

Control-based continuation maps out unstable equilibria in laboratory two-phase flow

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We apply control-based continuation, an experimental method used to explore the bifurcation structure of nonlinear systems, to a spatially extended system by examining the equilibrium configurations of a bubble placed at the center of a straining flow. Feedback control is delivered by injecting/withdrawing fluid at actuation points and is chosen according to real-time observations of the bubble as viewed from above. The amplitude of actuation is used to locate its unstable equilibria, and thus enable their observation. We map the unstable steady shapes of the bubble as a function of straining flow rate and demonstrate excellent agreement of the resulting experimental bifurcation diagram with numerical predictions.

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Complex systems formed of multiple interacting components are ubiquitous across the natural world, science, and society [1–3], but their dynamics can be challenging to unravel and predict because their nonlinear nature allows for multiple equilibria at the same system parameters [4]. Dynamically, the system may converge onto stable equilibria, which can be measured in experiments, or evolve unsteadily in the absence of these [5]. In many scenarios, unstable equilibria are key to the dynamics in that they guide the time-evolution of the system as it briefly visits and is then repelled between unstable states. However, unstable equilibria can be at best glimpsed fleetingly in experiments.

Control-based continuation (CBC) is a method to systematically discover, observe, and track unknown unstable equilibria in experiments [6]. The method does not depend on a mathematical model or require particular initial conditions. Instead, it relies on active feedback control, where real-time system measurements drive a system toward a "target" state. Crucially, the target state is varied to search for those cases where the control signal converges to zero (noninvasive control), recovering an invariant solution of the uncontrolled system. Inspired by the principles of numerical continuation [7], path-following techniques can then be used to track solution branches as experimental parameters are varied.

CBC has been used in experiments to identify steady or time-periodic unstable equilibria of mechanical models whose dynamics depend on relatively few degrees of freedom, such as periodically forced oscillators [8,9], pendula [10], and bending beams [11,12]. This includes tracking of stabilized periodic dynamics through saddle-node bifurcations [10], locating limit points [9] and isolas [12], mapping out natural frequencies [8], as well as identifying model parameters [11].

Relevant real-world applications include improving the stability analysis of aircraft landing gear to include structural deformations [13].

Recently, the use of CBC has been extended to numerical models of more intricate real-world applications, tracking equilibrium curves in simulations of synthetic gene regulatory networks [14], and unstable periodic states in atomic force microscopy simulations [15]. On the experimental front, however, further application of CBC has been extended to the noninvasive control of pedestrian motion around an obstacle [16], demonstrating remarkably systematic behavior as the position of the obstacle varied, despite stochasticity arising from human decision-making.

In this Letter, we broaden the applicability of CBC to spatially-extended experimental systems, which are modeled by partial differential equations. Fluid motion has long served as a test bed for complex nonlinear behavior, and we select a canonical two-dimensional straining flow to probe the instability of a bubble positioned at its center. The flow is confined to a very shallow channel (Fig. 1 inset), called a Hele-Shaw channel. The low Reynolds number of the straining flow means that non-linearities arise exclusively at the air-oil interface, and so the dynamical state of the system is encoded in the 2D shape of the flattened bubble as viewed from above [17].

In the absence of a straining flow, the bubble takes a circular capillary-static shape, neutrally stable to translation in either direction. When a straining flow is introduced, an equilibrium shape is only possible when the bubble is at its center; and the bubble is unstable to translation away from this point. As the strength of the straining flow is increased, the bubble becomes increasingly deformed, and can become inflected. Eventually, surface tension is insufficiently strong to maintain an equilibrium shape, and the bubble breaks up. A bifurcation diagram (Fig. 1) for equilibria in this system should contain two branches, both unstable, connected by a limit-point associated with bubble breakup [18]. Steady bubble shapes on the lower, less-deformed, branch of Fig. 1 are unstable only to translation, while the more deformed shapes

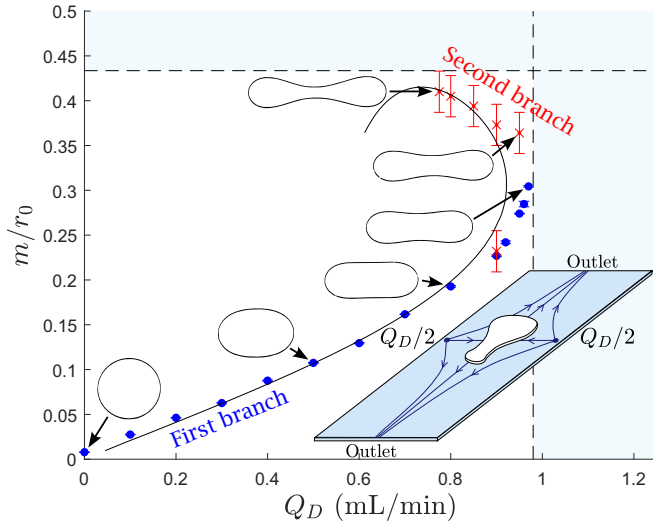


FIG. 1. Experimental bifurcation diagram for bubble system, plotting the shape measure m defined in (1) and flow rate Q_D of the straining flow for a bubble with fixed area πr_0^2 . The lower branch (blue circles) requires positional control only, while the upper branch (red crosses) also requires shape control. Insets show top-view images of the bubble from experiments. The black curve corresponds to numerical solutions of a depth-averaged model (see End Matter). The horizontal dashed line is the limit $m/r_0 \approx 0.43$ where the thinnest region of the dumbbell shape reaches zero width, which corresponds to bubble pinch-off. The vertical dashed line is the limit $Q_D = 0.97$ mL/min beyond which bubbles could no longer be stabilized under feedback control, indicating the absence of steady states past this limit point.

on the upper branch are unstable to both translation and deformation. The upper branch acts as an edge state, a boundary separating those symmetric initial bubble shapes which will evolve toward break up from those which contract toward the equilibria shapes of the first branch.

Previous experimental studies of drops and bubbles in 2D straining flows have been performed using Taylor's classic four-roll mill [18,19], in much deeper tanks than our Hele-Shaw geometry, though the qualitative bifurcation diagram is similar to Fig. 1. Here, feedback control was used to constrain the bubble position to the center of the flow by adjusting the individual speeds of the rollers used to produce the straining flow. This control strategy allowed experimental access to most of the first branch, but not observation of the second branch. Numerical simulations for the four-roll mill predict an analogous picture to that of Fig. 1 [20], where bubbles systematically evolve toward break-up beyond the limit point. Similar behavior persists in 3D straining flows, where the numerical model includes dependence on the vertical coordinate [21]. In this Letter, we demonstrate how CBC can be applied to locate both branches of unstable equilibria shown in Fig. 1, thus unlocking its potential for discovering and mapping unstable equilibria in spatially extended systems.

We produce both the straining flow and actuation for feedback control by fluid injection/suction. The experiments were performed in a milled Perspex channel of dimensions $H = 1.01 \pm 0.01$ mm, $W = 40 \pm 0.1$ mm and $L = 400 \pm 0.1$ mm filled with silicone oil (Dow[®], dynamic viscosity

$\mu = 0.112 \pm 0.002$ kg m⁻¹ s⁻¹, surface tension $\sigma = 21.1 \pm 0.1$ mN m⁻¹, laboratory temperature 21 ± 1 °C) and levelled horizontally to within 0.03° ; see Fig. 2(a). Prior to each experiment, an air bubble was injected manually at one end of the channel [Fig. 2(a)] using a 0.25 mL glass syringe (No. 1725, Hamilton) and slowly propagated to the center of the channel. The bubble was imaged in top-view using a CMOS camera (GO-2400M-PGE, Jai) with a 12 mm f/1.8 wide lens. The channel was backlit by a diffuse LED source so that the bubble-fluid interface appeared as a black line due to light refraction.

The straining flow was imposed by injecting fluid at a constant flow rate Q_D with a syringe pump (200 legacy series, KDS) through a pair of inlets positioned symmetrically about the center of the channel. Bubble control was actuated by injecting/withdrawing liquid with flow rates $Q_{1,2}$, through two pairs of access ports positioned at equal distances either side of the central inlets [Fig. 2(a)]. To enable actuation via fluid withdrawal as well as injection, the channel was pressurized to 400 mbar immediately after forming the air bubble.

We used a linear (proportional) control strategy to stabilize the bubble's position and shape. As inputs, we use the bubble's axial centroid coordinate $x_M(t)$ [Fig. 2(a)] and a "shape measure" m defined by

$$m(t) = \frac{1}{2\pi} \int_0^{2\pi} \cos(2\theta) r(\theta, t) d\theta, \quad (1)$$

where $r(\theta, t)$ is the radius of the bubble interface, relative to the bubble centroid (x_M, y_M) , when viewed from above (see Supplemental Material [22]). The weight function $\cos(2\theta)$ is the dominant Fourier mode of the unstable eigenmode associated with the limit point. The desired actuator flow rates are calculated according to

$$\tilde{Q}_{1,2}(t) = \mp \alpha(x_M(t) - x_T) + \beta(m(t) - m_T). \quad (2)$$

The first and second terms enable positional and shape control toward target values x_T and m_T respectively; these two scalar values are the only characteristics required from a target state. The feedback gains $\alpha = 1370$ mm²/min and $\beta = 8607$ mm²/min were chosen heuristically so that equilibrium bubbles would be stabilized across a reasonable range of parameters (see Supplemental Material [23]).

The first branch of equilibria can be accessed without the use of CBC. We used control only to maintain the centroid of the bubble at the center of the channel $x_T = 0$ (Fig. 2). In contrast, bubbles on the second branch are unstable to both shape deformation and translational modes, so control of both shape and position is required. However, the bubble shape on the second branch is not known *a priori*, and so we must determine the value of m_T corresponding to noninvasive control; a central task of CBC. We seek values of m_T , and the corresponding bubble shapes, where feedback actuation fluctuates about zero; in these cases control is noninvasive, and so the controlled equilibria would also be (perhaps unstable) equilibria in the absence of control.

Each pair of actuator ports was connected to separate channels of a pressure controller (OB1 MK3+TM, Elveflow), which adjusted pressure on the millisecond scale. To achieve flow rate actuation on the same time scale as pressure, we

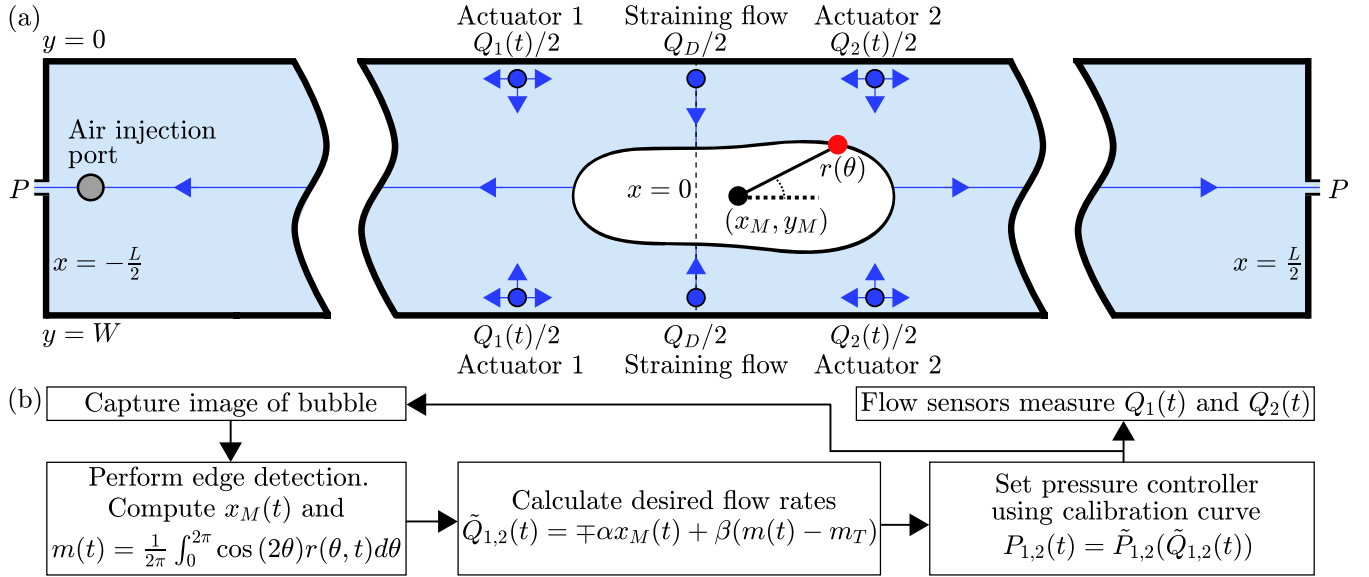


FIG. 2. (a) Top-view schematic diagram of the Hele-Shaw channel. The axial center of the channel is defined as $x = 0$. All fluid injection holes have a diameter 1.5 ± 0.01 mm. Holes are spaced by 25 ± 0.1 mm in the x - direction and inset by 3 ± 0.2 mm from the channel side walls. (b) The feedback control loop used to control the bubble is automated in Matlab.

calibrated actuator flow rates $Q_{1,2}$ as a function of actuator pressure $P_{1,2}$ in the absence of a bubble, using in-line flow sensors for each pair of actuators (MFS4TM, Elveflow), and applied this mapping within the feedback loop [Fig. 2(b)] (see Supplemental Material [24]). The time-dependent flow rates delivered through each pair of actuators were measured directly for accurate *a posteriori* assessment of actuation amplitude. The feedback control loop was implemented in Matlab (R2021a), interfacing via USB3 to both the pressure controller and the CMOS camera.

The real-time feedback control loop is shown in Fig. 2(b). In each loop iteration, we recorded and analyzed an image of the bubble to calculate $x_M(t)$ and $m(t)$ (see Supplemental Material [25]). These values were input into (2) to calculate the desired actuator flow rates. Images were taken at 3.7 frames per second to match the processing time of the feedback control loop, with resolution 0.073 mm per pixel in the laboratory frame.

While seeking to construct a bifurcation diagram at fixed bubble volume by varying Q_D , we found that our measurements were hampered by diffusion of air into silicone oil, especially for this pressurized channel, which caused the bubble to slowly lose volume [26] (see Supplemental Material [27]). To ensure consistency of bubble volume across experiments, we formed larger bubbles which we controlled toward a state of equilibrium, but only recorded measurements once the bubble area had decreased to 293 ± 0.1 mm². Observations of m and $Q_{1,2}$ used for data processing were taken for one further minute, over which period the bubble area decreased by no more than 0.6%, before flushing the bubble from the system (see Fig. 3 and Supplemental Material [28]).

Figures 3(a) and 3(b) show time series of the actuator flow rates and the shape measure of the bubble during stabilization under positional control for $Q_D = 0.9$ mL/min (see

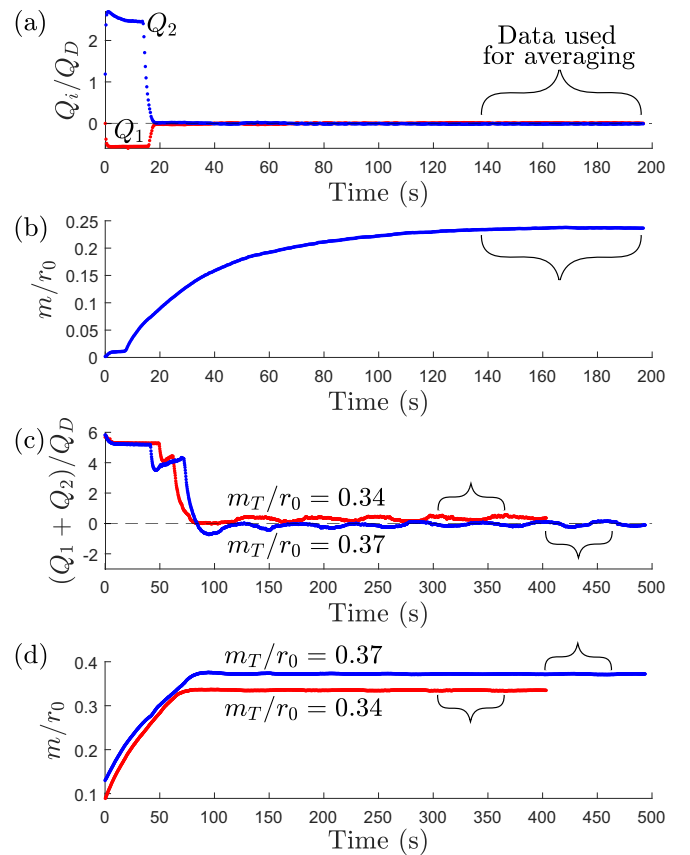


FIG. 3. Time series for actuator flow rate (a), (c) and shape measure (b), (d) when using (a), (b) positional control, and (c), (d) both positional and shape control, at fixed $Q_D = 0.9$ mL/min. Time-averages of the flow rate and shape measure are calculated over the 60-s periods highlighted.

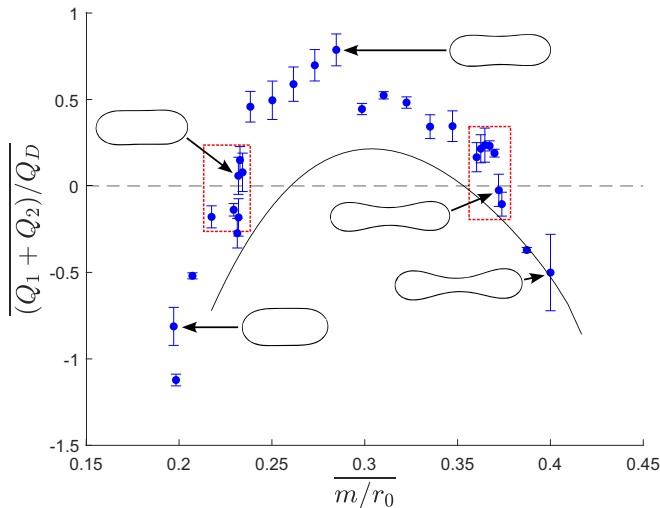


FIG. 4. Plot of time-averaged (barred) total actuator flow rate versus shape measure at $Q_D = 0.9$ mL/min. A linear fit is performed on the points in the dashed boxes. The zero crossing of this fit generates one point in the bifurcation diagram (Fig. 1). The solid line is the equivalent data for the depth-averaged model (see End Matter).

Supplemental Material [29]). The bubble is initially slightly offset, but feedback control returns the bubble to the center of the channel. Likewise, the actuator flow rates, which should be proportional to the displacement from the center of the channel [Fig. 3(a)], decay to approximately zero within 20 s. Hence, the centered bubble is noninvasively controlled. The bubble shape is not explicitly controlled, and the measure $m(t)$ reaches an equilibrium over a slower timescale [Fig. 3(b)]. We mapped the first branch in Fig. 1 by carrying out a sequence of experiments at different values of Q_D . As expected, bubbles become increasingly deformed as Q_D is increased. When $Q_D > Q_D^* \approx 0.97$ mL/min, all bubbles either evolved toward breakup or escaped the flow and thus, we did not find any equilibria.

Searching for the unstable bubble equilibria on the second branch in Fig. 1 requires both positional control to the center of the channel along with shape control toward the unknown value m_T for the second branch equilibrium state. For most values of m_T we choose, positional and shape control enables an approximately steady bubble shape to be reached [see Fig. 3(d)], but maintaining this shape usually requires ongoing injection or suction, and hence invasive control. To identify the unknown target states that yield noninvasive control, we use CBC by varying m_T for fixed $Q_D = 0.9$ mL/min, and measuring the actuation delivered once a controlled equilibrium state is reached [Fig. 3(c)]. The time period of the oscillations present in the control signal varies with the size of the bubble but is independent of Q_D , β , and m_T . Hence, it is uncertain whether the addition of integral/derivative terms to (2) would suppress these oscillations (see Supplemental Material [30]). Figure 4 shows the time-averaged total actuator flow rate plotted against the observed shape measure for such equilibria. The data indicates two zero-crossings, which correspond to noninvasive control. We determine the value of m at each crossing by fitting a straight

line to nearby data points (dashed boxes in Fig. 4). For the first zero crossing ($m = 0.23$), corresponding to the unstable equilibrium on the first branch of Fig. 1, this value is within 2% of the value found without CBC. The second zero crossing ($m = 0.37$) gives a new unstable equilibrium shape located on the second branch, which can only be accessed via CBC.

We repeated this procedure for other values of Q_D , with the previous values of m_T for noninvasive control acting as the initial guess for the next step. This CBC protocol allowed us to reach a substantial portion of the second branch of unstable equilibria. However, we were unable to access equilibria on the second branch for $Q_D < 0.775$ mL/min. By this point, the bubbles are highly deformed and often pinch off under actuation.

Although we did not rely on a mathematical model to design our control system, there are well-established depth-averaged models for bubbles confined within a Hele-Shaw channel. We used a lubrication model (see End Matter), including a factor of $\pi/4$ proposed as a strong surface-tension thin-film correction, to calculate equivalent curves to Figs. 1 and 4. The bifurcation diagram shown in Fig. 1 (solid line) and associated bubble shapes are in remarkably good quantitative agreement with the experimental data, given that there are no fitting parameters. The relative difference in Q_D for the limit point is around 5%. The comparison in Fig. 4 for actuation amplitude vs shape measure at fixed $Q_D = 0.9$ mL/min may appear less compelling, but this is an expected consequence of the discrepancy in the location of a limit point, with $Q_D = 0.9$ mL/min much closer to its limit point value in the model than in the experiment. The presence of slip conditions on the channel side walls in the Hele-Shaw model, which could promote a small enhancement of the recirculation around the bubble, could be a contributing factor to the discrepancy in Fig. 4. This would be consistent with the need for slightly reduced numerical actuation flow rates compared to the experiments. Finally, in our model, we can calculate a larger portion of the second branch curve until the branch terminates due to self-intersection. However, the model may lose validity before this point, with our 2D model unable to capture 3D effects such as the Rayleigh-Plateau instability, which may lead to earlier bubble pinch-off in the experiment.

In conclusion, we have successfully used CBC in a spatially extended experimental system, specifically the free-surface fluid mechanics problem of a bubble in a straining flow. We were able to map two branches of unstable equilibria, including navigating close to a limit point associated with bubble breakup. We used symmetries of the system and its expected equilibria to devise a suitable control strategy, without the need for a mathematical model. Our next step is to use CBC to probe a spatially extended system in which bubble propagation necessitates an array of actuators. In this context, the control problem becomes time-dependent and more difficult to analyze [31]; overcoming this complexity in experiments would represent a further significant increase in the applicability of CBC.

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Appendix: Mathematical model. Under the assumption of very slow flow (vanishing Reynolds number), the flow in the fluid domain $-L/2 < x < L/2$, $-W/2 < y < W/2$, $0 < z < H$ and excluding the bubble, is governed by the 3D Stokes equation:

$$\mu(\mathbf{u}_{xx} + \mathbf{u}_{yy} + \mathbf{u}_{zz}) = \nabla p, \quad \nabla \cdot \mathbf{u} = 0$$

subject to $\mathbf{u} = 0$ at $z = 0$, and

$$\mathbf{u} = F(x, y, t)\mathbf{e}_z = -\sum_{i=1}^6 q_i(t)f_i(x, y)\mathbf{e}_z, \quad \text{at } z = H$$

where $q_i(t)$ is the injection/actuation rate:

$$\begin{aligned} q_{1,2} &= Q_D/2, & x_{1,2} &= 0, & y_{1,2} &= \pm 17 \text{ mm} \\ q_{3,4} &= Q_1/2, & x_{3,4} &= -25 \text{ mm}, & y_{3,4} &= \pm 17 \text{ mm} \\ q_{5,6} &= Q_2/2, & x_{5,6} &= 25 \text{ mm}, & y_{5,6} &= \pm 17 \text{ mm}. \end{aligned}$$

Here 17 mm = $W/2 - 3$ mm corresponds to the inset of each injection point from the channel side walls. The functions $f_i(x, y)$ are chosen as Gaussian distributions:

$$f_i(x, y) = \exp[-((x - x_i)^2 + (y - y_i)^2)/\ell^2]/(\pi\ell^2)$$

and we use $\ell = 1.5$ mm for the smoothing lengthscale.

Under the assumption of a very thin cell ($H \ll W, L$), the horizontal velocity $\mathbf{u}_{2D}$ and depth-averaged fluid velocity $\mathbf{u}_H$ can be approximated by lubrication flow:

$$\mathbf{u}_{2D} = -\frac{1}{2\mu}(z(H-z))\nabla_H p, \quad \mathbf{u}_H = \frac{1}{H} \int_0^H \mathbf{u}_{2D} dz$$

where at leading order $p = p(x, y)$ is independent of z , and the out-of-plane velocity w is defined by

$$\nabla_H \cdot \mathbf{u}_{2D} + \frac{\partial w}{\partial z} = 0$$

with the boundary conditions $w = 0$ at $z = 0$, $w = F(x, y, t)$ at $z = H$. These combine to give the mass conservation equation:

$$F(x, y, t) = -\frac{H^3}{12\mu}\nabla^2 p$$

which is the single PDE to be solved in the fluid domain.

The shape of the bubble boundary $\partial\Omega$ is determined implicitly so that two boundary conditions are satisfied, corresponding to mass conservation and a version of the Young-Laplace equation, respectively:

$$\mathbf{u}_H \cdot \mathbf{n} = \frac{\partial \mathbf{R}}{\partial t} \cdot \mathbf{n}, \quad p - p_b = \sigma\left(\frac{2}{b} + \frac{\pi}{4}\kappa_H\right). \quad (\text{A1})$$

Here $2/b$ is the curvature of a semi-circular interface confined within the height of the channel, and κ_H is the curvature of the bubble when viewed in the (x, y) plane. The factor $\pi/4$ in (A1) is the leading order correction for slowly moving interfaces (small local $Ca = \mu U/\sigma$, where U is the interface speed in the laboratory frame) for bubbles spreading or propagating in Hele-Shaw cells, as proposed by researchers including [32,33]. In equilibrium, $U = 0$ so the $\pi/4$ term is the only relevant correction. Omitting this correction factor is equivalent to scaling the model predictions in Fig. 1 by $\pi/4$ in the Q_D direction, while the data in Fig. 4 would need to be recalculated.

The bubble pressure p_b is set by the volume constraint:

$$\int_{\Omega} H dx dy = \pi r_0^2 H$$

where we assume that the bubble fills the whole height of the channel. The remaining boundary conditions are (i) no fluid flow through the side walls of the channel: $\mathbf{u}_H \cdot \mathbf{e}_y = 0$ at $y = \pm W/2$, and (ii) the boundary condition $p = 0$ at $x = \pm L/2$ as an approximation of the localized outlet ports in Fig. 2.

We formulate the system in terms of dimensionless variables, and discretize and solve using a finite-element method implemented in the finite-element library `oomph-lib` with a boundary-fitted deformable mesh, in a similar manner to Refs. [31,34].

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- [23] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/nznz-zc63> for discussion of the choice of α and β .
- [24] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/nznz-zc63> for design and validation of this calibration curve.
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- [28] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/nznz-zc63> for details of post-experiment data processing.
- [29] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/nznz-zc63> for animations.
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