

Experimental observations of cluster synchronization in a biologically inspired neuronal network of chaotic electronic oscillators

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This study investigates the emergence of cluster synchronization in a biologically inspired neuronal network composed of 59 coupled chaotic electronic oscillators that is modeled after a dissociated rat cortical culture. A recently introduced spectral block method is employed to predict the formation and sequence of synchronization clusters based solely on the network's structural connectivity, although the exact transition points depend on the values at which the master stability function changes sign. The analysis is conducted through numerical simulations of Rössler oscillators, which allow for a controlled numerical investigation, followed by a second part devoted to the experimental study of a circuit board composed of single-transistor chaotic oscillators. In both cases, the results confirm the expected hierarchical dynamical transition from incoherence to global

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synchronization via an intermediate cluster state. Furthermore, the analysis of the network of single-transistor chaotic units provides the first experimental validation of the spectral block method and demonstrates that cluster synchronization arises naturally in networks without intentionally designed symmetries. The experimental platform, which features a scalable and reconfigurable electronic network, thus bridges theoretical predictions and real-world observations, highlighting the robustness of cluster synchronization in the presence of nonidealities, noise, and parameter variations, while also emphasizing the practical considerations required when applying the method to real experiments, such as robust cluster identification and calibration of the critical points of the master stability function. These findings advance our understanding of self-organized synchronization in complex systems and have important implications for neuroscience, adaptive computing, and network engineering.

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I. INTRODUCTION

A. Synchronization in complex networks: from global to cluster

Synchronization is a pervasive phenomenon that is observed across a wide variety of complex systems and encompasses diverse disciplines such as physics, biology, engineering, and social sciences [1–4]. Specific examples of it are the synchronized flashing of fireflies [5], the coordinated firing of neuronal populations under normal and pathological conditions [6,7], and phase locking in power grids and communication systems [8,9]. The ubiquity of synchronization has motivated the substantial attention devoted over the past three decades to the study of oscillator synchronization in large-scale networks that model interactions within complex systems. While early studies primarily focused on global synchronization, i.e., the regime in which all nodes in the network converge to a common trajectory, recent research has increasingly explored cluster synchronization, where distinct subsets of nodes preferentially synchronize internally while remaining asynchronous with the rest of the network, in a configuration that does not trivially reflect the structural adjacency [10–13]. Cluster synchronization has profound implications for the function and performance of both natural and man-made systems, as it embodies a balance between integration and decoupling among units, which is a key condition for effective information processing, task differentiation, and adaptive behavior. This form of synchronization belongs to a broader class of phenomena in which synchronization does not encompass all units, including chimera states, where synchronized and unsynchronized groups coexist. Such states arise from spontaneous symmetry breaking in networks of identical oscillators [14–22] and are associated with various real-world phenomena, such as unihemispheric sleep in migratory birds [23,24] and the stationary bump state in neuronal networks [25,26].

The types of synchronization that can ensue are driven by the structural arrangements of interactions among coupled units. For instance, modular, scale-free, or core-periphery topologies dictate the stability of particular synchronization manifolds, resulting in the establishment of hierarchical or clustered synchronization patterns [27–29]. In global synchronization, the conditions for existence and stability have been elucidated through the master stability function (MSF) formalism [30], which provides a compact criterion on the Laplacian eigenvalues depending both on the dynamics of the individual units and the coupling functions. For cluster synchronization, it was recognized that symmetries

determine which cluster configurations are dynamically realizable [31–35]. While such symmetries are easily identifiable in small or highly regular networks, they become increasingly difficult to detect in large-scale or heterogeneous topologies. To address this challenge, fundamental for predicting the emergence of specific cluster synchronization patterns, several theoretical methods based on group theory and symmetry analysis have been developed, particularly for networks of identical oscillators [36–39]. To accommodate more general network structures, group-theoretic approaches have been extended through the groupoid formalism, which relaxes the requirement of global symmetries through introducing bijections between input node sets. This framework enables the classification of synchrony patterns even in systems lacking strict symmetry [40]. Other approaches, including external equitable partitions and eigenvector-based analysis, have been proposed and facilitate the identification of dynamically invariant subspaces and functional modules in large and possibly asymmetric graphs [41–43]. Other efforts have addressed how specific features, such as those characterizing multilayer networks and time delayed coupling, or directed interactions, influence the formation, stability, and transitions of synchronous clusters [44–46].

B. Prediction, design, and control of cluster synchronization

A recent important theoretical advance in the study of cluster synchronization is the development of a computationally efficient method capable of predicting the entire sequence of cluster formation during the transition to global synchronization, as the coupling strength among units increases [47]. The method, based on the notion of spectral blocks of the network Laplacian, first enables the identification of specific nodes involved in each cluster, second, determines the sequence in which these transitions unfold as the coupling strength increases, and, finally, provides the exact critical coupling strengths at which synchronization transitions occur. Crucially, this method does not rely on the presence of exact symmetries. Instead, it identifies clusters as groups of nodes that receive indistinguishable inputs from the rest of the network. This generalization significantly broadens its applicability to a larger range of network topologies, laying the foundation for predictive models of cluster synchronization in real-world systems. The method plays a central role in the study presented in this paper, as will be discussed in detail below.

Understanding the principles that govern the formation of synchronous clusters is fundamental to not only analyzing

existing systems but also designing the networks that exhibit deliberate synchronization patterns. For example, Klickstein and Sorrentino [48] proposed an algorithm to generate networks with prescribed symmetries. However, scaling this method to large networks presents substantial computational challenges. Khanra *et al.* [49] overcame this limitation in their framework, which leverages the distribution of eigenvector centrality to construct networks with predefined modular structures, thereby establishing a direct correspondence between the network's structure and its modular dynamical behavior, and thus finally offering a scalable and analytically grounded solution. A complementary approach proposed by Gambuzza *et al.* [50] introduced minimal structural modifications to an existing network in order to endow it with a desired set of symmetries. Following the same rationale, Tomaselli *et al.* [51] achieved control of cluster synchronization by introducing structural modifications to the network that specifically target changes in the structure of spectral blocks. Other approaches for dynamically regulating cluster size and stability rely on adjusting the system parameters. For instance, it has been experimentally demonstrated that reducing the coupling resistance between nodes in a large ring network of 30 single-transistor chaotic oscillators (similar to those used in this work) leads to various synchronization phenomena, including phase synchronization, partial synchronization, and ultimately complete synchronization [52]. Moreover, he demonstrated that the addition of long-distance connections between a node within a cluster and a distant one can result in the obliteration of the cluster. Similarly, Fan *et al.* [53] demonstrated the precise tuning of the size of the synchronous clusters by adjusting a single system parameter for networks of coupled chaotic oscillators.

C. Earlier experimental works on synchronization and their significance

While theoretical and computational studies provide profound insights into cluster synchronization and related phenomena in complex networks, experimental validation remains elusive. Real-world systems are inherently subject to factors such as component variability, nonideal behaviors, noise, delays, and structural imperfections, which can significantly affect the synchronization dynamics and also the stability of cluster states. These considerations are crucial for evaluating the robustness and practicality of theoretical predictions, ensuring that the observed phenomena are not simply artifacts of idealized assumptions. Consequently, a variety of experimental platforms have been employed to explore different types of synchronization. Optoelectronic and photonic systems, including semiconductor laser arrays, have successfully demonstrated complex patterns such as chimera states and regimes of partial synchronization [54–58]. Chemical reaction-diffusion systems, which are based on the Belousov-Zhabotinsky reaction in particular, have also been used to study pattern formation and synchronization in excitable media [59,60]. Mechanical systems, such as coupled metronomes and simple pendulums, provide more accessible demonstrations of partial synchronization and phase clustering [61–65]. Electronic circuits play a central role in experimental platforms for studying synchronization patterns in networks

of coupled oscillators because they enable precise control over their parameters and coupling strengths, enabling more detailed studies of system dynamics [52,66–68]. In addition, they are programmable and inherently easy to integrate, thus supporting dynamic reconfiguration of the network topologies. Despite previous progress and the enormous potential of electronic systems, most existing experimental platforms still face significant limitations in scalability and flexibility. For example, optoelectronic systems typically rely on small-scale setups or time-delay architectures, while mechanical and electronic networks typically support a large number of oscillators but are typically limited to conventional topologies such as rings, stars, or lattices. Therefore, exploring cluster synchronization in large-scale physical oscillator networks with heterogeneity, irregular structures, and long-range couplings (e.g., small-world topologies) remains an open challenge.

At the same time, a deeper understanding of the mechanisms underlying cluster synchronization is of both theoretical and practical significance. For example, in neuroscience, cluster synchronization has been recognized as a fundamental dynamical mechanism underpinning the functional organization of the brain. During specific cognitive tasks or brain states, anatomically segregated brain regions coordinate their activity via transient synchronization patterns, thus forming dynamic, functional modules that enable specialized processing while supporting global integration. This modular synchronization is believed to be fundamental to core processes such as attention, perception, and memory, enabling the brain to seamlessly alternate between segregated and integrated functional modes [69–72]. Modeling such dynamical mechanisms remains challenging due to the complex interplay between heterogeneous topologies, nonlinear and often chaotic or excitable node behaviors, and varying interaction strengths. A recent study presents the first attempt to design an experimental electronic neuronal culture model (NCM), which is based on 59 single-transistor chaotic oscillators [73]. The topology of this electronic network, derived from a mesoscopic representation of a real dissociated neuronal culture, demonstrates a complex interplay between structural connectivity and node dynamics, leading to a nontrivial correlation between topological properties of nodes and signal complexity during synchronization. Here, we complement that study by investigating the onset of synchronous clusters in the electronic model of the neuronal culture—an aspect not explored in Ref. [73]. While Ref. [73] introduced the hardware platform and primarily focused on the degree of global synchronization across the entire ensemble of units, the present work concentrates on cluster synchronization. This analysis is not merely an extension demonstrating that a different synchronization regime can occur in the same network (in cluster synchronization, nodes split into synchronized groups, whereas in global synchronization all nodes converge to the same trajectory). Rather, as noted above, the emergence of clusters is particularly significant in the context of neuroscience, as it reflects a state in which the network becomes functionally organized into groups exhibiting differentiated activity. Therefore, although Ref. [73] and the present study rely on the same experimental platform, the phenomena investigated are fundamentally different.

D. Brief insight of our work

Our work provides the first observation and systematic characterization of spontaneous cluster synchronization in a large-scale (as compared with previous works), biologically inspired electronic network. By leveraging the spectral block approach [47], we demonstrate that it is possible to predict both the composition and sequence of synchronous clusters in the neuronal culture model [73], as well as the critical coupling strengths at which each cluster synchronizes, thereby validating the method on experimental data. Crucially, the technique relies solely on the network's structural connectivity, making it applicable across a wide range of unit dynamics. In fact, our analysis is first numerically benchmarked using an ensemble of Rössler systems coupled according to the topology of the neuronal culture [73]. It is then experimentally validated using signals recorded from a custom-built electronic oscillator network, where system dynamics are measured over a systematically varied range of coupling strengths. Our results show that the spectral block approach consistently predicts the cluster composition and the correct sequence of synchronization, as the coupling strength varies, in both the numerical model and the real electronic system. However, while, for the network of Rössler systems, the exact threshold values at which the clusters synchronize can be theoretically derived using the system MSF, our results for the electronic circuit highlight the importance of estimating these thresholds based on critical parameters obtained from experimental data or circuit-level simulations.

The remainder of the paper is organized as follows. Section II describes the experimental setup, data acquisition, and preprocessing, as well as the theoretical methods used for analysis. Section III presents the results. First, the topological and spectral block analysis of the network modeling the interactions between the electronic circuits is presented. Then, the dynamical behavior of synchronous clusters in simulations of Rössler units and, subsequently, the experimental results obtained from the electronic platform are provided. Section IV offers a discussion of the results and draws general conclusions. Finally, a detailed description of the experimental setup, data collection, and data preprocessing is provided in Appendix A, while the topology and working principle of the single-transistor chaotic circuit are briefly summarized in Appendix B.

II. MATERIALS AND METHODS

A. Experimental platform and network topology

A conceptual overview of our study is presented in Fig. 1. The dashed boxes indicate the steps carried out in Ref. [74], where the spike activity of neurons within a self-organized network formed in *in vitro* cultures of dissociated neurons is recorded using multielectrode arrays (MEAs). The following steps in the outlook represent the procedures related to the extraction of functional connectivity from spiking activity, physical network realization, and preliminary synchronization analysis, which are addressed in Ref. [73]. The last two steps of the outlook, which involve a detailed investigation of cluster synchronization and its theoretical prediction via spectral block analysis, are the focus and unique

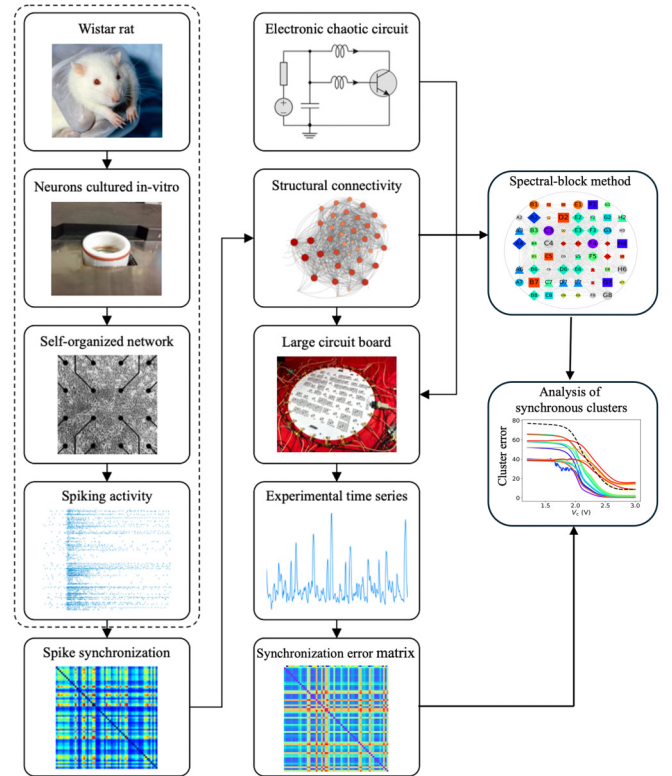


FIG. 1. Conceptual outlook of the study, depicting the steps from the *in vitro* neuronal culture to the analysis of the time series generated by the experimental setup. Dashed box denotes the steps undertaken in previous work, namely, public data drawn from Ref. [74]. Reprinted and adapted from Ref. [73], with permission from Elsevier.

contribution of the present paper. Specifically, to investigate the emergence of cluster synchronization as the coupling strength varies, the time series recorded from the physical system are preprocessed and analyzed to construct a series of synchronization matrices, each corresponding to a value of the coupling strength. From there, we identify the synchronous clusters at each coupling level. The resulting sequence of cluster formations is then compared to the theoretical predictions obtained by applying the spectral block method directly to the Laplacian matrix that encodes the structural connectivity of the physical network.

The network investigated in this work consists of 59 chaotic electronic oscillators coupled according to a topology derived from a dissociated rat cortical culture (see Appendix A for details). Termed the neuronal culture model, the network is undirected, connected, and contains no self-loops. Accordingly, its adjacency matrix is irreducible, symmetric, and has a zero diagonal. The associated Laplacian matrix is therefore positive semidefinite, with a single zero eigenvalue ($\lambda_1 = 0$) and all remaining eigenvalues real and strictly positive. The network exhibits small-world properties and a bimodal degree distribution, as discussed in Appendix A. Each node is implemented using a single-transistor chaotic oscillator, that is, a minimalistic transistor-based circuit introduced in Ref. [75] and capable of exhibiting rich nonlinear dynamics, including spiking and bursting behaviors similar to

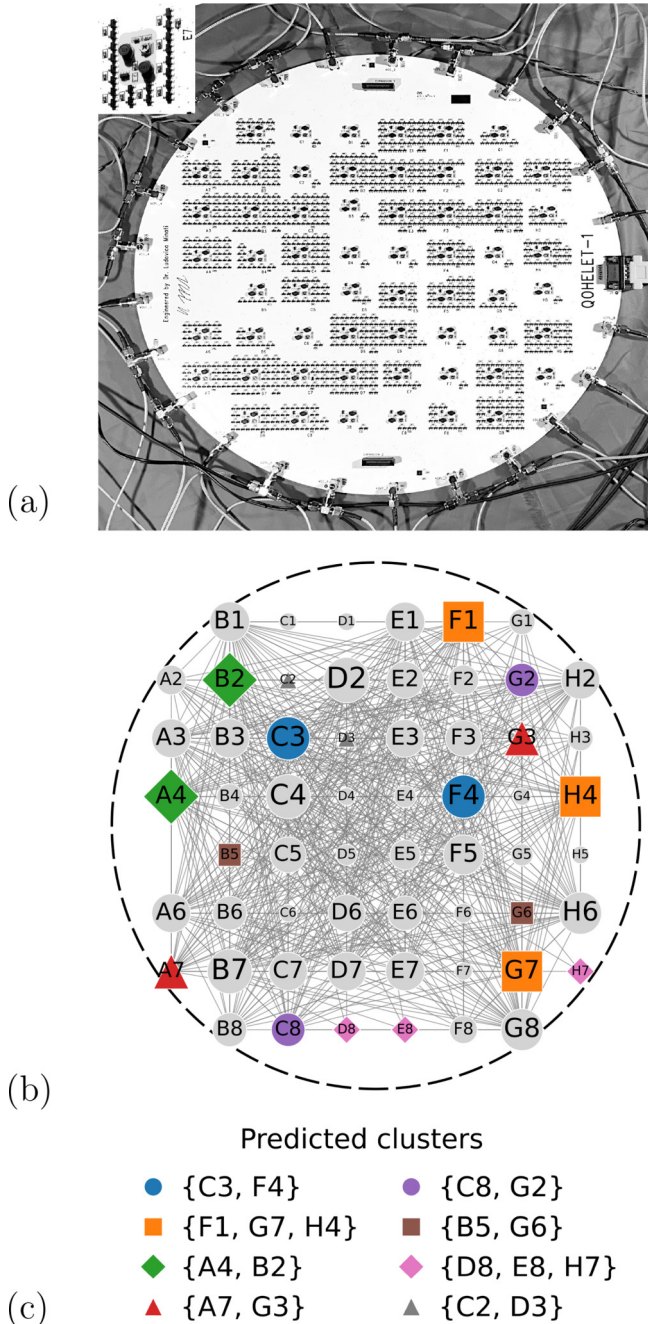


FIG. 2. (a) Experimental setup of the NCM network with each node (represented by letter and number format) corresponding to a single-transistor chaotic oscillator. Reprinted from Ref. [73], with permission from Elsevier. Additional wiring for simultaneous signal acquisition not shown. (b) Visualization of the NCM network overlaid on the circuit board with color coding scheme. The node size is proportional to its degree, while the marker shapes with colors indicate the clusters predicted by the method [47]. (c) Marker shape and color coding scheme for the predicted clusters.

biological neurons. A detailed description of the circuit and its mathematical model is provided in Appendix B. Figure 2(a) illustrates the electronic board made of 59 single-transistor chaotic circuits coupled according to the NCM network, while Fig. 2(b) depicts the nodes and links of the structure. In

order to maintain correspondence between the physical board and the spatial organization of the original neuronal culture, each node in the network is labeled using an alphanumeric coordinate system based on an 8×8 grid, where columns are indexed by letters (A–H) and rows by numbers (1–8). For example, the node label A2 denotes the element located in column A and row 2. Since the network contains only 59 nodes, corresponding to the active electrodes in the neuronal culture, unused labels (A1, A5, A8, H1, H8) are excluded.

In our experiments, signals from all oscillators were acquired simultaneously at 250 kHz using a high-performance data acquisition card. The coupling strength between nodes was controlled by an external voltage V_c , which modulates the effective internode resistance. For each configuration of the circuit parameters (R , V_S), V_c was varied from 1.2 to 3.0 V in increments of 0.01 V. Synchronization errors were computed from the recorded time series and averaged over multiple experimental runs to reduce noise and transient effects. Full details of the experimental setup, data acquisition, and pre-processing are provided in Appendix A.

B. Networks of nonlinear oscillators

The network of circuits implemented in the experimental setup is here described using a general formulation that allows for a broader treatment and can be readily adapted to accommodate different nodal dynamics. The network is represented by a system of $N = 59$ coupled dynamical units and each unit is described by an n -dimensional state vector $\mathbf{x}_i = [u_{C1}^{(i)}, u_{C2}^{(i)}, i_{L1}^{(i)}, i_{L2}^{(i)}]^T \in \mathbb{R}^n$, with $n = 4$ and $i = 1, \dots, N$ (here, $u_{C1}^{(i)}$, $u_{C2}^{(i)}$, $i_{L1}^{(i)}$, and $i_{L2}^{(i)}$ denote the key voltage and current signals that form the state vector of the single-transistor chaotic circuit, described in Appendix B). The isolated dynamics of each unit are governed by a nonlinear vector field

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i), \quad (1)$$

where $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ describes the intrinsic dynamics of the system, and, for the single-transistor chaotic circuit, is given by

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} \frac{V_S - u_{C1}}{RC_1} - \frac{i_{L1} + i_{L2}}{C_1} \\ i_{L2} - \beta \Gamma(i_{L1}) \tanh\left(\frac{u_{C2}}{2V_{th}}\right) \\ \frac{u_{C1} - V_{th}}{L_1} \\ \frac{u_{C1} - u_{C2}}{L_2} \end{bmatrix}. \quad (2)$$

The units are coupled through a network represented by an undirected graph with adjacency matrix A , whose elements take binary values: the generic element A_{ij} is set to $A_{ij} = 1$ when the units i and j are connected, and to $A_{ij} = 0$ otherwise. The coupling reflects the structural connectivity derived from the functional analysis of neuronal culture data, and is physically implemented by a pair of metal-oxide-semiconductor field-effect transistors (MOSFETs) controlled by a voltage input V_c (the coupling circuitry is described in more detail in Appendix B). The coupling occurs via the variables $u_{C1}^{(i)}$ and $u_{C1}^{(j)}$ and, to simplify the analysis, is here modeled as a diffusive term $\frac{u_{C1}^{(j)} - u_{C1}^{(i)}}{RC_{C1}}$ appearing (with opposite signs) in the

equations of both variables. Here, R_C represents the equivalent resistance of the coupling circuitry, controlled by the voltage V_C .

Taking into account this equivalent formulation of the coupling, the equations of the coupled system can be compactly written as

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) + d \sum_{j=1}^N A_{ij} H(\mathbf{x}_j - \mathbf{x}_i), \quad i = 1, \dots, N, \quad (3)$$

where $d > 0$ is the coupling strength, assumed to be uniform despite unavoidable parameter tolerances in the physical system, and $H \in \mathbb{R}^{n \times n}$ is a matrix of constant terms encoding which variables are used for coupling. In the case of the network of single-transistor chaotic circuits, the coupling coefficient d is given by $d = \frac{1}{R_C C_1}$, while the matrix H is given by

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (4)$$

1. Global synchronization

For convenience, Eq. (3) can be written using the Laplacian matrix L defined by

$$L_{ij} = \begin{cases} -A_{ij}, & i \neq j, \\ \sum_{k=1}^N A_{ik}, & i = j. \end{cases}$$

In doing so, one obtains

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) - d \sum_{j=1}^N L_{ij} H(\mathbf{x}_j), \quad i = 1, \dots, N, \quad (5)$$

The matrix L is zero-row sum by construction. In addition, since the graph is undirected and connected, then L is symmetric, positive semidefinite, and admits an orthonormal eigenbasis with eigenvalues (sorted in increasing order): $0 = \lambda_1 < \lambda_2 \leq \dots \leq \lambda_N$. The corresponding orthonormal eigenvectors are indicated as $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_N$, with $\mathbf{v}_1 = \frac{1}{\sqrt{N}}[1, 1, \dots, 1]^T$. These properties will be useful in the following.

The zero-row sum property of the Laplacian matrix guarantees the existence and invariance of a solution where all nodes converge to the same trajectory

$$\mathbf{x}_1(t) = \mathbf{x}_2(t) = \dots = \mathbf{x}_N(t) = \mathbf{x}_s(t). \quad (6)$$

This equation represents the global synchronization manifold, which is the subspace of the high-dimensional space determined by the state variables of all oscillators, defining the properties of a synchronous trajectory. Namely, each trajectory followed in the regime of global synchronization lies in this subspace. The synchronization manifold always exists, but can be either stable or not. The stability analysis determines whether small deviations from this manifold (such as tiny perturbations to one oscillator) will decay or grow in time. In the first case, global synchronization is stable, while in the second case, it is unstable. A remarkable property of the synchronous solution, due to the zero-row sum property of L , is that, on the synchronization manifold, the dynamics

reduces to that of the isolated system

$$\dot{\mathbf{x}}_s = \mathbf{f}(\mathbf{x}_s). \quad (7)$$

To analyze the stability of the synchronization manifold, following the approach introduced in Ref. [30], we consider small deviations $\delta \mathbf{x}_i = \mathbf{x}_i - \mathbf{x}_s$ and linearize Eq. (5) around $\mathbf{x}_s$. This yields the following set of linear, time-varying equations for the deviations:

$$\delta \dot{\mathbf{x}} = [I_N \otimes \mathbf{J}\mathbf{f}(\mathbf{x}_s) - d L \otimes H] \delta \mathbf{x}, \quad (8)$$

where $\delta \mathbf{x} = (\delta \mathbf{x}_1^T, \dots, \delta \mathbf{x}_N^T)^T$, I_N is the $N \times N$ identity matrix, and $\mathbf{J}\mathbf{f}(\mathbf{x}_s)$ is the Jacobian matrix of $\mathbf{f}$ evaluated at $\mathbf{x}_s$.

For the single-transistor chaotic circuit, $\mathbf{J}\mathbf{f}(\mathbf{x}_s)$ is given by

$$\mathbf{J}\mathbf{f}(\mathbf{x}_s) = \begin{bmatrix} -\frac{1}{RC_1} & 0 & -\frac{1}{C_1} & -\frac{1}{C_1} \\ 0 & \frac{\beta \Gamma(i_{L1}) (\tanh^2(\frac{u_{C2}}{2V_{th}}) - 1)}{2C_2 V_{th}} & -\frac{\beta \Theta(i_{L1}) \tanh(\frac{u_{C2}}{2V_{th}})}{C_2} & \frac{1}{C_2} \\ \frac{1}{L_1} & 0 & 0 & 0 \\ \frac{1}{L_2} & -\frac{1}{L_2} & 0 & 0 \end{bmatrix}_{\mathbf{x}_s} \quad (9)$$

where $\Theta(x)$ is the Heaviside step function defined as $\Theta(x) = 1$ if $x > 0$, and $\Theta(x) = 0$ otherwise.

Let T be the $N \times N$ matrix that diagonalizes the Laplacian matrix L , that is, $T^{-1} L T = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_N)$. By applying a linear transformation that introduces the new variables $\xi = (I_N \otimes T^{-1}) \delta \mathbf{x}$, the dynamics of the deviations decouples into N blocks that differ only for a single parameter (the eigenvalue of L appearing in them), that is,

$$\dot{\xi}_i = [\mathbf{J}\mathbf{f}(\mathbf{x}_s) - d \lambda_i H] \xi_i, \quad i = 1, \dots, N, \quad (10)$$

where ξ_i is the perturbation mode associated with eigenvalue λ_i . The mode corresponding to $\lambda_1 = 0$ describes perturbations along the synchronization manifold and is neutral. The transverse modes $i = 2, \dots, N$ determine the stability of synchronization. Specifically, stability requires that each of these transverse modes asymptotically converges to zero.

From Eq. (10), one can parametrize the dependence on the Laplacian eigenvalues using $\nu = d \lambda_i$, which yields the following master stability equation (MSE):

$$\dot{\xi} = [\mathbf{J}\mathbf{f}(\mathbf{x}_s) - \nu H] \xi, \quad (11)$$

where $\nu \in \mathbb{R}$ is a scalar parameter. This parametrization is a crucial step as it separates the dynamics (determined by $\mathbf{f}$ and H) of a single unit from the network structure (encoded by the eigenvalues of the Laplacian matrix) appearing in Eq. (8).

From the MSE, we can calculate the master stability function, denoted by $\Lambda(\nu)$ and defined as the largest Lyapunov exponent computed from the MSE (11) as a function of ν . This provides a complete characterization of the stability of the synchronization manifold $\mathbf{x}_s$ for given $\mathbf{f}$ and H . In fact, stability requires that

$$\Lambda(d \lambda_i) < 0, \quad \forall i = 2, \dots, N. \quad (12)$$

Depending on the shape of $\Lambda(\nu)$, three classes of MSF can be distinguished:

(1) Class I: $\Lambda(\nu) > 0$ for all ν ; synchronization is never stable.

(2) Class II: There exists a critical v^* such that $\Lambda(v) < 0$ for all $v > v^*$; synchronization is stable provided that the coupling is selected such that $d\lambda_2 > v^*$.

(3) Class III: $\Lambda(v) < 0$ only for v in a finite interval $[v_1^*, v_2^*]$; synchronization requires that $d\lambda_i$ for $i = 2, \dots, N$ lie within this interval.

2. Cluster synchronization

In this section, we illustrate how the MSF-based approach, described in the previous section that applies to global synchronization, can be extended to address the case in which units synchronize in clusters. As mentioned in the Introduction, the emergence of synchronous clusters is fundamentally linked to structural properties of the network. The analysis carried out in this work relies on the theoretical result that connects the formation of synchronous clusters to the presence of spectral blocks in the network's Laplacian matrix [47].

A spectral block $\mathcal{S}$ localized at k nodes indexed as $\{i_1, i_2, \dots, i_k\}$ is defined as a subset $\mathcal{S}$ of the set of vectors $\mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_N$, namely, $\mathcal{S} \subseteq \{\mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_N\}$, that contains $k - 1$ eigenvectors of L satisfying the following properties:

(1) Each $\mathbf{v} \in \mathcal{S}$ is such that $v_j = 0$ for all $j \notin \{i_1, i_2, \dots, i_k\}$. This means that, for each eigenvector of the spectral block, only the elements at the index positions given by $\{i_1, i_2, \dots, i_k\}$ are nonzero.

(2) Each $\mathbf{v} \notin \mathcal{S}$ has identical entries for $j \in \{i_1, i_2, \dots, i_k\}$ (i.e., $v_{i_1} = v_{i_2} = \dots = v_{i_k}$). This means that any eigenvector of L that does not belong to the spectral block has equal entries at the index positions $\{i_1, i_2, \dots, i_k\}$.

The above definition of spectral blocks implies that the nodes $\{i_1, i_2, \dots, i_k\}$ form a cluster with the property that

$$L_{pj} = L_{qj}, \quad \forall p, q \in \{i_1, \dots, i_k\}, \quad \forall j \notin \{i_1, \dots, i_k\}. \quad (13)$$

This condition ensures that all nodes in the cluster are indistinguishable in terms of the input they receive from the rest of the network (i.e., the nodes not in the cluster). Quite interestingly, this condition does not give any constraint on the connections among the nodes within the cluster, which can be arbitrary. Clusters so defined constitute the cells of an *external equitable partition* [42]. In fact, the equivalence between spectral blocks and the cells of an external equitable partition has been proved (see the Supplementary Material of Ref. [47]). Furthermore, using spectral blocks has the advantage of enabling the analysis of the synchronization stability of the clusters, thus allowing prediction of their sequence of synchronization/desynchronization.

Having defined the structural condition for the emergence of clusters, we now turn to the analysis of the stability of cluster synchronization. To this end, we consider an arbitrary cluster C induced by a spectral block, consisting of $|C| = k$ nodes. The nodes $\{i_1, i_2, \dots, i_k\}$ of C synchronize if

$$\mathbf{x}_{i_1}(t) = \mathbf{x}_{i_2}(t) = \dots = \mathbf{x}_{i_k}(t) = \mathbf{x}_C(t). \quad (14)$$

Following steps similar to those leading to the definition of the MSF for global synchronization in Sec. II B 1, the stability of the synchronous cluster can be analyzed by linearizing the dynamics around the solution $\mathbf{x}_C(t)$. In more detail, from Eq. (5), we can write the dynamics of a node of the cluster,

$i \in C$, as follows:

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) - d \sum_{j \in C} L_{ij} H(\mathbf{x}_j) + d \sum_{j \notin C} A_{ij} H(\mathbf{x}_j), \quad (15)$$

where the coupling term is split into an intracluster and an intercluster contribution. The synchronous cluster solution obeys

$$\dot{\mathbf{x}}_C = \mathbf{f}(\mathbf{x}_C) + d \sum_{j \notin C} A_{ij} (H(\mathbf{x}_j) - H(\mathbf{x}_C)). \quad (16)$$

With calculations similar to the ones done for the analysis of stability of global synchronization, one obtains that the deviations around the synchronous cluster state, i.e., $\delta \mathbf{x}_{i,C} = \mathbf{x}_i - \mathbf{x}_C$ with $i \in C$, evolve according to

$$\delta \dot{\mathbf{x}}_C = [I_k \otimes J\mathbf{f}(\mathbf{x}_C) - d L \otimes H] \delta \mathbf{x}_C, \quad (17)$$

where $\delta \mathbf{x}_C \in \mathbb{R}^{kn}$ and $k = |C|$.

However, in contrast to what is found in global synchronization, the dynamics of $\mathbf{x}_C(t)$, as described by Eq. (16), are not isolated, but are influenced by inputs from nodes outside the cluster that are not synchronized. As a result, the trajectory of $\mathbf{x}_C(t)$ generally differs from that of the isolated system $\dot{\mathbf{x}}_C = \mathbf{f}(\mathbf{x}_C)$. Consequently, the synchronous cluster solution $\mathbf{x}_C(t)$ is not invariant and depends on the extra term $d \sum_{j \notin C} A_{ij} (H(\mathbf{x}_j) - H(\mathbf{x}_C))$ in Eq. (16).

Following the steps described in Ref. [47], the influence of the extra-cluster coupling term on the cluster trajectory is assumed to be negligible. More precisely, we posit that, for the purpose of computing the maximum Lyapunov exponent of the transverse modes, the trajectory of the synchronous cluster can be approximated by that of global synchronization, with only a minor impact on the critical coupling strengths at which synchronization transitions occur. Under this assumption, the stability of each synchronous cluster state can thus be assessed using the MSF $\Lambda(v)$, as in Eq. (11).

The approximation we use is justified by the following considerations. First, during the transition to synchronization, the coupling strength d is relatively small, so the extra-cluster term in Eq. (16) is of order d . Consequently, its influence on the cluster trajectory is relatively small. Second, the inputs from nodes outside the cluster are typically uncorrelated, leading to a net effect that fluctuates around zero over time. Since the maximum Lyapunov exponent is an average measure computed over a long trajectory, these fluctuations are smoothed out. Therefore, the maximum Lyapunov exponent computed using the global synchronous trajectory provides a good approximation to that computed with the actual cluster trajectory, particularly for determining the critical coupling strengths at which the cluster synchronizes. This approximation, originally introduced in Ref. [47], is a crucial step that enables stability analysis for each cluster, thereby providing a unique insight into the dynamical processes underlying cluster synchronization. It has also been successfully applied in subsequent studies, for instance, in the design of controllers for cluster synchronization [51].

3. Cluster analysis based on spectral blocks

In this section, we outline the spectral block method for analyzing synchronous clusters. We build on the methodology

introduced in Ref. [47] and add an extra step to improve robustness against numerical inaccuracies. Starting from the graph Laplacian matrix, the method predicts both the formation of synchronous clusters and the range of coupling strength values over which synchronization within each cluster remains stable. This means that, for systems with a class II MSF, the method predicts the coupling strength at which each cluster synchronizes, while, for systems with a class III MSF, it identifies two critical values: the first marks the onset of synchronization within the cluster, while the second indicates the loss of that synchronized state.

The method consists of the following steps:

(1) *Compute the spectrum of the Laplacian matrix L .* Given the Laplacian matrix L of the graph G , we extract the spectrum of its eigenvalues $\{\lambda_i : i = \{1, 2, \dots, N\}\}$ and their corresponding eigenvectors $\{\mathbf{v}_i : i = \{1, 2, \dots, N\}\}$ (notice that eigenvalues are sorted in ascending order, i.e., $\lambda_1 < \lambda_2 \leq \dots \leq \lambda_N$).

(2) *Calculate the matrices S_n .* Let V be the matrix containing the eigenvectors $\mathbf{v}_i = [v_{i1}, v_{i2}, \dots, v_{iN}]^T$ as columns, that is,

$$V = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_N] = \begin{bmatrix} v_{11} & v_{21} & \cdots & v_{N1} \\ v_{12} & v_{22} & \cdots & v_{N2} \\ \vdots & \vdots & \ddots & \vdots \\ v_{1N} & v_{2N} & \cdots & v_{NN} \end{bmatrix}.$$

Notice that, since L is symmetric, $V^T V = I = V V^T$ and $V^T = V^{-1}$.

Now let us define for each eigenvalue λ_i the following matrix E_{λ_i} :

$$\begin{aligned} E_{\lambda_i} &= \left(\begin{bmatrix} \mathbf{v}_i^T \\ \mathbf{v}_i^T \\ \vdots \\ \mathbf{v}_i^T \end{bmatrix} - [\mathbf{v}_i \quad \mathbf{v}_i \quad \cdots \quad \mathbf{v}_i] \right) \\ &\quad \odot \left(\begin{bmatrix} \mathbf{v}_i^T \\ \mathbf{v}_i^T \\ \vdots \\ \mathbf{v}_i^T \end{bmatrix} - [\mathbf{v}_i \quad \mathbf{v}_i \quad \cdots \quad \mathbf{v}_i] \right) \\ &= \begin{bmatrix} (v_{i1} - v_{i1})^2 & (v_{i2} - v_{i1})^2 & \cdots & (v_{iN} - v_{i1})^2 \\ (v_{i1} - v_{i2})^2 & (v_{i2} - v_{i2})^2 & \cdots & (v_{iN} - v_{i2})^2 \\ \vdots & \vdots & \ddots & \vdots \\ (v_{i1} - v_{iN})^2 & (v_{i2} - v_{iN})^2 & \cdots & (v_{iN} - v_{iN})^2 \end{bmatrix}, \end{aligned} \quad (18)$$

where $\odot$ indicate the elementwise (or Hadamard) product of the two matrices.

Then, the matrices S_n are computed in decreasing order of $n \in \{N, N-1, \dots, 1\}$ as follows:

$$\begin{aligned} S_N &= E_{\lambda_N}, \\ S_{N-1} &= S_N + E_{\lambda_{N-1}}, \\ &\vdots \\ &= \end{aligned}$$

$$\begin{aligned} S_2 &= S_3 + E_{\lambda_2}, \\ S_1 &= S_2 + E_{\lambda_1}, \end{aligned} \quad (19)$$

(3) *Determine the clusters.* The matrices S_n are analyzed following the same order of computation, i.e., from the largest eigenvalue λ_N to the smallest one λ_1 . During this iterative procedure, one monitors the first occurrence of the index n , say, $n = \bar{n}$, at which one or multiple entries of S_n are equal to 2. A cluster C is formed by the nodes with the indices corresponding to these entries, while the eigenvalue $\lambda_{\bar{n}}$ is a quantity fundamental to determine when the cluster becomes synchronized. We note that the procedure involves several steps with numerical calculations, which may be affected by accuracy and precision. For instance, the condition that entries of the matrix S_n equal 2 must be checked within a numerical tolerance. To mitigate potential numerical errors in the procedure, we incorporate an additional check compared to the original methodology. Let $\pi = \{C_1, C_2, C_3, \dots\}$ be the partition or set of clusters in sequence obtained so far. We explicitly check that each of the clusters identified satisfies the condition for equitable partitions [35,76]. That is, given a cluster C_l , the nodes inside it, $v_i \in C_l$, must have the same out-degree pattern with respect to any other cluster C_k (or exactly the same number of neighbors in C_k) such that $l \neq k$.

(4) *Compute the critical values for cluster synchronization stability.* The clusters determined at the previous step are then analyzed one by one. For each cluster C , we have that for a class II MSF system, it will synchronize at the coupling strength

$$d_{\bar{n}} = v^*/\lambda_{\bar{n}}, \quad (20)$$

while for a class III MSF system, the cluster will synchronize at

$$d_{\bar{n},1} = v_1^*/\lambda_{\bar{n}} \quad (21)$$

and desynchronize at

$$d_{\bar{n},2} = v_2^*/\lambda_{\bar{n}}. \quad (22)$$

Once all matrices S_n have been examined, the full sequence of transitions to or from synchronization can be determined, along with the critical coupling strengths at which the units of the cluster synchronize/desynchronize. Notice that these transitions are not always associated with the onset (or disappearance) of an isolated synchronous cluster, but they mark either the emergence of new synchronized clusters (single or multiple) or the fusion of existing clusters into a larger synchronized group. We will further discuss this point while analyzing numerical simulations of the network realized in the experimental setup.

III. RESULTS

A. Analysis of spectral blocks in the neuronal culture model network

To identify the spectral blocks in the NCM network, the procedure described in Sec. II B 3 was applied to the network's Laplacian matrix L (the full matrix is provided in the repository associated with this paper [77]). The clusters, obtained in this way, and their sequence of appearance are

TABLE I. Analysis of the spectral blocks in the NCM network. The first column labels the detected clusters in increasing order. The second column shows the index n at which the cluster emerges. The third column reports the corresponding eigenvalue. The last column illustrates the composition of the predicted cluster.

Cluster No.	n	λ	Predicted cluster
1	55	46.0000	{C3, F4}
2	50	41.0000	{F1, G7, H4}
3	45	37.0000	{A4, B2}
4	32	27.0000	{A7, G3}
5	28	24.0000	{C8, G2}
6	18	7.0000	{B5, G6}
7	11	3.0000	{D8, E8, H7}
8	9	1.0000	{C2, D3}

reported in Table I along with the index n at which the cluster is found and the corresponding eigenvalue λ_n . The nodes of these clusters are visually identified with different colors and markers, as specified in Fig. 2(c).

Specifically, we have first calculated the matrices S_n as in Eq. (19), and then have examined them in decreasing order of $n \in \{59, 58, \dots, 2, 1\}$. At each iteration n , we locate the entries of S_n that change to 2 from a different previous value. The indices of the entries identify the pairs, or groups, of nodes expected to synchronize at that stage. More specifically, the numerical criterion we use is to check whether the generic element $S_{n,ij}$ satisfies $|S_{n,ij} - 2| < \theta$, where θ represents the tolerance threshold. By inspecting the results at different settings of θ (in the range between 10^{-4} and 10^{-12}), we have found that for θ between 10^{-4} and 10^{-6} slight changes in the cluster composition with a few nodes moving from one group to another may appear, while the majority of clusters remain unchanged. In contrast, the cluster composition is unaffected by values of θ below 10^{-6} . The results here illustrated are obtained with an accuracy of $\theta = 10^{-8}$. For the groups of nodes identified in this way, we then check that they satisfy the condition for cells of an external equitable partition; i.e., all nodes within a candidate cluster have identical numbers of connections to every other cluster. In more detail, we proceed as follows: For each candidate cluster C_i and every other cluster C_j obtained from the analysis up to the current matrix S_n (with $i \neq j$), we first construct the submatrix of the adjacency matrix A corresponding to the rows in C_i and the columns in C_j . We then verify that the row sums, representing the total number of neighbors in C_j , since A is unweighted, are equal for all nodes in C_i (equality is checked within a numerical tolerance of 10^{-8}).

As Table I shows, the first cluster is found in correspondence of the matrix S_{55} and includes nodes C3 and F4. This means that the matrices S_{59} , S_{58} , S_{57} , and S_{56} do not have entries equal to 2, indicating that no clusters are detected in correspondence of the indices $n = \{59, 58, 57, 56\}$. Figure 3 showcases heatmap visualization of a few selected matrices S_n including S_{55} [Fig. 3(a)]. In such visualization, the entries equal to 2 are marked in green. In the matrices S_{54} through S_{51} , which are examined immediately after S_{55} , no

additional entries equal to 2 are found. In contrast, the analysis of S_{50} reveals the emergence of a new cluster {F1, G7, H4}. However, this is not an isolated cluster, as its nodes merge with C3 and F4 to form a larger cluster comprising five units. This conclusion derives from the inspection of the matrix S_{50} [see Fig. 3(b)], which displays elements equal to 2 also in correspondence of the indices associated with one element of the cluster {F1, G7, H4} and one of the cluster {C3, F4}. We will examine in more detail this aspect of the cluster evolution in Sec. III B 3 with the help of numerical simulations of the NCM network. As we progress further with the analysis of the matrices S_n , additional clusters are identified: At S_{45} [Fig. 3(c)] and S_{32} [Fig. 3(d)], we find clusters formed by the nodes {A4, B2} and {A7, G3}, respectively. These nodes also merge with the existing group, finally contributing to the progressive formation of a larger synchronous cluster. Altogether, the spectral block analysis of all matrices S_n for the NCM network showcases eight clusters: six clusters formed by two nodes and two clusters formed by three nodes.

Finally, we notice that some steps of the iterative procedure indicate the appearance of individual nodes. However, these instances correspond to situations where the nodes actually merge into existing clusters. Therefore, such steps do not signify the formation of new clusters but rather reflect a change in the composition of existing ones. In fact, suppose that the matrix $S_{\bar{n}}$ contains an entry equal to 2 at position (i, j) . It indicates that at this stage, the nodes i and j form a cluster $\{i, j\}$. In the subsequent matrix $S_{\bar{n}-1}$, let us assume that a new entry equal to 2 is found at position (j, k) in addition to the existing (i, j) th entry. It implies that node k is now predicted to synchronize with node j , thereby expanding the existing cluster $\{i, j\}$ to include k . In our analysis, this behavior was observed for eight nodes: A2, C4, C6, D5, F8, G8, H3, and H6. Since the inclusion of these nodes does not affect the stability properties of the clusters but only their composition, they have been excluded from the present analysis.

Next, we visually illustrate the progressive onset of synchronous clusters as the coupling strength d increases from zero. The results, shown in Fig. 4, use the same shapes and colors as in Fig. 2(c) to highlight which nodes become synchronized at different values of d . The emergence of each cluster is marked by the transition points $d_1, d_2, \dots, d_8$, at which an increasing number of nodes become synchronized. Here, the predicted transition points are given by $d_i \propto 1/\lambda_j$, where i is the cluster number and j the eigenvalue index associated to cluster i . In more detail, we have the following associations: (i, j) : (1, 55), (2, 50), (3, 45), (4, 32), (5, 28), (6, 18), (7, 11), and (8, 9) (see also Table I). We observe that the structure of the emerging clusters becomes progressively more intricate, exhibiting a nontrivial relationship with the underlying NCM network topology.

The analysis performed here relies solely on the network structure. It serves as the foundation for predicting the transitions into and out of the synchronous clusters, which, in contrast, also depend on the dynamics of the units and the specific form of the coupling function. In fact, while the sequence of clusters depends only on the structure, the critical values of the transition to synchronization (for class II MSF systems) or to synchronization and from synchronization (for

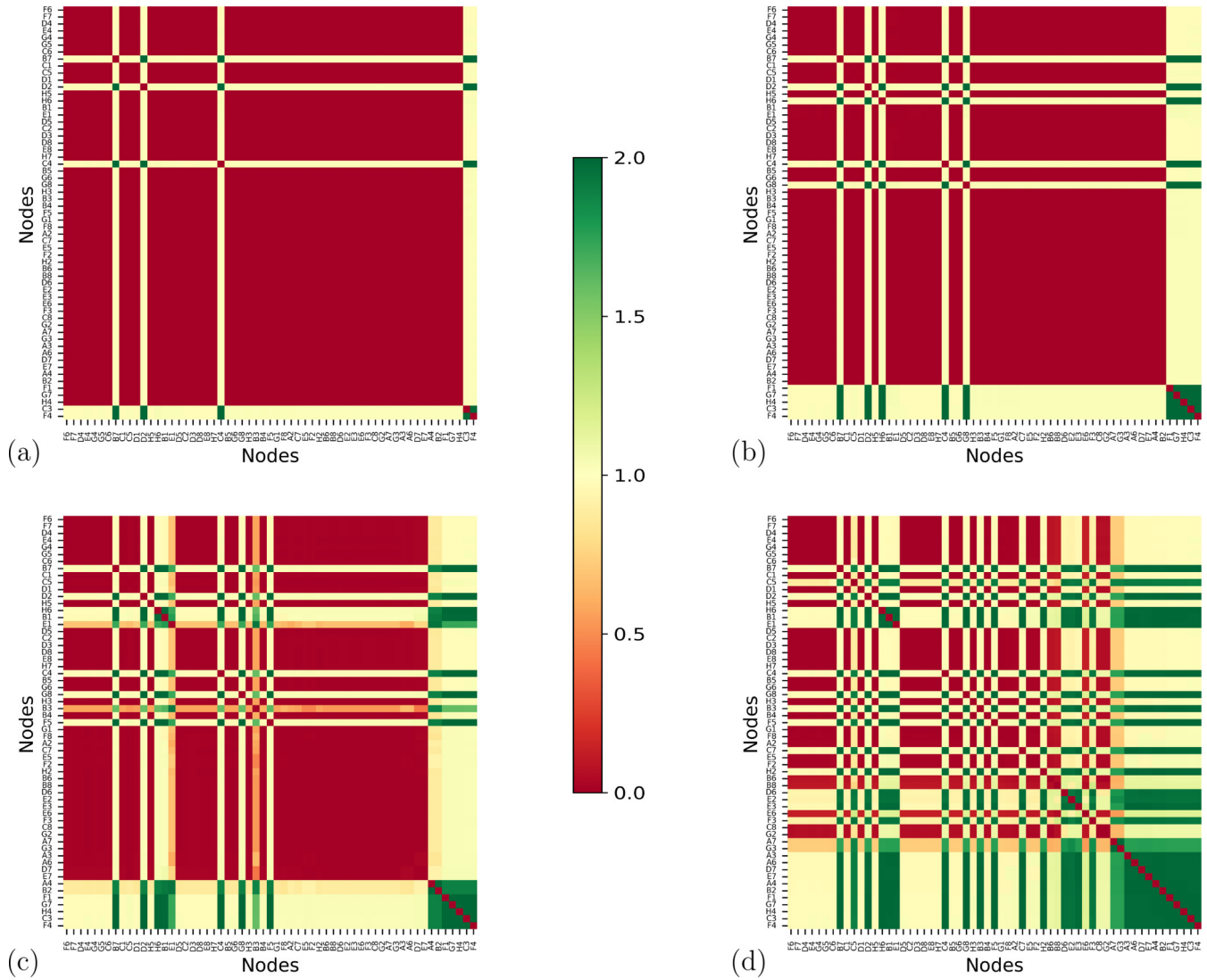


FIG. 3. Heatmap visualization of selected S_n matrices from the spectral block analysis of the NCM network. (a) S_{55} (corresponding to $\lambda_{55} = 46$), (b) S_{50} (corresponding to $\lambda_{50} = 41$), (c) S_{45} (corresponding to $\lambda_{45} = 37$), and (d) S_{32} (corresponding to $\lambda_{32} = 27$). Matrix indices are arranged from right to left and bottom to top according to the order of nodes' predicted clusters transitioning to synchronization (see Table I). Color bar in the center spans the range $[0, 2]$, corresponding to values observed in S_n matrices.

class III MSF systems) also depend on the dynamics $\mathbf{f}$ of the units and the coupling matrix H —a dependence that is ultimately unveiled by the MSF analysis. In the following sections, we discuss this analysis for the Rössler system and for the single-transistor chaotic circuit.

B. Network of Rössler systems

1. Analysis of spectral blocks

In this section, we study the formation of synchronous clusters via spectral block analysis in a mathematical model of nonlinear oscillators coupled via the NCM network. To set the theoretical framework for the analysis, we consider the Rössler system that is a paradigmatic model widely used in the study of synchronization phenomena due to its simple structure and rich chaotic dynamics [1,78]. The Rössler system is described by the following nonlinear dynamical

equations:

$$\begin{aligned}\dot{x} &= -y - z, \\ \dot{y} &= x + ay, \\ \dot{z} &= b + z(x - c),\end{aligned}\quad (23)$$

where a , b , and c are system parameters. Here, we use the system to model the dynamics of the units in a network whose topology is defined by the NCM. Specifically, we consider the scenario where diffusive coupling acts via the variable y , such that the generic oscillator i of the network (with $i = 1, \dots, 59$) is described by the following equations:

$$\begin{aligned}\dot{x}_i &= -y_i - z_i, \\ \dot{y}_i &= x_i + ay_i + d \sum_{j=1}^N A_{ij}(y_j - y_i), \\ \dot{z}_i &= b + z_i(x_i - c),\end{aligned}\quad (24)$$

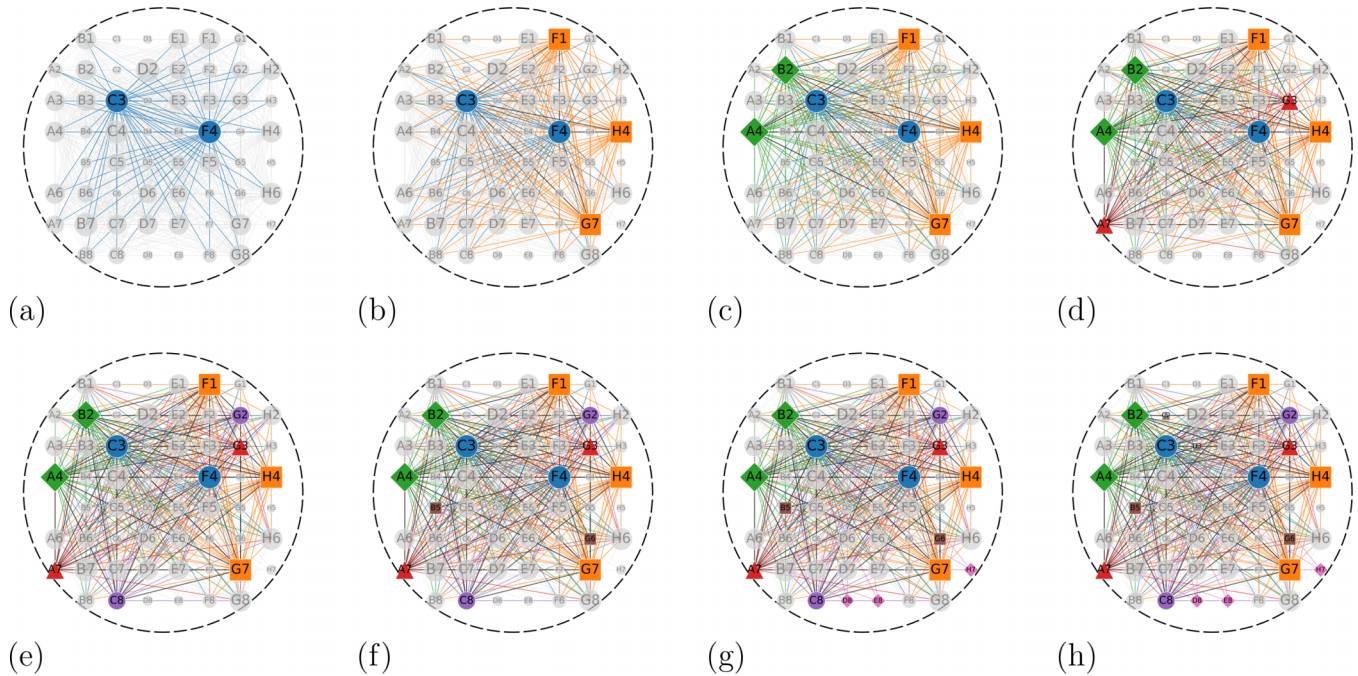


FIG. 4. Progressive emergence of synchronous clusters, from C_1 to C_8 , at increasing coupling strengths $d_i \propto 1/\lambda_j$, where i is the cluster number and j is eigenvalue index when the cluster emerges, with the index pairs (i, j) given by (1, 55), (2, 50), (3, 45), (4, 32), (5, 28), (6, 18), (7, 11), and (8, 9) (see also Table I). Each panel corresponds to a coupling value $d \simeq d_i$ and highlights the nodes that synchronize at that threshold, thereby forming the new cluster C_i : (a) $d \simeq d_1$; (b) $d \simeq d_2$; (c) $d \simeq d_3$; (d) $d \simeq d_4$; (e) $d \simeq d_5$; (f) $d \simeq d_6$; (g) $d \simeq d_7$; (h) $d \simeq d_8$. Marker shapes and colors for the cluster nodes are consistent with those used in Fig. 2(c).

where A is the adjacency matrix of the NCM network and d is the coupling strength, uniform across the network. Equation (24) aligns with the compact formulation provided by Eq. (3), by considering $n = 3$, state vectors defined by $\mathbf{x}_i = [x_i \ y_i \ z_i]^T$ for $i = 1, \dots, 59$, and $\mathbf{f}(\mathbf{x}_i)$, which models the isolated dynamics of each Rössler system, given by

$$\mathbf{f}(\mathbf{x}_i) = \begin{bmatrix} -y_i - z_i \\ x_i + ay_i \\ b + z_i(x_i - c) \end{bmatrix}.$$

The matrix $H \in \mathbb{R}^{3 \times 3}$, which specifies the variables used for coupling, is given by

$$H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (25)$$

The analysis of synchronous clusters in the NCM network composed of Rössler systems is performed by applying the procedure described in Sec. II B to two case studies, each characterized by different values of the system parameters a , b , and c : $(a, b, c) = (0.2, 0.2, 9)$ and $(a, b, c) = (0.1, 0.1, 18)$. As will be shown in the following, the two scenarios showcase different classes of MSFs. Specifically, we calculate the maximum Lyapunov exponent from Eq. (11) with H as in Eq. (25) and $\mathbf{Jf}(\mathbf{x}_s)$ given by

$$\mathbf{Jf}(\mathbf{x}_s) = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ z & 0 & x - c \end{bmatrix}_{\mathbf{x}=\mathbf{x}_s}. \quad (26)$$

For the calculation of $\Lambda(\nu)$, we use the algorithm reported in Ref. [79] (pp. 116–117) with the following parameters: integration step size $\delta t = 10^{-3}$, number of iterations per cycle $I = 150\,000$, and number of cycles $C = 15$. Figure 5 shows the results for the two sets of parameters considered, illustrating the curves of the MSF $\Lambda(\nu)$ as a function of the coupling strength ν varied in the range $[0, 30]$ with a step size of 0.01. We find that, for $(a, b, c) = (0.2, 0.2, 9)$, the system exhibits a

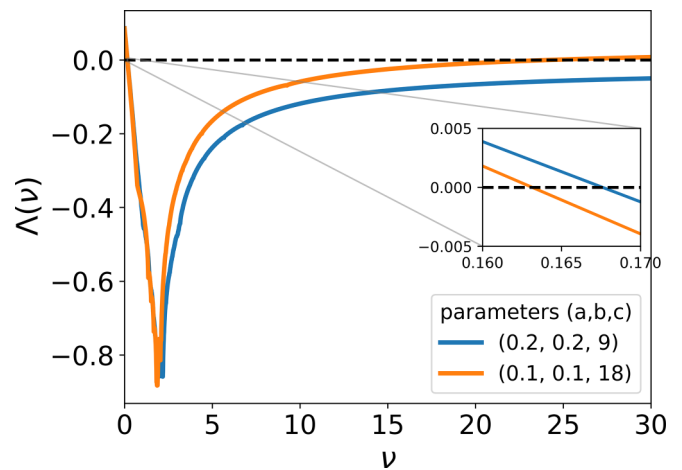


FIG. 5. Curve of the maximum Lyapunov exponent $\Lambda(\nu)$ vs parameter ν for the NCM network of Rössler systems for two sets of parameters: $(a, b, c) = (0.2, 0.2, 9)$ (blue continuous line) and $(a, b, c) = (0.1, 0.1, 18)$ (orange continuous line). The insets show magnifications of the curves for $\nu \in [0.16, 0.17]$ to better highlight the transition points.

TABLE II. Critical values of the coupling strengths predicted by the spectral-block method where clusters transition to/from synchronization in an NCM network of Rössler systems. For parameters (0.2, 0.2, 9) (MSF class II), $d_c = 0.168/\lambda_i$; for (0.1, 0.1, 18) (MSF class III), $d_{c1} = 0.163/\lambda_i$ and $d_{c2} = 24.33/\lambda_i$.

Cluster No.	λ	d_c (MSF II)	d_{c1} (MSF III)	d_{c2} (MSF III)
1	46.0000	0.0037	0.0035	0.5289
2	41.0000	0.0041	0.0040	0.5934
3	37.0000	0.0045	0.0044	0.6576
4	27.0000	0.0062	0.0060	0.9011
5	24.0000	0.0070	0.0068	1.0138
6	7.0000	0.0240	0.0233	3.4757
7	3.0000	0.0560	0.0543	8.1100
8	1.0000	0.1680	0.1630	24.3300

class II MSF, with $\Lambda(\nu)$ intersecting the x axis at $\nu^* = 0.168$ (here and in the following, threshold values reported with three digits are obtained via interpolation of the calculated data). This point marks the transition from positive values of $\Lambda(\nu)$ found for $\nu < \nu^*$ to negative values that occur for $\nu > \nu^*$. For the parameter setting $(a, b, c) = (0.1, 0.1, 18)$, instead, the system exhibits a class III MSF, with $\Lambda(\nu)$ crossing the x axis at two critical values, $\nu_1^* = 0.163$ and $\nu_2^* = 24.330$. These values mark two transitions as ν increases: first, from positive to negative values of $\Lambda(\nu)$, and then, from negative to positive values of $\Lambda(\nu)$, resulting in a limited interval of values of ν where synchronization is stable.

Once the critical value ν^* for the class II MSF case and the two critical values ν_1^* and ν_2^* for the class III MSF case are obtained, the predicted values of the coupling strength at which synchronous clusters gain or lose stability can be calculated by applying Eqs. (20)–(22). The results of this analysis are summarized in Table II. For the class II MSF case, d_c , calculated by Eq. (20), denotes the critical coupling strength at which a given cluster, as predicted by the spectral block method, transitions to synchronization. As d increases from zero, we note an increasing number of clusters that synchronize up to $d = 0.1680$, where all clusters entered the region of synchronization stability. For the class III MSF case, two values are associated with each cluster: d_{c1} and d_{c2} , computed using Eqs. (21) and (22), respectively. The first critical point d_{c1} marks the onset of synchronization stability for the cluster, while the second critical point d_{c2} indicates the point at which the cluster loses stability. Since d_{c1} of cluster 8 is smaller than d_{c2} of cluster 1, the spectral block analysis predicts that the clusters synchronize sequentially, from cluster 1 to cluster 8. This is followed by an interval of coupling values, $d \in [0.1630, 0.5289]$, where all clusters remain synchronized. Beyond this range, the clusters lose synchrony in the same order, again from cluster 1 to cluster 8, until $d = 24.3300$, at which point no cluster remains synchronized.

2. Numerical simulations

The theoretical predictions derived from the spectral block analysis are here validated by comparing them with results from numerical simulations. To this end, we simulate the NCM network of Rössler systems represented by

Eq. (24). Specifically, Eq. (24) is integrated using the explicit Runge-Kutta method of order 5, with a relative tolerance of 10^{-3} and an absolute tolerance of 10^{-6} , for $T = 1500$ time units. For use in subsequent computations, all state variables are sampled at uniform intervals of 0.25 time units. We monitor the degree of synchronization in each cluster by defining a cluster-specific synchronization error

$$E_{C_i} = \left\langle \left(\frac{1}{N_{C_i}} \sum_{i \in C_i} (\|\mathbf{x}_i - \bar{\mathbf{x}}\|^2)^{\frac{1}{2}} \right) \right\rangle_{\Delta T}, \quad (27)$$

where N_{C_i} denotes the number of nodes in the cluster C_i , namely, its cardinality, $\bar{\mathbf{x}}$ is defined as the average state vector of the units in cluster C_i , i.e., $\bar{\mathbf{x}}(t) = \frac{1}{N_{C_i}} \sum_{i \in C_i} \mathbf{x}_i(t)$, and ΔT is the interval where the error is averaged (in all simulations we have discarded the first 1400 time units of the numerical integration and calculated the synchronization error considering $\Delta T = 100$ time units). Finally, the error is averaged over 500 runs, each obtained by randomizing the initial conditions of the oscillators. For each oscillator, at each run, the initial conditions have been set by considering $(x_i(0), y_i(0), z_i(0)) = (0.1 + \epsilon_1, 0.1 + \epsilon_2, 0.1 + \epsilon_3)$, with $\epsilon_1 = 0.01 \times U_1$, $\epsilon_2 = 0.01 \times U_2$, $\epsilon_3 = 0.01 \times U_3$, where $U_1 \sim \mathcal{U}(0, 1)$, $U_2 \sim \mathcal{U}(0, 1)$, $U_3 \sim \mathcal{U}(0, 1)$ are uniformly distributed random variables on the interval $[0, 1]$.

Two different scenarios are considered for the analysis: identical oscillators and oscillators with slight parameter mismatches. For each scenario, we consider two settings of the unit dynamics parameters: $(a, b, c) = (0.2, 0.2, 9)$ and $(a, b, c) = (0.1, 0.1, 18)$. We begin by considering the case of identical oscillators, which is illustrated in Fig. 6. The figure shows the synchronization error of each cluster, as well as that of the entire network, plotted against the coupling strength d , which varies from 0 to 0.40 in increments of 0.001. Specifically, Fig. 6(a) illustrates the behavior for $(a, b, c) = (0.2, 0.2, 9)$, while Fig. 6(b) corresponds to the case $(a, b, c) = (0.1, 0.1, 18)$. In both panels, the transition points predicted by the spectral block analysis are marked with circles, colored to match the corresponding numerical curves. For the Rössler system with parameters $(a, b, c) = (0.2, 0.2, 9)$ [Fig. 6(a)], consistent with theoretical expectations for a class II MSF system, each cluster undergoes a single transition from incoherent to coherent dynamics. Although the predicted transition points (d_c in Table II) slightly underestimate the coupling strengths at which the synchronization error numerically vanishes, the sequence of synchronized clusters closely matches the theoretical prediction. For the Rössler system with parameters $(a, b, c) = (0.1, 0.1, 18)$ [Fig. 6(b)], consistent with the theoretical expectations for a class III MSF system, each cluster exhibits two transitions: first from asynchronous to synchronous behavior, and subsequently back to incoherent dynamics. For the first transition, the critical coupling strengths are slightly underestimated, whereas those associated with the second transition are slightly overestimated. Consequently, the synchronization interval for each cluster is generally narrower than the one predicted by the spectral block analysis. Also in this case, the sequence in which the clusters synchronize, as well as the sequence in which they desynchronize, aligns well with the theoretical predictions.

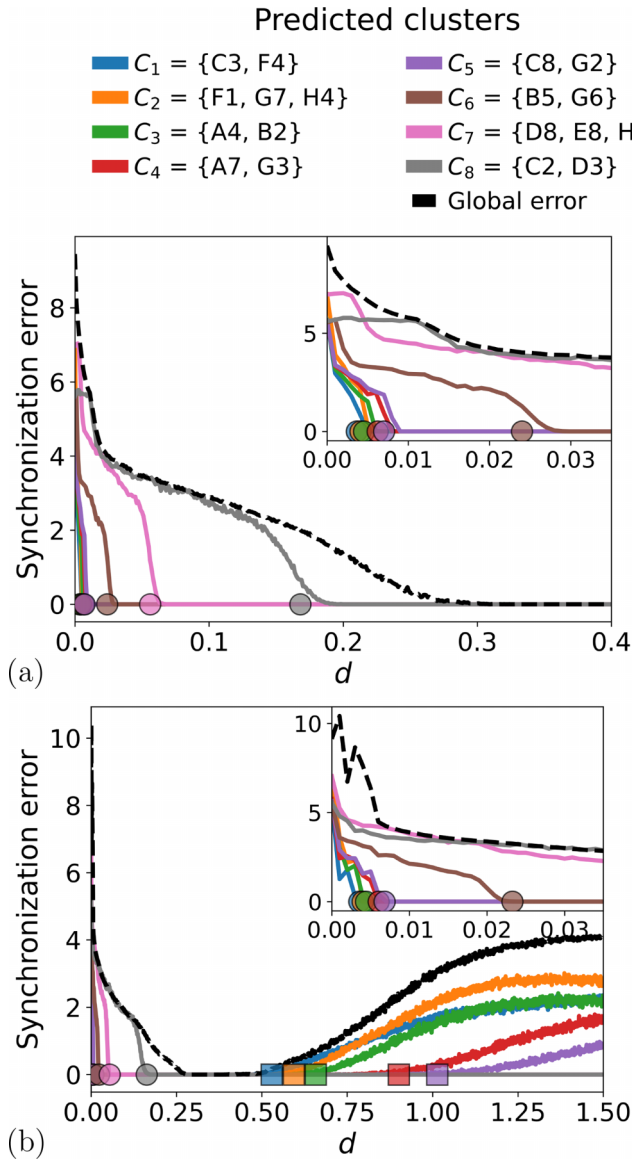


FIG. 6. Numerical simulations of an NCM network of identical Rössler systems, as in Eq. (24): Synchronization error for the clusters predicted by the spectral block analysis vs coupling strength d . Two sets of parameters are considered for the units: (a) $(a, b, c) = (0.2, 0.2, 9)$, which corresponds to a class II MSF, and (b) $(a, b, c) = (0.1, 0.1, 18)$, which corresponds to a class III MSF. The synchronization error curves for the different clusters use the same colors as in the legend of Fig. 2(c). The insets show magnification of the curves for $d \in [0, 0.035]$. The transition points predicted by the spectral block analysis are marked with circles and squares in the same colors as the corresponding clusters shown in the above legend, where d_c corresponds to the circle markers in panel (a), d_{c1} to the circle markers in panel (b), and d_{c2} to the square markers in panel (b).

To quantify the accuracy of the spectral block predictions, we compare the theoretically predicted transition points of Table II with those obtained from numerical simulations. In the simulations, transition points are determined from the curve of the cluster synchronization error as a function of the coupling strength. Specifically, we identify the extremes of the interval in which $E_{C_i}(d) \leq \varepsilon$, with $\varepsilon = 10^{-2}$. The

TABLE III. Transition points of the predicted clusters, obtained from numerical simulations of NCM networks of class II and class III MSF Rössler systems.

Cluster No.	Predicted cluster	d_c^{sim}	d_{c1}^{sim}	d_{c2}^{sim}
1	{C3, F4}	0.005	0.004	0.480
2	{F1, G7, H4}	0.006	0.004	0.518
3	{A4, B2}	0.007	0.005	0.556
4	{A7, G3}	0.009	0.006	0.768
5	{C8, G2}	0.009	0.007	0.840
6	{B5, G6}	0.030	0.024	4.01
7	{D8, E8, H7}	0.066	0.054	7.6
8	{C2, D3}	0.193	0.171	24.1

corresponding values are denoted as d_c^{sim} (or d_{c1}^{sim} , d_{c2}^{sim} in the case of a class III MSF). Tables III and IV illustrate the results. Table III lists the transition points extracted from the simulations, while Table IV shows the relative errors $\Delta d_c = \frac{|d_c - d_c^{\text{sim}}|}{d_c}$ (for class II MSF), $\Delta d_{c1} = \frac{|d_{c1} - d_{c1}^{\text{sim}}|}{d_{c1}}$ (for class III MSF), and $\Delta d_{c2} = \frac{|d_{c2} - d_{c2}^{\text{sim}}|}{d_{c2}}$ (MSF III) between the spectral block predictions (Table II) and the values extracted from simulations. The results of Tables II and III confirm that the spectral block method correctly predicts the sequence of cluster transitions. However, as it can be observed from the analysis of Table IV, the exact critical coupling strengths may deviate from the simulated values. While for the class III MSF system these errors typically remain below 15%, for the class II MSF system they are typically larger, with some value ranging up to $\sim 54\%$. These discrepancies are expected due to the approximations inherent both in the MSF-based approach and in the numerical estimation of the transition points. The MSF framework relies on linear stability analysis and, in the specific case of the spectral block method, further assumes that the maximum Lyapunov exponent can be computed from the global synchronous trajectory rather than from the actual cluster trajectory. Moreover, the numerically determined transition points are themselves affected by uncertainties, since they depend on the chosen synchronization threshold and on the initial conditions of the finite-time trajectories from which they are calculated.

Despite these quantitative discrepancies, both the simulations and the quantitative analysis consistently confirm the

TABLE IV. Relative error computed between the transition points predicted by the spectral block analysis and those obtained from numerical simulations of NCM networks of Rössler systems.

Cluster No.	Predicted cluster	Δd_c	Δd_{c1}	Δd_{c2}
1	{C3, F4}	0.369	0.129	0.092
2	{F1, G7, H4}	0.464	0.006	0.127
3	{A4, B2}	0.542	0.135	0.154
4	{A7, G3}	0.446	0.006	0.148
5	{C8, G2}	0.286	0.031	0.171
6	{B5, G6}	0.250	0.031	0.154
7	{D8, E8, H7}	0.179	0.006	0.063
8	{C2, D3}	0.149	0.049	0.009

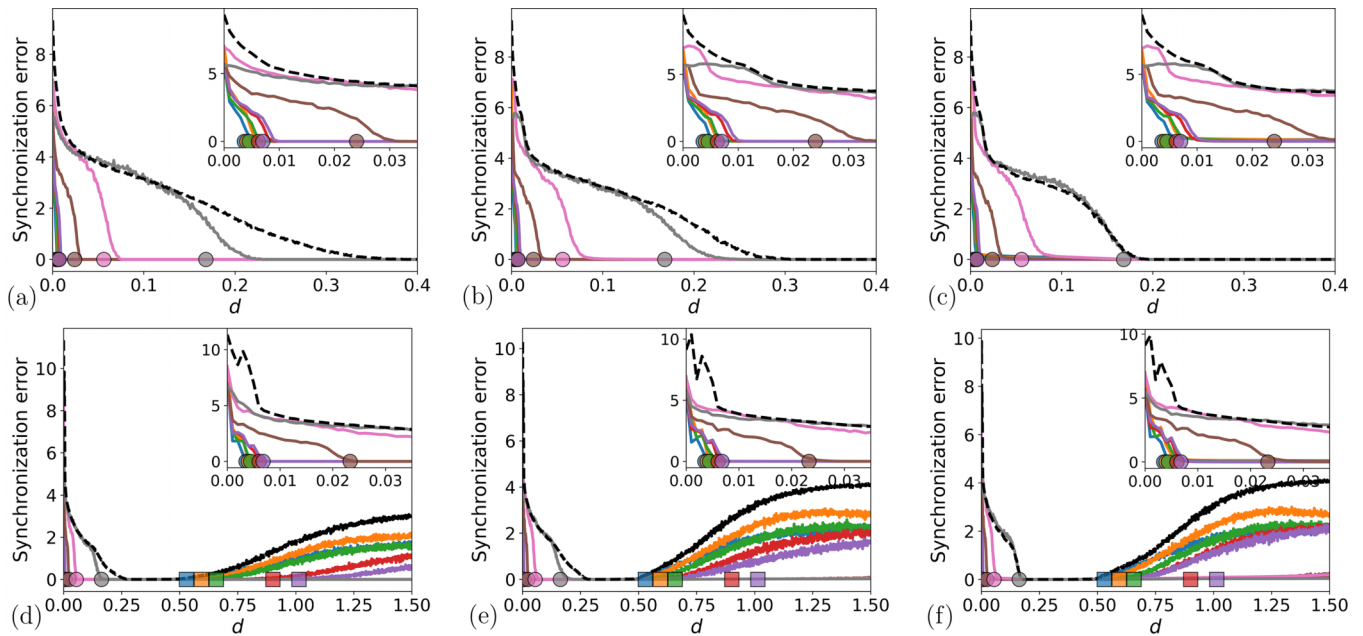


FIG. 7. Numerical simulations of a heterogeneous NCM network of Rössler systems: synchronization error for the clusters predicted by the spectral block analysis vs coupling strength d . (a), (d) Heterogeneity in the unit parameters. (b), (e) Structural heterogeneity introduced in the form of link weight perturbations. (c), (f) Structural heterogeneity introduced via degree-preserving rewiring. Two sets of nominal values for the unit parameters are considered: In panels (a)–(c), $(a, b, c) = (0.2, 0.2, 9)$, which corresponds to a class II MSF; in panels (d)–(f), $(a, b, c) = (0.1, 0.1, 18)$, which corresponds to a class III MSF. The synchronization error curves for the different clusters use the same colors as in the legend of Fig. 2(c). The insets show magnification of the curves for $d \in [0, 0.035]$. The transition points predicted by the spectral block analysis are marked with circles and squares in the same colors as the corresponding clusters [see Table II, where d_c corresponds to the circle markers in panels (a)–(c); d_{c1} to the circle markers in panels (d)–(f); and finally d_{c2} to the square markers in panels (d)–(f)].

ability of the spectral block method to accurately predict which synchronous clusters emerge and the sequence of their transitions. Furthermore, despite its approximate nature, the method provides predictions of the transition points that are frequently in good agreement with the observed values (with errors typically below 15%, and, in the remaining cases, sufficiently accurate for a first theoretical estimate, which can then be refined through targeted numerical analysis). This result is particularly significant in light of the intrinsic complexity of the cluster synchronization problem, especially with regard to deriving the conditions for cluster stability.

So far, we have analyzed the case of homogeneous Rössler systems. We now turn to the scenario in which heterogeneity is introduced. This is particularly relevant considering that the experimental setup inherently contains heterogeneity. In our numerical simulations of the NCM network of Rössler systems, heterogeneity is introduced by perturbing either the unit dynamics or the network links. Specifically, we consider two types of perturbations in the network links, resulting in a total of three distinct scenarios. In the first case, the parameters of the units (a, b, c) are perturbed with random values uniformly sampled within $\pm 10\%$ of their nominal values. In the second case, structural heterogeneity is introduced by modifying the network adjacency matrix: With a probability of 50%, a random value $U \sim \mathcal{U}(0, 1)$ is added to selected matrix elements. In the third case, structural heterogeneity is introduced by degree-preserving reconnection of the network edges. Specifically, we use a double-edge swapping algorithm that maintains the degree of each node while randomizing the connections. The algorithm iteratively

selects two random edges (i, j) and (k, l) involving four distinct nodes i, j, k , and l . Then, rewiring is performed, resulting in (i, k) and (j, l) or (i, l) and (j, k) , provided that the new edges do not already exist. This process is repeated for 10 000 attempts such that it progressively randomizes the network topology while keeping the degree distribution identical to the original network. This results in an average of approximately 65 edge swaps (about 10% of the total edges) over 500 realizations. This method ensures that the resulting network maintains the same local connectivity properties (individual node degrees) while randomizing the global wiring pattern.

The three scenarios are illustrated in Fig. 7. Panels (a) and (d) correspond to simulations with parameter mismatches, panels (b) and (e) depict the case of link weight perturbation, while panels (c) and (f) depict the case of degree-preserving link rewiring. The class II MSF case, with parameters $(a, b, c) = (0.2, 0.2, 9)$, is shown in panels (a)–(c), while the class III MSF case, with parameters $(a, b, c) = (0.1, 0.1, 18)$, in panels (d)–(f). Across all simulations, conducted under the same conditions as the homogeneous case, we find that the clusters synchronize and desynchronize in the same sequence predicted by the spectral block analysis, regardless of how heterogeneity is introduced. Even in the presence of heterogeneity, the coupling values at which clusters achieve synchronization are slightly underestimated, while those at which clusters lose synchrony are slightly overestimated. As expected, since the theoretical predictions assume identical units, the discrepancies between theory and the observed numerical thresholds are slightly larger than in the homogeneous case.

3. Joint versus independent cluster dynamics

As discussed in Sec. III A, the spectral block analysis of the NCM network may reveal clusters that do not emerge as isolated groups. Instead, the nodes identified in these steps merge with those of an existing cluster, resulting in the formation of a joint, larger cluster of synchronous nodes. While such situations can be identified by inspecting the elements of the matrix S_n at which a cluster first emerges, confirming the actual formation of the synchronous clusters requires complementing the structural analysis with a numerical simulation of the network dynamics, which is the focus of the present section.

Two different scenarios can emerge by analyzing the matrices S_n (recall that these matrices are sorted in descending order from $n = N$ to $n = 1$). Let us suppose that the first block of elements equal to 2, indicating the emergence of a cluster denoted as C_j , appears in matrix S_j , and that a second distinct block, suggesting the formation of another cluster indicated by C_i (that involves a different subset of nodes), is found in matrix S_i , with $i < j$. To determine whether C_i forms independently or merges with an existing cluster (for instance, with C_j), one must inspect the elements of S_i with one index in C_i and the other in C_j . If these entries are different from 2, then the two clusters are dynamically distinct and oscillate independently of each other. Conversely, finding that the elements are equal to 2 indicates that C_i is not independent from C_j , but instead that its nodes merge with those of C_j to form a larger synchronous cluster. Analogously, by inspecting the time series of the node variables obtained from numerical simulations of the network, or, equivalently, by tracking the time evolution of the synchronization error between representative nodes of each predicted cluster, we can clearly distinguish between the two cases: a nonzero synchronization error indicates that the two clusters are dynamically independent, while a zero error implies they have merged into a single synchronous group.

To illustrate this approach, we consider again the two clusters $C_2 = \{F1, G7, H4\}$, which first appear at S_{50} [Fig. 3(b)], and $C_3 = \{A4, B2\}$, which emerges at S_{45} [Fig. 3(c)]. Figure 8(a) shows the subblock of the matrix S_{45} corresponding to the indices of the nodes in the two clusters. In this matrix, the entries corresponding to the clusters C_2 and C_3 are exactly equal to 2, indicating synchronization within each group. Since the cross-entries between the nodes of C_2 and C_3 are not equal to 2 (taking the value 1.8969), the two clusters are functionally distinct and synchronize independently. We validate this prediction through numerical simulations by monitoring the synchronization error over the time interval $[0, 1000]$, both within each cluster and between pairs of nodes drawn from the two clusters. Specifically, we consider the following groups of nodes from the two clusters that we refer to as cross-clusters and label as CC : $CC_3 = \{F1, A4\}$, $CC_4 = \{F1, B2\}$, $CC_5 = \{H4, A4\}$, $CC_6 = \{H4, B2\}$, and finally a group encompassing all nodes of both clusters, i.e., $CC_7 = \{F1, G7, H4, A4, B2\}$. For each of these groups, we calculate the synchronization error within the group using Eq. (27) with $C_i = CC_3, CC_4, \dots$, and then average the results over 500 simulation runs. For the numerical simulations, we use an NCM network of identical Rössler systems with parameters $(a, b, c) = (0.1, 0.1, 18)$, which corresponds to a class III MSF behavior, and initial

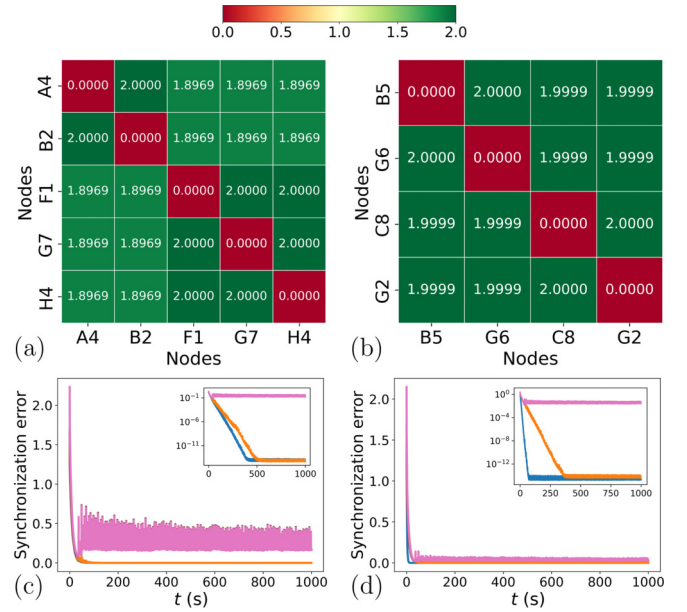


FIG. 8. Subblocks of the matrices S_n from the spectral block method, and corresponding synchronization errors for a network of Rössler systems with $(a, b, c) = (0.1, 0.1, 18)$ (MSF class III). (a) Subblock of the matrix S_{45} [the full matrix is shown Fig. 3(c)] corresponding to nodes belonging to predicted clusters $C_2 = \{F1, G7, H4\}$ and $C_3 = \{A4, B2\}$, and (b) subblock of the matrix S_{18} corresponding to nodes belonging to predicted clusters $C_5 = \{C8, G2\}$ and $C_6 = \{B5, G6\}$. (c) Synchronization error for clusters C_2 (blue curve), C_3 (orange curve), and for the cross-clusters CC_3 – CC_7 (remaining colors, overlapping under the pink curve). (d) Synchronization error for clusters C_5 (blue curve), C_6 (orange curve), and for the cross-clusters (remaining colors, overlapping under the pink curve). The inset shows the synchronization error (y axis) in log scale.

conditions set as $(x_i(0), y_i(0), z_i(0)) = (0.1 + \epsilon_1, 1 + \epsilon_2, 5 + \epsilon_3)$ with $\epsilon_1 = 5 \times U_1$, $\epsilon_2 = 5 \times U_2$, and $\epsilon_3 = 5 \times U_3$, where $U_1 \sim \mathcal{U}(0, 1)$, $U_2 \sim \mathcal{U}(0, 1)$, and $U_3 \sim \mathcal{U}(0, 1)$ are uniformly distributed random variables on the interval $[0, 1]$. For the coupling coefficient, we select $d = 0.008$, larger than $d_{c1} = 0.0044$, which, as shown in Table II, represents the predicted transition point for cluster C_3 . So, at $d = 0.008$, both clusters are predicted to synchronize. This is confirmed by the time evolution of the synchronization error within the two clusters, shown in Fig. 8(c). Both values quickly converge to zero, reaching orders of 10^{-12} (as seen from the inset). Moreover, the synchronization errors associated with the node groups CC_3 – CC_7 remain distinct from zero, i.e., varying around a mean value of ~ 0.4 and remaining always greater than 0.2. In addition, they display nearly overlapping temporal dynamics. This means that the nodes of cluster C_2 are not synchronized with those of cluster C_3 , ultimately confirming that clusters C_2 and C_3 synchronize independently of each other.

As a second example, we now consider cluster $C_5 = \{C8, G2\}$ that, as shown in Table I, appears at S_{28} , and cluster $C_6 = \{B5, G6\}$ that appears at S_{18} . Figure 8(b) shows the subblock extracted from the S_{18} matrix (corresponding to λ_{18}), which contains the nodes belonging to these two clusters. In this matrix, the entries corresponding to clusters

C_5 and C_6 are exactly equal to 2, indicating synchronization within each group. However, the cross-entries between nodes belonging to C_5 and C_6 are found to be very close to 2, but not exactly equal (taking the value 1.9999). This observation indicates that the two clusters, C_5 and C_6 , synchronize perfectly within their respective groups, and that, when considered together, they are nearly, but not fully, synchronized. For numerical simulations, we again consider an NCM network of Rössler systems with parameters and initial conditions set as in the first example and monitor the synchronization error over the time interval $[0, 1000]$, both within each cluster and between pairs of nodes drawn from the two clusters. Specifically, we focus on the following groups of nodes from the two clusters C_5 and C_6 forming cross-clusters (CC 's): $CC_1 = \{C8, B5\}$, $CC_2 = \{C8, G6\}$, $CC_3 = \{G2, B5\}$, $CC_4 = \{G2, G6\}$, and $CC_5 = \{C8, G2, B5, G6\}$. We fix the coupling coefficient to $d = 0.05$ that is larger than $d_{c1} = 0.0233$, i.e., the predicted transition point for cluster C_6 (Table II). As shown in Fig. 8(d), the synchronization errors for clusters C_5 and C_6 rapidly converge to zero, reaching magnitudes on the order of 10^{-14} . However, the synchronization errors for the node groups CC_1 – CC_5 remain clearly above zero, fluctuating around a mean value of approximately 0.04. These curves clearly indicate that although the two clusters C_5 and C_6 synchronize perfectly within their respective groups, their joint synchronization is not complete, thus ultimately validating the predictions of the spectral block method.

We conclude this section by noting that we have thoroughly searched for subblock matrices in which the entries corresponding to cross-indices of the clusters are exactly equal to 2. This analysis, thus, indicates that the NCM network does contain examples of exact joint clusters.

C. Network of single-transistor chaotic (Minati-Frasca) circuits

1. Analysis of spectral blocks

In this section, we conduct the spectral block analysis of the NCM network composed of single-transistor chaotic circuits. Since the cluster composition and their sequence of synchronization and desynchronization are solely determined by the network topology, they remain unchanged from the case of Rössler systems. However, the specific transition points depend on the values at which the MSF changes sign, namely, v^* for class II systems, and v_1^* and v_2^* for class III systems. In the following, we first compute these parameters using the simplified mathematical model of the single-transistor chaotic circuit described in Appendix B. We then propose a method that leverages a more realistic, circuit-level model, implemented using a Simulation Program with Integrated Circuit Emphasis (SPICE) for replicating the experimental configuration, so as to directly evaluate the critical

point in terms of the input used in the experimental setup that controls the coupling coefficient.

We begin by illustrating the theoretical computation of the MSF based on the mathematical model of the single-transistor chaotic circuit. To facilitate this analysis, we rescale Eq. (B1) using the following dimensionless variables:

$$x_1 = k_V u_{C1}, \quad x_2 = k_V u_{C2}, \quad x_3 = k_I i_{L1}, \\ x_4 = k_I i_{L2}, \quad t' = \tau t, \quad (28)$$

where $k_V = 1 \text{ V}^{-1}$, $k_I = 1000 \text{ A}^{-1}$, and $\tau = 10^3 \text{ s}^{-1}$. Similarly, the system parameters (see Appendix B for the component values used in the experimental setup) are rescaled as

$$a_{MF} = k_V V_S, \quad b_{MF} = \tau k_I C_1 = 0.025, \\ c_{MF} = \tau k_I C_2 = 5 \times 10^{-6}, \quad d_{MF} = k_V V_{th} = 0.6, \\ e_{MF} = L_1 \tau k_V / k_I = 0.005, \quad f_{MF} = L_2 \tau k_V / k_I = 0.010, \\ g_{MF} = R k_V / k_I. \quad (29)$$

Using the dimensionless variables x_1, x_2, x_3 , and x_4 , the equations of the circuits, i.e., Eq. (B1), are rewritten as

$$\dot{x}_1 = \frac{a_{MF} - x_1}{b_{MF} g_{MF}} - \frac{x_3 + x_4}{b_{MF}}, \\ \dot{x}_2 = \frac{x_4 - \beta \Gamma(x_3) \tanh(x_2 / (2d_{MF}))}{c_{MF}}, \\ \dot{x}_3 = \frac{x_1 - d_{MF}}{e_{MF}}, \quad \dot{x}_4 = \frac{x_1 - x_2}{f_{MF}}, \quad (30)$$

where $\dot{x} = dx/dt'$ denotes differentiation with respect to the dimensionless time variable t' . Analogously, the equations of a network of single-transistor chaotic circuits can be rewritten as follows:

$$\dot{x}_1^{(i)} = \frac{a_{MF} - x_1^{(i)}}{b_{MF} g_{MF}} - \frac{x_3^{(i)} + x_4^{(i)}}{b_{MF}} + d \sum_{j=1}^N A_{ij} \frac{x_1^{(j)} - x_1^{(i)}}{b_{MF}}, \\ \dot{x}_2^{(i)} = \frac{x_4^{(i)} - \beta \Gamma(x_3^{(i)}) \tanh(x_2^{(i)} / (2d_{MF}))}{c_{MF}}, \\ \dot{x}_3^{(i)} = \frac{x_1^{(i)} - d_{MF}}{e_{MF}}, \quad \dot{x}_4^{(i)} = \frac{x_1^{(i)} - x_2^{(i)}}{f_{MF}}, \quad (31)$$

where $d = \frac{k_I}{R_C k_V}$ is the rescaled coupling strength, and, as usual, A is the adjacency matrix encoding the network links.

Equation (31) is consistent with the general network representation given in Eq. (3), thereby enabling the application of the procedure outlined in Sec. II B to analyze cluster synchronization. In particular, the maximum Lyapunov exponent is calculated from Eq. (11) with the following Jacobian matrix $J\mathbf{f}(\mathbf{x}_s)$:

$$J\mathbf{f}(\mathbf{x}_s) = \begin{bmatrix} -\frac{1}{b_{MF} g_{MF}} & 0 & -\frac{1}{b_{MF}} & -\frac{1}{b_{MF}} \\ 0 & \frac{\beta \Gamma(x_{3,s}) \left(\tanh^2\left(\frac{x_{2,s}}{2d_{MF}}\right) - 1 \right)}{2c_{MF} d_{MF}} & -\frac{\beta \Theta(x_{3,s}) \tanh\left(\frac{x_{2,s}}{2d_{MF}}\right)}{c_{MF}} & \frac{1}{c_{MF}} \\ \frac{1}{e_{MF}} & 0 & 0 & 0 \\ \frac{1}{f_{MF}} & -\frac{1}{f_{MF}} & 0 & 0 \end{bmatrix}, \quad (32)$$

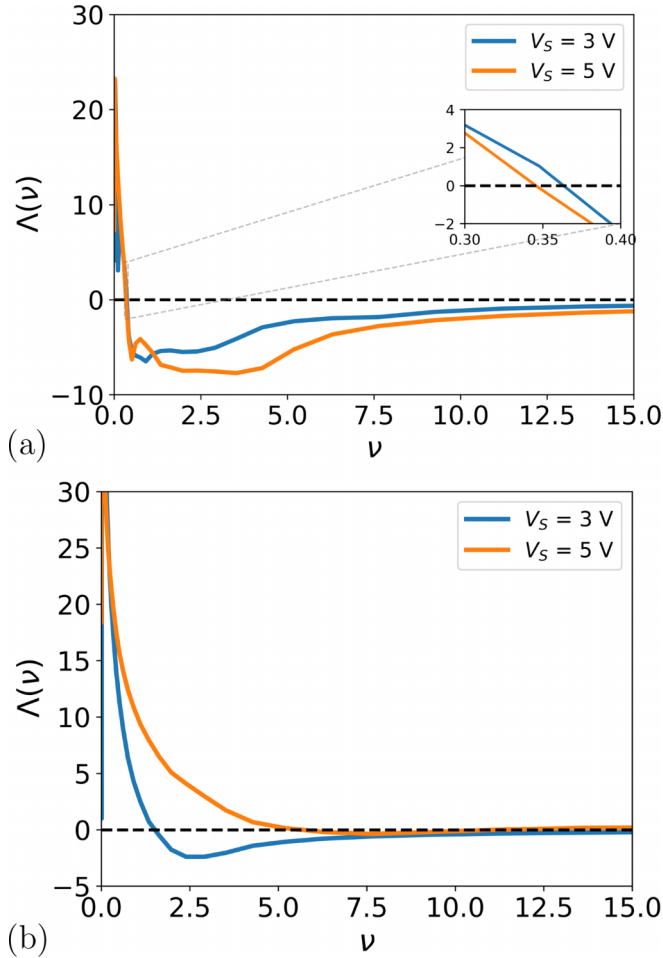


FIG. 9. MSF (Λ) vs coupling parameter (ν) for the single-transistor chaotic oscillator for (a) $R = 470 \, \Omega$. The inset enlarges the region $[0.3, 0.4]$ to distinguish clearly the intersection points. (b) $R = 1000 \, \Omega$, under supply voltages $V_S = 3 \, \text{V}$ (blue curve) and $V_S = 5 \, \text{V}$ (orange curve).

where $\Theta(x)$ is the Heaviside function, i.e., $\Theta(x) = 0$ for $x < 0$ and $\Theta(x) = 1$ for $x > 0$. In more detail, to calculate the maximum Lyapunov exponent we use the algorithm provided in Ref. [79] (pp. 116–117), with the following parameters: integration step size $\delta t = 10^{-8}$, number of iterations per cycle $I = 5 \times 10^8$, and number of cycles $N_C = 15$.

The analysis is conducted using different configurations of the parameters a_{MF} and g_{MF} , that is $a_{\text{MF}} \in \{3, 5\}$ and $g_{\text{MF}} \in \{0.47, 1\}$, while all other parameters are fixed to the values specified in Eq. (29). To facilitate the interpretation of the experimental results, we directly relate these configurations to the corresponding values of physical components: the supply voltage $V_S \in \{3, 5\} \, \text{V}$ and the resistance $R \in \{470, 1000\} \, \Omega$. The results are presented in Fig. 9, with panel (a) corresponding to the case $R = 470 \, \Omega$ and panel (b) to the case $R = 1000 \, \Omega$. Each panel illustrates two values of V_S , namely, $V_S = 3 \, \text{V}$ and $V_S = 5 \, \text{V}$. While for $V_S = 5 \, \text{V}$ and $R = 1000 \, \Omega$ we find a class III MSF, in all other cases the system exhibits class II behavior. The analysis of the experimental results does not provide evidence of a transition from synchronization back to incoherent behavior. Therefore, in the following, we

TABLE V. Values of the critical coupling strengths, predicted by the spectral block analysis, where clusters transition to synchronization in the single-transistor chaotic systems for various configurations of the parameters. For $R = 470 \, \Omega$ and $V_S = 3 \, \text{V}$, $d_c^{(1)} = 0.363/\lambda_i$; for $R = 470 \, \Omega$ and $V_S = 5 \, \text{V}$, $d_c^{(2)} = 0.345/\lambda_i$; for $R = 1000 \, \Omega$ and $V_S = 3 \, \text{V}$, $d_c^{(3)} = 1.514/\lambda_i$; and for $R = 1000 \, \Omega$ and $V_S = 5 \, \text{V}$, $d_c^{(4)} = 5.645/\lambda_i$.

Cluster No.	λ	$d_c^{(1)}$	$d_c^{(2)}$	$d_c^{(3)}$	$d_c^{(4)}$
1	46.0000	0.0079	0.0075	0.0329	0.1227
2	41.0000	0.0089	0.0084	0.0369	0.1377
3	37.0000	0.0098	0.0093	0.0409	0.1526
4	27.0000	0.0134	0.0128	0.0561	0.2091
5	24.0000	0.0151	0.0144	0.0631	0.2352
6	7.0000	0.0519	0.0493	0.2163	0.8064
7	3.0000	0.1210	0.1150	0.5047	1.8817
8	1.0000	0.3630	0.3450	1.5140	5.6450

focus only on the first transition, namely that from incoherence to synchronization, effectively treating this case as a type II MSF. Further analyses, performed using alternative approaches and discussed below, confirm that this behavior can be attributed to discrepancies between the experimental system and the corresponding abstract mathematical model. The transition from incoherent to coherent behavior occurs at $\nu^* = 0.363$ for $V_S = 3 \, \text{V}$ and $R = 470 \, \Omega$; $\nu^* = 0.345$ for $V_S = 5 \, \text{V}$ and $R = 470 \, \Omega$; $\nu^* = 1.514$ for $V_S = 3 \, \text{V}$ and $R = 1000 \, \Omega$; and $\nu^* = 5.645$ for $V_S = 5 \, \text{V}$ and $R = 1000 \, \Omega$. From these values, the critical coupling strengths marking the transition to synchronization stability for each cluster are calculated. The resulting values are reported in Table V, where for each configuration of the parameters (R, V_S), the critical coupling d_c , computed as $d_c = \nu^*/\lambda_i$, is listed. As expected, the sequence of cluster formation replicates that of Table I, since it is determined solely by the network topology and is therefore independent of the node dynamics.

The MSF analysis presented above is performed on the mathematical model of the single-transistor chaotic circuit, as described by the set of ordinary differential equations in Eq. (30). The model is effective in capturing the fundamental mechanisms of chaos generation but can only provide a qualitative description of the circuit's behavior. This limitation arises mainly from the phenomenological representation of the transistor, which does not fully account for its real operating conditions. Specifically, the transistor is approximated by a memoryless nonlinearity with *ad hoc* switching functions [e.g., the ramp function $\Gamma(\cdot)$ and the Heaviside function $\Theta(\cdot)$], and it ignores parasitic elements, bias-dependent capacitances, and detailed device physics. Furthermore, the MSF analysis links the synchronization stability to the parameter ν (or the coupling strength d), and indirectly to the equivalent resistance R_C through the relation $d = \frac{k_I}{R_C k_V}$. However, it does not establish a direct relationship with the tunable coupling voltage V_c , which is the actual control input used in the experiments.

For this reason, we present here an alternative practical approach based on SPICE to determine the critical coupling strength, expressed in terms of V_c , by leveraging circuit-level

simulations of a pair of single-transistor chaotic circuits connected via the coupling device used in the experimental setup [the circuitry is shown in Appendix B, Fig. 14(b)]. This approach builds upon the idea of using an elementary network as a probe to study synchronization stability, as originally proposed in Ref. [80]. In more detail, the “universal probe” introduced in Ref. [80] consists of three coupled oscillators to experimentally evaluate the MSF of networks of chaotic circuits. This configuration is sufficiently general to characterize the maximum transverse Lyapunov exponent as a function of two parameters: the real and imaginary parts of the eigenvalues of the Laplacian matrix. In our case, by exploiting the fact that the network under investigation is undirected, so that the eigenvalues of its Laplacian matrix are always real, we reduce the probe to a simpler configuration of only two coupled nodes. Thanks to the separation between network topology and node dynamics afforded by the MSF framework, this two-node probe is sufficient to determine the critical transition point for the network of single-transistor chaotic circuits. Specifically, by simulating the elementary two-node configuration, we numerically determine the minimum coupling voltage $\bar{V}_c$ required for synchronization between the two circuits. While this value is specific to the simulated configuration, we can use the general MSF relationship, $\nu^* = d\lambda_2$, to derive V_c^* , namely, the circuit equivalent of ν^* , as follows:

$$V_c^* = \lambda_2 \bar{V}_c, \quad (33)$$

where $\lambda_2 = 2$ since the Laplacian matrix of the elementary configuration of two coupled nodes is $L = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and so its eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 2$.

For this analysis, we consider circuit-level SPICE simulations that, in contrast to the simplified mathematical model of Eq. (31), solve the full set of differential-algebraic equations that enforce charge/flux conservation and device physics. The bipolar transistor is evaluated with a compact semiconductor model (e.g., Gummel-Poon) that includes (1) the exponential base-current law $i_B(v_{BE})$, (2) bias-dependent current gain $\beta(I_C, V_{CE}, T)$, (3) junction capacitances (C_{BE} , C_{BC} , C_{CE}) that vary with voltage, and (4) charge-storage effects, early effect, and saturation behavior. Its nonlinear, bias-dependent capacitances (C_{BE} , C_{BC} , C_{CE} , and diffusion charge) are updated pointwise, capturing Miller feedback and timescale modulation. All passive components are modeled with their real-world imperfections (direct-current resistance (DCR), equivalent series resistance (ESR), parasitic inductance/capacitance, finite Q), and interconnect parasitics are naturally accounted for. Consequently, SPICE simulations capture waveform details, timing, and spectral content that the simplified model cannot. Moreover, they avoid the hard, user-imposed discontinuities (ramps/Heaviside switches) that can induce stiffness in the integration of the ordinary differential equations.

To ensure that the circuit simulations reflect the experimental variability on the circuit board, they have been carried out after introducing a realistic mismatch of approximately 4–5 mH in the inductance parameters L_1 and L_2 of the two single-transistor chaotic circuits. For one of the two circuits, we use $(L_1^{(1)}, L_2^{(1)}) = (9 \text{ mH}, 15 \text{ mH})$, while for the second circuit of the pair, the parameters are set to

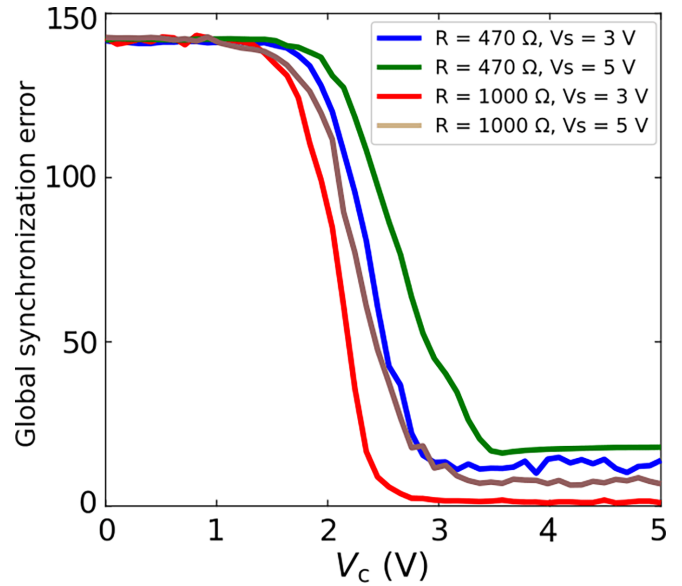


FIG. 10. SPICE simulations of a pair of single-transistor chaotic circuits: global synchronization error E^{global} as a function of coupling voltage V_c for four different configurations of R and V_S .

$(L_1^{(2)}, L_2^{(2)}) = (5 \text{ mH}, 10 \text{ mH})$. All other parameters are kept constant to the values specified in Appendix B. Simulations are performed for a period of time $T = 10 \text{ ms}$ with the first 1 ms discarded to remove transient dynamics. The coupling voltage V_c is varied from 0 to 5 V in increments of 0.1 V. For each value of V_c , the synchronization error between the two MF circuits is computed by taking the time-averaged absolute value of the difference between the two circuit voltages $u_{C1}^{(1)}(t)$ and $u_{C1}^{(2)}(t)$ namely, $E = \langle |u_{C1}^{(1)}(t) - u_{C1}^{(2)}(t)| \rangle_T$. Figure 10 presents the results of the SPICE simulations for the parameter combinations defined by $R \in \{470, 1000\} \Omega$ and $V_S \in \{3, 5\} \text{ V}$. For each configuration, the critical coupling voltage $\bar{V}_c$ is determined as the value where the synchronization error reaches a minimum and plateaus. Using Eq. (33), the corresponding critical coupling strength V_c^* for each configuration is then obtained: For $R = 470 \Omega$ and $V_S = 3 \text{ V}$, $\bar{V}_c = 3 \text{ V}$ and $V_c^* = 6 \text{ V}$; for $R = 470 \Omega$ and $V_S = 5 \text{ V}$, $\bar{V}_c = 3.5 \text{ V}$ and $V_c^* = 7 \text{ V}$; for $R = 1000 \Omega$ and $V_S = 3 \text{ V}$, $\bar{V}_c = 2.8 \text{ V}$ and $V_c^* = 5.6 \text{ V}$; and for $R = 1000 \Omega$ and $V_S = 5 \text{ V}$, $\bar{V}_c = 3.3 \text{ V}$ and $V_c^* = 6.6 \text{ V}$. These values are used in Table VI to obtain the critical coupling voltages $V_{c,\text{crit}}^{(1)}$, $V_{c,\text{crit}}^{(2)}$, $V_{c,\text{crit}}^{(3)}$, $V_{c,\text{crit}}^{(4)}$ (expressed in V) for each of the clusters predicted by the spectral block analysis at the different configurations of R and V_S .

2. Experimental results

In this section, we analyze the experimental data recorded from the electronic board implementing the NCM network of single-transistor chaotic circuits. Four cases are considered, corresponding to the combinations of two values of the resistor, $R \in \{470, 1000\} \Omega$, and two values of the supply voltage, $V_S \in \{3, 5\} \text{ V}$. For each configuration, the input voltage controlling the coupling strength among the circuits, V_c , is varied from 1.2 to 3.0 V in increments of 0.01 V. For each recording, the signals, one per oscillator, are preprocessed following the procedure described in Appendix A. In particular, two

TABLE VI. Values of the critical coupling thresholds (in V) for each cluster computed as $V_{c,\text{crit}}^{(h)} = V_c^*/\lambda_i$, with $h = 1, \dots, 4$ indicating one of the four parameter sets considered. Specifically, for $R = 470 \Omega$ and $V_S = 3 \text{ V}$, $V_{c,\text{crit}}^{(1)} = 6 \text{ V}/\lambda_i$; for $R = 470 \Omega$ and $V_S = 5 \text{ V}$, $V_{c,\text{crit}}^{(2)} = 7 \text{ V}/\lambda_i$; for $R = 1000 \Omega$ and $V_S = 3 \text{ V}$, $V_{c,\text{crit}}^{(3)} = 5.6 \text{ V}/\lambda_i$; and for $R = 1000 \Omega$ and $V_S = 5 \text{ V}$, $V_{c,\text{crit}}^{(4)} = 6.6 \text{ V}/\lambda_i$.

n	λ	$V_{c,\text{crit}}^{(1)}$	$V_{c,\text{crit}}^{(2)}$	$V_{c,\text{crit}}^{(3)}$	$V_{c,\text{crit}}^{(4)}$
1	46.0000	0.1304	0.1522	0.1217	0.1435
2	41.0000	0.1463	0.1707	0.1366	0.1610
3	37.0000	0.1622	0.1892	0.1514	0.1784
4	27.0000	0.2222	0.2593	0.2074	0.2444
5	24.0000	0.2500	0.2917	0.2333	0.2750
6	7.0000	0.8571	1.0000	0.8000	0.9429
7	3.0000	2.0000	2.3333	1.8667	2.2000
8	1.0000	6.0000	7.0000	5.6000	6.6000

synchronization matrices, E^{raw} and E^{norm} , are computed by averaging the synchronization errors over a time window of $\Delta T = 0.4 \text{ s}$, which contains $M = 100\,000$ samples per signal. From such matrices, we compute the synchronization error for the clusters and for the entire network.

We begin by analyzing global synchronization in the network. To this aim, we compute the global synchronization error from the normalized synchronization matrix E^{norm} by averaging all pairwise synchronization errors across the network as follows:

$$E^{\text{global}} = \frac{1}{N(N-1)} \sum_{i,j=1}^N e_{ij}^{\text{norm}}. \quad (34)$$

Figure 11 shows the curve of E^{global} as a function of V_c for each configuration of R and V_S . Although a direct comparison with Fig. 10 is not possible, as it refers to a SPICE simulation of a different configuration (two coupled circuits versus the NCM network), we expect a consistent qualitative behavior in

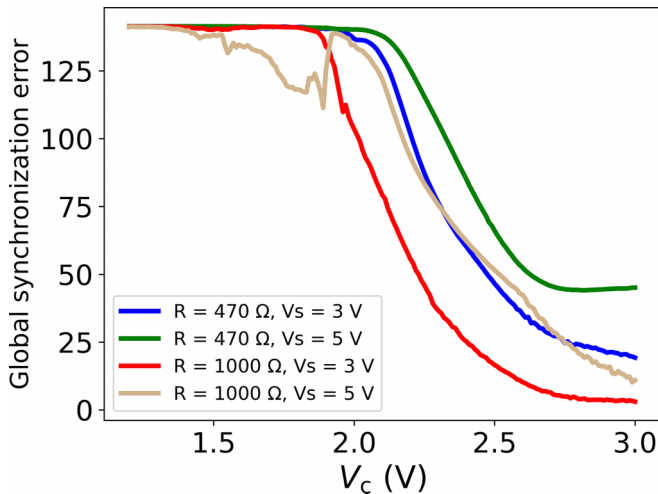


FIG. 11. Experimental results for the NCM network of single-transistor chaotic circuits: global synchronization error E^{global} as a function of coupling voltage V_c for four different configurations of R and V_S .

terms of the overall trend of the errors and the influence of the parameters R and V_S . This expectation is indeed confirmed, as the two figures exhibit qualitatively similar features. In particular, both the experimental and simulated curves for $R = 470 \Omega$ and $V_S = 5 \text{ V}$ exhibit a smooth decrease followed by a plateau, settling at the highest error level among all the configurations. Similarly, the curves corresponding to the configurations $R = 470 \Omega$, $V_S = 3 \text{ V}$ and $R = 1000 \Omega$, $V_S = 5 \text{ V}$ reach low synchronization error values as V_c approaches 3 V . Finally, both the experimental and simulated curves for $R = 470 \Omega$, $V_S = 3 \text{ V}$ attain the lowest error values among all configurations.

In Fig. 11, the sequence in which the curves of E^{global} for different parameter configurations reach a plateau (indicating the highest level of synchronization) is as follows: First, the configuration $(R, V_S) = (1000 \Omega, 3 \text{ V})$, next $(R, V_S) = (1000 \Omega, 5 \text{ V})$, followed by $(R, V_S) = (470 \Omega, 3 \text{ V})$, and finally $(R, V_S) = (470 \Omega, 5 \text{ V})$. This sequence aligns with the SPICE simulations (Fig. 10) but not with the MSF analysis (Fig. 9), which, for example, predicts that the configuration with $V_S = 5 \text{ V}$ and $R = 470 \Omega$, having $\nu^* = 0.345$, should synchronize before the others that have a higher value of ν^* . Similarly, according to the MSF, the configuration with $V_S = 5 \text{ V}$ and $R = 1000 \Omega$, having the largest $\nu^* = 5.645$, should be the last to reach synchronization, which is not in agreement with the experimental findings. We also observe that the large disparity between the smallest ($\nu^* = 0.345$) and the largest ($\nu^* = 5.645$) values of ν^* predicted by the MSF analysis is not reflected in the experimental results, where the differences between the critical points for synchronization are significantly smaller. Altogether, these observations indicate that the synchronization threshold predicted by the MSF lacks accuracy, likely due to the approximations involved in deriving the mathematical model of the single-transistor chaotic circuit. For this reason, in the subsequent analyses we rely on the critical point values predicted through the SPICE-based simulation approach, as reported in Table VI.

For the analysis of cluster synchronization, we monitored the degree of synchronization within each of the predicted clusters by calculating the error

$$E_{C_i} = \frac{1}{N_{C_i}(N_{C_i} - 1)} \sum_{i,j \in C_i} e_{ij}^{\text{raw}} \quad (35)$$

for each cluster C_i of the NCM network of single-transistor chaotic circuits. These quantities have been calculated using the unnormalized errors [see Eq. (A1) with $\Delta T = 0.4 \text{ s}$]. Each error has then been further averaged across five repetitions of measurements performed under the same experimental conditions.

The behavior of the synchronization error E_{C_i} for each cluster C_i as a function of V_c has been analyzed across the four parameter configurations obtained by considering $V_S \in \{3, 5\} \text{ V}$ and $R \in \{470, 1000\} \Omega$. The results are illustrated in Fig. 12, where each panel corresponds to a different configuration of R and V_S : $R = 470 \Omega$, $V_S = 3 \text{ V}$ [Fig. 12(a)]; $R = 470 \Omega$, $V_S = 5 \text{ V}$ [Fig. 12(b)]; $R = 1000 \Omega$, $V_S = 3 \text{ V}$ [Fig. 12(c)]; and $R = 1000 \Omega$, $V_S = 5 \text{ V}$ [Fig. 12(d)]. We observe that, across all panels, the sequence in which the

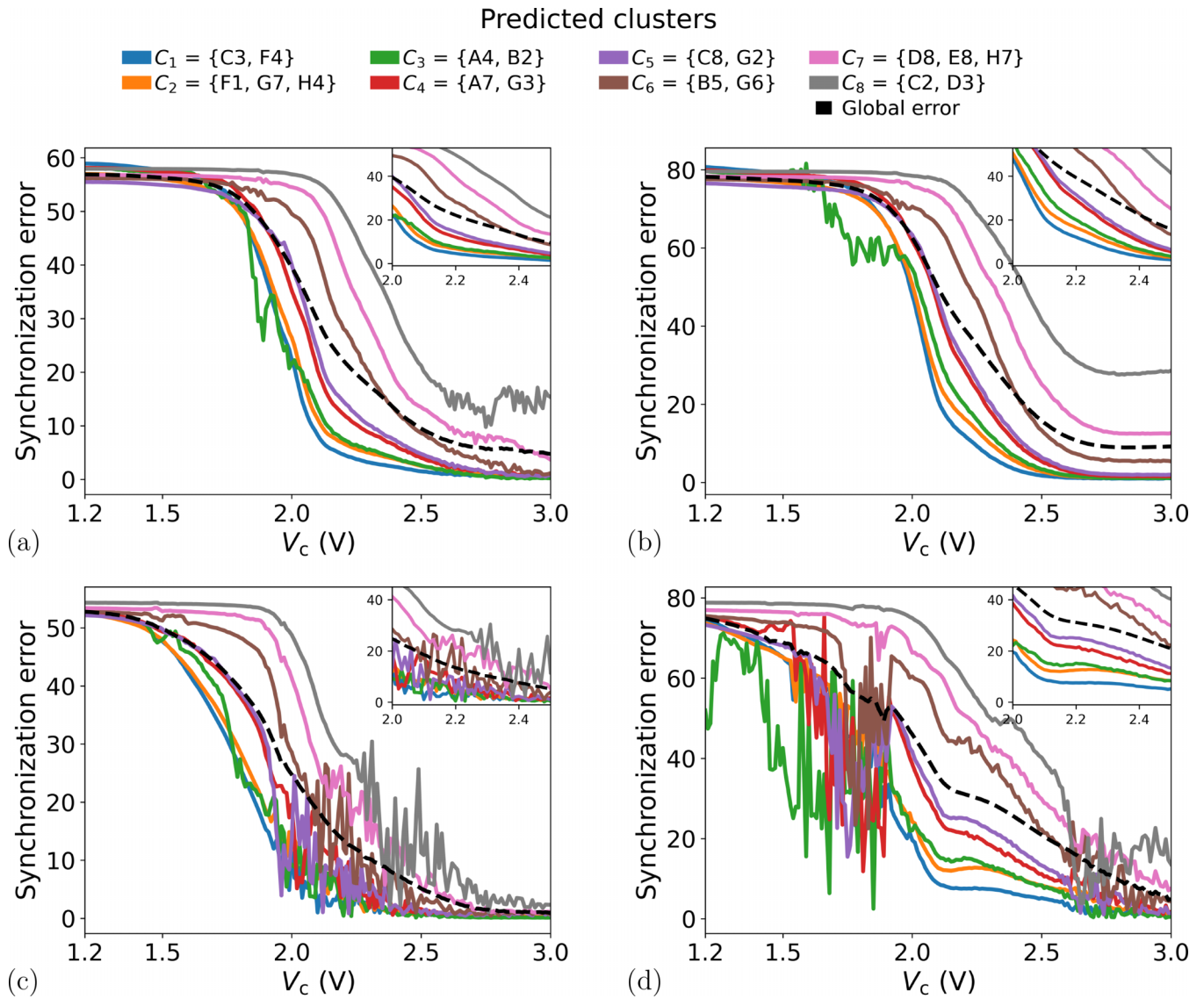


FIG. 12. Synchronization error in experimental data (averaged over five runs) for predicted clusters vs coupling voltage $V_c \in [1.2, 3]$ V. The insets show a close-up into the range $V_c \in [2, 2.5]$ V. The panels refer to different configurations of the parameters R and V_S : (a) $R = 470 \Omega$, $V_S = 3$ V; (b) $R = 470 \Omega$, $V_S = 5$ V; (c) $R = 1000 \Omega$, $V_S = 3$ V; and (d) $R = 1000 \Omega$, $V_S = 5$ V.

clusters reach synchronization as V_c increases is consistent with the predictions of the spectral block analysis based on the critical value V_c^* obtained from SPICE simulations (Table VI), thereby confirming the validity of the approach.

Regarding the transition points, for the two configurations with $R = 470 \Omega$, a clear rightward shift in the synchronization error curves was observed when compared from $V_S = 3$ V [Fig. 12(a)] to $V_S = 5$ V [Fig. 12(b)]. This trend is consistent with the predictions of the spectral block analysis, which shows that the critical values $V_{c,\text{crit}}$ for $V_S = 5$ V are higher than those for $V_S = 3$ V. A similar shift is observed for the configurations with $R = 1000 \Omega$, where the synchronization error curves also move rightward when increasing the supply voltage from $V_S = 3$ V [Fig. 12(c)] to $V_S = 5$ V [Fig. 12(d)]. This observation is consistent with the values reported in Table VI, which show that the critical parameters $V_{c,\text{crit}}^{(3)}$ and

$V_{c,\text{crit}}^{(4)}$ increase with V_S . Here, due to the observed rightward shift, we do not attempt to perform a quantitative comparison between the transition points predicted by the SPICE-based analysis and those extracted from the experimental data. Several factors may contribute to this bias, including noise, component variability, and parameter mismatches among the 59 single-transistor chaotic circuits in the network. All these aspects influence both the precise coupling values at which clusters synchronize and the degree of synchronization within and across clusters. A more reliable estimate could be obtained by physically implementing the two-node probe within the experimental setup, thereby incorporating a dedicated calibration unit directly into the electronic board.

The overall analysis of the synchronization errors across all configurations shows that, although the exact transition

points are not predicted with perfect quantitative accuracy, their sequence and general dependence on the parameters of the unit dynamics are reliably captured by the spectral block analysis. At the same time, these results suggest that dedicated experimental solutions could be developed to improve the accuracy of the predictions.

IV. DISCUSSION AND CONCLUSION

A. Brief summary of our work

This study presents an experimental demonstration of the spontaneous emergence of cluster synchronization in a network of 59 coupled transistor-based chaotic oscillators, whose topology was derived from a neuronal culture. We show that the spectral block analysis of the network's Laplacian matrix can successfully predict both the composition of the synchronous clusters and the sequence in which they synchronize. This study advances beyond earlier experimental work on the same electronic neuronal culture model [73] and related investigations in several key ways. First, it presents the first experimental observation of spontaneous cluster synchronization in a large-scale, biologically inspired electronic network, achieved without deliberately engineered symmetries or imposed parameter differences. Second, it represents the first application of the spectral block method to predict cluster formation in a real experimental system, thereby validating the theoretical framework under realistic conditions, including component variability and noise. Third, it demonstrates a hierarchical transition from incoherence to global synchronization via intermediate cluster states, revealing a refined multiscale organization of the underlying dynamics.

It is important to note that the synchronous clusters observed in our work are solely determined by network topology and, therefore, invariant with respect to specific nodal dynamics. To support the latter point, in this work we have also performed numerical simulations using Rössler systems coupled via the same topology, which confirmed the predicted cluster composition and the sequence of synchronization and desynchronization. Although the spectral block method successfully captures synchronization patterns, which are validated by both numerical and experimental results, it does not precisely predict the transition points. This limitation arises from inherent assumptions, such as the homogeneity of the units and the negligible influence of intercluster terms in the MSF derivation. Although they constitute a crucial step in enabling the analysis, they ultimately introduce approximations in the estimation of the critical thresholds. In an experimental setup such as ours, which inherently includes multiple sources of heterogeneity (e.g., parameter mismatches among units and variability in network links) as well as nonidealities (e.g., imperfections in electronic component behavior), observing the robustness of the method is particularly significant. This, in turn, demonstrates its practicality and reliability for the analysis of real-world systems. Our results also suggest possible improvements for future experiments aimed at a more precise identification of the transition points. In particular, the implementation of a dedicated calibration probe, physically realized on the electronic board as a pair of coupled units, could provide a more realistic estimate of the critical values,

thereby enabling more accurate predictions of the cluster transition points. Finally, it is important to note that the method is also capable of distinguishing between joint and independent clusters.

The spectral block analysis shows that the sequence of cluster formation is purely a topological property, independent of the specific node dynamics. This explains why the same cluster sequence is observed in both the simulations on Rössler systems and the experiments on single-transistor chaotic circuits: both systems share the identical NCM network topology. The method identifies groups of nodes that receive equivalent inputs from the rest of the network [Eq. (13)], a structural condition that does not depend on the functional form of the oscillator equations. While the sequence is therefore universal for a given network, the exact coupling strengths at which synchronization transitions occur are determined by the oscillator-specific MSF. The consistent characterization of cluster synchronization patterns across multiple scenarios underscores the robustness and generality of the spectral block approach.

Our analysis has focused on clusters that can be considered robust, as they exactly satisfy the equitable partition condition. By relaxing this condition and considering entries of S_n that are close to 2 within a specified tolerance, it becomes possible to identify groups of nodes that approximately satisfy the equitable partition condition, for instance, differing by only a few links. The dynamics of such node groups is characterized by a high, though not perfect, degree of synchronization, a situation that is reminiscent of quasisymmetries that give rise to nearly synchronized states [81,82]. Clearly, these node groups may appear more frequently than the robust clusters, especially in real networks where the topology is often inferred from noisy data. Investigating the properties and stability of these groups, as well as their transitions toward perfect synchronization or desynchronization, represents an important direction for future research, to which the spectral block method can provide a valuable contribution.

B. Cluster synchronization in the NCM: spontaneous emergence versus engineered control

A first key aspect of the work is how cluster synchronization emerges in the network. Cluster synchronization has been studied under two main contexts: (1) spontaneous emergence, where clusters form naturally as a result of the network topology [36,37,47]; and (2) engineered control, where synchronization patterns are deliberately induced through specific topological designs or external inputs [50,51]. In the context of engineered control of cluster synchronization, several strategies have been proposed to enforce desired synchronization patterns selectively. For example, it has been shown that network symmetries can be exploited to induce desynchronization in specific clusters while preserving synchronization in others by carefully designed perturbations to selected nodes or connections [36]. In experiments using laser networks, synchronous clusters have been shown to be engineered by modifying the coupling topology through the introduction of directed loops and self-feedback links [83]. Furthermore, the

task of identifying suitable perturbations to achieve a target clustering configuration can be systematically addressed by formulating it as an optimization problem [50]. In contrast to these engineered approaches, our findings highlight that cluster synchronization can spontaneously emerge in a network of chaotic oscillators, without the need for externally imposed parameter tuning or specifically designed structural features. In this context, it is quite significant that the phenomenon can be observed in a real network whose topology models a neuronal culture. Remarkably, the results hold whether the network is assumed to be homogeneous (as in most of our simulations) or heterogeneous (as in some numerical simulations and in the experimental setup), indicating the robustness of the observed synchronization patterns and of the technique used for their study. This result reinforces that cluster synchronization is not a rare phenomenon confined to special cases, but rather a naturally attainable state that occurs pervasively in complex networks of nonlinear systems. It may, in fact, be a common occurrence in systems where global synchrony is not immediately realized.

C. Topological drivers of synchronization and transitional dynamics

A second aspect to consider pertains to the topological determinants of cluster synchronization. While traditional approaches that relate synchronization patterns to network symmetries [36] are powerful for small or regular networks, they become computationally impractical for large networks. In contrast, our results demonstrate that efficient methods based on spectral blocks successfully predict cluster formation, providing a more scalable alternative for analyzing complex topologies. The spectral block method [47] is grounded on the notion that synchronization clusters correspond to localized eigenmodes of the network Laplacian, that is, node groups receiving equivalent dynamical inputs from the rest of the network. A notable aspect of this method is that it is an extension of the MSF framework, originally developed for global synchronization, to the study of cluster stability. Specifically, it enables the identification of which Laplacian eigenvalues govern the stability of each cluster's synchronization, in turn enabling the prediction of the sequence in which clusters synchronize or desynchronize.

The predicted cluster sequence matched with the transitions observed in a physical electronic experimental setup, implying that the Laplacian eigenmodes govern the synchronization dynamics even in imperfect and noisy real-world systems. This result is particularly important from a practical standpoint. Notably, the fact that these patterns continue to exist in spite of the intrinsic variability and nonideality of components indicates that spectral block-based predictors remain robust even when exact network symmetries are broken. In simulations and experiments alike, as the coupling strength increased, clusters emerged in sequence, following the order of the Laplacian eigenvalues, and eventually merged with other already synchronized groups of nodes. This reflects the hierarchical organization of the Laplacian eigenmodes, where finer partitions coalesce into coarser ones as coupling strengthens.

This stepwise progression from localized clusters to global synchronization demonstrates how network topology naturally guides the system through a hierarchy of partially synchronized states, providing insights for controlling synchronization patterns in real-world networks [51]. In this respect, our experimental demonstration of spontaneous cluster synchronization in the NCM network of transistor-based chaotic oscillators has multifaceted implications for understanding how coordination arises in complex systems. Specifically, such a network progresses through a sequence of intermediate states in which subsets of nodes gradually synchronize before merging into larger assemblies. This sequential transition reveals the underlying multiscale synchronization process of the network. This is in line with recent work that posits global synchronization ensues through a cascade of partial, cluster-based transitions [47]. Each synchronized cluster marks a reduction in the system's effective dimensionality, as groups of oscillators begin to behave coherently and can be regarded as collective units. Observing such transitions in the presence of uniform coupling and in the absence of *ab initio* engineered asymmetries underscores the fact that cluster synchronization and community formation, far from being a fragile or rare occurrence, are a pervasive emergent phenomenon that is shaped by the underlying network structure and intrinsic dynamics of the units.

D. Comparison across experimental platforms

A third aspect to consider regards the construction of the experimental platform under consideration. Cluster and partial synchronization have been found to arise across countless scenarios, including networks of mechanical, chemical, optoelectronic, and photonic oscillators. Mechanical systems such as coupled pendula and chemical systems have shown self-organized synchronization patterns and transient clustering dynamics [59,61,63]. While these experimental setups are viable and widely available, their scalability and configurability tend to be limited. On the other hand, photonic systems, particularly arrays of coupled lasers and time-delayed feedback systems, have enabled more versatile explorations of synchronization pattern formation. For instance, Nixon *et al.* [54] were among the first to report the emergence of synchronized clusters in semiconductor laser networks with delays. In subsequent work, they demonstrated how tuning feedback loops of varying lengths can generate multiple clusters [83]. Our study extends the investigation of cluster synchronization to the purely electronic domain, offering advantages in scalability, controllability, and engineering relevance. First, our system employs real, physically realized nodes, which are realized by means of elementary analog oscillators operating in parallel with effectively instantaneous coupling, unlike many photonic setups that rely on virtual nodes or time-multiplexed dynamics. Second, the cluster states in our network emerge spontaneously, without engineered heterogeneities, designed symmetry breaking, or external intervention, therefore highlighting the intrinsic capacity of the network structure and chaotic oscillator dynamics to support modular synchronization.

To our knowledge, this work represents one of the first experimental demonstrations of spontaneous cluster

synchronization in a purely electronic network of chaotic oscillators. This distinguishes our findings from previous electronic experiments, where the latter have typically observed global synchronization or engineered two- or multicluster states by imposing large parameter mismatches so as to create asymmetric couplings [52,56]. Furthermore, our platform provides continuous, fine-grained control of the coupling strength through analog voltages, thus enabling real-time tracking of synchronization transitions. This level of experimental flexibility is difficult to achieve in laser systems [83], where coupling is adjusted discretely using interferometric techniques and often exhibits hysteresis. Furthermore, the simplicity and low component count of the transistor-based chaotic oscillators used here facilitate scalable implementation, paving the way for studying larger networks. Finally, the biological relevance of our platform further distinguishes it. By constructing the network topology from a neuronal culture, our system naturally incorporates key features of cortical networks, such as small-world properties and degree heterogeneity. This biologically grounded framework, in combination with precise control over coupling, makes our platform particularly well suited for investigating cluster synchronization in realistic neurallike architectures. Our results hold potential applications in neuromorphic computing and brain-inspired models, and, more broadly, highlight the analysis of cluster synchronization as a powerful probe of collective dynamics in complex, real-world electronic systems.

E. Conclusion

In summary, our work has provided a comprehensive experimental and theoretical investigation of cluster synchronization in a biologically inspired network of chaotic electronic oscillators. We have performed numerical simulations and experimental validation in combination with spectral block analysis. As a result, the spontaneous emergence of complex synchronization patterns was demonstrated with network topology alone, i.e., without the requirement for engineered symmetries or mismatched parameters as in earlier works. Importantly, our results have shown that, through a well-defined sequence of clustered states, complete transitions to global synchronization in class II systems and both to and away from global synchronization in class III systems were achieved. Moreover, the order of such a sequence of clusters is dictated by the network's Laplacian spectrum. Despite the presence of experimental imperfections, the robustness of such synchronization patterns persists, thus highlighting the fundamental role of network structure in organizing collective dynamics. Overall, our results advance the experimental frontier of cluster synchronization by demonstrating its emergence in a biologically inspired, physically implemented, and dynamically rich system, furthering our fundamental understanding of spontaneous pattern formation in complex networks while providing a versatile platform for studying emergent synchronization phenomena. They pave the way for using similar networks as realistic analog models for neural dynamics, and, more broadly, as experimental platforms for studying emergent coordination in natural and engineered systems. Consequently, such platforms can be promising candidates for neuromorphic computing architectures or adaptive

reservoirs, where modularity and flexible coordination are important functional characteristics. Future research could investigate the emergence of synchronous clusters in experimental platforms with time-varying network topologies and adaptive couplings. In addition, the interplay between noise, heterogeneity, and cluster stability, as well as strategies for controlling and stabilizing desired cluster states, represents an important avenue for further exploration.

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DATA AVAILABILITY

The code used for numerical simulations and analysis, along with the data and experimental results, is publicly available in Ref. [77]. Additional data supporting the findings of this study are available from the corresponding author upon reasonable request.

APPENDIX A: EXPERIMENTAL SETUP, DATA COLLECTION, AND DATA PREPROCESSING

In this Appendix, we briefly summarize the steps, from the recorded spontaneous bursting activity of the embryonic rat neuronal culture to the derivation of the synchronization matrices analyzed in this study. Neuronal activity was collected from *in vitro* dissociated cortical cultures of embryonic rat neurons plated on multielectrode arrays. These MEAs consisted of 59 recording electrodes arranged in a square grid (8×8), missing corner electrodes and one edge electrode. Thereafter, spike trains were extracted using standard threshold-based detection techniques [74], ensuring accurate discrimination of action potentials from background noise. For this study, the recordings from the culture referred to as "4-1," were curated to capture the development of collective neuronal dynamics over several weeks [84]. Notably, only signals from the mature stage of the culture, characterized by high spiking variability and network-wide bursting activity, were considered [73]. From these experimentally acquired spike trains, a pairwise synchronization matrix was computed as a proxy of the functional connectivity of the culture's neuronal activities, using the cross-correlation of spike events. The coincidence index (CI) was finally used to quantify syn-

chronization between electrode pairs as a scalar value [73]. This calculation provided a quantitative representation of the degree of synchronous activity and functional interactions within the neuronal network [73].

The topology of the electronic network was directly mapped from the binarized pairwise synchronization matrix. Initially, the synchronization matrix, computed using the CI [73], was thresholded at $CI > 0.075$, resulting in a symmetric matrix with 1432 nonzero entries (density ~ 0.4). To ensure the feasibility of the printed circuit board (PCB) layout, manual refinements were applied: 96 links were removed (86 due to excessive connections at certain nodes that could not be routed given the density constraints of the board, and an additional 10 links were removed due to routing congestion and wire-crossing issues as they increased the complexity of design management), while 16 links were added to guarantee global connectedness. Finally, a symmetric synchronization matrix containing 1352 nonzero entries was obtained. These manual modifications did not significantly alter the network's statistical properties from the original culture-derived network, preserving the biological culture's heterogeneous degree distribution and small-world characteristics. Specifically, the structure retained a density of 0.4, and a mean node degree of 23 with a standard deviation of 16 (coefficient of variation 0.7). Although this matrix is initially derived from dynamical activity, it is henceforth considered as structural connectivity in the context of this work, as it determines the couplings between the oscillators.

This structural connectivity, which, as mentioned in Sec. II, is referred to as the Neuronal Culture Model, is a network that is a low-resolution, biologically inspired connectome model consisting of 59 nodes and 676 undirected edges (density $\sim 39.5\%$). The topological properties of this network are discussed in detail in Ref. [73] and here we briefly summarize as follows. The NCM network has an average clustering coefficient of 0.73, an average shortest path length of 1.75, a link density of 0.395, and a diameter of 4. It exhibits hallmark features of small-world organization. The small-worldness coefficient S [85] was computed as $S = (C/C_{\text{rand}})/(L/L_{\text{rand}})$, where C and L are the clustering coefficient and average shortest path length of the NCM, and C_{rand} and L_{rand} are the corresponding values from an ensemble of 100 Erdős-Rényi (ER) random networks [86] with matched size and density. The resulting small worldness was $S = 1.71 \pm 0.07$ ($p < 0.001$), confirming a notable small-world organization. In comparison, the ER networks showed significantly lower clustering coefficients (0.40 ± 0.10 , $p < 0.001$) and slightly shorter path lengths (1.60 ± 0.10 , $p < 0.001$). Furthermore, the node degree distribution of the ER networks was more homogeneous (average degree 22.9 ± 3.7 , coefficient of variation 0.16) compared to the NCM (coefficient of variation 0.7), reflecting the higher degree heterogeneity typical of biological systems. The properties of the NCM network are also compared with those of an ensemble of 100 Watts-Strogatz (WS) small-world networks [87] of identical density. The WS networks exhibited high clustering (0.68 ± 0.01) but substantially shorter path lengths (0.75 ± 0.02) and extremely low heterogeneity (average degree 22 ± 0.74 , coefficient of variation 0.03). In conclusion, the NCM combines small-world

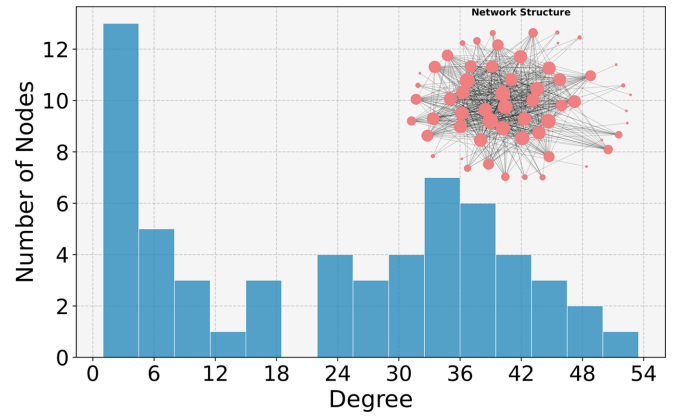


FIG. 13. Degree distribution of the NCM network (bin size 3.5). The inset shows the network visualization. The bimodal distribution highlights the heterogeneity in nodal degree, indicative of a core-periphery architecture.

properties with marked degree heterogeneity, distinguishing it from both ER and WS canonical models.

The degree distribution of the NCM is bimodal (Fig. 13), with one mode corresponding to peripheral low-degree nodes and a second mode capturing a subset of highly connected hubs. This bimodality is inconsistent with the narrow unimodal distributions of ER and WS networks and suggests a core-periphery organization, as visualized in the inset of Fig. 13, highlighting dense interconnections among hub nodes and sparser peripheral connectivity. Such a structure underpins the emergence of differentiated functional dynamics, including the spontaneous formation of synchronization clusters explored in this study.

Due to their combination of simple topology and rich dynamical behavior, single-transistor chaotic circuits were employed as the nodes of the network [88]. These electronic oscillators were configured to operate in a sensitive dynamical regime (near the boundary between periodic and chaotic states), facilitating responsiveness to internode coupling. Further details and the complete design materials can be found in Ref. [73].

The physical implementation of the electronic network on a four-layer printed circuit board involved oscillator nodes arranged in a regular grid layout and interconnections routed in two orthogonal inner layers. Unlike in Ref. [73], all the time series corresponding to the voltage u_{C1} were simultaneously acquired at a sampling frequency of 250 kHz with a high-performance data acquisition card connected directly to a workstation. The recorded signals were subsequently analyzed through the following steps:

(1) *Normalization and error computation.* The recorded signals were first mean centered, yielding what is hereafter referred to as “raw” or unnormalized signals. These were then scaled by their respective standard deviations so as to obtain normalized signals, allowing amplitude-invariant comparisons. Accordingly, both unnormalized and normalized synchronization error matrices, $E^{\text{raw}} = \{e_{ij}^{\text{raw}}\}$ and $E^{\text{norm}} = \{e_{ij}^{\text{norm}}\}$, were computed. The generic ij element of these matrices represents the synchronization error between the

oscillators i and j :

$$e_{ij}^{\text{raw}} = \langle |u_i^{\text{raw}}(t) - u_j^{\text{raw}}(t)| \rangle_{\Delta T} \quad (\text{A1})$$

or

$$e_{ij}^{\text{norm}} = \langle |u_i^{\text{norm}}(t) - u_j^{\text{norm}}(t)| \rangle_{\Delta T}, \quad (\text{A2})$$

where $\langle \rangle_{\Delta T}$ represents averaging over time interval ΔT and $u_i^{\text{raw}}(t)$ [$u_i^{\text{norm}}(t)$] denotes the raw (normalized) signal obtained from the measured variable, specifically the voltage across capacitor C_1 from circuits i , namely, $u_{C1,i}(t)$ (see Appendix B).

(2) *Averaging over number of repetitions and aggregation.* The experimental protocol was repeated for each set of the circuit's parameters to ensure the robustness of the synchronization analysis. To be precise, the resistance R (see Appendix B) was set to two different values (either 470 or 1000 Ω), and the supply voltage V_s was assigned one of two fixed values (3 or 5 V). For each configuration, the coupling voltage V_c —a parameter controlling the coupling strength among the network nodes (see Appendix B)—was incrementally varied from 1.2 to 3.0 V in steps of 0.01 V. Five independent experimental runs were performed, and for each run of the experiment, the synchronization matrices were computed. These were then averaged across 19 repeated measurements to mitigate the impact of noise and transient effects.

APPENDIX B: TOPOLOGY AND WORKING PRINCIPLE OF THE SINGLE-TRANSISTOR CHAOTIC CIRCUIT

The chaotic oscillator instantiated at each network node is a minimalistic autonomous circuit, consisting of one Bipolar Junction Transistor, two inductors (denoted by L_1 and L_2), a single capacitor C_1 , and a resistor R . The circuit is powered by a single-ended voltage supply V_s . The circuit diagram is shown in Fig. 14(a). The base and collector terminals of the transistor are each connected to a common central node via separate inductors (L_1 and L_2 , respectively), which in turn is grounded through a capacitor C_1 and is also connected to the direct current (DC) voltage source V_s via the resistor R .

As detailed in Ref. [88], chaotic oscillations arise due to the complex interplay between a relaxation-type oscillator subcircuit and a damped resonator subcircuit. The relaxation oscillation primarily arises from the interaction between the inductor L_2 , the resistor R , and the nonlinear characteristics of the transistor. The subcircuit made by these elements generates pulses that periodically “reset” a resonant subcircuit primarily involving the inductor L_1 and capacitor C_1 . The damped resonator that gets excited by each reset event produces oscillations whose decay interacts reciprocally with the relaxation process. Under certain parameter conditions and in particular when neither relaxation nor resonance entirely dominates, chaos emerges. This results in persistent and unpredictable variation from one cycle to the next. The circuit is known to exhibit diverse and rich nonlinear dynamics including spiking and burstlike behaviors, similar to those observed in biological neurons for specific sets of the parameter values chosen. By tuning the resistor R and inductors L_1, L_2 , a clear route to chaos emerges via period-doubling bifurcations. Notably, chaotic behavior predominantly occurs when L_2 is substantially greater than L_1 , providing distinct timescales for relaxation and resonance to interact in the form of fast and

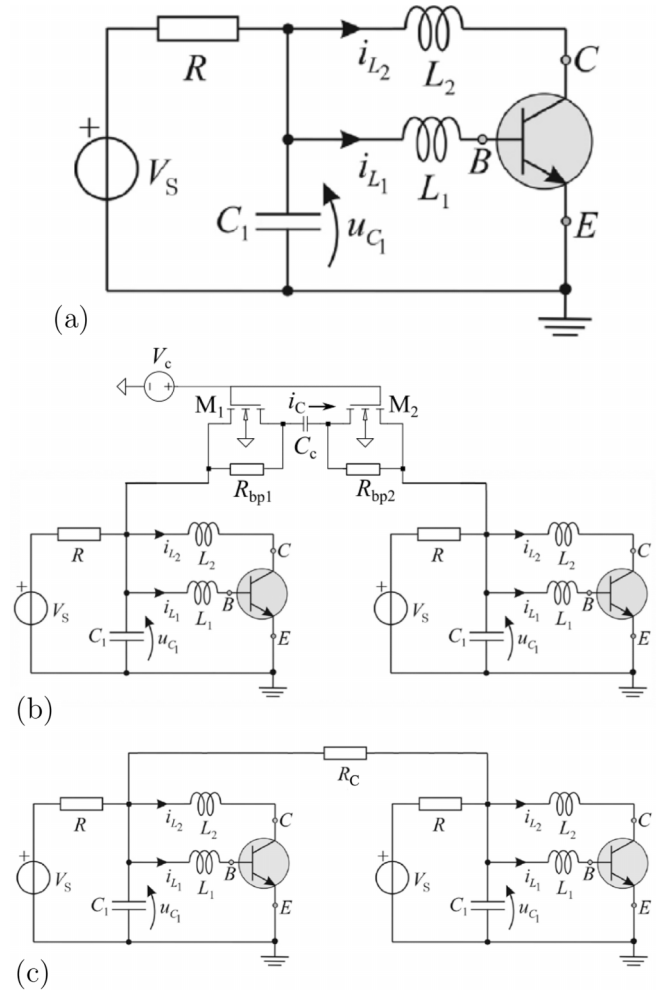


FIG. 14. Single-transistor chaotic circuit: (a) electrical scheme of the circuit [88] (reprinted from Ref. [73], with permission from Elsevier). (b) configuration used in the experimental board for coupling two units via circuitry realizing a voltage-dependent resistor. (c) Equivalent electrical scheme of two coupled circuits.

slow dynamics, without trivially synchronizing. Additionally, V_s and R , which can be easily tuned experimentally, are typically taken as the main control parameters of the circuit.

The circuit behavior can be captured through a simplified mathematical model that retains the essence of its nonlinear dynamics [88]. This consists of approximating the transistor nonlinearity and then introducing a capacitor C_2 as an additional element to account for the capacitive effects between the collector and the emitter. The system is described by four state variables: u_{C1} and u_{C2} , which are the voltages across the two capacitors, and i_{L1} and i_{L2} , which are the currents through the two inductors. These evolve according to

$$\begin{aligned} \frac{du_{C1}}{dt} &= \frac{V_s - u_{C1}}{RC_1} - \frac{i_{L1} + i_{L2}}{C_1}, \\ \frac{du_{C2}}{dt} &= \frac{i_{L2} - \beta \Gamma(i_{L1}) \tanh\left(\frac{u_{C2}}{2V_{th}}\right)}{C_2}, \\ \frac{di_{L1}}{dt} &= \frac{u_{C1} - V_{th}}{L_1}, \quad \frac{di_{L2}}{dt} = \frac{u_{C1} - u_{C2}}{L_2}, \end{aligned} \quad (\text{B1})$$

where $\Gamma(i_{L1})$ is a ramp function defined as $\Gamma(i_{L1}) = i_{L1}$ for $i_{L1} > 0$, and $\Gamma(i_{L1}) = 0$ otherwise. This formulation ensures the absence of negative base-current amplification and, together with the term $\beta \tanh(\frac{u_{C2}}{2V_{th}})$, captures the nonlinearity that approximates the behavior of the transistor. The parameters in Eq. (B1) have the following meaning: V_S is the DC supply voltage, V_{th} the transistor base-emitter voltage assumed to be constant (typically, $V_{th} \sim 0.6$ V), R is the series resistor, C_1 is the capacitor at the common node, C_2 is the equivalent transistor capacitance between collector and emitter assumed to be stationary, L_1 is the base-side inductor, L_2 is the collector-side inductor, and β is the effective current gain of the transistor. Throughout our paper, the following component values were used: $L_1 = 5$ mH, $L_2 = 10$ mH, $C_1 = 25$ nF, $V_{th} = 0.6$ V, $\beta = 100$, and $C_2 = 5$ pF.

Figure 14(b) describes the coupling circuitry adopted to physically implement the structural connectivity derived from the functional analysis of neuronal culture data. The circuitry comprises a pair of MOSFETs controlled by a voltage input V_C . Hence, a link (i, j) in the network corresponds to a pair of nodes i and j connected as in Fig. 14(b). The

coupling circuitry effectively implements a voltage-dependent resistor between the C_1 capacitors at nodes i and j . To simplify the analysis, we model the circuitry as an equivalent resistance, R_C , controlled by the voltage V_C , as illustrated in Fig. 14(c). In this framework, the coupling corresponds to a diffusive term between the capacitor voltages, specifically $\frac{u_{C1}^{(j)} - u_{C1}^{(i)}}{R_C C_1}$, which appears with opposite signs in the equations for both variables, yielding the network dynamics described by Eq. (3). The relationship between V_C and R_C (or equivalently the conductance $G = 1/R_C$) is nonlinear, as shown in Fig. 3(d) of Ref. [73]. Consequently, there is no simple analytical expression directly mapping V_C to the theoretical coupling strength d ; instead, the mapping follows a smooth nonlinear curve. However, within the experimental range $V_C \in [1.2, 3.0]$ V, the conductance varies approximately linearly with V_C , providing a nearly linear correspondence between the control voltage and $d = 1/(R_C(V_C)C_1)$. This linear approximation, together with averaging over multiple experimental runs to account for component tolerances, allows V_C to serve as a reliable proxy for the coupling strength in our analysis.

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