor beyond these values is the major technical challenge of our method. At these frequencies and cryogenic temperatures, cavities made of normal metals such as copper are limited to around $Q = 4 \times 10^4$ [54]. Cavities coated in superconducting NbSn₃ have reached $Q = 1 \times 10^5$ at 1 T, falling to $Q = 6 \times 10^4$ at 2 T [54]. Superior performance can be achieved with a hybrid copper/NbTi cavity coating, achieving $Q = 6 \times 10^5$ at 1 T, falling to $Q = 4 \times 10^5$ at 2 T for an unloaded cavity [55]. Both these cavities were measured at resonant frequencies close to 9 GHz, so we would expect the Q factor to be lower at 30–60 GHz. Even higher quality factors can be reached using thin superconducting NbTi coatings on a thick, low-loss substrate like alumina [56] on all surfaces parallel to the magnetic field lines. By these methods, we think Q factors in the range $Q = 10^5$ – 10^6 could eventually be achieved. In this paper, we consider $Q = 10^4$ – 10^6 .

So far, we have assumed that the cavity frequency exactly matches the modified cyclotron frequency. In the sections that follow, we will see how the motion of the electron can introduce frequency shifts that may lead to a relative detuning $\Delta = \omega - \omega_+$ between these frequencies. To understand how this affects the electron-cavity coupling, we numerically calculate P_e for a range of detunings Δ and then find the full width at half maximum of P_e with respect to Δ , which we call the absorption linewidth of the electron-cavity system. The electron-cavity linewidths obtained are shown in Fig. 3. In this

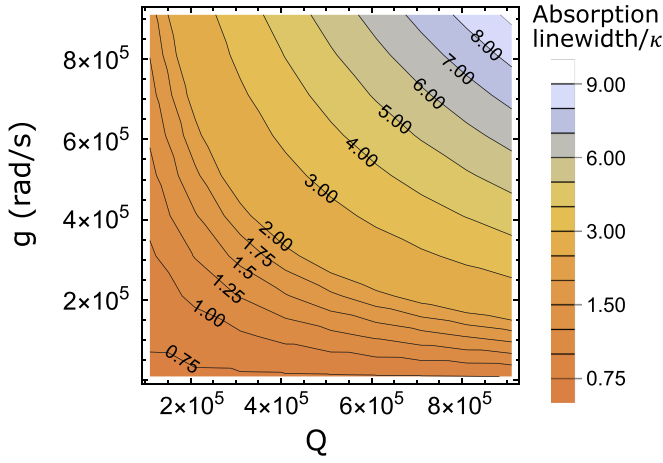


FIG. 3. The absorption linewidth of the electron and cavity when a single electron is placed in a cavity, divided by the cavity linewidth κ . Here, $\omega = 2\pi \times 30$ GHz.

figure, we have divided the electron-cavity linewidth by κ , the cavity linewidth without an electron. The contours represent the extent to which the electron-cavity linewidth is broader or narrower than the cavity's linewidth κ . For the electron photon counter, the typical operating parameters are in the bottom section of this figure, in the low g region, where we can approximate the absorption linewidth by the cavity linewidth κ .

We now consider the phase accrued by the electron as a result of occupying the excited state. First, to get a sense of the typical magnitude of the effect, we calculate the mean phase shift, $\langle\phi\rangle = \Delta\omega_z \langle t_{\text{ex}} \rangle$. This can be calculated from the average time spent in the cyclotron excited state $\langle\phi\rangle = \Delta\omega_z \int_0^\infty \varrho_{++}(t) dt = \frac{\Delta\omega_z}{\kappa}$. Interestingly, this mean shift is independent of g . For the parameters in Table II, with $Q = 10^6$, $\langle\phi\rangle = 1^\circ$, indicating the typical size of phase shifts that will need to be resolved.

To proceed further, we need to derive the form of the probability density function (PDF) $p_\phi(\phi)$ for the phase accrued. Once a decay into the $|0\rangle$ state occurs, the excitation is gone from the cavity-electron system and can no longer contribute to the final axial phase via the continuous Stern-Gerlach effect. Given that the system definitely starts with an excitation in the cavity, and it can only decay into the $|0\rangle$ state from the cavity excited state, the PDF for the time t to decay into the $|0\rangle$ state is

$$p_t(t) = \kappa \varrho_{\gamma\gamma}(t). \quad (27)$$

The extra phase accumulated by the system in the time t while it remains in the excited state is

$$\phi = \Phi(t) = \Delta\omega_z \int_0^t \frac{\varrho_{++}(t')}{\varrho_{++}(t') + \varrho_{\gamma\gamma}(t')} dt'. \quad (28)$$

We can use Eq. (28) to transform $p_t(t)$ into $p_\phi(\phi)$ using the normal rules for a change of variables of a PDF

$$\begin{aligned} p_\phi(\phi) &= p_t(\Phi^{-1}(\phi)) \left| \frac{dt}{d\phi} \right| \\ &= p_t(\Phi^{-1}(\phi)) \frac{\varrho_{++}(\Phi^{-1}(\phi)) + \varrho_{\gamma\gamma}(\Phi^{-1}(\phi))}{\Delta\omega_z \varrho_{++}(\Phi^{-1}(\phi))}, \end{aligned} \quad (29)$$

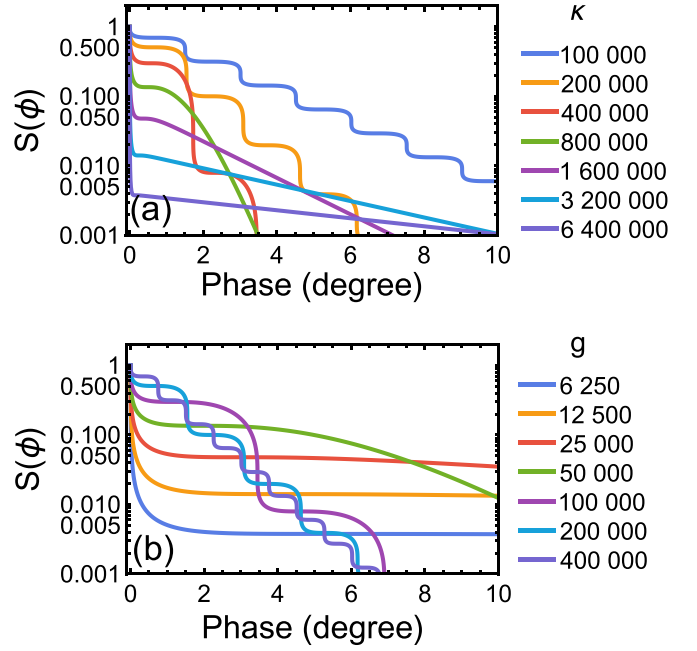


FIG. 4. (a) Survival function $S(\phi)$ for $g = 2 \times 10^5$ and various values for κ . (b) Survival function $S(\phi)$ for $\kappa = 2 \times 10^5$ and various values of g . In both graphs $\omega = 2\pi \times 30$ GHz and $B_2 = 10^5$, other parameters as in Appendix A. The green curves in each graph correspond to the boundary between underdamped cyclotron-cavity population oscillations and overdamped decay.

where $\Phi^{-1}(\phi)$ is the inverse function of $\Phi(t)$. The probability distribution $p_\phi(\phi)$ should be folded into an interval spanning 2π . However, restricting to a single photon in the cavity, a realistic electron single-photon counter rarely results in a phase exceeding π , with a single absorption. Where multiple photons are present, the effects of phase wrapping must be considered.

More informative than the PDF is the survival function $S(\phi) = 1 - C(\phi)$, where $C(\phi)$ is the cumulative density function $C(\phi) = \int_0^\phi p_\phi(\phi') d\phi' = \int_0^{\Phi^{-1}(\phi)} p_t(t') dt'$. The survival function $S(\phi)$ gives the fraction of the distribution that has a phase greater than ϕ . There will typically be a phase uncertainty σ_ϕ associated with measuring the axial phase, which implies that ϕ should exceed some minimum value ϕ_m in order for the phase shift to be unambiguously detected. Hence, we are interested in maximizing $S(\phi_m)$. Figures 4(a) and 4(b) plot $S(\phi)$ for various values of g or κ , respectively. Considering Fig. 4(a), we find that generally, for small phases, $S(\phi)$ is maximized if κ is made as small as possible. At larger phases, it can sometimes be beneficial to choose a larger value of κ , e.g., at $\phi = 4^\circ$, $\kappa = 1.6 \times 10^6$ outperforms $\kappa = 4 \times 10^5$. Figure 4(b) shows that if κ is fixed, the optimum value of g depends on the choice of ϕ_m . If the phase threshold for detection is small, then generally it is beneficial to have $g \gg \kappa$. However, if ϕ_m is large, it may be better to have g of similar size to or smaller than κ . As a general rule, for high efficiency detection, κ should be as small as possible and g should be optimized once the phase resolution associated with the detection of the axial phase is known.

In this section, we have derived a method for calculating the phase shift from a transient occupation of the cyclotron level, and how this depends on g and κ . To calculate whether this phase is resolvable, we need to know the phase noise associated with a measurement of the axial mode. In the next section, we consider the dynamics of the axial and magnetron motion and use these results to derive the limits of axial phase resolution and hence the overall detection probability.

V. AXIAL AND MAGNETRON MOTION

A. Wigner function description of the motion

As we are interested in tracking the motion of an electron that begins in the absolute ground state, we must initially treat the motion quantum mechanically. The interaction Hamiltonians for the excitation, amplification, and coupling pulses are found by using $H_{\text{int}} = e\nabla\hat{\phi}$, with the electrostatic potential $\hat{\phi}$ operator chosen appropriately for electric fields $\mathbf{E}_d = \mathcal{E}_d \cos(\omega_z t) \hat{\mathbf{z}}$ for the excitation pulse and $\mathbf{E}_i = \mathcal{E}_i \cos(\omega_i t)(x\hat{\mathbf{x}} + z\hat{\mathbf{z}})$, where $i = p$, $\omega_p = \eta + \omega_z - \omega_-$ during the amplification, $i = \pi$, $\omega_\pi = \eta + \omega_z + \omega_-$ during the coupling pulse, and η represents a possible frequency detuning. Where operators $\hat{x}$ and $\hat{z}$ appear, these are replaced by functions of the creation and annihilation operators for the axial and magnetron modes using the expressions in Ref. [49]. The Hamiltonians are, for the excitation pulse,

$$\hat{H}_d = \frac{e|\mathcal{E}_d|}{2} \sqrt{\frac{\hbar}{2m\omega_z}} (a_z^\dagger e^{-i\omega_z t} + a_z e^{i\omega_z t}), \quad (30)$$

for the amplification pulse

$$\hat{H}_p = \frac{ie\hbar|\mathcal{E}_p|}{4m\sqrt{(\omega_+ - \omega_-)\omega_z}} (a_- a_z e^{i\omega_p t} - a_-^\dagger a_z^\dagger e^{-i\omega_p t}), \quad (31)$$

and for the coupling pulse

$$\hat{H}_\pi = \frac{ie\hbar|\mathcal{E}_\pi|}{4m\sqrt{(\omega_+ - \omega_-)\omega_z}} (a_- a_z^\dagger e^{i\omega_\pi t} - a_-^\dagger a_z e^{-i\omega_\pi t}). \quad (32)$$

We now describe the evolution of the system under the sequence of pulses described in Sec. III. The change of the state under the excitation pulse can be described using standard unitary operators; however, the amplification and remaining steps in the sequence are best treated using Wigner functions. During our analysis of these steps, we introduce various standard results from Refs. [57,58] for Wigner functions as required.

1. Excitation

The excitation pulse Hamiltonian causes the state to evolve under the displacement operator $\mathcal{D}(\alpha) = e^{-i\hat{H}_d t_d/\hbar}$, where $\alpha = \frac{-ie\mathcal{E}_d t_d}{2} \sqrt{\frac{1}{2\hbar m\omega_z}}$. We label the state of the axial and magnetron modes by their complex coherent state amplitudes α and β for the axial and magnetron modes, respectively, $|\psi\rangle = |\alpha\beta\rangle$, where $a_z|\alpha\beta\rangle = \alpha|\alpha\beta\rangle$ and $a_-|\alpha\beta\rangle = \beta|\alpha\beta\rangle$. We label the coherent axial amplitude state created by the displacement $\mathcal{D}(\alpha_i)|00\rangle = |\alpha_i 0\rangle$ when acting on the particle that is initially in the ground magnetron and axial state.

2. Free evolution

During the free evolution following the axial excitation, the state gains a phase factor $\phi_T = \omega_z t_{\text{ev}} + \Delta\omega_z t_{\text{ex}}$, where $\Delta\omega_z = 0$ if the cyclotron state remains unchanged. The state just before the amplification pulse is therefore $|\alpha' 0\rangle$, where $\alpha' = e^{-i\phi_T} \alpha_i$.

3. Amplification

The dynamics of a system that begins in a thermal or coherent state and undergoes evolution according to the two-mode nondegenerate parametric amplification Hamiltonian $\hat{H}_a$ has been studied extensively by Mollow and Glauber [57,58], and we reproduce much of the following analysis from their work. We define $k = \frac{e\hbar|\mathcal{E}_a|}{4m\sqrt{(\omega_+ - \omega_-)\omega_z}}$ and adjust the initial time to remove the complex prefactor. We note that as the magnetron energy is negative, to match our expression with Refs. [57,58], we would need to set $\omega_a = \omega_z$ and $\omega_b = -\omega_-$, explaining why in our case parametric amplification occurs at $\omega_z - \omega_-$ rather than $\omega_z + \omega_-$. The time evolution is solved in the Heisenberg picture, where the time-dependent creation operators can be expressed in terms of the time-independent creation operators and various hyperbolic functions according to

$$a_z(t) = a_z c_z(t) + a_-^\dagger s_z(t), \quad (33)$$

$$a_-(t) = a_- c_-(t) + a_z^\dagger s_-(t), \quad (34)$$

where

$$c_z(t) = e^{-i\omega_z t} \cosh(kt), \quad (35)$$

$$s_z(t) = ie^{-i\omega_z t} \sinh(kt), \quad (36)$$

$$c_-(t) = e^{i\omega_- t} \cosh(kt), \quad (37)$$

$$s_-(t) = ie^{i\omega_- t} \sinh(kt). \quad (38)$$

Since we want to make a connection between the quantum description of the system and a classical phase space probability distribution, we use the Wigner function to keep track of the state of the system. Although the magnetron energy is negative, this does not prevent a Wigner function description of this mode; the magnetron mode is metastable and spontaneous decay to lower-energy states can be neglected, and whenever the magnetron mode needs to be thermalized or cooled, this occurs via mode coupling to the (positive energy) axial mode or cyclotron modes. The Wigner function for this two-mode system is given by

$$W(\alpha, \beta, t) = \frac{1}{\pi^4} \int e^{\alpha\eta^* + \beta\xi^* - \text{c.c.}} \chi(\eta, \xi, t) d^2\eta d^2\xi, \quad (39)$$

where the characteristic function is

$$\chi(\eta, \xi, t) = \text{Tr}\{\rho(t) e^{\eta a_z^\dagger + \xi a_-^\dagger - \eta^* a_z - \xi^* a_-}\}. \quad (40)$$

For our system, the Wigner function after amplification for a time t_a is given in terms of the functions $\alpha_{0,c}$, $\beta_{0,c}$:

$$\begin{aligned} W(\alpha, \beta, t) &= W(\alpha_{0,c}(\alpha, \beta, t_a), \beta_{0,c}(\alpha, \beta, t_a), 0) \\ &= \left(\frac{2}{\pi}\right)^2 e^{-2|\alpha_{0,c}(t_a) - \alpha'|^2} e^{-2|\beta_{0,c}(t_a)|^2}, \end{aligned} \quad (41)$$

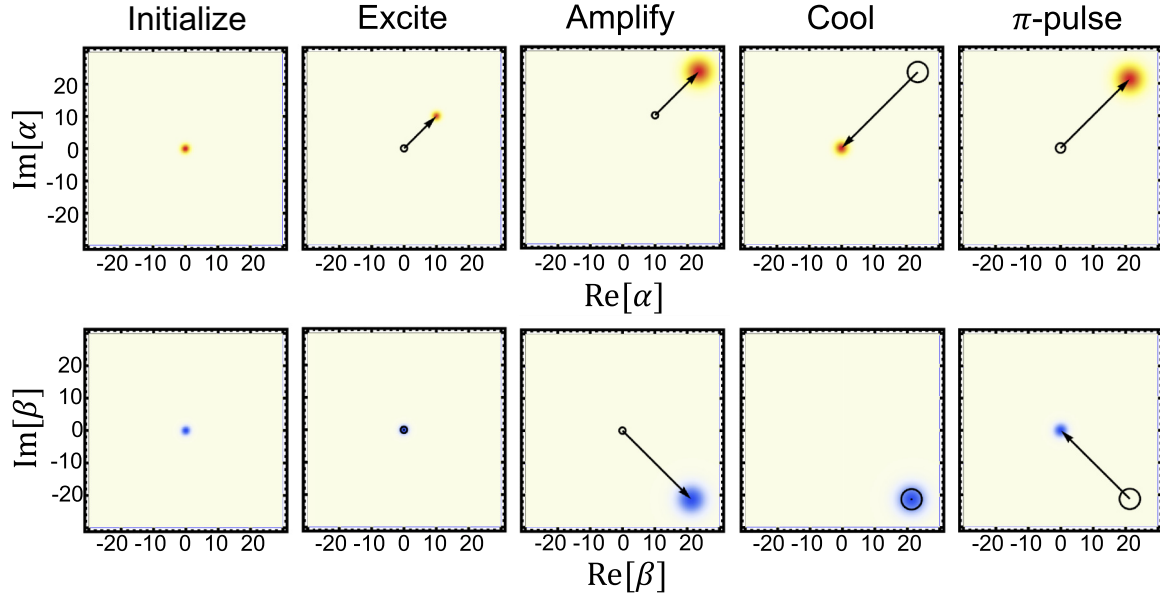


FIG. 5. Wigner function during the photon counting sequence for the reduced density matrix for the axial and magnetron modes, expressed in terms of the real and imaginary parts of the coherent amplitudes α and β , respectively, which appear in the definition of the Wigner function [Eq. (39)]. The black circle shows the standard deviation of the Wigner function at the preceding step for comparison.

where $\alpha_{0,c}(t) = \alpha c_z^*(t) - \beta^* s_-(t)$ and $\beta_{0,c}(t) = \beta c_-^*(t) - \alpha^* s_z(t)$. The Wigner function for the reduced density matrix is plotted in Fig. 5. Here, the top row corresponds to the axial mode (coherent amplitude α) and the bottom row to the magnetron mode (coherent amplitude β). The two-mode Wigner function begins with both modes in their ground state. After the excitation drive, it can be written as the product of two exponentials, which are functions of only α and β , respectively, corresponding to the product of a coherent axial state of amplitude α' and the magnetron ground state. The parametric drive couples together the axial and magnetron modes, as shown in the “Amplify” step in Fig. 5, and prevents the joint Wigner function from being expressed as a separable product of functions that depend on only α and β , respectively. The amplitudes become increasingly correlated as the parametric drive continues. This follows from the fact that the Heisenberg operator $\hat{X}(t) = a_z^\dagger(t)a_z(t) - a_-^\dagger(t)a_-(t)$, representing the amplitude difference, is a constant of motion [57]. Its expectation value can be calculated at $t = 0$,

$$\begin{aligned} \langle a_z^\dagger a_z - a_-^\dagger a_- \rangle &= \iint (|\alpha|^2 - |\beta|^2) W(\alpha, \beta, t) d^2\alpha d^2\beta \\ &= |\alpha'|^2. \end{aligned} \quad (42)$$

As the individual amplitudes grow coherently according to hyperbolic functions, but their amplitude difference remains constant, the fractional amplitude difference becomes closely correlated.

Functions of $\hat{X}$ are also constants of motion. This means the standard deviation of the amplitude difference $(\langle \hat{X}^\dagger(t)\hat{X}(t) \rangle - \langle \hat{X}(t) \rangle^2)^{1/2}$ does not grow and can be calculated at the start of the amplification pulse as

$$\sqrt{\langle \hat{X}^\dagger \hat{X} \rangle - \langle \hat{X} \rangle^2} = |\alpha'|. \quad (43)$$

This helpful correlation will enable the most troublesome frequency shifts imposed by the magnetic bottle during the detection period to be removed.

We note that, despite the amplification, the magnetron radius is small compared to the axial oscillation amplitude after the parametric amplification pulse is completed. The absolute value of the most probable magnetron radius at this point is

$$|\rho_-| = \sqrt{2}\rho_0\bar{\beta} = \sqrt{\frac{2\hbar}{m\omega_z}} \sqrt{\frac{2\omega_-}{\omega_z}} s^2(t_a) |\alpha'|, \quad (44)$$

where $\rho_0 = \sqrt{\frac{2\hbar\omega_-}{m\omega_z^2}}$. This can be compared to the most probable axial amplitude after amplification

$$|z| = \sqrt{2}z_0\bar{\alpha} = \sqrt{\frac{2\hbar}{m\omega_z}} c^2(t_a) |\alpha'|, \quad (45)$$

where $z_0 = \sqrt{\frac{\hbar}{m\omega_z}}$. Note that for large amplifications where $c^2(t_a) \simeq s^2(t_a)$, the magnetron radius remains smaller by a factor $\sqrt{\frac{2\omega_-}{\omega_z}}$, which means that in most geometries, the magnetron mode will be well away from the trap walls.

4. Axial cooling

When the axial resonant circuit is connected in its cooling mode, the effect is to rapidly damp the axial motion, as pictured in Fig. 5. The resulting Wigner function can be expressed as the product of a thermal state for the axial mode, while the magnetron mode is the Wigner distribution associated with the reduced density matrix $\rho_B = \text{Tr}_A\{\rho\}$, where the axial state vectors have been traced over. The resulting Wigner

function after a time t_c when the axial cooling is complete is

$$W(\alpha, \beta) = \frac{2e^{\frac{-|\alpha|^2}{\langle n_z \rangle + \frac{1}{2}}}}{\pi^2(\langle n_z \rangle + \frac{1}{2})} \int e^{-2|\beta - \beta'|^2} P(\beta', \bar{\beta}) d^2\beta'. \quad (46)$$

Here, $\langle n_z \rangle$ is the average axial quantum number after the mode is cooled to T_e . We have left the portion of the Wigner function for the magnetron mode in terms of the P function, which can be used to expand the reduced density matrix in terms of the coherent states of the magnetron mode $|\beta\rangle$ according to

$$\rho_B = \int P(\beta, \bar{\beta}) |\beta\rangle\langle\beta| d^2\beta, \quad (47)$$

where in our case

$$P(\beta, \bar{\beta}) = \frac{1}{\pi s^2(t_a)} \exp\left[-\frac{|\beta - \bar{\beta}|^2}{s^2(t_a)}\right]. \quad (48)$$

Here, $s^2(t) = \sinh^2 kt$, $c^2(t) = \cosh^2 kt$, and $\bar{\beta} = e^{-i\omega_- t_c} (\alpha')^* s_-(t_a)$. Using the P representation emphasizes that the density matrix for the system corresponds to the classical phase space probability distribution for complex axial position $\bar{\beta}(t)$. We will spell this connection out more clearly at the end of the detection sequence.

5. Axial-magnetron π flip

We now use an rf pulse at $\omega = \omega_z + \omega_-$ to exchange the axial and magnetron amplitudes. The basic treatment of Refs. [57,58] can easily be expanded to deal with the Hamiltonian H_π . The solutions for the time-dependent operators under this Hamiltonian are

$$a_z(t) = a_z \tilde{c}_z(t) + a_- \tilde{s}_z(t), \quad (49)$$

$$a_-(t) = a_- \tilde{c}_-(t) + a_z \tilde{s}_-(t), \quad (50)$$

where

$$\tilde{c}_z = e^{-i\omega_z t} \cos(kt), \quad (51)$$

$$\tilde{s}_z = ie^{-i\omega_z t} \sin(kt), \quad (52)$$

$$\tilde{c}_- = e^{i\omega_- t} \cos(kt), \quad (53)$$

$$\tilde{s}_- = ie^{i\omega_- t} \sin(kt). \quad (54)$$

Once again, the time evolution of the Wigner function can be expressed in terms of the time evolution of two time-dependent functions $\alpha_{0,\pi}$ and $\beta_{0,\pi}$ of the complex variables α and β :

$$W(\alpha, \beta, t) = W(\alpha_{0,\pi}(\alpha, \beta, t), \beta_{0,\pi}(\alpha, \beta, t), 0), \quad (55)$$

where in this case the functions are given by $\alpha_{0,\pi} = \alpha \tilde{c}_z^* + \beta \tilde{s}_-$ and $\beta_{0,\pi} = \alpha \tilde{s}_z^* + \beta \tilde{c}_-$.

We can see that, up to some phase factors, this corresponds to the periodic exchange of $\alpha \leftrightarrow \beta$ with a Rabi rate $\Omega = 2k$. In particular, if we set the pulse duration to give a π pulse, $t = \frac{\pi}{2k}$, then if the initial state is $\alpha = \alpha_0$ and $\beta = \beta_0$ the Wigner function becomes

$$W\left(\alpha, \beta, t = \frac{\pi}{2k}\right) = W\left(\beta_0 e^{-i\pi(\frac{\omega_-}{2k} + \frac{1}{2})}, \alpha_0 e^{i\pi(\frac{\omega_-}{2k} - \frac{1}{2})}, 0\right). \quad (56)$$

Using the form of the Wigner function after the axial cooling [Eq. (46)], we have, after the axial-magnetron π pulse,

$$W(\alpha, \beta) = \frac{2e^{\frac{-|\beta|^2}{\langle n_z \rangle + \frac{1}{2}}}}{\pi^2(\langle n_z \rangle + \frac{1}{2})} \int e^{-2|\alpha - \alpha'|^2} P(\alpha', \bar{\alpha}'') d^2\alpha'. \quad (57)$$

We can write out $\bar{\alpha}''$ fully to show that the final phase depends on $\Delta\omega_z$ as required: $\bar{\alpha}'' = \alpha_i^* \exp[-i\phi_f] s_-(t_a)$ with $\phi_f = \pi(\frac{\omega_z}{2k} - \frac{1}{2}) + \omega_- t_c + \omega_z t_{ev} + \Delta\omega_z t_{ex}$

B. Axial detection

Finally, we now close the switch between the resonant circuit and the axial amplifier and detect the axial amplitude. As the magnetron mode is not detected, we can take a partial trace over the states of the magnetron mode. The resulting single-mode Wigner distribution can be written in the P representation, using the expression for P from Eq. (48):

$$W(\alpha) = \frac{2}{\pi} \int e^{-2|\alpha - \alpha'|^2} P(\alpha', \bar{\alpha}'') d^2\alpha'. \quad (58)$$

This Wigner function has a natural interpretation as a classical probability distribution over coherent states, where each coherent state with amplitude α' is given a weight $P(\alpha', \bar{\alpha}'')$. By using Eq. (47), the classical probability density function for the amplitude $|\bar{z}|$ and phase ϕ_z of the electron's complex position $\bar{z} = |\bar{z}|e^{i\phi_z}$ corresponding to Eq. (58) is

$$\mathcal{F}(|\bar{z}|, \phi_z) = \frac{1}{2\pi\sigma_{2,z}^2} \exp\left[-\frac{(|\bar{z}| \cos(\phi_z) - |z_f| \sin(\phi_f))^2 + (|\bar{z}| \sin(\phi_z) - |z_f| \cos(\phi_f))^2}{2\sigma_{2,z}^2}\right]. \quad (59)$$

Here,

$$\sigma_{2,z}^2 = \frac{z_0^2}{2} (c^2(t_a) + s^2(t_a)) \quad (60)$$

and

$$|z_f| = z_0 s(t_a) |\alpha_i| \sqrt{2}. \quad (61)$$

For the remainder of this paper, we use this classical PDF to analyze the detection noise. This gives a good approxi-

mation to all quantities of interest provided that $|z_f| \gg \sigma_{2,z}$. The form of expressions derived from Eq. (59) is not correct if the initial excitation amplitude is small, $|\alpha_i| \lesssim \frac{1}{\sqrt{2}}$; however, this is not the situation in the photon counting application.

The detection system adds Johnson noise, which we incorporate into our analysis classically. When interacting with the detection system, the instantaneous amplitude of the voltage across the impedance $Z(\omega_z)$ of the detection circuit at the

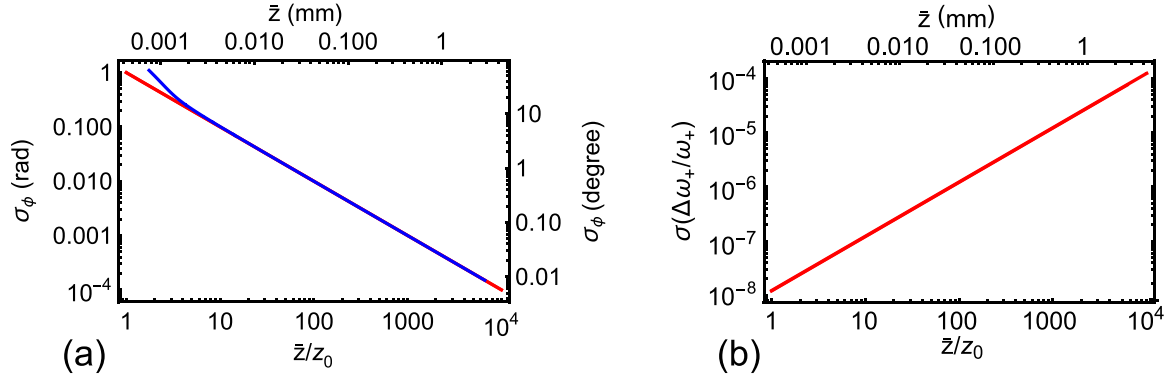


FIG. 6. (a) Standard deviation of measured phases as a function of axial amplitude; blue is calculated numerically from Eq. (66), and red is the approximation for σ_ϕ from Appendix D. (b) Standard deviation of cyclotron shift as a function of axial amplitude, evaluating Eq. (71) using values from Appendix A.

frequency ω_z is

$$\bar{V}_f = \frac{e\omega_z |Z(\omega_z)| |\bar{z}|}{D_{\text{eff},z}}. \quad (62)$$

Here, $D_{\text{eff},z}$ is the effective electrode distance for the axial detection, which can be calculated analytically for a typical trap geometry and is of the same order of magnitude as the trap radius. The resulting signals are typically Fourier transformed prior to detection, with a signal bandwidth $2\pi \times \delta\nu = 1/\tau$ specified by the measurement time τ . As we will later show, the frequency shifts during detection are typically within the bandwidth $\delta\nu$, which means we can neglect them. During detection, the amplitude decays according to $|\bar{z}(t)| = |\bar{z}_2| e^{-\frac{\gamma_z t}{2}}$, where $\bar{z}_2$ is the axial amplitude at the start of the acquisition time, which means that the average signal amplitude after a time τ is

$$\bar{V}_f(\tau) = \frac{2m\omega_z D_{\text{eff},z} |\bar{z}_2| (1 - e^{-\frac{\gamma_z \tau}{2}})}{e\tau}. \quad (63)$$

The axial decay rate due to the particle-detector interaction is

$$\gamma_z = \frac{e^2 |Z(\omega_z)|}{m D_{\text{eff},z}^2}. \quad (64)$$

The rms value of the voltage amplitude of the detector averaged over at time τ is given by

$$V_n = \sqrt{\frac{4k_B T_D \text{Re}[Z(\omega_z)]}{\tau} + \frac{e_n^2}{\tau}}. \quad (65)$$

Here, the first term in the square root is the Johnson noise from the finite detector temperature T_D , and the second term is a possible equivalent input noise e_n intrinsic to the amplifier.

We could, in principle, convert all the axial amplitudes into voltages across the detection circuit and continue our discussion in the voltage/phase space. However, it is more physically meaningful for the discussion of the experimental imperfections in the sections that follow to continue working in terms of particle amplitudes and phases, rather than detector voltages. We therefore use Eq. (63) to convert the detector noise voltage back into an additional fictitious axial amplitude noise. This means we treat the noise sources associated with the detector as an additional, Gaussian distributed vector with random phase and an amplitude chosen such that, when this vector is converted into a voltage using Eq. (62), the resulting voltage noise is V_n . The final distribution including particle and detector fluctuations is a displaced Gaussian

$$\mathcal{F}(|\bar{z}|, \phi_z) = \frac{1}{2\pi\sigma_{3,z}^2} \exp \left[-\frac{(|\bar{z}| \cos(\phi_z) - |z_f| \sin(\phi_f))^2 + (|\bar{z}| \sin(\phi_z) - |z_f| \cos(\phi_f))^2}{2\sigma_{3,z}^2} \right], \quad (66)$$

with

$$\sigma_{3,z}^2 = \sigma_{2,z}^2 + \sigma_D^2, \quad (67)$$

$$\sigma_D^2 = \frac{e^2 \tau (e_n^2 + 4k_B \text{Re}[Z(\omega_z)] T_D)}{4m^2 \omega_z^2 D_{\text{eff},z}^2 (1 - e^{-\frac{\gamma_z \tau}{2}})^2}. \quad (68)$$

This is the final distribution that we work with in the remainder of the paper. Note that, although we derived this expression for a particle cooled to the ground state, essentially the same result follows in the case of a particle with an initial axial temperature T_z and magnetron temperature T_- . In this

case, we replace $\sigma_{2,z}^2 \rightarrow \sigma_{\text{Th}}^2$, where

$$\sigma_{\text{Th}}^2 = \frac{k_B}{m\omega_z^2} \left(T_z c^2(t_a) + \frac{\omega_z}{\omega_-} T_- s^2(t_a) \right). \quad (69)$$

C. Zero-point noise after the initial axial excitation

We consider the axial phase scatter immediately before the amplification pulse is applied. In Fig. 6(a), we plot, in blue, the standard deviation of the axial phase at this time, $\sigma_\phi = \sqrt{\langle \phi^2 \rangle - \langle \phi \rangle^2}$ as a function of the initial axial excitation amplitude $|z_i|$ calculated numerically from Eq. (66). On the lower axis, this is expressed as a ratio of the ground state amplitude $z_0 = \sqrt{\frac{\hbar}{m\omega_z}}$ and on the upper axis this is expressed

in millimeters using numbers from Appendix A. Also plotted in red is the approximation for σ_ϕ from Appendix D, which is

$$\sigma_\phi \simeq \sqrt{\frac{\hbar}{m\omega_z \bar{z}^2}}. \quad (70)$$

This approximation matches the true value when $\bar{z}/z_0 \gg 1$. As expected, increasing the initial axial amplitude results in lower phase uncertainty, so if there were no other considerations, it would be advantageous to use as large a value of $|\bar{z}_i|$ as possible.

The existence of the B_2 term, while crucial for resolving cyclotron phase jumps, also effectively places an upper limit on $|\bar{z}_i|$ and hence the achievable phase resolution. The B_2 term causes the cyclotron frequency to depend on the axial amplitude. The frequency shift is $\Delta\omega_+/\omega_+ = \frac{\bar{z}^2 B_2}{2B_0}$. The direct effect of this shift can be compensated: once the desired excitation amplitude is chosen, the change in cyclotron frequency can be removed by shifting the magnetic field B_0 or making the cavity resonant with the shifted cyclotron frequency. However, there is a residual effect on the cyclotron linewidth. This can be calculated by finding the standard deviation of the frequency shift $\Delta\omega_+/\omega_+$, $\sigma(\Delta\omega_+/\omega_+) = \frac{B_2}{2B_0} \sqrt{\langle \bar{z}^4 \rangle - \langle \bar{z}^2 \rangle \langle \bar{z}^2 \rangle}$, which can be evaluated directly using the equations in Appendix D, when $\bar{z} \gg \bar{z}_0$:

$$\sigma\left(\frac{\Delta\omega_+}{\omega_+}\right) \simeq \frac{B_2 \bar{z} z_0}{\sqrt{2} B_0}. \quad (71)$$

The evaluation of Eq. (71) is plotted in Fig. 6(b). We showed in Sec. IV that the electron-cavity linewidth in our region of interest was around the same as the bare cavity linewidth κ . This means that to avoid broadening the cyclotron resonance beyond this width, the presence of the B_2 term implies a maximum initial axial amplitude

$$\bar{z}_i \simeq \frac{B_0}{nQB_2} \sqrt{\frac{2m\omega_z}{\hbar}}, \quad (72)$$

where n is the ratio between the cyclotron linewidth due to natural decay and the linewidth due to B_2 broadening. We normally set $n = 2$ in the following discussion, which, as the linewidths combine in quadrature, leads to a 10% reduction in absorption probability.

D. Including detector Johnson noise

We now consider the phase uncertainty at the end of the photon counting sequence and incorporate the effect of the detector Johnson noise. The optimum detection time in this situation, considering the form of σ_D from Eq. (68), is $\tau \simeq 2.5/\gamma_z$. At this optimum, the noise associated with detection is

$$\sigma_D^2 \simeq \frac{\frac{e_n^2}{4|Z(\omega_z)|} + \frac{k_B \text{Re}[Z(\omega_z)] T_D}{|Z(\omega_z)|}}{m\omega_z^2}. \quad (73)$$

A parallel LCR circuit will be used for the detection, in which case $\text{Re}[Z(\omega)] = R_p \mathcal{L}(\omega)$ and $|Z(\omega)| = R_p \sqrt{\mathcal{L}(\omega)}$, where $\mathcal{L}(\omega)$ is a Lorentzian function of unit height

$$\mathcal{L}(\omega) = \frac{1}{1 + \frac{4Q_D^2(\omega - \omega_D)^2}{\omega_D^2}}. \quad (74)$$

Here, Q_D is the quality factor of the LCR circuit and ω_D is its resonant frequency. So, σ_D^2 becomes

$$\sigma_D^2 \simeq \frac{\frac{e_n^2}{4R_p \sqrt{\mathcal{L}(\omega)}} + k_B T_D \sqrt{\mathcal{L}(\omega)}}{m\omega_z^2}. \quad (75)$$

If the particle oscillation frequency is close to ω_D , the second term dominates, while if the particle is far from resonance, the first term is more significant. At the optimum detunings of the particle from the resonator,

$$\omega = \omega_D \pm \sqrt{\frac{\omega_D^2 (4k_B T_D R_p - e_n^2)}{4e_n^2 Q_D^2}}, \quad (76)$$

the noise reaches its minimum value and takes on the form

$$\sigma_D^2 \simeq \frac{T_{\text{eff}} k_B}{m\omega_z^2}, \quad (77)$$

where $T_{\text{eff}} = \sqrt{\frac{e_n^2 T_D}{k_B R_p}}$. We have measured GaAs metal-semiconductor field-effect transistor (MESFET) amplifiers with $e_n = 0.7 \text{ nV}/\sqrt{\text{Hz}}$, while copper coils can have $R_p \simeq 300 \text{ k}\Omega$, which makes $T_{\text{eff}} = 0.35 \text{ K}$ at $T_D = 1 \text{ K}$. Put another way, an optimized detection process with these parameters adds an effective noise source to the axial amplitude equivalent to the thermal motion of a particle at 0.35 K. If the measurement had been conducted on resonance, with ω_z matching ω_D , the noise would be fixed at $\sigma_D^2 = \frac{T_D k_B}{m\omega_z^2}$, almost three times as high, and could only be further reduced by cooling the detection system. This is challenging as the detectors all dissipate some power and so cannot typically operate below 1 K. Detecting slightly off resonance gives us more scope to reduce this noise contribution by increasing R_p further using superconducting resonant coils or decreasing e_n by switching from GaAs MESFETs to SQUIDS, in which case the benefits of measuring off resonance would be more pronounced.

While the foregoing analysis shows how the noise contribution of the detection system can be minimized, it still remains unacceptably high. To further suppress this noise, it is necessary to amplify the axial and magnetron modes to satisfy the condition

$$\sigma_{2,z}^2 \gg \sigma_D^2. \quad (78)$$

If we let $\sinh(kt_a) = 27$, which amounts to an amplification of the axial amplitude $|\bar{z}_i|$ by a factor 27, then using $\sigma_D^2 = \frac{T_{\text{eff}} k_B}{m\omega_z^2}$ and $T_{\text{eff}} = 0.35 \text{ K}$, we have $\sigma_{2,z}^2 = 10\sigma_D^2$. The resulting amplified distribution corresponds closely to Eq. (59), with $\sigma_{z,2}$ increased by 5%. Since the amplification is large, $c^2(t) \simeq s^2(t)$. In this case, the moments calculated from the amplified distribution become almost the same as those calculated from the distribution before amplification. The only difference is that σ is increased by a factor of $\sqrt{2} \times 1.05$, the factor $\sqrt{2}$ from the extra noise associated with the additional zero-point energy from the second mode, and the factor 1.05 from the residual influence of σ_D .

The amplification pulse places an additional constraint on $\bar{z}_i$, namely, that $|\bar{z}_f| = 27|\bar{z}_i| < |\bar{z}_{\text{max}}|$, where $\bar{z}_{\text{max}}$ is the maximum feasible axial amplitude that can be used in the trap, limited by the trap geometry. Taking into account

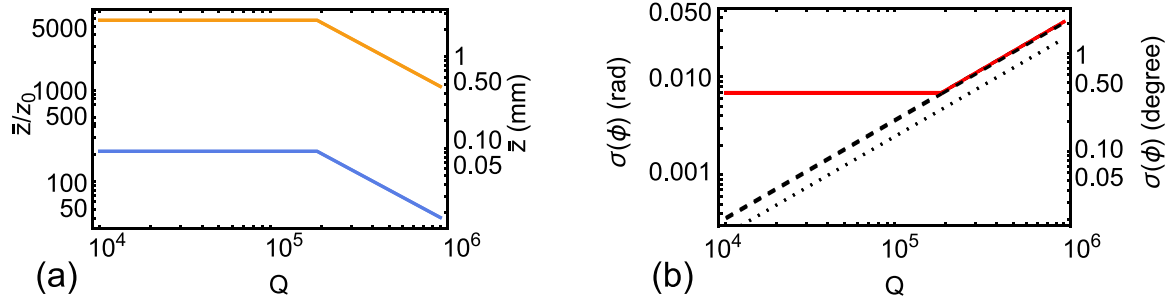


FIG. 7. (a) Initial axial amplitude $|\bar{z}_i|$ from Eq. (79) in blue, and final axial amplitude $|\bar{z}_i|s(t_a)$, assuming $|s(t_a)| = 27$, in orange. (b) σ_ϕ based on $|\bar{z}_i|$ from Eq. (79), plotted in red. Also denoted by a dashed line is σ_ϕ , if there were no limit on $\bar{z}_{\max}$. A dotted line plots σ_ϕ for the hypothetical case where the detector noise is negligible and therefore no amplification is needed.

the constraints from both B_2 and the amplification, the initial axial amplitude can therefore be no greater than

$$|\bar{z}_i| = \sqrt{2z_0}|\alpha_i| = \min\left(\frac{\bar{z}_{\max}}{|s(t_a)|}, \frac{B_0}{nQB_2}\sqrt{\frac{2m\omega_z}{\hbar}}\right). \quad (79)$$

The associated phase uncertainty including the increased noise caused by amplification is then given, to a good approximation, by

$$\sigma_\phi \simeq \sqrt{\frac{2\hbar}{m\omega_z\bar{z}_i^2}}. \quad (80)$$

A plot of $|\bar{z}_i|$ and $|\bar{z}_i|s(t_a)$ is shown in Fig. 7(a) in blue and orange, respectively, as a function of the trap Q factor. In this case, we have assumed $\bar{z}_{\max} = 2.5$ mm. At Q factors lower than $Q = 10^5$, where the cyclotron linewidth of the trap is comparatively broad, $|\bar{z}_i|$ is limited by the condition $\bar{z}_f < \bar{z}_{\max}$. Above $Q = 10^5$, the overriding limitation is that $\bar{z}_i$ should not lead to excess broadening by B_2 , which places a more stringent limit on the initial amplitude.

Figure 7(b) plots the phase uncertainty σ_ϕ found using the final axial amplitude after amplification, in red. This uncertainty shows a dependence on Q that is the inverse of the blue trend shown in Fig. 7(a). Also shown as a black dashed line is the phase uncertainty that would be achievable if there were no $\bar{z}_{\max}$ limit. The dotted line corresponds to σ_ϕ when σ_D is so small that the amplification stage can be dispensed with, and there is no $\bar{z}_{\max}$ limit. This dotted line is a factor of $\sqrt{2}$ lower than the dashed line. We note in passing that this figure also shows the important difference between the pulse and phase (PnP) [50] and pulse and amplify (PnA) [59] methods often used in Penning trap experiments. If detector noise σ_D is negligible compared to the particle temperature noise, the phase noise achievable with PnP is a factor of $\sqrt{2}$ lower than PnA. The PnA method allows signals to be amplified to the level where σ_D can be neglected, at the cost of increasing the noise by a factor of $\sqrt{2}$, if both modes used in the PnA amplification stage are thermalized to the same average quantum state prior to the measurement.

In this section, we have considered the phase noise we would expect in a simple realization of the photon counting experiment, and we have explored the limits imposed by B_2 and the need to amplify the signals above the detector Johnson

noise. In the next section, we consider additional contributions to phase uncertainty imparted by further frequency shifts.

E. Frequency shifts

Electrostatic and magnetic imperfections, the magnetic bottle term B_2 , and the relativistic mass increase all lead to frequency shifts that could potentially disrupt the photon detection method. We have already discussed how the B_2 term, in combination with axial amplitude noise, places limits on the initial excitation amplitude. In this section, we review the effect of other frequency shifts in detail. The form of the first-order shifts is described in Ref. [49], while higher-order shifts are summarized in, e.g., Refs. [60–62]. The pertinent points to note are as follows:

(1) The complete form of the electrostatic potential V and the magnetic scalar potential Ψ can be expanded in cylindrical coordinates r and ϕ using Legendre polynomials $P_k(\cos(\phi))$ from the center of the trap according to

$$V = \frac{1}{2}V_0 \sum_{k=0}^{\infty} \left(\frac{r}{d}\right)^k C_k P_k(\cos(\phi)), \quad (81)$$

$$\Psi = - \sum_{k=1}^{\infty} k^{-1} B_{k-1} r^k P_k(\cos(\phi)), \quad (82)$$

where d and V_0 have been defined previously. The coefficients C_k and B_k for $k > 2$ are unwanted imperfections, while C_2 is the term responsible for axial trapping and B_2 is the deliberately introduced bottle.

(2) The C_n coefficients are dimensionless and depend only on the trap geometry, while the B_n coefficients have units of $T\,m^{-n}$ and so depend on both the geometry and material properties through the magnetic field strength and magnetization.

(3) The axial reflection symmetry means that only even powers need to be considered in these sums, so we neglect odd k from now on.

(4) In general, each term C_k for $k > 2$ and $B_{k'}$ for $k' > 0$ leads to separate shifts of ω_+ , ω_z , and ω_- , where each shift depends on the total energy in the axial, magnetron, and modified cyclotron modes.

(5) Any increase in the particle energy in any mode leads to some increase in speed that causes a relativistic mass increase. This shift is most significant when the axial and

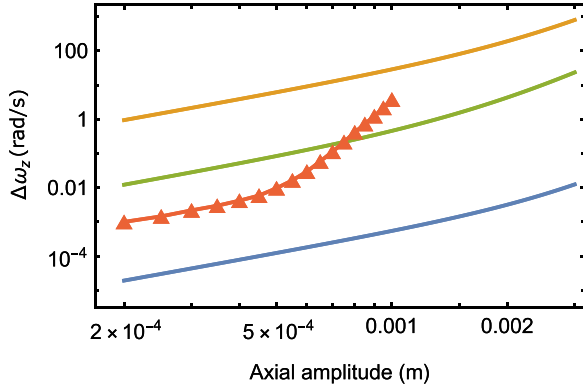


FIG. 8. Axial frequency shift due to electrostatic imperfections in a 3.7 mm radius 7-pole trap. Blue line represents an upper limit from an ideal trap, green a trap with 5 μm manufacturing tolerances, and yellow 15 μm tolerances. Red points are from Ref. [63], scaled to an axial frequency of 100 MHz.

modified cyclotron energy change and has the biggest impact on the axial and modified cyclotron frequencies.

In a compensated 7-pole trap, it should be possible to set all coefficients C_k with $k > 2$ to zero up to $k = 10$ by controlling five trap voltages and grounding two end electrodes. In a realistic trap, manufacturing tolerances mean that the cancellation may not be perfect and residual shifts from these terms may persist. Figure 8 shows the typical range of axial frequency shifts that arise from electrostatic and relativistic perturbations in a 7-pole trap as the axial amplitude is varied. The blue line corresponds to our simulation of an optimized 7-pole design for a 3.7 mm radius trap; it is slightly nonzero because of the time limit placed on the optimization. The green and orange lines represent a trap with manufacturing tolerances of 5 and 15 μm , respectively. The red data correspond to an actual 7-pole trap [63], scaled to an axial frequency of 100 MHz. We use the data from Ref. [63] as a reasonable estimate for what can be achieved in a Penning trap. From these data, we see there is a residual shift $\Delta\omega_z = c_E |z|^2$, $c_E = 2.4 \times 10^4 \text{ rad s}^{-1} \text{ m}^{-2}$ expected in our trap. We can attribute this to a residual C_4 contribution $c_E = \frac{3\omega_z C_4}{4d^2 C_2}$.

Similarly to the electrostatic imperfections, an ideal magnetic bottle produces a pure B_2 field, while a realistic bottle also produces an unwanted B_4 contribution and higher-order terms. Our numerical calculations indicate that without much care, values of $B_4 \approx 10^9 \text{ T/m}^4$ are typical. With some careful design of the location of magnetic material and potentially some shimming, simulations show it should be possible to limit $B_4 < 10^7 \text{ T/m}^4$. We now discuss the unwanted effects that B_2 , B_4 , and C_4 have on the detection process. Throughout, we are primarily concerned with the variance in the frequency shifts, which exists because of the axial and magnetron zero-point motion. This frequency noise can lead to phase noise in the final phase we wish to measure.

1. Shifts at the point of axial excitation

The zero-point fluctuations in the axial motion in the ground state mean that, after excitation, there is an inevitable spread in the axial energy

$$\sigma(E_z) = \frac{\hbar\omega_z \bar{z}_i}{\sqrt{2}z_0}. \quad (83)$$

Inserting this energy variance into the expressions for the shifts caused by $c_E \propto C_4$ and B_4 (the B_2 term does not cause an axial energy-dependent axial frequency shift), we can determine the associated frequency noise using the parameters in Appendix A. To relate this to a worst-case phase noise at the end of the counting sequence, we assume a maximum integration time of 100 s. The resulting phase noise is plotted in Fig. 9 in blue and orange. The phase noise arising from the axial zero-point fluctuations and the finite size of $\bar{z}_i$ is also plotted as a black dashed line—the same as the red line in Fig. 7(b). It can be seen that the phase noise from the amplitude noise of initial axial excitation is far smaller than the phase scatter associated with the axial zero-point noise and can be neglected.

2. Frequency shifts during amplification, axial cooling, and axial-magnetron π flip

The axial amplitude grows significantly during amplification, and the magnetron radius is boosted from its ground

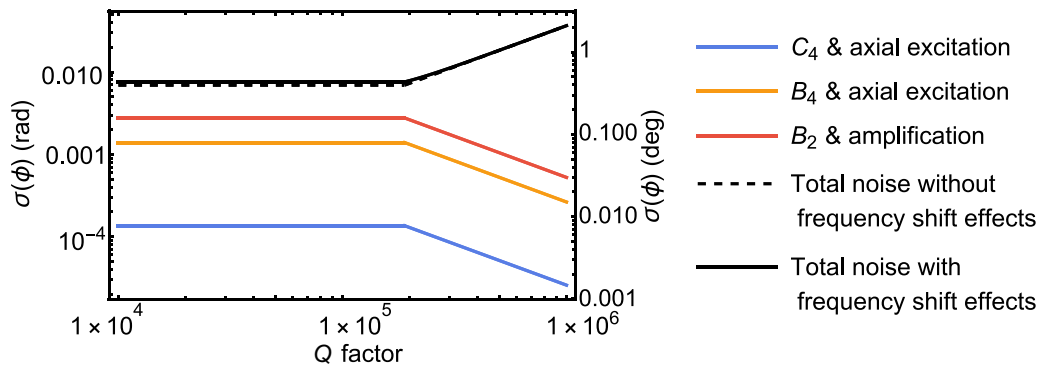


FIG. 9. Various sources of phase noise, as a function of the cavity Q factor for the parameters listed in Appendix A. The blue line shows the noise contribution from C_4 and the initial axial amplitude noise, the orange line shows the noise contribution from B_4 and the initial axial amplitude noise, and the red line shows the noise contribution from B_2 and the axial and magnetron variance created during the amplification stage discussed in Sec. VE 2. Also plotted is the expected phase noise, ignoring the additional noise sources discussed in this section (black dashed), and including these noise sources (black solid).

state. Both effects impart additional phase shifts onto the final phase detected after the π flip. The electrostatic effects arise due to the variance in axial and magnetron amplitude, in combination with C_4 , which leads to a variance in the magnetron frequency after excitation. As the axial cooling takes a finite time and there is a deliberate delay between the start of the axial cooling and the π flip, this frequency variance translates into phase noise. Fortunately, for the parameters in Appendix A, these effects lead to negligible additional phase noise. More troublesome is the equivalent magnetostatic effect, where the variance in axial and magnetron amplitudes after amplification combines with the large B_2 term to produce additional noise. The magnetron frequency shift after the amplification is given by [49]

$$\Delta\omega_- = \Delta\omega_{-,z} + \Delta\omega_{-,-}, \quad (84)$$

$$\Delta\omega_{-,z} = -\frac{\omega_z^2 B_2}{4\omega_+ B_0} \bar{z}^2, \quad (85)$$

$$\Delta\omega_{-,-} = \frac{\omega_z^2 B_2}{4\omega_+ B_0} \rho_-^2. \quad (86)$$

Notice how an increase in the magnetron radius leads to a shift of the opposite sign to the shift when the axial amplitude increases. This is as expected, since the shift is driven by the quadrupolar B_2 term, which has opposite curvature in the axial and radial directions. We saw above that, after amplification, the axial and magnetron amplitudes are strongly correlated. We can use this correlation, in combination with a carefully chosen wait time, to almost completely cancel this phase shift and reduce its variance to an absolute minimum.

We assume for simplicity that the amplification time t_a is much shorter than the inverse of the energy damping constant γ_{fast} in the detector's rapid cooling mode. In this case, the total magnetron phase shift imparted by the axial excitation after the axial amplitude has been completely cooled is $\Delta\omega_{-,z}/\gamma_{\text{fast}}$. We choose the waiting time between the end of the parametric amplification pulse and the π pulse to be equal to $t_{\text{wait}} = \frac{\omega_z}{2\omega_+ \gamma_{\text{fast}}}$, so that at the moment of the π pulse the magnetron shift caused by the magnetron amplitude is $\Delta\omega_{-,-} t_{\text{wait}}$. The total magnetron phase shift after amplification due to the finite cooling time is then

$$\Delta\phi_- = \frac{B_2 \omega_z^2}{4B_0 \gamma_{\text{fast}} \omega_+} \left(\frac{\rho_-^2}{\frac{2\omega_-}{\omega_z}} - \bar{z}^2 \right). \quad (87)$$

This can be rewritten in terms of the amplitudes at the end of the parametric amplification α_f and β_f :

$$\Delta\phi_- = -\frac{\omega_z^2 B_2}{2\gamma_{\text{fast}} \omega_+ B_0} \frac{\hbar}{m\omega_z} (|\alpha_f|^2 - |\beta_f|^2). \quad (88)$$

The standard deviation of this phase shift can be immediately calculated using Eq. (43):

$$\sigma(\Delta\phi_-) = -\frac{\hbar|\alpha'| \omega_z B_2}{2m\gamma_{\text{fast}} \omega_+ B_0}. \quad (89)$$

This noise is plotted in Fig. 9 in red. Although this is the largest shift, its size is manageable for the values of $|\bar{z}_i|$ we envision using, and it does not add a significant amount of extra noise.

3. Frequency shifts during detection

During the detection period, the axial amplitude decreases as the particle is cooled by the detection system. If the axial frequency changes dramatically during the cooling period, it is more challenging for the required phase information to be recovered, and this may have an impact on the observed phase noise. In the discussion of the phase noise during the detection period, we considered only small frequency shifts during the detection. The most stringent requirement for this to be valid is that the frequency drifts by no more than one frequency bin of the Fourier transformed image current signal. As the optimum acquisition time is $\tau \simeq 2.5/\gamma_z$, this means that the drifts in frequency should be no more than $\Delta\omega_z = \gamma_z/2.5$. Using $R_p = 300 \text{ k}\Omega$ and $D_{\text{eff},z} = 12.75 \text{ mm}$, we find $\gamma_z = 2\pi \times 8.4 \text{ Hz}$. A typical final maximum axial amplitude after amplification will be around 2.5 mm. For the electrostatic imperfections, again using the measured trap data from Ref. [63], this corresponds to a frequency shift of 1.8 Hz, well within the detection bandwidth $1/\tau$.

F. Other experimental imperfections

1. Voltage noise

Fluctuations $\sigma(\omega_z)$ in the trapping frequency ω_z during the axial free evolution period t_{ev} due to voltage fluctuations can lead to phase noise $\sigma(\phi) = \sigma(\omega_z) t_{\text{ev}}$. For our parameters, we need the fractional voltage noise $\sigma(\frac{\Delta V_0}{V})$ to be controlled better than $\sigma(\frac{\Delta V_0}{V}) < \max(Q, 10^5) \times 1.6 \times 10^{-16}$ per second of evolution time t_{ev} . The best axial stability in a Penning trap has been achieved using three stacked Stahl Electronics UM-14 supplies and a 2000 s time constant low-pass filter [64]. The long-term drift in this experiment is $\sigma(\frac{\Delta V_0}{V}) = t \times 5 \times 10^{-12}$, which would allow a free evolution time of around 1 s. Higher voltage stability with shorter filter time constants can be achieved with a programmable Josephson voltage standard [65], which should allow longer averaging times to be reached. A complementary solution is to connect the voltage source to a second trap with a second electron and use measurements in this auxiliary trap to correct for voltage drifts [66].

2. Heating rates

Heating of the modified cyclotron, axial, and magnetron modes needs to be carefully controlled in the experiment. The thermal microwave photons in the cavity are a background noise source that adds dark counts. The cavity occupation number is

$$\bar{n} = \frac{1}{\exp\left(\frac{\hbar\omega_+}{k_B T_e}\right) - 1}. \quad (90)$$

For a critically coupled cavity, this cavity occupation number is equivalent to an incoming photon rate $R_a = \frac{1}{2} \kappa \bar{n}$. For $Q > 10^4$, $T < 70 \text{ mK}$, and $\omega_+ > 2\pi \times 30 \text{ GHz}$, the dark count rate is always less than 0.01 counts/s. As we will see later, a count rate from axion conversion of $R_a > 0.1 \text{ counts/s}$ is needed for a feasible axion search experiment, much larger than the dark count rate. Hence, cyclotron dark counts do not

contribute meaningfully to the noise compared to other noise sources.

The experiment is more robust with respect to axial or magnetron heating than it is to cyclotron heating. Although a full description of this is beyond the scope of this paper, we expect that any heating that occurs will add phase noise rather than destroying the measurement completely. Starting with the axial mode, we note that, except when cooling is desired, the detection system is far detuned from the axial resonance. The requirement is that $\gamma_z \ll 1/t_{\text{ev}}$, which can be achieved by switching a capacitance parallel to the LCR circuit. To estimate the axial heating rates, we can use the cyclotron heating rates achieved in other cryogenic Penning trap experiments and scale them to our trapping frequencies and particle mass. Reference [67] demonstrated a cryogenic Penning trap with an electric field spectral density of $S_E \leq 7.5^{+3.4}_{-2.8} \times 10^{-20} \text{ V m}^{-2} \text{ Hz}^{-2}$ at a frequency of $\omega_+ = 2\pi \times 17.845 \text{ MHz}$. This electric field density would lead to an axial transition rate out of the ground state for the electron at $\omega_z = 2\pi \times 100 \text{ MHz}$,

$$\zeta_z = \frac{q^2}{4m\hbar\omega_z} S_E, \quad (91)$$

of $\zeta_z \lesssim 0.01 \text{ s}^{-1}$, which would allow evolution times up to 100 s before, on average, a photon absorption event occurs. We might also expect that S_E might scale favorably in our proposed trap compared to Ref. [67]. The precise scaling of S_E with trap dimensions d , frequency ω , and temperature T is complicated and shows large experiment-to-experiment variation [68]. However, using a scaling $S_E \propto \omega^{-0.6} d^{-2} T^{1/2}$, which appears to be empirically justified for a cryogenic trap [68], we might expect the electric field density to be further reduced to $S_E \simeq 1 \times 10^{-21} \text{ V m}^{-2} \text{ Hz}^{-2}$, which would lead to $\zeta_z \lesssim 0.6$ per hour, which is negligible.

The magnetron mode, since it is at a much lower frequency, is more susceptible to noise. Repeating the transition rate calculation for this mode yields $\zeta_- \leq 7 \text{ s}^{-1}$ for unscaled S_E and $\zeta_- \leq 4 \text{ s}^{-1}$ assuming that S_E varies as expected with frequency, temperature, and distance. These rates would be somewhat limiting for the experiment's free evolution, so to match the limit imposed by voltage noise, further reduction of magnetron heating rates by at least an order of magnitude would be necessary. This could potentially be achieved by using notch filters tuned specifically to the magnetron frequency on the lines providing the trap voltages.

VI. CALCULATING THE DETECTION EFFICIENCY

A. Figure of merit

In this section, we calculate the performance of the single-photon counter, in other words, how well we can infer the photon state of the cavity from the measured axial phase, given that this phase is drawn from a distribution.

Before conducting detailed calculations using the full form of the phase distributions, it is interesting to consider a simple proxy for the performance of the detector. This is $\frac{\langle \phi \rangle}{\sigma_\phi}$, the average mean phase shift when a photon is present in the cavity, divided by the standard deviation of the axial phase. Maximizing this figure of merit would be expected to maximize

the detection efficiency. For large initial excitation amplitudes, and where the parametric amplification is sufficiently large that the thermal detection noise is negligible, the standard deviation of the final axial phase is approximately

$$\sigma_\phi \simeq \frac{\hbar B_2 n Q}{B_0 m \omega_z}, \quad (92)$$

hence, using the previously calculated value for $\langle \phi \rangle$:

$$\frac{\langle \phi \rangle}{\sigma_\phi} = \frac{1}{n} \kappa \langle t_{\text{ex}} \rangle = \frac{1}{n}. \quad (93)$$

Surprisingly, this calculation suggests that almost all the choices of trap parameters do not have a fundamental influence on the ability of the device to resolve phase shifts and can be chosen to allow other unwanted frequency shifts to be manageable. Recall that n is the ratio of the cavity absorption linewidth to magnetic broadening and cannot be reduced without compromising the absorption probability—In the following sections, we choose $n = 2$.

The more sophisticated treatment we now perform indicates that the expected performance of the device *does* depend on parameters such as g and κ . These parameters influence the shape of the phase distribution, and the performance of the device depends more critically on the proportion of the distribution shifted to large phases above a given cutoff related to σ_ϕ , than it does on the ratio $\frac{\langle \phi \rangle}{\sigma_\phi}$.

To characterize the performance of the detector, we introduce two different efficiencies, the detection efficiency η_d and exclusion efficiency η_e . The detection efficiency is the expected fraction of runs where the photon counter detects a microwave single photon if it is present in the cavity—This is the same as the photon detection efficiency of a traditional optical photon counter, and for an ideal device $\eta_d = 1$. For axion experiments, we expect that most of the time there will be no photons in the cavity. Therefore, it makes sense to also characterize how much longer the device needs to operate in order to exclude the presence of any photons in the cavity, compared to an ideal photon counter. The exclusion efficiency is the inverse of this extra waiting time. We now determine these efficiencies.

B. Detection efficiency

The detection efficiency η_d is defined as the probability that a photon detection event is registered, if a photon is initially present in the cavity: $\eta_d = P(\text{detection} | \text{photon present in cavity})$. In a detector with readout noise, such as ours, defining the detection efficiency first requires establishing the significance level α_s with which we want to reject the hypothesis that the count was actually caused by readout noise and not an incoming photon. In our electron single-photon counter, the axial phase is the single number reported at the end of a photon counting sequence, referenced to the average axial phase recorded when no photons are incident on the cavity. We assume that this reference phase is stable and known sufficiently well from preliminary measurements that it contributes no additional uncertainty. When there are no incident photons, the distribution of axial phase differences from this mean is a

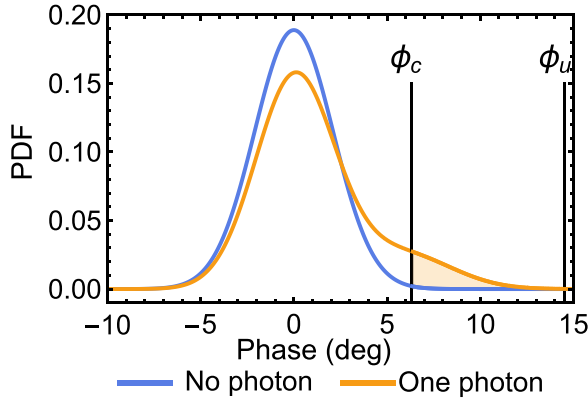


FIG. 10. An illustration of the portion of $h_1(\phi)$ (plotted in yellow) falling between ϕ_c and ϕ_u , which is used to define the detection efficiency η_d . The false positive rate is found by integrating $h_0(\phi)$ (plotted in blue) between the same limits. For this plot, $Q = 10^6$ and $\sigma_c = 3\sigma_\phi$; other parameters are as in Table II.

Gaussian $h_0(\phi)$, with zero mean and a standard deviation σ_ϕ given by Eq. (80).

If a photon is present in the cavity, the distribution of final phases $h_1(\phi)$ is given by convolving the Gaussian noise distribution $h_0(\phi)$ with $p_\phi(\phi)$, the probability density function for the axial phase advance found in Sec. IV,

$$h_1(\phi) = (h_0 * p_\phi)(\phi) = \int_{-\pi}^{\pi} h_0(\phi - \phi') p_\phi(\phi') d\phi'. \quad (94)$$

The $*$ notation is a standard way of denoting the convolution of two functions. An example of the distributions $h_0(\phi)$ and $h_1(\phi)$ is shown in Fig. 10; here, $h_1(\phi)$ is the PDF when one photon is present at some point in the cavity and $h_0(\phi)$ when no photons are present. We define two phases ϕ_u and ϕ_c , also shown in Fig. 10, with $\phi_c \ll \phi_u \ll \pi$ such that if the measured phase ϕ satisfies $\phi_c < \phi < \phi_u$ after a photon counting sequence, we consider that a photon has been detected. The value of ϕ_c is found by requiring the probability of a false positive to be α_s , which, considering that $h_0(\phi)$ is Gaussian distributed and so long as $\sigma_\phi \ll \pi$ is approximately given by

$$\phi_c = \sqrt{2}\sigma_\phi \text{erf}^{-1}(1 - 2\alpha_s), \quad (95)$$

where erf^{-1} is the inverse error function. The detection probability η_d is then simply given by

$$\eta_d = \int_{\phi_c}^{\phi_u} h_1(\phi) d\phi. \quad (96)$$

An illustration of the integrated portion of $h_1(\phi)$ between ϕ_c and ϕ_u is shown in Fig. 10.

In general, $p_\phi(\phi)$ is found by numerically integrating the equations for the cavity occupation as described in Sec. IV. In the case that $g \ll \kappa$, it is possible to use an approximate form $p_\phi(\phi) \simeq \tilde{p}_\phi(\phi)$ valid for $\phi \geq 0$ and hence provide an analytical expression for η_d . When $g \ll \kappa$, we model $p_\phi(\phi)$ as the sum of a delta function distribution centered at $\phi = 0$ and a single-sided exponential distribution

$$\tilde{p}_\phi(\phi) = \begin{cases} (1 - P_e)\delta(\phi) + P_e\lambda e^{-\lambda\phi}, & \phi \geq 0, \\ 0, & \phi < 0, \end{cases} \quad (97)$$

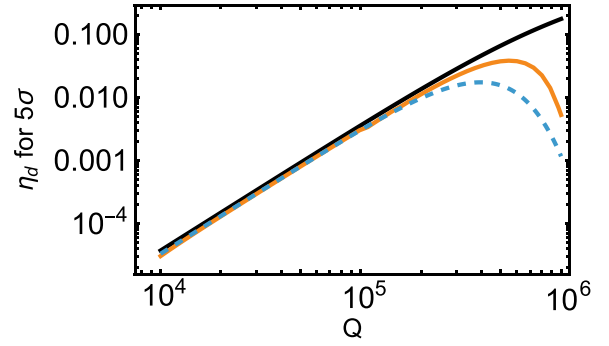


FIG. 11. The detection efficiency for a 5-sigma CL, plotted in orange. Also shown in black is the maximum possible detection efficiency given by Eq. (25). The blue dashed lines show the detection efficiency calculated from the analytical approximation to the phase distribution function $\tilde{h}_1(\phi)$.

with $\lambda = \frac{4g^2}{\kappa} \frac{m\omega_c B_0}{\hbar\omega_+ B_2}$, and under the assumption $\lambda^{-1} \ll \pi$ as is the case for the situations we consider. This probability distribution has a simple physical interpretation: In the weak coupling limit $g \ll \kappa$, a photon is either absorbed by the electron, with probability P_e or is not absorbed, with probability $1 - P_e$. If there is no absorption, then no additional phase is imparted, while if the absorption occurs, the excited state lifetime follows an exponential decay with a time constant $\tau = \frac{\kappa}{4g^2}$. Relating the time in the excited state to the phase accrued leads to the expression for λ . The convolution of $\tilde{p}_\phi(\phi)$ with $h_0(\phi)$ can also be computed analytically, with the result

$$\tilde{h}_1(\phi) = (1 - P_e) \frac{e^{-\frac{\phi^2}{2\sigma_\phi^2}}}{\sqrt{2\pi}\sigma_\phi} + \frac{1}{2}\lambda P_e e^{\frac{1}{2}\lambda(\lambda\sigma_\phi^2 - 2\phi)} \left(\text{erf}\left(\frac{\phi - \lambda\sigma_\phi^2}{\sqrt{2}\sigma_\phi}\right) + 1 \right). \quad (98)$$

This approximate form allows for a rapid calculation of η_d in the weak coupling case, but breaks down when $g \simeq \kappa$. The results of the full calculation of $h_1(\phi)$ and the approximation $\tilde{h}_1(\phi)$ are plotted in Fig. 11. Both these efficiencies have been multiplied by $(1 + n^{-2})^{-1/2} = \sqrt{4/5}$ to account for the loss in efficiency due to B_2 induced broadening. Here, Fig. 11(a) shows η_d requiring a 5-sigma significance for detection, ($\alpha_s \simeq 2.9 \times 10^{-7}$). The dashed blue line represents $\tilde{h}_1(\phi)$, while the solid orange line shows the full form of $h_1(\phi)$ without approximation. Also shown in black is P_e , which represents the theoretical maximum efficiency if every photon absorption led to a phase advance in the photon counter of sufficient size that it could unambiguously be attributed to an absorption event. The performance of the counter tracks this line at low Q , but deviates at higher Q . This is the region where the axial phase noise leads to a loss in detection efficiency. The decrease in efficiency once Q passes an optimum value can be attributed to the reduction in excited state lifetime due to enhanced spontaneous emission rate, a lower excited state lifetime outweighing the benefit of a higher absorption probability. Figure 12 plots η_d calculated using $h_1(\phi)$ for 3- and 5-sigma significance as a function of κ and g , where the

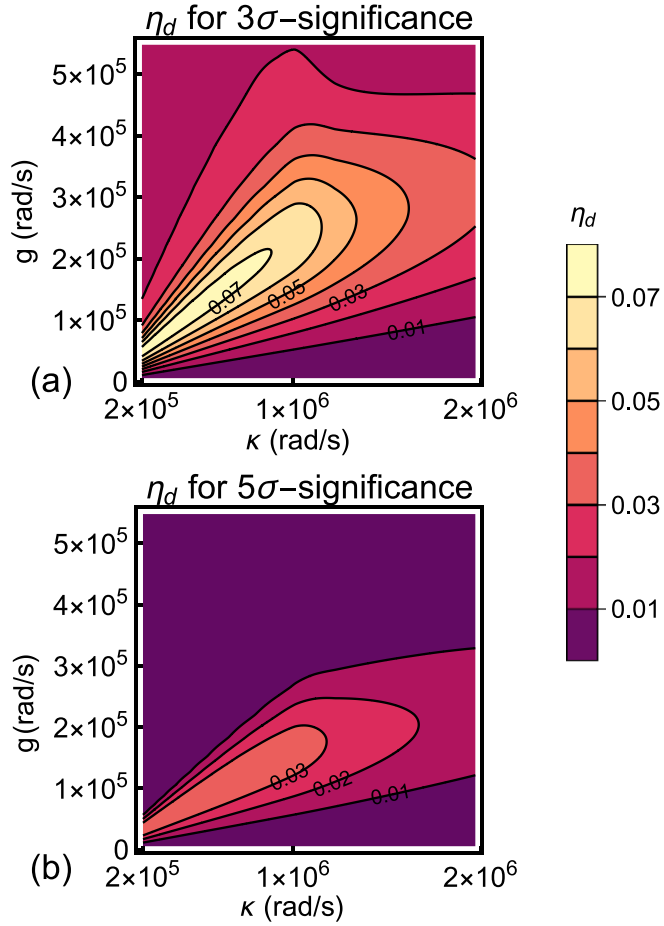


FIG. 12. Detection efficiency η_d calculated from $h_1(\phi)$ as a function of κ and g . Panel (a) is for 3-sigma significance and panel (b) is for 5-sigma significance.

plotted η_d is again multiplied by $\sqrt{4/5}$. These plots show that the efficiency peaks in a diagonal region at around $g = \kappa/4$ for the 3-sigma confidence and $g = \kappa/6$ for 5-sigma confidence at lower values of g and κ , but then declines at higher values of both parameters. This decline is caused by the lower limit on σ_ϕ set by the finite trap size as plotted in Fig. 9. At the optimum values of g and κ , the maximum detection efficiency is $\eta_d = 7.4\%$ for 3-sigma and $\eta_d = 3.8\%$ for 5-sigma confidence, respectively.

C. Exclusion efficiency

We now turn to the exclusion efficiency η_e , which captures how well the device performs at excluding the presence of photons in the cavity. We define the exclusion efficiency $\eta_e(C_L) = \frac{t_{\text{ideal}}(C_L)}{t_{\text{real}}(C_L)}$. Here, $t_{\text{ideal}}(C_L)$ is the waiting time required for an ideal single-photon counter to exclude an incident photon rate R_a with a certain confidence level C_L and $t_{\text{real}}(C_L)$ is the waiting time for the electron photon counter we have been considering. The inverse η_e^{-1} tells us, on average, how much longer we would have to wait to exclude a certain count rate using the electron single-photon counter, compared to an ideal device.

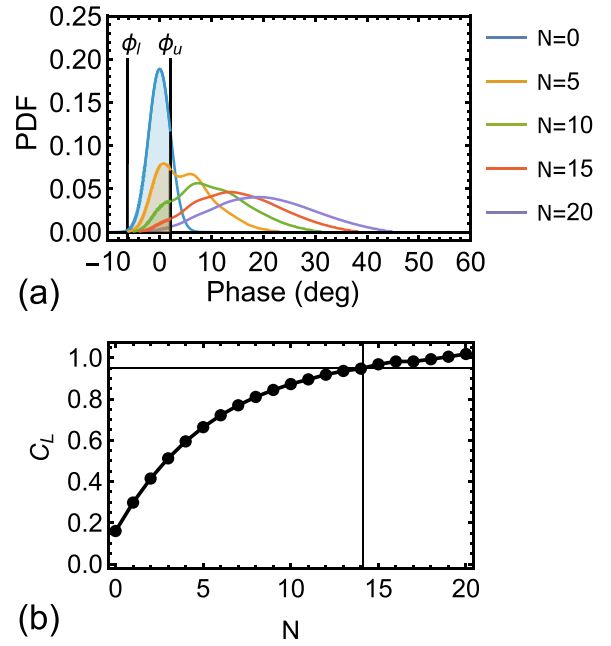


FIG. 13. The procedure to define the exclusion efficiency η_e , for $Q = 10^6$; other parameters are listed in Table II. (a) $h_N(\phi)$ is plotted for several values of N . The segment of $h_N(\phi)$ between ϕ_l and ϕ_u is shown with color filled to the x axis. Panel (b) shows C_L , and the point at which $I(N) = 0.95$ is marked by lines.

For low photon count rates such as we expect in axion experiments, the number of photons appearing in an interval t should conform to a Poisson distribution. For an ideal photon counter, the wait time required can be found from the probability of zero counts being observed after a wait time t , $e^{-R_a t} = 1 - C_L$. Typically, a 95% exclusion interval $C_L = 0.95$ is required, so that $t_{\text{ideal}} \simeq 3/R_a$.

We now must find the equivalent time $t_{\text{real}}(C_L)$ for our photon counter. If the rate is R_a , we would expect a total number of $N = R_a t_{\text{real}}$ photons to have occupied the cavity, on average. We imagine that N photons come sequentially, and the phase readout occurs after all of them have arrived. The presence of these N photons modifies the final distribution of phases, now given by $h_N(\phi)$. This can be found by performing repeated convolutions of the phase PDF $p_\phi(\phi)$ and the noise PDF $h_0(\phi)$:

$$h_N(\phi) = (h_0 * \underbrace{p_\phi * \dots * p_\phi}_N)(\phi). \quad (99)$$

An illustration of $h_N(\phi)$ for various values of N is shown in Fig. 13(a). Here, the blue distribution corresponds to $N = 0$, and larger values of $R_a t_{\text{real}}$ shift this distribution to larger phase values and broaden it. The goal is to determine how large $N = R_a t_{\text{real}}$ must be before a typical phase from the $N = 0$ distribution is unlikely (with confidence level C_L) to have come from the distribution $h_N(\phi)$. To formalize this concept, we specify a region of phases between ϕ_l and ϕ_u , such that after a wait time t_{real} any measured phase that falls between ϕ_l and ϕ_u should exclude the rate R_a with a confidence C_L . This

means that the value of N satisfying the following equation:

$$C_L = 1 - \int_{\phi_l}^{\phi_u} h_N(\phi) d\phi \quad (100)$$

defines $t_{\text{real}}(C_L) = N/R_a$. Figure 13(b) shows C_L as N increases for $\phi_u = \sigma_\phi$, $\phi_l = -3\sigma_\phi$. The point where $C_L = 0.95$ is also indicated.

The phases ϕ_l and ϕ_u are chosen to include most of the distribution at $N = 0$, so that the resulting exclusion efficiency is representative of the typical performance of the device. The proportion of runs where the $N = 0$ distribution produces a phase within ϕ_l and ϕ_u is simply

$$\frac{1}{\sqrt{2\pi}\sigma_\phi} \int_{\phi_l}^{\phi_u} e^{-\frac{\phi^2}{2\sigma_\phi^2}} d\phi. \quad (101)$$

Using the phase that falls between ϕ_l and ϕ_u rather than the phase larger than a single bound allows us to manage the situation where the phase wraps around from π to $-\pi$. Any measured phase that is greater than ϕ_u could be used to place a lower confidence limit on R_a or exclude a higher rate at the same confidence.

As with consideration of η_d , an analytical integral $\tilde{h}_N(\phi)$ found by evaluating Eq. (99) with $\tilde{p}_\phi(\phi)$ instead of $p_\phi(\phi)$ can be derived, and it is expected to be valid in the limit that $g \ll \kappa$ and provided that effects related to the wrapping around of the axial phase are not significant. The expression for $\tilde{h}_N$ is provided in Appendix E.

Figure 14 shows the 95% exclusion efficiency calculated for the electron single-photon counter, setting $\phi_u = \sigma_\phi$, $\phi_l = -3\sigma_\phi$, with other parameters listed in the caption. This choice of ϕ_u and ϕ_l means that at least 84% of measurements would be expected to set a 95% confidence limit or better on a given photon rate. The rates derived from $\tilde{h}_N(\phi)$ and $h_N(\phi)$ (dashed blue and orange) have been multiplied by $0.84(1 + n^{-2})^{-1/2} = 0.84\sqrt{4/5}$ to account for the loss in efficiency due to B_2 induced broadening and the fact that this efficiency is only achieved on 84% of measurements.

Comparing Figs. 11 and 14(a), we see that the exclusion efficiency is generally higher than the detection efficiency and that the optimum Q factor for the most efficient detection is higher. This is because phase shifts from each photon absorption, which are individually too small to be resolved against the phase noise, can nevertheless add together and lead to a measurable shift when $R_a t_{\text{real}}$ is large. Figure 14(b) plots η_e as a function of g when $Q = 10^6$ (equivalently, $\kappa = 188496$). This shows that the exclusion efficiency continues to rise with increasing g and saturates at about $\eta_e = 0.30$ when g and κ become comparable.

VII. DISCUSSION AND CONCLUSIONS

Here, we discuss the implications of what we have discovered for axion detection and compare the expected performance of our scheme to other proposals for photon counting with trapped electrons.

Our development of a single-photon counter is motivated by the aim of speeding up axion search experiments. In Sec. I, we noted that for axion experiments where $Q_c \approx Q_a$, at 30 GHz, and assuming $\eta_{\text{la}} = 1$ a single-photon counter with

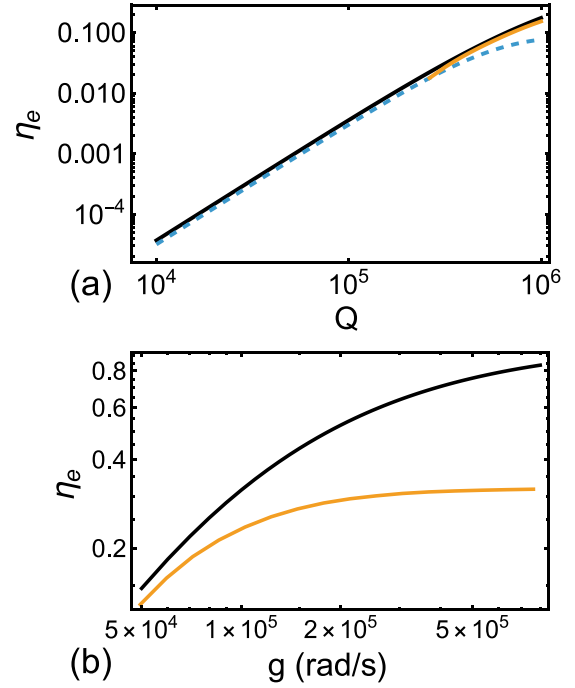


FIG. 14. Exclusion efficiency η_e for $\phi_l = -3\sigma_\phi$, $\phi_u = \sigma_\phi$. Panel (a) plots the efficiency as a function of Q and $\omega_+ = 2\pi \times 30$ GHz, and panel (b) as a function of g for $Q = 10^6$ and $\omega_+ = 2\pi \times 30$ GHz. The orange lines plot the full calculation using $h_N(\phi)$. The blue dashed line gives the approximation derived from $\tilde{h}_N(\phi)$, and the black line is the maximum possible efficiency using the absorption probability from Eq. (12). The orange and blue efficiencies have been multiplied by $0.84\sqrt{4/5}$ as discussed in the text.

an efficiency of $\eta_{\text{sp}} > 3 \times 10^{-5} R_a$ will outperform a linear amplifier. Assuming axions solve the strong CP problem and make up all of dark matter, we could expect a haloscope to produce of order 0.001–1 counts/s at 30 GHz depending on the design. We can consider five different scenarios for the electron-photon counter depending on the electron trap construction materials and design. The corresponding Q factors, coupling strength g , exclusion efficiencies, and speedup times are shown in Table I, assuming an additional $\eta_c = 0.5$ coupling loss between the amplifier and the axion-to-photon conversion device of the type introduced in Sec. II, so that the total exclusion efficiency is $\eta_t = \eta_c \eta_e$.

TABLE I. Parameters for various potential photon counters. For the last column, $R_a = 1$ counts/s, $\eta_{\text{la}} = 1$, $Q_c \simeq Q_a$, $\eta_c = 0.5$.

Cavity type	Q	g (krad/s)	η_e (%)	Speedup vs linear amplifier
(1) Copper	2×10^4	$2\pi \times 9.1$	0.01	1.4
(2) NbSn ₃	6×10^4	$2\pi \times 9.1$	0.08	13
(3) Copper/NbTi	4×10^5	$2\pi \times 9.1$	4.2	630
(4) Thin NbTi	1×10^6	$2\pi \times 9.1$	15	2300
(5) Thin NbTi	1×10^6	$2\pi \times 30$	29	4300

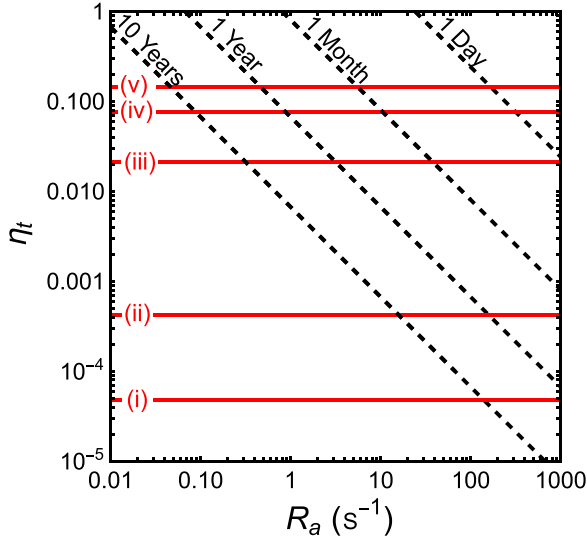


FIG. 15. Total integration time required to exclude an axion signal at 95% confidence over a frequency octave $m_a \rightarrow 2m_a$ as a function of R_a and η_t , indicated by dashed lines. The red horizontal lines indicate the photon counting efficiencies envisioned in Table I.

We see that even a modestly efficient single-photon counter speeds up the axion search, while the ultimate device leads to a 4300-fold increase for a 1 Hz axion count rate.

Another way in which the usefulness of the photon counter can be put in context is by considering the total integration time required to scan a particular range of axion masses, given that the axion haloscope produces a count rate R_a . Assuming once again that $Q_c \simeq Q_a$ and $\eta_c = 0.5$, the time required to scan between an axion mass m_a and $2m_a$ assuming that the central frequency is increased by a factor $1 + 1/Q_a$ each time, is

$$t_s = \frac{3 \ln(2)}{R_a \eta_t \ln\left(\frac{1}{Q_a} + 1\right)}. \quad (102)$$

The dashed lines in Fig. 15 correspond to t_s for one day, one month, one year, and ten years, as a function of η_t and R_a . Also shown on the plot are horizontal lines representing the five photon counting scenarios considered in Table I. We consider that an experiment that takes 10 years of measurement time to measure an octave from m_a to $2m_a$ is the limit of feasibility. We see from this figure that scenario (1) will only be feasible with an axion haloscope that produces count rates in excess of $R_a = 140 \text{ s}^{-1}$. On the other hand, scenario (2) can allow experiments with count rates $R_a > 16 \text{ s}^{-1}$, while (3), (4), and (5) allow experiments with count rates $R_a > 0.32 \text{ s}^{-1}$, $R_a > 0.08 \text{ s}^{-1}$, and $R_a > 0.05 \text{ s}^{-1}$, respectively. The count rates for scenarios (3)–(5) could be achieved with haloscope designs.

Our ultimate goal is a Penning trap with $Q = 10^6$, and with a g controllable between $g \simeq 2\pi \times 6 \times 10^3$ (for high detection efficiency) and $g > 2\pi \times 3 \times 10^4$ (for high exclusion efficiency). A variable g could potentially be achieved by moving the electron position within the electric field of the cavity. Alternatively, we could vary the number of electrons in the trap and work with the collective axial mode of small electron clouds, which boosts g by the square root of the number of electrons.

We now compare our method to other proposals for counting single photons using trapped electrons. Cridland *et al.* [30] propose trapping a single electron above a surface Penning trap that doubles as a microwave coplanar waveguide. The basics of the method are similar to ours: A trap with a B_2 is used to cause the electron axial frequency to shift on absorption of the photon. The electron-waveguide mode coupling is analyzed using transmission line theory. To boost the coupling between the trapped electron and the microwave photon traveling in the waveguide, the electron is trapped at a position where the admittance of the waveguide (the reciprocal of the impedance) is as small as possible compared to the effective admittance of the electron, which is typically a few tens of pS. It is proposed that low waveguide impedance can be achieved by a shorted stub microwave termination at an electrical distance corresponding to $\lambda/4$ away from the electron position. It is estimated that dielectric losses are due to the sapphire substrate of the coplanar waveguide at cryogenic temperatures, and an admittance of 900 pS can be achieved, which gives a few % absorption efficiency if the electron is around 100 μm above the trap surface. This is similar to the midrange efficiencies achievable with our method. The method of Ref. [30] has the principal advantage that, as it uses a waveguide rather than a cavity, the sensitive frequency of the device can be adjusted by simply adjusting the magnetic field of the trap, rather than, in our case, also having to move the endcaps to keep the cavity resonant with the modified cyclotron mode. The surface trap method also has technical challenges, chief among these being maintaining a low admittance at the electron position. We note that a small displacement away from the optimized position can dramatically increase the admittance, which reduces the absorption efficiency. Excited axial amplitudes are needed for the types of phase-sensitive detection methods proposed in Ref. [30], which may be challenging to reconcile with the narrow range of permissible electron positions. Trapping electrons above surface traps is challenging, as yet single electrons have not been observed in such traps due to complications arising from the nonideal trapping potential. If these challenges are overcome, we note that the method of Ref. [30] could be combined with our approach by using a coplanar waveguide cavity rather than a transmission line. This structure would have a much higher coupling constant g than a cylindrical trap, enabling the $g = \kappa$ strong coupling regime to be reached at a lower Q factor.

A more recent proposal by Fan *et al.* [31] describes a complete experiment to search for dark matter axions using a mirror in a magnetic field of the broadband reflector experiment for axion detection (BREAD) design [34] to convert axions into photons, and a Penning-trapped electron to detect those photons. In their work, the orbital electric dipole moment of the electron is increased by placing the particle into a highly excited cyclotron state $n_+ = 10^6 (\frac{\omega_a}{0.1 \text{ meV}})^2$. The electron then behaves similarly to a Rydberg atom, with an enhanced interaction to electric fields, and a coupling rate that scales like n_+ . As in our proposal and Ref. [30], the cyclotron state in Ref. [31] is detected using a B_2 bottle and by measuring the axial frequency. The signature in the Fan *et al.* proposal is a one-quantum increase in the cyclotron state, superimposed over the decaying cyclotron signal. In Appendix B of Ref. [31], it is estimated that the photon detection efficiency

TABLE II. Trap parameters used to evaluate expressions throughout this paper, unless otherwise stated.

Parameter name	Symbol	Value	Unit
Modified cyclotron frequency	$\frac{\omega_+}{2\pi}$	30	GHz
Axial frequency	$\frac{\omega_z}{2\pi}$	100	MHz
Magnetron frequency	$\frac{\omega_-}{2\pi}$	167	kHz
Axial frequency shift from C_4	C_E	2.4×10^4	rad m ⁻²
Static magnetic field	B_0	1.07	T
Magnetic bottle strength	B_2	10^5	T m ²
Quartic magnetic field correction	B_4	$<10^7$	T m ⁻⁴
Trap length	l	24.55	mm
Trap radius	r	3.7	mm
Trap electric volume	$\tilde{V}$	2.4×10^{-7}	m ³
Cavity frequency	$\frac{\omega}{2\pi}$	30	GHz
Cavity coupling constant	$\frac{g}{2\pi}$	9.2	kHz
Free phase evolution time	t_{ev}	1–100	s
Detection system parallel resistance	R_p	300	k Ω
Detection system effective electrode distance	$D_{\text{eff},z}$	12.75	mm
B_2 broadening to cavity linewidth ratio	n	2	None
Maximum axial amplitude	z_{max}	2.5	mm
Axial amplification factor	$\sinh kt_a$	27	None
Detector noise	σ_D^2	$\frac{0.35k_B}{m\omega_z^2}$	m ²

of the highly excited electron photon counter is approximately 10^{-4} , which is lower than the maximum achievable in our method. However, the photon counting method in Ref. [31] is considerably more broadband than our method, with an effective detection bandwidth of $\frac{\omega}{Q}$, where $20 < Q < 200$, compared to our bandwidth of $\frac{\omega}{Q}$ with $Q > 10^5$ for reasonably high detection efficiencies. This broadband sensitivity to incoming photons makes good use of the BREAD-style antenna, which is able to convert axions with a wide range of possible masses into photons without adjustment. In contrast, our method, for the typical parameters considered in Table II, would yield higher efficiency detection of photons emitted from a narrow-band cavity-based axion-to-photon converter, provided the conversion cavity's quality factor was approximately larger than $Q_c \gtrsim 10^3$.

In this work, we have described a technique for counting single photons using an electron in a Penning trap. The method is optimized for use in axion searches, where a photon counter can speed up experiments operating above 30 GHz by several orders of magnitude. The method will enable axion searches above $\sim 100 \mu\text{eV}$ in a reasonable experimental time. In future work, we will describe how this electron single-

photon counter can be combined with high Q -factor axion conversion cavities under development in our laboratory to make an axion haloscope able to probe DFSZ axions above $120 \mu\text{eV}$.

ACKNOWLEDGMENTS

This work was supported by the Royal Society (URF/R1/21105), STFC (ST/W006650/1), and UKRI under the UK government's Horizon Europe funding Guarantee (EP/Y036263/1).

DATA AVAILABILITY

The data that support the findings of this article are openly available [69].

APPENDIX A: PARAMETERS AND VALUES USED THROUGHOUT THE PAPER

The typical experimental parameters used in this paper are given in Table II.

APPENDIX B: CONSISTENCY BETWEEN DERIVED MATRIX ELEMENTS AND CLASSICAL CYCLOTRON RADIATION RATE

In this Appendix, we confirm that the matrix elements [Eqs. (16) and (17)] are consistent with the familiar expression for the rate of cyclotron emission of an electron in a magnetic field [49,52],

$$\frac{dE}{dt} = -\frac{1}{4\pi\epsilon_0} \frac{4e^2\omega^2}{3mc^3} E. \quad (\text{B1})$$

Fermi's Golden Rule gives the transition rate from an initial state to a set of final states f :

$$\Gamma_{i \rightarrow f} = \frac{2\pi}{\hbar} |\langle i | H_{\text{int}} | f \rangle|^2 \rho(E_f), \quad (\text{B2})$$

where $\rho(E_f)$ is the density of final states corresponding to the final state energy E_f . To calculate the free-space emission, we need to consider that the photon can be emitted in any direction, so we need to integrate over all angles to get

$$\begin{aligned} \Gamma_{i \rightarrow f} &= \frac{2\pi}{\hbar} e^2 \frac{\hbar}{\epsilon_0 \omega \tilde{V}} \left(\frac{\omega_+}{\omega_+ - \omega_-} \right)^2 \rho(E_f) \\ &\times \int_0^\pi \int_0^{2\pi} |\langle i | \frac{1}{4\pi} (a_\gamma e^{ik \cdot r} + a_\gamma^\dagger e^{-ik \cdot r}) \\ &\times \cos(\theta) \hat{\mathbf{x}} \cdot \mathbf{V}^+ | f \rangle|^2 \sin(\theta) d\theta d\phi, \end{aligned} \quad (\text{B3})$$

where θ is the azimuthal angle, ϕ the polar angle, and the x axis is chosen to lie along the projection of the emitted photon's direction onto the radial plane. The result is

$$\begin{aligned} \Gamma_{i \rightarrow f} &= \frac{2\pi}{3\hbar} \rho(E_f) e^2 \frac{\hbar}{\epsilon_0 \omega \tilde{V}} \left(\frac{\omega_+}{\omega_+ - \omega_-} \right)^2 \\ &\times |\langle i' | (a_\gamma e^{ik \cdot r} + a_\gamma^\dagger e^{-ik \cdot r}) \hat{\mathbf{x}} \cdot \mathbf{V}^+ | f' \rangle|^2, \end{aligned} \quad (\text{B4})$$

where the prime index indicates that the angular dependence of the final states has been removed.

Now, consider a decay from n_+ to $n_+ - 1$, where the cavity begins empty $n_- = 0$. As a given cyclotron mode can decay by emitting photons of two orthogonal polarizations via couplings between $\hat{\mathbf{x}} \cdot \mathbf{V}^+$ as considered so far and $\hat{\mathbf{y}} \cdot \mathbf{V}^+$, up to now ignored, the total decay rate for the cyclotron mode needs to be multiplied by a factor of 2. The terms to the right of the density of states $\rho(E_f)$ can be replaced using the matrix element [Eq. (17)], yielding

$$\Gamma_{n_+ \rightarrow (n_+ - 1)} = \frac{4\pi}{3\hbar} \rho(E_f) e^2 \hbar^2 \frac{n_+}{4m\epsilon_0 \tilde{V}}. \quad (\text{B5})$$

The density of final states, considering that there are two polarizations, is

$$\rho(E_f) = \frac{\omega^2 V}{\pi^2 c^3 \hbar}. \quad (\text{B6})$$

Inserting this into Eq. (B5) and assuming that as the physical volume gets large, the mode volume $\tilde{V} \rightarrow V$, we arrive at the final expression for the transition rate out of the n_+ excited state as

$$\Gamma_{n_+ \rightarrow (n_+ - 1)} = \frac{1}{4\pi\epsilon_0} \frac{4e^2\omega^2}{3mc^3} n_+. \quad (\text{B7})$$

As $n_+ \propto E$, this is equivalent to the classical result we wanted to demonstrate.

APPENDIX C: EQUIVALENCE BETWEEN QUANTUM AND CLASSICAL CYCLOTRON DECAY RATES IN A CAVITY

In this Appendix, we show the equivalence of the quantum and classical decay rates for a charged particle interacting with the cavity in the weak coupling limit. We showed previously that the rate at which a particle in the excited cyclotron state decays and hence loses energy is given by

$$\gamma_q = \frac{4g^2}{\kappa} = \frac{q^2 Q}{m\tilde{V}\omega\epsilon_0}. \quad (\text{C1})$$

The classical rate at which a charged particle loses energy due to interacting with nearby conductors is given by the expression

$$\gamma_c = \frac{q^2 R}{mD_{\text{eff},+}^2}. \quad (\text{C2})$$

Here, R is the resistance experienced by the oscillating particle current as a result of nearby conductors and $D_{\text{eff},+}$ is the effective electrode distance for the cyclotron mode, which also satisfies the relation

$$D_{\text{eff},+}^2 = \frac{V_{\text{induced}}^2}{E_p^2}. \quad (\text{C3})$$

Here, E_p is the electric field at the position of the particle produced by the image currents in the conductors and V_{induced} is the induced voltage on those conductors.

computer algebra package, $\tilde{h}_n$ as

$$\tilde{h}_n(\phi) = \frac{2^{\frac{n}{2}-2} \lambda^n \sigma_\phi^{n-1} e^{-\frac{\phi^2}{2\sigma_\phi^2}}}{\sqrt{\pi} \Gamma(n)} \left[\sqrt{2} \Gamma\left(\frac{n}{2}\right) {}_1F_1\left(\frac{n}{2}; \frac{1}{2}; \frac{1}{2} \left(\lambda \sigma_\phi - \frac{\phi}{\sigma_\phi} \right)^2 \right) \right] \quad (\text{E2})$$

We recognize that the power loss can be written in two different ways, first in terms of the induced voltage

$$P = \frac{V_{\text{induced}}^2}{R}. \quad (\text{C4})$$

Second, in terms of the quality factor

$$P = \omega U Q. \quad (\text{C5})$$

Here, $U = \int \epsilon_0 |E|^2 dV$ is the total power stored in the cavity system, with associated quality factor Q . Putting this all together with Eq. (C2), we have

$$\gamma_c = \frac{q^2 Q}{m\omega\epsilon_0} \frac{E_p^2}{\int |E|^2 dV}, \quad (\text{C6})$$

which, given the previous definition for $\tilde{V}$, shows that $\gamma_c = \gamma_q$. Note that as discussed in the text, this equivalence no longer holds when κ becomes comparable or smaller than g and hence the classical expression for the radiated power should expect to break down when strong coupling occurs.

APPENDIX D: EXPECTATION VALUES OF THE AXIAL PHASE PROBABILITY DISTRIBUTION

The probability distributions for the axial position after excitation and at the point of detection both have the form

$$\mathcal{F}(r, \theta) = \frac{1}{2\pi\sigma^2} \exp \left[-\frac{(r \cos(\theta) - a)^2 + (r \sin(\theta) - b)^2}{2\sigma^2} \right], \quad (\text{D1})$$

where $a^2 + b^2 = c^2$ and $\langle X \rangle = \int_0^\infty \int_{-\pi}^\pi X r \mathcal{F} dr d\theta$. Here, we provide some evaluations used throughout the paper:

$$\langle r^2 \rangle = c^2 + 2\sigma^2 \simeq c^2, \quad (\text{D2})$$

$$\langle r^4 \rangle - \langle r^2 \rangle^2 = 4\sigma^2(c^2 + \sigma^2) \simeq 4\sigma^2 c^2, \quad (\text{D3})$$

where the approximation holds if $c^2 \gg \sigma^2$. In this case, it is also true that

$$\langle \theta^4 \rangle - \langle \theta^2 \rangle^2 = \frac{2\sigma^2}{c^2}. \quad (\text{D4})$$

APPENDIX E: EXPRESSIONS FOR CONVOLUTIONS OF $\tilde{p}_\phi$ AND THE FUNCTION $\tilde{h}_n(\phi)$

The repeated convolution of $\tilde{p}_\phi$ has a simple form in terms of an Erlang distribution

$$\underbrace{(\tilde{p}_\phi * \dots * \tilde{p}_\phi)}_n(\phi) = \frac{\lambda^n \phi^{n-1} e^{-\lambda\phi}}{\Gamma(n)}, \quad (\text{E1})$$

where $\Gamma(n)$ is the Euler gamma function. A further convolution with a normal distribution gives, via the Mathematica

$$-2\Gamma\left(\frac{n+1}{2}\right)\left(\lambda\sigma_\phi - \frac{\phi}{\sigma_\phi}\right){}_1F_1\left(\frac{n+1}{2}; \frac{3}{2}; \frac{1}{2}\left(\lambda\sigma_\phi - \frac{\phi}{\sigma_\phi}\right)^2\right)\Bigg]. \quad (\text{E3})$$

Here, ${}_1F_1(a; b; z)$ is the Kummer confluent hypergeometric function.

-
- [1] R. D. Peccei and H. R. Quinn, Constraints imposed by CP conservation in the presence of pseudoparticles, *Phys. Rev. D* **16**, 1791 (1977).
- [2] R. D. Peccei and H. R. Quinn, CP conservation in the presence of pseudoparticles, *Phys. Rev. Lett.* **38**, 1440 (1977).
- [3] F. Wilczek, Problem of strong P and T invariance in the presence of instantons, *Phys. Rev. Lett.* **40**, 279 (1978).
- [4] S. Weinberg, A new light boson? *Phys. Rev. Lett.* **40**, 223 (1978).
- [5] M. Gorghetto and G. Villadoro, Topological susceptibility and QCD axion mass: QED and NNLO corrections, *J. High Energy Phys.* **03** (2019) 033.
- [6] J. E. Kim, Weak-interaction singlet and strong CP invariance, *Phys. Rev. Lett.* **43**, 103 (1979).
- [7] M. Shifman, A. Vainshtein, and V. Zakharov, Can confinement ensure natural CP invariance of strong interactions? *Nucl. Phys. B* **166**, 493 (1980).
- [8] M. Dine, W. Fischler, and M. Srednicki, A simple solution to the strong CP problem with a harmless axion, *Phys. Lett. B* **104**, 199 (1981).
- [9] A. Zhitnitskij, On possible suppression of the axion-hadron interactions, *Sov. J. Nucl. Phys.* **31**, 260 (1980).
- [10] S. Navas, C. Amsler, T. Gutsche, C. Hanhart, J. Hernández-Rey, C. Lourenço, A. Masoni, M. Mikhasenko, R. E. Mitchell, C. Patrignani, *et al.*, Review of particle physics, *Phys. Rev. D* **110**, 030001 (2024).
- [11] J. Preskill, M. B. Wise, and F. Wilczek, Cosmology of the invisible axion, *Phys. Lett. B* **120**, 127 (1983).
- [12] L. Abbott and P. Sikivie, A cosmological bound on the invisible axion, *Phys. Lett. B* **120**, 133 (1983).
- [13] M. Dine and W. Fischler, The not-so-harmless axion, *Phys. Lett. B* **120**, 137 (1983).
- [14] D. J. Marsh, Axion cosmology, *Phys. Rep.* **643**, 1 (2016).
- [15] D. Hanneke, S. Fogwell, and G. Gabrielse, New measurement of the electron magnetic moment and the fine structure constant, *Phys. Rev. Lett.* **100**, 120801 (2008).
- [16] R. S. Van Dyck, P. B. Schwinberg, and H. G. Dehmelt, New high-precision comparison of electron and positron g factors, *Phys. Rev. Lett.* **59**, 26 (1987).
- [17] Q. Yu, A. M. Alonso, J. Caminiti, K. M. Beck, R. T. Sutherland, D. Leibfried, K. J. Rodriguez, M. Dhital, B. Hemmerling, and H. Häffner, Feasibility study of quantum computing using trapped electrons, *Phys. Rev. A* **105**, 022420 (2022).
- [18] P. Peng, C. Matthiesen, and H. Häffner, Spin readout of trapped electron qubits, *Phys. Rev. A* **95**, 012312 (2017).
- [19] N. Daniilidis, D. J. Gorman, L. Tian, and H. Häffner, Quantum information processing with trapped electrons and superconducting electronics, *New J. Phys.* **15**, 073017 (2013).
- [20] C. Matthiesen, Q. Yu, J. Guo, A. M. Alonso, and H. Häffner, Trapping electrons in a room-temperature microwave Paul trap, *Phys. Rev. X* **11**, 011019 (2021).
- [21] A. Osada, K. Taniguchi, M. Shigefuji, and A. Noguchi, Feasibility study on ground-state cooling and single-phonon readout of trapped electrons using hybrid quantum systems, *Phys. Rev. Res.* **4**, 033245 (2022).
- [22] P. Bushev, S. Stahl, R. Natali, G. Marx, E. Stachowska, G. Werth, M. Hellwig, and F. Schmidt-Kaler, Electrons in a cryogenic planar Penning trap and experimental challenges for quantum processing, *Eur. Phys. J. D* **50**, 97 (2008).
- [23] I. Marzoli, P. Tombesi, G. Ciaramicoli, G. Werth, P. Bushev, S. Stahl, F. Schmidt-Kaler, M. Hellwig, C. Henkel, G. Marx, *et al.*, Experimental and theoretical challenges for the trapped electron quantum computer, *J. Phys. B: At. Mol. Opt. Phys.* **42**, 154010 (2009).
- [24] J. Goldman and G. Gabrielse, Optimized planar Penning traps for quantum-information studies, *Phys. Rev. A* **81**, 052335 (2010).
- [25] D. Budker, P. W. Graham, H. Ramani, F. Schmidt-Kaler, C. Smorra, and S. Ulmer, Millicharged dark matter detection with ion traps, *PRX Quantum* **3**, 010330 (2022).
- [26] D. Carney, H. Häffner, D. C. Moore, and J. M. Taylor, Trapped electrons and ions as particle detectors, *Phys. Rev. Lett.* **127**, 061804 (2021).
- [27] X. Fan, G. Gabrielse, P. W. Graham, R. Harnik, T. G. Myers, H. Ramani, B. A. Sukra, S. S. Wong, and Y. Xiao, One-electron quantum cyclotron as a milli-eV dark-photon detector, *Phys. Rev. Lett.* **129**, 261801 (2022).
- [28] J. A. Devlin, Dark matter: What is it, and can quantum sensors help find it? *Contemp. Phys.* **65**, 239 (2024).
- [29] S. Peil and G. Gabrielse, Observing the quantum limit of an electron cyclotron: QND measurements of quantum jumps between Fock states, *Phys. Rev. Lett.* **83**, 1287 (1999).
- [30] A. Cridland, J. Lacy, J. Pinder, and J. Verdú, Single microwave photon detection with a trapped electron, *Photonics* **3**, 59 (2016).
- [31] X. Fan, G. Gabrielse, P. W. Graham, H. Ramani, S. S. Wong, and Y. Xiao, Highly excited electron cyclotron for QCD axion and dark-photon detection, *Phys. Rev. D* **111**, 075022 (2025).
- [32] D. Kim, J. Jeong, S. W. Youn, Y. Kim, and Y. K. Semertzidis, Revisiting the detection rate for axion haloscopes, *J. Cosmol. Astropart. Phys.* **03** (2020) 066.
- [33] E. Graham, S. Ghosh, Y. Zhu, X. Bai, S. B. Cahn, E. Durcan, M. J. Jewell, D. H. Speller, S. M. Zacarias, L. T. Zhou, and R. H. Maruyama, Rydberg-atom-based single-photon detection for haloscope axion searches, *Phys. Rev. D* **109**, 032009 (2024).
- [34] J. Liu *et al.* (BREAD Collaboration), Broadband solenoidal haloscope for terahertz axion detection, *Phys. Rev. Lett.* **128**, 131801 (2022).
- [35] P. Brun, A. Caldwell, L. Chevalier, G. Dvali, P. Freire, E. Garutti, S. Heyminck, J. Jochum, S. Knirck, M. Kramer, *et al.*, A new experimental approach to probe QCD axion dark matter in the mass range above 40 μeV , *Eur. Phys. J. C* **79**, 186 (2019).

- [36] A. J. Millar, S. M. Anlage, R. Balafendiev, P. Belov, K. van Bibber, J. Conrad, M. Demarteau, A. Droster, K. Dunne, A. G. Rosso, *et al.*, Searching for dark matter with plasma haloscopes, *Phys. Rev. D* **107**, 055013 (2023).
- [37] S. K. Lamoreaux, K. A. van Bibber, K. W. Lehnert, and G. Carosi, Analysis of single-photon and linear amplifier detectors for microwave cavity dark matter axion searches, *Phys. Rev. D* **88**, 035020 (2013).
- [38] D. Fukuda and T. Kikuchi, Single and few-photon detection using superconducting transition edge sensors, *Prog. Opt.* **69**, 135 (2024).
- [39] W. Yin, H. Wang, X. Wang, C. M. Natarajan, M. G. Tanner, and R. H. Hadfield, Superconducting nanowire single-photon detectors: Physics and applications, *Supercond. Sci. Technol.* **25**, 063001 (2012).
- [40] D. Alesini, D. Babusci, C. Barone, B. Buonomo, M. M. Beretta, L. Bianchini, G. Castellano, F. Chiarello, D. Di Gioacchino, P. Falferi, *et al.*, Status of the SIMP project: Toward the single microwave photon detection, *J. Low Temp. Phys.* **199**, 348 (2020).
- [41] A. Blais, A. L. Grimsmo, S. M. Girvin, and A. Wallraff, Circuit quantum electrodynamics, *Rev. Mod. Phys.* **93**, 025005 (2021).
- [42] A. V. Dixit, S. Chakram, K. He, A. Agrawal, R. K. Naik, D. I. Schuster, and A. Chou, Searching for dark matter with a superconducting qubit, *Phys. Rev. Lett.* **126**, 141302 (2021).
- [43] A. Anferov, S. P. Harvey, F. Wan, J. Simon, and D. I. Schuster, Superconducting qubits above 20 GHz operating over 200 mK, *PRX Quantum* **5**, 030347 (2024).
- [44] A. L. Pankratov, L. S. Revin, A. V. Gordeeva, A. A. Yablokov, L. S. Kuzmin, and E. Il'ichev, Towards a microwave single-photon counter for searching axions, *npj Quantum Inf.* **8**, 61 (2022).
- [45] A. Ghirri, S. Cornia, and M. Affronte, Microwave photon detectors based on semiconducting double quantum dots, *Sensors* **20**, 4010 (2020).
- [46] K. Zhang, F. Bariani, Y. Dong, W. Zhang, and P. Meystre, Proposal for an optomechanical microwave sensor at the subphoton level, *Phys. Rev. Lett.* **114**, 113601 (2015).
- [47] M. Tada, Y. Kishimoto, K. Kominato, M. Shibata, H. Funahashi, K. Yamamoto, A. Masaike, and S. Matsuki, CARRACK II—A new large-scale experiment to search for axions with Rydberg-atom cavity detector, *Nucl. Phys. B Proc. Suppl.* **72**, 164 (1999).
- [48] M. Tada, Y. Kishimoto, K. Kominato, M. Shibata, S. Yamada, T. Haseyama, I. Ogawa, H. Funahashi, K. Yamamoto, and S. Matsuki, Single-photon detection of microwave blackbody radiations in a low-temperature resonant-cavity with high Rydberg atoms, *Phys. Lett. A* **349**, 488 (2006).
- [49] L. S. Brown and G. Gabrielse, Geonium theory: Physics of a single electron or ion in a Penning trap, *Rev. Mod. Phys.* **58**, 233 (1986).
- [50] E. A. Cornell, R. M. Weisskoff, K. R. Boyce, and D. E. Pritchard, Mode coupling in a Penning trap: Pulses and a classical avoided crossing, *Phys. Rev. A* **41**, 312 (1990).
- [51] J. Herkenhoff, J. Notter, and K. Blaum, Sympathetic cooling of charged particles in Penning traps using electron cyclotron radiation, *Phys. Rev. Res.* **8**, 013071 (2026).
- [52] J. D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, Hoboken, NJ, 1998).
- [53] B. D'Urso, R. Van Handel, B. Odom, D. Hanneke, and G. Gabrielse, Single-particle self-excited oscillator, *Phys. Rev. Lett.* **94**, 113002 (2005).
- [54] J. Golm, S. Arguedas Cuendis, S. Calatroni, C. Cogollos, B. Döbrich, J. Gallego, J. García Barceló, X. Granados, J. Gutierrez, I. Irastorza, *et al.*, Thin film (high temperature) superconducting radiofrequency cavities for the search of axion dark matter, *IEEE Trans. Appl. Supercond.* **32**, 1 (2022).
- [55] D. Alesini, C. Braggio, G. Carugno, N. Crescini, D. D'Agostino, D. Di Gioacchino, R. Di Vora, P. Falferi, S. Gallo, U. Gambardella, *et al.*, Galactic axions search with a superconducting resonant cavity, *Phys. Rev. D* **99**, 101101 (2019).
- [56] X. Xi, J. Hwang, C. Martin, D. B. Tanner, and G. L. Carr, Far-infrared conductivity measurements of pair breaking in superconducting Nb_{0.5}Ti_{0.5}N thin films induced by an external magnetic field, *Phys. Rev. Lett.* **105**, 257006 (2010).
- [57] B. R. Mollow and R. J. Glauber, Quantum theory of parametric amplification. I, *Phys. Rev.* **160**, 1076 (1967).
- [58] B. R. Mollow and R. J. Glauber, Quantum theory of parametric amplification. II, *Phys. Rev.* **160**, 1097 (1967).
- [59] S. Sturm, A. Wagner, B. Schabinger, and K. Blaum, Phase-sensitive cyclotron frequency measurements at ultralow energies, *Phys. Rev. Lett.* **107**, 143003 (2011).
- [60] M. Borchert, Challenging the Standard Model by high precision comparisons of the fundamental properties of antiprotons and protons, Ph.D. thesis, Leibniz Universität Hannover, 2020.
- [61] F. G. Major, V. N. Gheorghe, and G. Werth, *Charged Particle Traps* (Springer, Berlin, 2005).
- [62] J. Ketter, T. Eronen, M. Höcker, S. Streubel, and K. Blaum, First-order perturbative calculation of the frequency-shifts caused by static cylindrically-symmetric electric and magnetic imperfections of a Penning trap, *Int. J. Mass. Spectrom.* **358**, 1 (2014).
- [63] F. T. Köhler, Bound-electron g-factor measurements for the determination of the electron mass and isotope shifts in highly charged ions, Ph.D. thesis, Ruperto-Carola-University of Heidelberg, Germany, 2015.
- [64] X. Fan, An improved measurement of the electron magnetic moment, Ph.D. thesis, Harvard University, 2022.
- [65] A. Kaiser, S. Dickopf, M. Door, R. Behr, U. Beutel, S. Eliseev, A. Kaushik, K. Kromer, M. Müller, L. Palafox, *et al.*, Josephson voltage standards as ultra-stable low-noise voltage sources for precision Penning-trap experiments, *Appl. Phys. Lett.* **124**, 224002 (2024).
- [66] D. B. Pinegar, K. Blaum, T. P. Biesiadzinski, S. L. Zafonte, and R. S. Van Dyck, Stable voltage source for Penning trap experiments, *Rev. Sci. Instrum.* **80**, 064701 (2009).
- [67] M. J. Borchert, P. E. Blessing, J. A. Devlin, J. A. Harrington, T. Higuchi, J. Morgner, C. Smorra, E. Wursten, M. Bohman, M. Wiesinger, *et al.*, Measurement of ultralow heating rates of a single antiproton in a cryogenic Penning trap, *Phys. Rev. Lett.* **122**, 043201 (2019).
- [68] M. Brownnutt, M. Kumph, P. Rabl, and R. Blatt, Ion-trap measurements of electric-field noise near surfaces, *Rev. Mod. Phys.* **87**, 1419 (2015).
- [69] J. A. Devlin, M. L. Wojtkowiak, S. Banhatti, H. Zhang, J. Shi, T. Dofher, J. Gosling, M. Tarbutt, and R. Thompson, Data presented in "A Penning trap single-photon counter for axion detection" [Data set], Zenodo, 2026, <https://doi.org/10.5281/zenodo.20634498>.