

Riccati equation perspective on Landau-Zener transitions

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We express the dynamics of the two probability amplitudes in the elementary Landau-Zener problem in terms of the solution of the corresponding Riccati differential equation and identify three key features: (i) The solution of the Riccati equation provides the bridge between the two probability amplitudes. (ii) Neglecting the nonlinearity in the Riccati equation is equivalent to the Markov approximation which yields the exact asymptotic expression for one of the probability amplitudes. (iii) The Riccati equation identifies the origin of the failure of the Markov approximation not being able to provide us, in general, with the correct asymptotic expression of the other probability amplitude. Our approach relies on approximate yet analytical solutions of the Riccati equation in different time regimes, highlighting the impact of its nonlinear nature on the time evolution of the system.

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I. INTRODUCTION

Nonadiabatic transitions like the Landau-Zener (LZ) transitions [1–5] play a crucial role in many areas of physics, including molecular collisions [6–8], quantum control [9–11], quantum information processing [12], atom interferometry [13–18], and other fields [19–21]. Among the simplest and most widely studied examples is the Landau-Zener model, which has served as a cornerstone for understanding time-dependent quantum dynamics. Although the LZ problem has had a complete analytic solution for nearly a century, it remains a valuable testbed for examining the validity and breakdown of various approximation methods [22–25], due to its conceptual clarity and wide applicability.

While the LZ model captures essential aspects of transition dynamics between two energy levels, it also has well-known limitations [26,27]. For instance, it assumes infinite interaction duration and unbounded energy levels, which are not realistic in many physical systems. Extensions such as the Demkov-Kunike models [28–30], the SLS model [29] and other models attempt to address these shortcomings. Nevertheless, the LZ framework continues to serve as a fertile ground for theoretical exploration.

A. Why the Riccati equation?

In this work, we revisit the LZ problem using the Riccati differential equation [31–35], which governs the time evolution of the ratio of diabatic probability amplitudes. This formulation offers an alternative to conventional methods based on parabolic cylinder functions [36,37], semiclassical, or analytic continuation techniques like Dykhne-Davis-Pechukas method [38–40] and the Zhu-Nakamura theory [41–44].

Our goal is not to re-derive the well-known LZ transition probability, but rather to use the Riccati equation to uncover structural and dynamical features that are less transparent in traditional formulations.

Specifically, the Riccati approach:

- (i) Reveals the role of nonlinearity and its relation to the so-called Markov approximation [45–47];
- (ii) Clarifies the asymmetry between the two probability amplitudes, one is directly proportional to the Riccati solution, while the other depends on its *integral*;
- (iii) Provides a framework for analytically connecting the negative and positive time domain via an approximate connection formula.

We show that the Markov approximation, originally introduced in Refs. [48,49] in the context of deriving the Landau-Zener transition probability, emerges as a linearization of the Riccati equation. This insight enables a systematic analysis of its range of validity and opens the door to improved approximations using asymptotic methods, all without relying on special-function representations. Moreover, we emphasize that the connection between the Markov approximation and the Riccati equation is particularly relevant for open quantum

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systems, where Landau–Zener transitions in non-Markovian environments have been actively investigated [50].

B. Physical insights

The Riccati equation reveals structural features of the transition dynamics that are often hidden in standard approaches.

Indeed, representations of the time dependence of the two probability amplitudes as trajectories in the complex plane bring out distinct differences in the behavior of their amplitudes and phases. One probability amplitude reverses its direction of rotation when the two levels cross, whereas the other one maintains its direction.

At the same time, their respective amplitudes change abruptly. The one starting from an amplitude of one quickly approaches its asymptotic value determined by the LZ formula while oscillating around it, giving rise to the Stueckelberg oscillations [5]. The other one spirals out of the starting point, that is the origin of the complex plane, and oscillates around its final radius, given by the combination of the LZ formula and the normalization condition.

These phenomena emerge from asymptotic solutions of the Riccati equation, thus providing a fresh analytical perspective on a classic problem. Moreover, the methods developed here offer the possibility for a generalization to more complex nonadiabatic scenarios, including time-dependent pulses, and multilevel systems or crossings [51–53]. By focusing on the interplay between the nonlinearity and the temporal evolution, our approach contributes to a more transparent understanding of approximate dynamical methods in quantum mechanics.

C. Overview and structure

The paper is organized as follows: In Sec. II, we briefly review the standard formulation of the Landau–Zener problem. Subsequently, we dedicate Sec. III to the derivation of the Riccati differential equation and analyze its structure and its relation to the probability amplitudes. In Sec. IV, we connect the Markov approximation to a linearization of the Riccati equation and investigate its range of validity. Section V analyzes the Markov solution in three different time domains and presents asymptotic expressions. In contrast to Sec. V, we investigate in Sec. VI the exact solution of the Riccati equation in three different time domains and derive asymptotic expressions which emphasize the role of the nonlinearity in the Riccati equation. In Sec. VII we return to the problem posed in Sec. II employing the results obtained in previous sections. Section VIII discusses the implications of our results and outlines possible extensions. Additional technical details and derivations are provided in the Appendixes.

II. FORMULATION OF THE PROBLEM: LANDAU-ZENER MODEL

In this section, we introduce the Landau–Zener model for a time-dependent two-level quantum system and present the corresponding coupled differential equations in dimensionless form, discuss their initial conditions, and summarize the known asymptotic solutions based on parabolic cylinder functions. For detailed derivations, see Appendix A and Ref. [24]. We conclude with a visualization of the time dependence of the probability amplitudes as trajectories in the complex

plane. The goal of our article is to bring out the key features of this time dependence by approximate and analytical solutions of the Riccati equation.

To set up the problem, we consider the coherent evolution of probability amplitudes $a \equiv a(\tau)$ and $b \equiv b(\tau)$ governed by the set of the two coupled differential equations

$$i\dot{a} = -\epsilon\tau a + b, \quad (1)$$

$$i\dot{b} = a + \epsilon\tau b, \quad (2)$$

where the dimensionless time $\tau = \Omega t$, and the inverse adiabaticity parameter $\epsilon = \alpha/\Omega^2$ includes the chirp rate α and the coupling constant Ω [24]. Here, the dot denotes differentiation with respect to τ . The amplitudes obey the normalization condition

$$|a(\tau)|^2 + |b(\tau)|^2 = 1, \quad (3)$$

and we impose the initial conditions

$$a(-\tau_0) = 1 \quad \text{and} \quad b(-\tau_0) = 0 \quad (4)$$

at a large but finite time $-\tau_0$. This approach regularizes the divergent derivative $\dot{a}$ at $\tau = -\infty$ and aligns with experimental and numerical treatments, where time evolution is naturally confined to finite intervals [54]. Throughout this work, simulations are performed for large but finite τ_0 , with the asymptotic limit $\tau_0 \rightarrow \infty$ taken analytically where required. Numerical integration of Eqs. (1) and (2) is carried out using standard libraries such as SciPy and Julia [55,56].

To understand the long-time behavior, we now summarize the asymptotic expressions for the amplitudes derived in Appendix A via asymptotic expansions of parabolic cylinder functions. The amplitude a approaches the exact Landau–Zener result [1–4]

$$a(\tau_0 \rightarrow \infty) = \exp\left(-\frac{\pi}{2\epsilon}\right), \quad (5)$$

which describes the survival probability in the initial diabatic state. This expression arises from a logarithmic phase singularity and the identity $-1 = \exp(-i\pi)$ highlighting the role of complex phases in transition amplitudes also shown in Appendix A.

The corresponding expression for b is given by [24,57,58]

$$b(\tau_0 \rightarrow \infty) = \sqrt{1 - |a(\tau_0 \rightarrow \infty)|^2} e^{i\phi}, \quad (6)$$

with the phase

$$\phi = \frac{3\pi}{4} - \epsilon\tau_0^2 - \frac{1}{\epsilon} \ln(\sqrt{2\epsilon}\tau_0) + \arg \Gamma\left(1 + \frac{i}{2\epsilon}\right), \quad (7)$$

where Γ denotes the Gamma function. This phase reflects accumulated dynamical and geometric contributions over the entire evolution.

To complement these analytical results, we numerically integrate the system and visualize the time evolution of a and b as trajectories in the complex plane under the initial conditions specified in Eq. (4). Figure 1 shows this evolution for both the Schrödinger picture (a) and the interaction picture (b).

In the Schrödinger picture Fig. 1(a), the amplitude a initially circles the unit circle, reverses its direction near $\tau = 0$, and approaches the asymptotic radius $\exp[-\pi/(2\epsilon)]$.

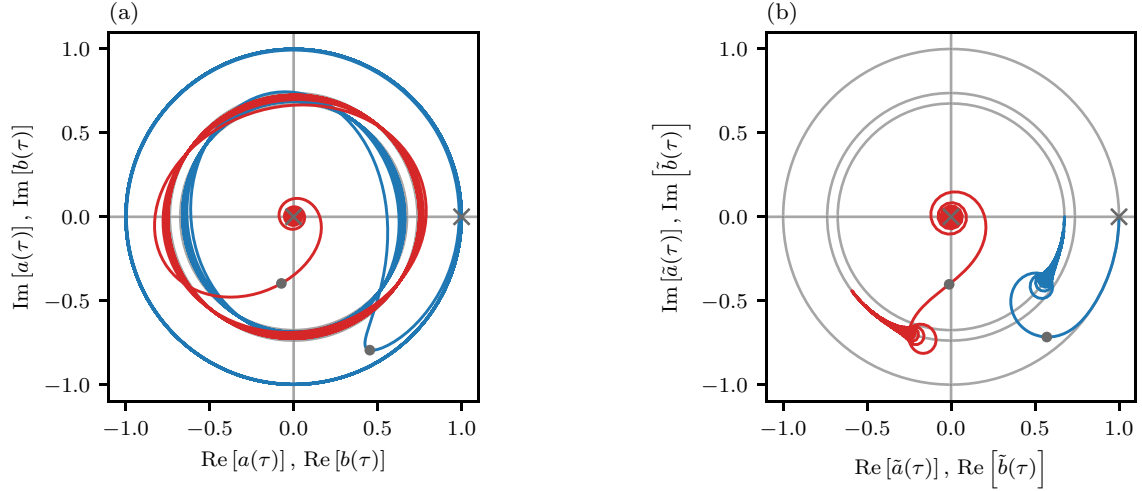


FIG. 1. Time evolution of the probability amplitudes a (blue) and b (red) for the Landau-Zener problem, shown as trajectories in the complex plane in the Schrödinger (a) and interaction pictures (b). Gray crosses mark the initial values at $\tau = -\tau_0$. Gray circles indicate: (i) the unit circle, (ii) the asymptotic value $\exp(-\pi/(2\epsilon))$ for a , and (iii) $\sqrt{1 - \exp(-\pi/\epsilon)}$ for b . In both panels, the blue trajectories start on the right-hand side, initially following the unit circle, and around $\tau = 0$ (dark gray dot), jump to their asymptotic circle, reversing direction and oscillating counterclockwise for $\tau > 0$ in panel (a). Red trajectories originate at the complex origin, spiral outward, and for $\tau > 0$ oscillate clockwise around their final value without a velocity sign change. In panel (b), the decay of Stueckelberg oscillations at positive times is visible for both amplitudes. Parameters are $\epsilon = 4.0$, $\tau_0 = 858.0855$, with τ spanning $-\tau_0 \leq \tau \leq \tau_0$.

Whereas the amplitude b begins near the origin, forming a Cornu spiral and transitions to a final circle of radius $\sqrt{1 - \exp(-\pi/\epsilon)}$, both amplitudes exhibiting Stueckelberg oscillations [5]. In the interaction picture Fig. 1(b), three differences are revealed: (i) rotational features diminish due to the transformation, (ii) the motion direction reverses, and (iii) the decay of Stueckelberg oscillations becomes more apparent for $\tau > 0$.

To bring out most clearly, the dependence of the probability amplitudes on the initial time, Fig. 2 compares the trajectories of a for three different values of τ_0 . While all paths originate from the same point (gray cross), they differ near $\tau = 0$ but converge to the same asymptotic radius, demonstrating the influence of initial conditions on the transient dynamics.

III. RICCATI EQUATION

In this section, we express the two probability amplitudes a and b in terms of a solution of a nonlinear differential equation of first order of the Riccati type. This equation will form the basis for elementary approximations employed to obtain asymptotic but exact results for the transition probability amplitudes.

A. Derivation

In order to derive the Riccati equation, we choose the ansatz

$$a(\tau) = e^{i\epsilon(\tau^2 - \tau_0^2)/2} e^{iH(\tau)}, \quad (8)$$

where we have defined the function

$$H(\tau) \equiv \int_{-\tau_0}^{\tau} d\tau' \eta(\tau') \quad (9)$$

for the integral of the function $\eta \equiv \eta(\tau)$ to be determined. We emphasize that the so-defined probability amplitude a satisfies the initial condition for a , Eq. (4).

This ansatz leads us to the expression

$$\dot{a} = i\epsilon\tau a + i\eta a \quad (10)$$

for the first derivative, and together with Eq. (1) to the formula

$$b = -\eta a, \quad (11)$$

which connects the probability amplitudes a and b .

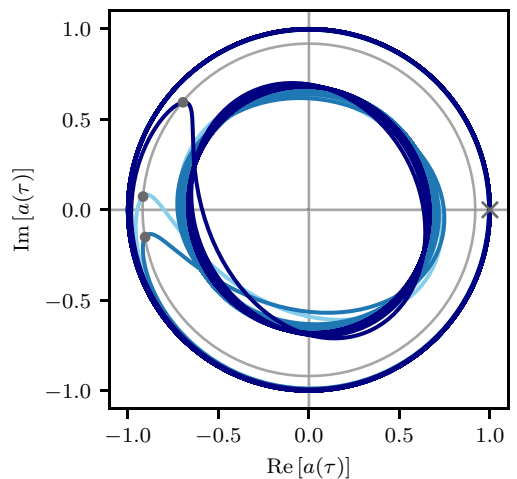


FIG. 2. Trajectories of the probability amplitude $a(\tau)$ for initial times $\tau_0 = 8.581$ (light blue), 27.135 (blue), and 54.270 (dark blue). All trajectories originate from the same point (gray cross) and asymptotically approach the radius $\exp(-\pi/(2\epsilon))$ predicted by the Landau-Zener formula. At $\tau = 0$ (dark gray dots), they differ in phase but share the same magnitude, lying on the gray circle. The 2π -periodicity of the phase induces nonmonotonic behavior near $\tau = 0$, with the trajectory for the smallest τ_0 (light blue) enclosed by those corresponding to larger τ_0 . Here, $\epsilon = 4.0$, and the trajectories span $-\tau_0 \leq \tau \leq \tau_0$.

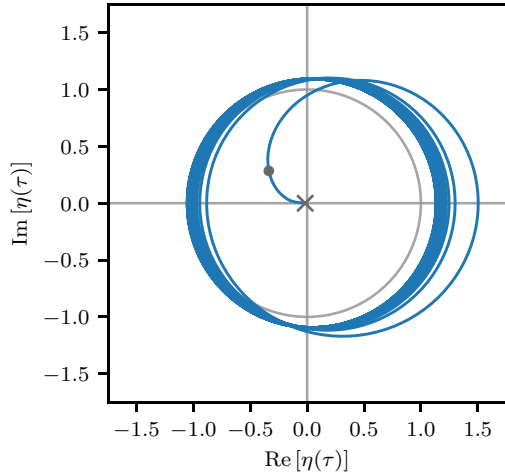


FIG. 3. Trajectory of the numerical solution $\eta(\tau)$ of the Riccati equation, Eq. (13), with initial condition, Eq. (14), in the complex plane. The trajectory originates at $\tau = -\tau_0$ (gray cross) and rapidly enters an oscillatory regime without encircling the origin; $\tau = 0$ is marked by a dark gray dot. The gray unit circle serves as a reference, illustrating that the motion does not remain on it. Parameters ϵ and τ_0 are as in Fig. 1.

Next, we differentiate Eq. (11) and find

$$\dot{b} = -\dot{\eta}a - \eta\dot{a}. \quad (12)$$

Thus, we insert Eq. (12) into the left-hand side, and Eq. (11) into the right-hand side of Eq. (2) which yields the nonlinear differential equation

$$\dot{\eta} + i\eta^2 + 2i\epsilon\tau\eta - i = 0 \quad (13)$$

of first order of the Riccati form with the initial condition

$$\eta(-\tau_0) = 0. \quad (14)$$

According to Eqs. (8) and (11), η given by the Riccati differential equation, Eq. (13), determines *both* probability amplitudes a and b . Whereas a is governed by the integral H of η , b is proportional to the product of a and η .

In Fig. 3 we present the numerical integration of the Riccati equation in the complex plane. The trajectory starts due to the initial condition at the origin and quickly transitions into a circular motion with oscillating amplitude. This behavior is in complete contrast to the dynamics of the probability amplitude b shown in Fig. 1, where b initially oscillates around the origin. This leads to the conclusion that η is almost zero for $\tau < 0$ and increases rapidly around $\tau = 0$.

B. Probability amplitudes in amplitude and phase

The solution η of the Riccati equation allows us to express the probability amplitudes a and b in amplitude and phase which brings out most clearly that there are only two independent degrees of freedom in the problem. Here, the decomposition

$$\eta \equiv \eta^{(R)} + i\eta^{(I)} \quad \text{or} \quad \eta \equiv |\eta|e^{i\phi_\eta} \quad (15)$$

of η in its real part $\eta^{(R)} \equiv \text{Re}[\eta]$ and an imaginary part $\eta^{(I)} \equiv \text{Im}[\eta]$ or in its amplitude $|\eta|$ and its phase ϕ_η plays

a crucial role. We now show that due to the ansatz Eq. (8) the representation Eq. (15) of η in its real and imaginary part is most convenient for a . However, due to the product of ηa defining b the representation Eq. (15) of amplitude and phase is more useful for b .

We start our analysis by considering the probability amplitude a and substitute the representation, Eq. (15), of η into the ansatz, Eq. (8), and arrive at the identity

$$a(\tau) = e^{i\Phi(\tau)}A(\tau), \quad (16)$$

where the amplitude

$$A(\tau) \equiv e^{-\text{Im}[H(\tau)]} \quad (17)$$

is given by the negative exponential of the integral of the imaginary part of η . The phase

$$\Phi(\tau) \equiv \epsilon(\tau^2 - \tau_0^2)/2 + \text{Re}[H(\tau)] \quad (18)$$

consists of a term quadratic in time and of the positive integrated real part of η given by $\text{Re}[H(\tau)]$. Hence, $\eta^{(R)}$ is a component of the phase velocity

$$\dot{\Phi}(\tau) = \epsilon\tau + \eta^{(R)}(\tau). \quad (19)$$

We note at this point that these expressions are related to the one we obtained in Ref. [49].

It is interesting to also represent b in terms of an amplitude and a phase. For this purpose, we cast Eq. (11) with the help of Eq. (16) into the form

$$b = -e^{i\Phi}A|\eta|e^{i\phi_\eta}. \quad (20)$$

We note at this point that the phase Φ is a global phase which is the same for the probability amplitude a and b .

When we substitute Eq. (11) into the normalization condition, Eq. (3), we obtain the relation

$$b(\tau) = e^{i\Phi(\tau)}\sqrt{1 - |A(\tau)|^2}e^{i\phi_\eta} \quad (21)$$

for the probability amplitude b .

Figures 4(a) and 4(d) show the real and imaginary parts of $\eta^{(R)}$ (blue) and $\eta^{(I)}$ (red), respectively. Both components vanishes as $\tau \rightarrow -\tau_0$, consistent with the initial condition, and enter an oscillatory regime with quadratically increasing frequency near $\tau = 0$. Notably, the growth of $\eta^{(I)}$ is more rapid than that of $\eta^{(R)}$ in this region.

Figure 4(e) presents the integral of $\eta^{(I)}$ (dark blue), plotted beneath the corresponding $\eta^{(I)}$ curve. As $\eta^{(I)}$ rises toward its first maximum, the integral increases rapidly. The oscillations in $\eta^{(I)}$ induce corresponding oscillations in its integral, known as Stueckelberg oscillations [5]. While the amplitude of $\eta^{(I)}$ remains roughly constant, the oscillations in its integral decay, approaching the asymptotic value $\pi/(2\epsilon)$, indicated by the horizontal gray line. The exponential of minus this integral (magenta curve) determines the absolute value of the amplitude a , denoted A which converges to the Landau-Zener limit Eq. (5) (second gray line).

Using the normalization condition Eq. (3) and A , Eq. (17), we obtain the absolute value of b , shown in Fig. 4(f) by the green curve. This value approaches the asymptotic expression $\sqrt{1 - \exp(-\pi/\epsilon)}$, indicated by the horizontal gray line.

The upper panels of Fig. 4 capture the phase dynamics. The global phase Φ , shared by both amplitudes, is shown as a light

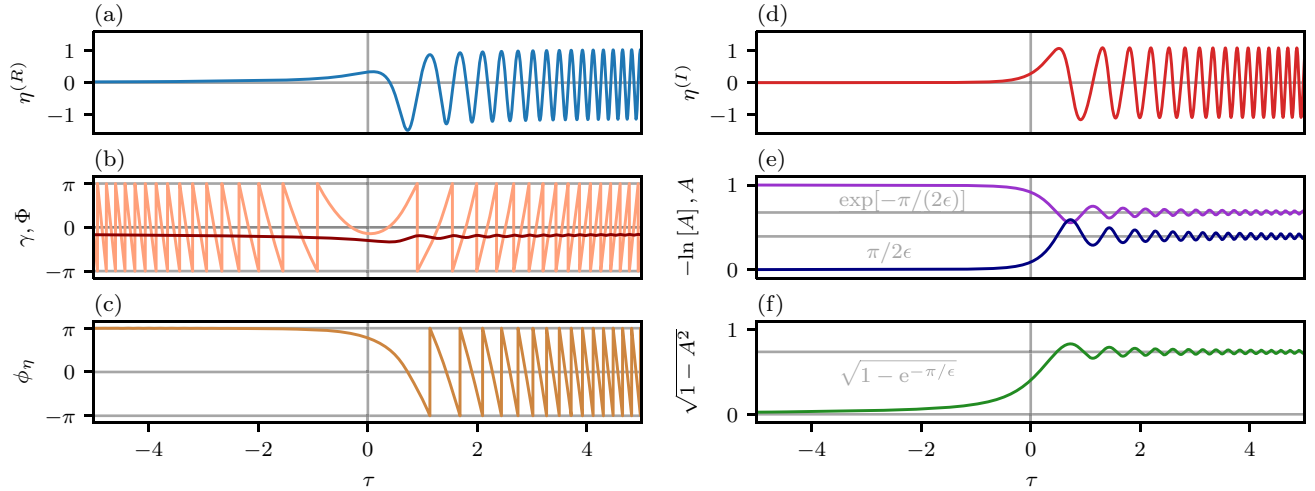


FIG. 4. Emergence of the time dependence of the probability amplitudes a and b from the numerical solution η of the Riccati equation Eq. (13), subject to the initial condition Eq. (14). Panel (a) shows the real part $\eta^{(R)}$ (blue), which determines the phases of a and b . Both share a common phase Φ , defined by Eq. (18) and shown in panel (b) (light red), featuring 2π -periodicity-induced jumps. Φ consists of a quadratic phase offset by the integral of the imaginary part $\eta^{(I)}$ (dark red). The phase ψ of η , governed by $\eta^{(R)}$ and $\eta^{(I)}$, is shown in panel (c) (brown). Panel (d) shows the imaginary part $\eta^{(I)}$ (red), which controls the absolute values of the probability amplitudes, shown in the right column. The dark blue curve in panel (e) displays the integrated imaginary part of η , whose oscillations reflect those of $\eta^{(I)}$ and decay as $\eta^{(I)}$ approaches $\pi/(2\epsilon)$ (lower gray horizontal line). The magenta curve represents the exponential of the dark blue curve, corresponding to the absolute value A of a , approaching the Landau-Zener result $\exp(-\pi/(2\epsilon))$ (upper gray horizontal line). Panel (f) shows the absolute value of b , obtained from the normalization condition Eq. (3) and A . Parameters ϵ and τ_0 are as in Figs. 1–3.

red curve in panel (b). It consists of a quadratic term minus $\gamma(\tau) \equiv \text{Re}[H(\tau')]$, the integral of $\eta^{(R)}$, which appears as the dark red curve. Initially monotonic, γ enters an oscillatory regime after the first zero of $\eta^{(R)}$ ultimately saturating as $\tau \rightarrow \tau_0$. These oscillations again reflect the rising frequency of the underlying $\eta^{(R)}$ near $\tau = 0$.

The phase Φ spans multiple 2π intervals but varies most significantly near $\tau = 0$. This behavior aligns with the complex-plane trajectory of a Fig. 1, where the amplitude encircles the origin several times before transitioning onto its final circle. The phase of b is governed by ϕ_η , shown in panel (c) (brown curve). Starting at π for $\tau = -\tau_0$, it remains flat until $\tau \approx 0$, then increases without bound, capturing the accumulated phase due to the transition.

We have shown that the dynamics of the two complex-valued probability amplitudes $a(\tau)$ and $b(\tau)$, representing four real degrees of freedom, can be compactly encoded in the complex Riccati equation Eq. (13), which describes only two. For the initial condition, Eq. (14), the modulus of a and the phase of b are determined, respectively, by the integral of the imaginary part $\eta^{(I)}$ and the phase ϕ_η , as given in Eq. (16) and Eq. (21). The amplitudes share a common global phase Φ , defined in Eq. (18), and are constrained by the normalization condition Eq. (3). These features are visualized in Fig. 4.

The Riccati amplitude η thus provides a minimal yet complete representation of the Landau-Zener transition. Its nonlinearity breaks the formal symmetry between a and b , capturing the asymmetry in their evolution. In addition to encoding the population ratio and relative phase, η offers a natural coordinate for interpreting the transition dynamics.

IV. MARKOV APPROXIMATION

In this section, we explore the Markov approximation in both the Schrödinger and interaction pictures, comparing and contrasting the results in each. We then focus on the probability amplitude a_M in Markov approximation and the exact amplitude a , which leads us to two different differential equations. By comparing them, we identify the nonlinearity in the Riccati equation for η as the source of non-Markovian behavior, and show that the Markov approximation is equivalent to neglecting this nonlinearity. Finally, we compare numerical simulations of η with those of η_M and discuss their resulting differences.

A. Markov solution

We start the discussion by solving Eq. (2) for the initial condition Eq. (4) and arrive at the formal solution

$$b(\tau) = -ie^{-i\epsilon\tau^2/2} \int_{-\tau_0}^{\tau} d\tau' e^{i\epsilon\tau'^2/2} a(\tau'). \quad (22)$$

When we insert Eq. (22) into the differential equation, Eq. (1), we obtain the integro-differential equation

$$\dot{a}(\tau) = i\epsilon\tau a(\tau) - e^{-i\epsilon\tau^2/2} \int_{-\tau_0}^{\tau} d\tau' e^{i\epsilon\tau'^2/2} a(\tau') \quad (23)$$

of Lippmann-Schwinger type [47] for the probability amplitude a . Up to this point, our derivation remains exact, and no approximation has been introduced.

Next, we perform the Markov approximation

$$e^{-i\epsilon\tau'^2/2} a(\tau') \approx e^{-i\epsilon\tau^2/2} a(\tau), \quad (24)$$

which assumes that the main contribution of the probability amplitude a to the integral in Eq. (23) arises from the upper limit, a condition that holds when $a(\tau)$ oscillates much more slowly than $e^{-i\epsilon\tau^2/2}$. This case occurs precisely when the condition $1 \ll |\epsilon\tau|$ is satisfied. The Markov approximation yields the differential equation

$$\dot{a}_M = [i\epsilon\tau + i\eta_M]a_M, \quad (25)$$

with the function

$$\eta_M(\tau) \equiv ie^{-i\epsilon\tau^2}\mathcal{F}(\tau) \quad (26)$$

and the abbreviation

$$\mathcal{F}(\tau) \equiv \int_{-\tau_0}^{\tau} d\tau' e^{i\epsilon\tau'^2}. \quad (27)$$

Here, we have introduced the subscript M to emphasize that the differential equation Eq. (25) is an approximation of the exact integro-differential equation Eq. (23), based on the Markov approximation, Eq. (24).

It is straightforward to integrate Eq. (25) and we find the Markov solution

$$a_M(\tau) = e^{i\epsilon(\tau^2 - \tau_0^2)/2} e^{iH_M(\tau)}, \quad (28)$$

with the abbreviation

$$H_M(\tau) \equiv \int_{-\tau_0}^{\tau} d\tau' \eta_M(\tau'). \quad (29)$$

Obviously, a_M satisfies the initial condition Eq. (4), that is $a_M(-\tau_0) = 1$.

In complete analogy to the exact expression, Eq. (16), for a , we can represent the Markov solution in amplitude and phase. We only have to add an subscript M to the existing expressions Eqs. (15)–(19) and we obtain them in the Markov approximation.

B. Markov approximation viewed from Riccati equation

The function η_M , defined by Eq. (26), and its role in determining the probability amplitude a_M via the Markov solution, Eq. (28), are closely related to the function η governed by the Riccati equation, Eq. (13). To highlight this analogy most clearly, we now differentiate the definition of η_M , given in Eq. (26), once more and obtain the *linear* differential equation

$$\dot{\eta}_M + 2i\epsilon\tau\eta_M - i = 0 \quad (30)$$

of first order. A comparison to the Riccati equation, Eq. (13), reveals that the two differential equations differ by the nonlinear term η^2 .

To emphasize this point, we now rewrite the Riccati equation, Eq. (13), in the form

$$\dot{\eta} + 2i\epsilon\tau\eta = i - i\eta^2, \quad (31)$$

which allows us to translate it into an integral equation

$$\eta(\tau) = \eta(-\tau_0)e^{-i\epsilon\tau^2} + ie^{-i\epsilon\tau^2} \int_{-\tau_0}^{\tau} d\tau' (1 - \eta(\tau')^2) e^{i\epsilon\tau'^2}, \quad (32)$$

which with the help of the initial condition, Eq. (14), and the definition, Eq. (26), of η_M reduces to

$$\eta(\tau) = \eta_M(\tau) - ie^{-i\epsilon\tau^2} \int_{-\tau_0}^{\tau} d\tau' \eta(\tau')^2 e^{i\epsilon\tau'^2}. \quad (33)$$

Hence, the Markov approximation is equivalent to neglecting the nonlinear term η^2 .

C. Limits of the Markov approximation

From the Riccati equation Eq. (13), two asymptotic regimes emerge: (i) $|\epsilon\tau| \gg 1$, where the linear term $\epsilon\tau$ dominates and the nonlinearity is negligible, and (ii) $|\epsilon\tau| \ll 1$, where the nonlinear term governs the dynamics. As a result, the Markov approximation, based on neglecting the nonlinearity, fails near the transition point $\tau = 0$. The validity of this approximation also depends on the inverse adiabaticity parameter ϵ . For $\epsilon \gg 1$ (nonadiabatic regime), the nonlinear region narrows, and the approximation becomes more accurate. Conversely, for $\epsilon \ll 1$, nonlinear effects extend over a broader time window, reducing accuracy. This behavior is clearly illustrated in Fig. 5, which shows both the real and imaginary parts of η and η_M for three different values of ϵ . Remarkably, despite its local inaccuracy near $\tau = 0$, the Markov approximation successfully connects the asymptotic time domains. This has different implications for the two amplitudes a and b . Since a depends only on the time integral of η , local deviations near $\tau = 0$ are averaged out. Consequently, a_M converges to the correct asymptotic value at large times, even though η and η_M differ in the transition region. However, the situation is more delicate for b . As shown in Eq. (11), the amplitude b depends directly on η as well as on a . Therefore, any local error in η_M directly affects b_M , especially near $\tau = 0$. To quantify this, we apply the Markov approximation to the formal solution Eq. (22), yielding

$$b_M(\tau) = -\eta_M(\tau)a_M(\tau), \quad (34)$$

in analogy to Eq. (11), but within the Markov framework.

Evaluating this expression in the asymptotic limit $\tau_0 \rightarrow \infty$ using Eqs. (26) to (29) and $\mathcal{F}(\tau_0 \rightarrow \infty) = \sqrt{i\pi/\epsilon}$, we find

$$b_M(\tau_0 \rightarrow \infty) = \sqrt{\frac{\pi}{\epsilon}} e^{-\frac{\pi}{2\epsilon}} e^{-i(\epsilon\tau_0^2 + \frac{\pi}{4})}. \quad (35)$$

This result deviates from the exact expression given by Eq. (6). In particular, the absolute value of $|b_M| = \sqrt{\pi/\epsilon} \exp[-\pi/(2\epsilon)]$ does not match the exact Landau-Zener expression $|b| = \sqrt{1 - \exp(-\pi/\epsilon)}$ obtained from Eq. (6). Only in the limit $\epsilon \gg 1$ do the two results agree approximately, as can be seen by expanding $\exp(-\pi/\epsilon) \approx 1 - \pi/\epsilon$, which yields

$$|b(\tau_0 \rightarrow \infty)| \approx |b_M(\tau_0 \rightarrow \infty)|. \quad (36)$$

Moreover, the phase of b_M differs from the exact result in Eq. (7). Although both contain the quadratic term $\epsilon\tau_0^2$, the Markov approximation omits the logarithmic contribution and the argument of the Gamma function, predicting a phase shift of $-\pi/4$ instead of $3\pi/4$.

Despite this, the Markov approximation correctly reproduces the modulus of b in the limit $1 \ll \epsilon$, capturing the nonadiabatic behavior but missing the full coherence of the quantum evolution.

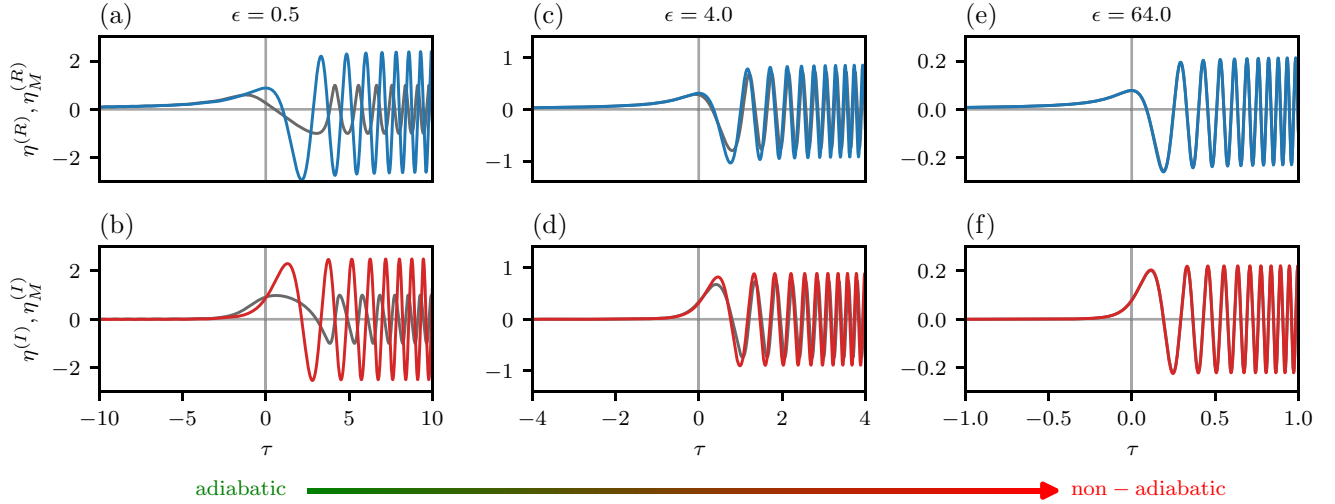


FIG. 5. Comparison between the exact numerical solution η of the Riccati equation and the analytic Markov approximation η_M . Here, we depict the real (a), (c), (e) and imaginary part (b), (d), (f) of η (solid gray line) and the approximated solution η_M (colored solid line) for different ϵ .

Finally, the Riccati formulation naturally connects to the concept of adiabatic elimination [47]. Neglecting the nonlinear term η^2 yields a linear equation decoupling the evolution of b , consistent with removing a weakly populated or fast-evolving state. This simplification underpins the Markov approximation and is valid only while η remains small. As η^2 grows during the evolution, this assumption breaks down. Thus, the Riccati formalism not only unifies exact and approximate treatments but also provides a dynamic criterion for the breakdown of adiabatic elimination and the onset of fully coherent quantum dynamics.

V. ANALYTIC EXPRESSIONS IN DIFFERENT TIME DOMAINS

Figure 5 demonstrates in a vivid way that η_M reproduces the essential features of η . For this reason, we now study η_M in more detail for different time domains. Here, we only focus on the results and refer for the derivations to Appendix D.

A. Large negative times

Throughout this article, we have analyzed the Landau-Zener dynamics with initial conditions set at a large but finite negative time, $\tau = -\tau_0$, before taking the limit $\tau_0 \rightarrow \infty$. The difference between setting a finite τ_0 and taking $\tau_0 = \infty$ directly is most significant near the initial time, $-\tau_0 \lesssim \tau$. In this section, we compare these two approaches, focusing on their implications for the behavior of η and explain their origin.

We start our discussion with the case of a vanishing initial condition, Eq. (14), at a finite but large negative time $\tau = -\tau_0$ and derive in Appendix D the expression

$$a_M(-|\tau|) = e^{i\epsilon(\tau^2 - \tau_0^2)/2} e^{iH_M(-|\tau|)} \quad (37)$$

for the probability amplitude a_M with

$$H_M(-|\tau|) \cong \frac{1}{2\epsilon} \ln\left(\frac{|\tau|}{\tau_0}\right) + \frac{i}{4\epsilon^2 \tau_0^2} - \frac{ie^{i\epsilon(\tau_0^2 - \tau^2)}}{4\epsilon^2 \tau_0 |\tau|}. \quad (38)$$

From Eq. (38), the last term can be directly identified as the source of the initial oscillatory behavior in η_M , suppressed by $1/\tau_0$. This behavior is confirmed numerically in Fig. 6, which shows the real and imaginary parts of η and η_M for three different values of τ_0 . The oscillations are clearly visible, and their amplitudes decay with increasing τ_0 , consistent with Eq. (38). On first sight, the appearance of these oscillations is surprising since they manifest themselves already for large negative times, well before the coupling between the two levels becomes significant. Indeed, the coupling is expected to be most effective near the level crossing at $\tau = 0$, leading to a modulation at positive times, but not at negative times.

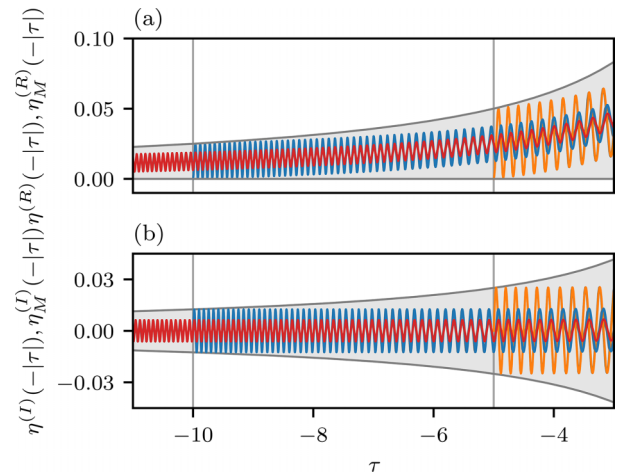


FIG. 6. Oscillations of η and η_M , defined by Eqs. (13) and (30), at large negative times for three starting times τ_0 : (i) 20.0 (red), (ii) 10.0 (blue), and (iii) 5.0 (orange). The upper panel (a) shows the real parts $\eta^{(R)}$ and $\eta_M^{(R)}$, while the lower panel (b) shows the imaginary parts $\eta^{(I)}$ and $\eta_M^{(I)}$. Gray solid lines mark the envelopes determined by τ_0 , with each point corresponding to a potential starting point of the oscillations. Vertical gray lines in both panels indicate the respective initial times. Here, $\epsilon = 4$.

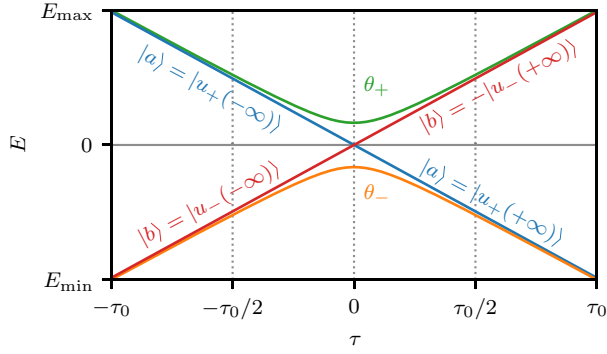


FIG. 7. Early-time oscillations of the probability amplitude a in the Landau-Zener problem, arising from a superposition of instantaneous energy eigenstates. The time-dependent diabatic energy levels associated with $|a\rangle$ (blue) and $|b\rangle$ (red) form the diabatic basis and vary linearly with time at rate ϵ over the interval $-\tau_0 \leq \tau \leq \tau_0$. In the asymptotic limit $\tau_0 \rightarrow \infty$, the diabatic and adiabatic bases coincide, with the adiabatic eigenvalues θ_+ (green) and θ_- (orange) matching the diabatic levels. At finite τ_0 , an initial condition confined to a single diabatic level excites both adiabatic states $|u_+\rangle$ and $|u_-\rangle$, leading to oscillations. The energy separation at $\tau = 0$ reflects the coupling strength between the two states.

However, this interpretation is misleading, as the coupling is always present and generates two instantaneous eigenstates, u_+ and u_- , with corresponding eigenvalues θ_+ and θ_- , as shown in Fig. 7. Only in the limit $\tau \rightarrow \infty$ do these eigenvalues approach the diabatic levels associated with a and b , as discussed in Appendix E.

As a result, the initial condition, Eq. (4), for a at finite τ can only be satisfied by a superposition of *both* eigenstates, leading to the observed oscillations. In particular, we show in Appendix E that the interference between the two instantaneous eigenstates yields Eq. (38).

B. At the time of the transition

Next, we derive an approximate analytic expression for η_M near $\tau = 0$ by performing a Taylor expansion around this point, as detailed shown in Appendix D. The expansion coefficients are obtained by differentiating the corresponding differential equation, Eq. (30). This yields the expansion

$$\eta_M(\tau) = i\mathcal{F}(0)(1 - i\epsilon\tau^2) + i\tau + \dots, \quad (39)$$

valid to second order in τ . The corresponding behavior in the Markov approximation is shown in Fig. 8 as a red solid line.

C. Large positive times

We now address the behavior of η_M for large positive times. In Appendix D, we establish in the connection formula

$$\eta_M(\tau) = -\eta_M(-\tau) + i\mathcal{F}(\tau_0)e^{-i\epsilon\tau^2}, \quad (40)$$

which relates the values of η_M at positive and negative times and shows that η_M is not antisymmetric, but acquires an additional phase factor with a quadratically growing phase.

We substitute the connection formula, Eq. (40), into the Markov solution, Eq. (28), which yields

$$a_M(|\tau|) = a_M(-|\tau|) \exp\left(-\mathcal{F}(\tau_0) \int_0^\tau d\tau' e^{-i\epsilon\tau'^2}\right), \quad (41)$$

demonstrating that the behavior of a_M for positive times is determined by its value at negative times. This dependence originates from the $-\eta_M(-|\tau|)$ term in the connection formula. Moreover, the term $\mathcal{F}(\tau_0) \exp(-i\epsilon\tau^2)$ induces oscillations in both the phase and amplitude of a_M . These oscillations are called Stueckelberg oscillations and are clearly visible in Fig. 1(b).

We are now in the position to rederive the exact Landau-Zener result [1–5,24,48], Eq. (5). For this purpose, we set in Eq. (41) $\tau = \tau_0$ and use the initial condition $a_M(-\tau_0) = 1$, which yields

$$a_M(\tau_0) = \exp\left[-\frac{1}{2}|\mathcal{F}(\tau_0)|^2\right]. \quad (42)$$

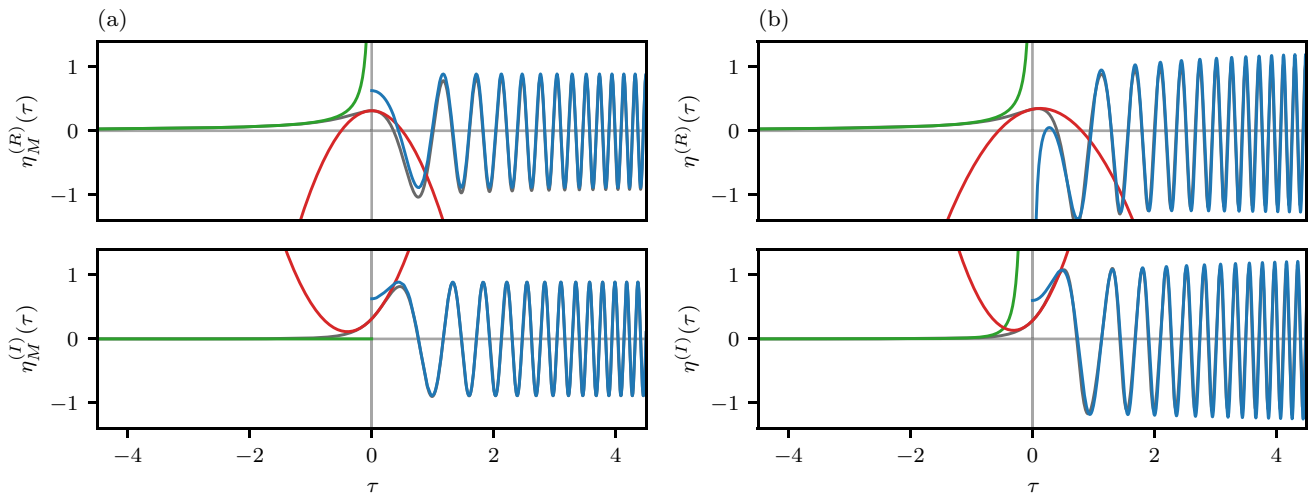


FIG. 8. Comparison between the numerical solution of the Markov approximation η_M (a) and the Riccati equation η (b) and its analytical approximations across three time regimes: (i) large negative times (green), (ii) near the transition (red), and (iii) large positive times (blue). Parameters are set to $\epsilon = 4.0$.

Next, we perform the limit $\tau_0 \rightarrow \infty$ and use the relation $\mathcal{F}(\tau_0 \rightarrow \infty) = \sqrt{i\pi/\epsilon}$ leading us to the expression

$$a_M(\tau_0 \rightarrow \infty) = \exp\left(-\frac{\pi}{2\epsilon}\right) = a(\tau_0 \rightarrow \infty), \quad (43)$$

which is identical to the exact Landau-Zener formula for the probability amplitude a .

VI. RICCATI EQUATION: ASYMPTOTICS

In Sec. III, we have obtained the Riccati equation for the integrand η of our exponential ansatz, Eq. (8). In the present section, we derive approximate analytic results for η in three domains of time, and compare and contrast them to the corresponding expressions for η_M . This analysis brings out most clearly the influence of the nonlinearity of the Riccati equation compared to the linearized Markov solution.

A. Large negative times

In the preceding section, we have concentrated on the low-est approximation of the Riccati equation, Eq. (13), given by the Markov solution, Eq. (28). We now discuss the influence of the nonlinearity contained in the quadratic term of the Riccati equation. For the sake of simplicity, we demonstrate this feature for the initial condition, $\eta(\tau = -\infty) = 0$.

For this purpose, we insert this ansatz, Eq. (D9), into the right-hand side of Eq. (D8) leading us to the approximation

$$\eta(\tau) \cong \frac{1}{2\epsilon\tau} + \frac{1}{8\epsilon^3\tau^3} - \frac{i}{4\epsilon^2\tau^3}. \quad (44)$$

We emphasize that the cubic correction to the real part

$$\eta_R^{(I)}(\tau) = -\frac{1}{2\epsilon\tau} + \frac{1}{8\epsilon^3\tau^3} \quad (45)$$

is a consequence of the nonlinearity.

Hence, for large negative times neglecting the cubic terms the phase velocity calculated from η and η_M agree. However, close to $\tau = 0$ the cubic correction to η_M becomes important and marks the onset of the transition. In contrast to the phase velocity, the absolute value $|a|$ given by the imaginary part, that is

$$\eta^{(I)}(\tau) \cong -\frac{1}{4\epsilon^2\tau^3} \quad (46)$$

is not influenced by the nonlinearity in this order of the approximation. The behavior of the real and imaginary part of η_R are depicted in Figs. 8(a) and 8(b) as a green solid line.

B. At the time of the transition

The previous analysis in Sec. VI A has already indicated that the nonlinearity becomes important in the neighborhood of $\tau = 0$, that is at the time of the transition. To bring this feature to light, we perform a Taylor expansion

$$\eta(\tau) = \eta(0) + \dot{\eta}(0)\tau + \frac{1}{2}\ddot{\eta}(0)\tau^2 + \dots \quad (47)$$

of η around $\tau = 0$ and use derivatives of the Riccati equation, Eq. (13), to evaluate the expansion coefficients. We compare and contrast these coefficients to the corresponding ones of η_M .

From the Riccati equation, Eq. (13), we find immediately

$$\dot{\eta}(0) = i[1 + \eta(0)^2]. \quad (48)$$

When we differentiate the Riccati equation one more time we arrive at the identity

$$\ddot{\eta} + 2i\eta\dot{\eta} + 2i\epsilon\eta + 2i\epsilon\tau\dot{\eta} = 0, \quad (49)$$

which with the help of Eq. (48) yields

$$\ddot{\eta}(0) = 2\eta(0)[1 - \eta(0)^2 - i\epsilon]. \quad (50)$$

In contrast, the corresponding expression for η_M reads, which clearly shows the important role of the nonlinearity in the neighborhood of $\tau = 0$.

Since η_M is given explicitly, we can obtain the expression for $\eta_M(0) = \mathcal{F}(\tau_0)/2$. In contrast, no such expression exists for $\eta(0)$.

The behavior of the Taylor expansion of second order in Markov approximation is depicted in Fig. 8 as the red solid line.

C. Large positive times

In Fig. 5, we have found a qualitative agreement between the functions η_M and η . However, there is a slight shift in the phase, and the amplitudes of the oscillations for positive times are smaller in η_M than in η . This modification of η_M stands out most clearly in the integral form, Eq. (33), of the Riccati equation. Indeed, η_M is only the lowest approximation neglecting the nonlinearity. Hence, an improved solution follows by replacing η^2 in the integral by η_M^2 , that is

$$\eta(\tau) \cong \eta_M(\tau) - ie^{-i\epsilon\tau^2}R(\tau), \quad (51)$$

where we have introduced the abbreviation

$$R(\tau) \equiv \int_{-\tau_0}^{\tau} d\tau' \eta_M(\tau')^2 e^{i\epsilon\tau'^2}. \quad (52)$$

We gain more insight into this expression when we recall the connection formula, Eq. (40), which casts Eq. (51) into the form

$$\eta_I(\tau) = -\eta_M(-|\tau|) + ie^{-i\epsilon\tau^2}[\mathcal{F}(\tau_0) - R(\tau)]. \quad (53)$$

Since according to Eq. (D9) η_M is of the form, $\eta_M(-|\tau|) = -1/(2\epsilon|\tau|)$ that it decays for increasing time we find the approximation

$$\eta(\tau) \cong e^{-i\epsilon\tau^2}[\mathcal{F}(\tau_0) - R(\tau)]. \quad (54)$$

Hence, the strength of the oscillatory term which in η_M is solely given by $\mathcal{F}(\tau_0)$ in the brackets, is modified by the integral containing η_M^2 and a quadratic phase factor. Since this is a complex number with an amplitude and phase, we find a modification of the amplitude as well as the phase. The behavior of $\eta_I(\tau)$ is portrayed in Fig. 4 as a red solid line.

VII. BACK TO THE PROBLEM

To understand the trajectories in the complex plane, shown in Fig. 1, and representing the time dependence of the probability amplitudes of the LZ problem, constitutes the central problem of our article. For this purpose, we have derived approximate but analytical solutions of the Riccati equation and

TABLE I. Asymptotic behavior of the phase velocity of the two probability amplitudes a and b in the limit of large negative and large positive times. Whereas in a the sign of the phase velocity changes from negative to positive, in b it is always negative.

Phase velocity	
$\tau \rightarrow -\tau_0$	$\tau \rightarrow \tau_0$
$a(\tau) - \epsilon \tau < 0$	$\epsilon \tau > 0$
$b(\tau) - \epsilon \tau < 0$	$-\epsilon \tau < 0$

have obtained expressions for the two probability amplitudes. We are now in the position to return to this problem and employ the various formulas to bring out the key features.

At large negative times, it suffices to employ the expression, Eq. (37), for a obtained by the Markov approximation since $a \approx a_M$. According to Eq. (38) we can approximate $H_M(-|\tau|) \approx 0$. Hence, $a_M(\tau)$ consists solely of a phase factor with a quadratic time dependence. Since the prefactor ϵ of τ^2 is positive, the phase velocity given by the time derivative of the phase is linear in τ , and thus negative for $\tau < 0$. A negative phase velocity corresponds to a clockwise rotation in the complex plane, in complete agreement with Table I.

For large positive times, Eq. (41) shows that $a(|\tau|)$ is given by the product of its negative-time counterpart and an exponential factor determining the final LZ probability as demonstrated by Eq. (43). In this time domain, the quadratic phase is still positive and hence, the phase velocity is now positive, reversing the direction of rotation to a counterclockwise motion in the complex plane.

The dynamics of the amplitude b emerges when we substitute the ansatz, Eq. (8), for a into the integral equation, Eq. (22), for b which yields

$$b(\tau) = -i\epsilon e^{-i\epsilon(\tau^2 + \tau_0^2)/2} \int_{-\tau_0}^{\tau} d\tau' e^{i\epsilon\tau'^2} e^{iH(\tau')}. \quad (55)$$

Since for large negative times, $H_M(-|\tau|) \approx 0$, we find with the help of the asymptotic expression, Eq. (D1), for the Fresnel integral, the result

$$b(-|\tau|) \cong -\frac{i}{2\epsilon\tau_0} e^{-i\epsilon(\tau^2 - \tau_0^2)/2} + \frac{i}{2\epsilon|\tau|} e^{i\epsilon(\tau^2 - \tau_0^2)/2}. \quad (56)$$

To satisfy the initial condition, Eq. (4), we have to have two contributions of opposite phases. However, in the limit of $\tau_0 \rightarrow \infty$ the second term is dominant. For decreasing τ the amplitude increases and the phase is positive. Hence, the phase velocity is again negative for negative times, leading to a clockwise rotation in the complex plane. This behavior is clearly visible in Fig. 1(a) and is displayed in Table I.

In contrast to a , the direction of rotation of b remains unchanged when transitioning from negative to positive times. This behavior follows from the structure of Eq. (55), where the integral approaches a fixed complex value for large positive τ , but the exponential prefactor dominates the phase dynamics. As this prefactor maintains a negative phase velocity, the clockwise rotation persists.

VIII. CONCLUSION AND OUTLOOK

In this work, we have reanalyzed the LZ transition problem using the Riccati differential equation governing the evolution of the amplitude ratio $\eta = -b/a$. This formulation, while equivalent in principle to the standard treatment involving parabolic cylinder functions, exposes a different structural and physical perspective, particularly in highlighting the role of nonlinearity and phase structure.

A key insight from this approach is the asymmetry between the probability amplitudes a and b . Their magnitudes are both determined by the time integral of the imaginary part $\eta^{(I)}$, while their phases differ due to an additional phase $\phi_\eta = \arg[\eta]$ which only appears in b . Interestingly the absolute value of the Riccati solution $|\eta|$ does not directly enter the expressions for their amplitude, reflecting the underlying normalization and the multiplicative structure $b = -\eta a$.

Hence, this approach has also clarified the structure and limitations of the Markov approximation previously introduced in Ref. [48]. From the Riccati perspective, this approximation corresponds to linearizing the equation by neglecting its nonlinear term. This yields a concise interpretation of when the Markov approximation is expected to hold and allows for systematic iterative improvements. In particular, we find that this nonlinearity has surprisingly little effect on the final transition probability $|a|^2$, but it results in a drastically incorrect prediction for the dynamics of b , highlighting the non-Markovian nature.

In addition to providing local approximations, we derived a connection formula that links the asymptotic behavior at large negative times with that at large positive times. This global insight is particularly valuable for understanding the full evolution across the avoided crossing and contributes to the completeness of the Riccati-based framework. Furthermore, our analysis revealed the presence of initial oscillations in the early-time dynamics of the transition, arising due to the sudden switch-on of the coupling when the system begins in an eigenstate of the uncoupled Hamiltonian. While these oscillations decay at later times, they reflect subtle interference effects that are naturally captured by our formalism.

In addition to its analytic clarity in the standard Landau Zener setting, the Riccati approach also offers a natural framework for generalization to a broader class of level-crossing models. In particular, it can be extended to treat situations beyond linear time-dependence of the diabatic energies, such as the Demkov-Kunike model, which features finite-time interactions and bounded asymptotics. Since the Riccati formulation relies only on the structure of the underlying Schrödinger equation, it remains applicable in scenarios with more intricate coupling profiles or energy landscapes. Furthermore, this approach may provide useful analytical insights into models where conventional techniques become cumbersome or where exact solutions are not available. We expect that its flexibility in dealing with nonlinear dynamics and approximate iterative schemes makes it a promising tool for studying a wide variety of nonadiabatic transitions in two-level and even multilevel systems.

It is worth noting that the nonlinearity of the Riccati equation, while key to capturing dynamical effects such as memory and phase corrections, can sometimes make direct numerical

integration or even finding an analytical solution challenging. In such cases, our iterative and asymptotic strategies provide controlled alternatives to exact integration, balancing insight with practicality.

These observations raise several intriguing questions. Why does the nonlinearity in the Riccati equation, which is important in the short- and intermediate-time dynamics, not influence the long-time transition probability of a ? Is it possible to recover the correct asymptotic phase structure from the nonlinear integral equations shown in Appendix C? And could one rederive the Landau-Zener formula from a classical oscillator analogy, such as the Newtonian equation with a time-dependent frequency and complex offset described in Ref. [59]?

These insights may also support modeling efforts in systems such as atom interferometers [16–18,60,61] or superconducting qubits [62,63], where nonadiabatic transitions and memory effects play an increasingly central role. Applications to quantum control protocols [64] or transition suppression schemes [65] could also benefit from the analytic transparency of the Riccati approach.

At its core, this work shows that the nonlinear Riccati structure offers not just a rephrasing of the Landau-Zener problem, but a physically transparent way to classify, improve, and generalize the dynamics of nonadiabatic transitions. As a final remark, we note that while the Riccati equation has long been recognized in mathematical treatments of quantum dynamics, its full potential as a physically interpretable and broadly applicable framework has yet to be fully realized. We hope this work contributes to that direction and encourages further exploration of its use in more complex quantum systems.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: APPROACH BASED ON PARABOLIC CYLINDER FUNCTIONS

In this Appendix, we rederive the full time-dependent expressions for the complex-valued amplitudes $a(\tau)$ and $b(\tau)$ in terms of parabolic cylinder functions (PCFs) [1]. An analogous solution for the Riccati variable $\eta(\tau)$, introduced in Sec. III, is also obtained. We recover the known asymptotic results for a and b , including the phase ϕ in Eq. (7), using standard PCF asymptotics [36,37]. Our derivation clarifies several ambiguities in the literature, particularly regarding scaling conventions and initial conditions. For completeness, we include explicit derivations for a , b , and η , and discuss

both general and specific initial conditions considered in the main text.

1. Expression for probability amplitude a

One way to decouple the equations, Eqs. (1) and (2), for the two probability amplitudes, is to differentiate the equation for a once more, and use the equation for the derivative of b which yields

$$\ddot{a}(\tau) + [\epsilon^2 \tau^2 + 1 - i\epsilon]a(\tau) = 0 \quad (\text{A1})$$

subjected to the initial conditions given by Eq. (4) and

$$\dot{a}(\tau = -\tau_0) = -i\epsilon\tau_0 \quad (\text{A2})$$

which follows from Eq. (1) in combination with initial condition, Eq. (4), for b . The solutions of Eq. (A1) are well known [36,37] to be the parabolic cylinder functions, provided Eq. (A1) is first converted to the standard form

$$\frac{d^2}{dz^2}a(z) - \left[\frac{z^2}{4} + c\right]a(z) = 0, \quad (\text{A3})$$

where we have used the complex scaling

$$z \equiv \sqrt{2\epsilon}\tau e^{i\frac{\pi}{4}} \quad (\text{A4})$$

and introduced the constant

$$c \equiv \frac{i}{2\epsilon} + \frac{1}{2}. \quad (\text{A5})$$

We emphasize, that z differs from the corresponding quantity in Ref. [24] by the factor $\sqrt{\epsilon}$. According to Refs. [36,37] the combination

$$a(z) = \mathcal{A}_+ \mathcal{D}_\nu(z) + \mathcal{A}_- \mathcal{D}_\nu(-z) \quad (\text{A6})$$

of PCFs $\mathcal{D}_\nu$ with arguments z and $-z$ and order

$$\nu \equiv -1 - \frac{i}{2\epsilon} \quad (\text{A7})$$

is a solution to Eq. (A3). Here, $\mathcal{A}_\pm$ are constants to match the initial conditions Eqs. (4) and (A2) which in the complex scaling Eq. (A4) read

$$a(-z_0) = 1 \quad \text{and} \quad \frac{d}{dz}a(-z_0) = -\frac{z_0}{2}. \quad (\text{A8})$$

We first derive expressions for $\mathcal{A}_\pm$ for arbitrary initial conditions and then reduce them to the ones given by Eq. (A8). From the ansatz, Eq. (A6), for a we find the relation

$$a(-z_0) = \mathcal{A}_+ \mathcal{D}_\nu(-z_0) + \mathcal{A}_- \mathcal{D}_\nu(z_0), \quad (\text{A9})$$

and differentiation of Eq. (A6) leads us to the expression

$$\frac{d}{dz}a(z) = \mathcal{A}_+ \dot{\mathcal{D}}_\nu(z) - \mathcal{A}_- \dot{\mathcal{D}}_\nu(-z), \quad (\text{A10})$$

where

$$\dot{\mathcal{D}}_\nu(z) \equiv \left. \frac{d\mathcal{D}_\nu(\xi)}{d\xi} \right|_{\xi=z}. \quad (\text{A11})$$

We note that the second term in Eq. (A10) results from the chain rule

$$\frac{d}{dz}\mathcal{D}_\nu(-z) = (-1) \cdot \left. \frac{d\mathcal{D}_\nu(\xi)}{d\xi} \right|_{\xi=-z} \equiv -\dot{\mathcal{D}}_\nu(-z). \quad (\text{A12})$$

As a consequence, we find

$$\frac{d}{dz}a(-z_0) = \mathcal{A}_+ \dot{\mathcal{D}}_v(-z_0) - \mathcal{A}_- \dot{\mathcal{D}}_v(z_0), \quad (\text{A13})$$

which together with Eq. (A9) can be cast into the vector form

$$\begin{pmatrix} a(-z_0) \\ \frac{d}{dz}a(-z_0) \end{pmatrix} = M \begin{pmatrix} \mathcal{A}_+ \\ \mathcal{A}_- \end{pmatrix}, \quad (\text{A14})$$

where

$$M \equiv \begin{pmatrix} \mathcal{D}_v(-z_0) & \mathcal{D}_v(z_0) \\ \dot{\mathcal{D}}_v(-z_0) & -\dot{\mathcal{D}}_v(z_0) \end{pmatrix}. \quad (\text{A15})$$

Next, we solve Eq. (A14) for the constants $\mathcal{A}_\pm$ by inverting M which yields the identity

$$\begin{pmatrix} \mathcal{A}_+ \\ \mathcal{A}_- \end{pmatrix} = M^{-1} \begin{pmatrix} a(-z_0) \\ \frac{d}{dz}a(-z_0) \end{pmatrix}, \quad (\text{A16})$$

where

$$M^{-1} = \frac{1}{\mathcal{M}} \begin{pmatrix} -\dot{\mathcal{D}}_v(z_0) & -\mathcal{D}_v(z_0) \\ -\dot{\mathcal{D}}_v(-z_0) & \mathcal{D}_v(-z_0) \end{pmatrix} \quad (\text{A17})$$

denotes the inverse matrix of M and $\mathcal{M} \equiv \det(M)$ is given by

$$\mathcal{M} = -[\mathcal{D}_v(-z_0)\dot{\mathcal{D}}_v(z_0) + \mathcal{D}_v(z_0)\dot{\mathcal{D}}_v(-z_0)] \quad (\text{A18})$$

and its determinant. As a result, we find the constants

$$\mathcal{A}_\pm = -\frac{1}{\mathcal{M}} \left[\dot{\mathcal{D}}_v(\pm z_0)a(-z_0) \pm \mathcal{D}_v(\pm z_0)\frac{d}{dz}a(-z_0) \right]. \quad (\text{A19})$$

When we replace the derivatives of the PCF using the relation [37]

$$\dot{\mathcal{D}}_v(z) = \frac{z}{2}\mathcal{D}_v(z) - \mathcal{D}_{v+1}(z), \quad (\text{A20})$$

the determinant $\mathcal{M}$ and the constants $\mathcal{A}_\pm$ take the form

$$\mathcal{M} = \mathcal{D}_v(-z_0)\mathcal{D}_{v+1}(z_0) + \mathcal{D}_v(z_0)\mathcal{D}_{v+1}(-z_0) \quad (\text{A21})$$

and

$$\mathcal{A}_\pm = \frac{1}{\mathcal{M}} \left\{ \mathcal{D}_{v+1}(\pm z_0)a(-z_0) \mp \mathcal{D}_v(\pm z_0) \left[\frac{z_0}{2} + \frac{d}{dz}a(-z_0) \right] \right\}. \quad (\text{A22})$$

So far, we have found an expression for the probability amplitude a for arbitrary initial conditions. We now specify the solution for the initial conditions, Eq. (A8), and Eq. (A22) reduce to

$$\mathcal{A}_\pm = \frac{\mathcal{D}_{v+1}(\pm z_0)}{\mathcal{M}}. \quad (\text{A23})$$

Thus, we obtain the expression

$$a(z) = \frac{\mathcal{D}_{v+1}(z_0)\mathcal{D}_v(z) + \mathcal{D}_{v+1}(-z_0)\mathcal{D}_v(-z)}{\mathcal{D}_v(-z_0)\mathcal{D}_{v+1}(z_0) + \mathcal{D}_v(z_0)\mathcal{D}_{v+1}(-z_0)} \quad (\text{A24})$$

for the probability amplitude a . Here, we have recalled Eq. (A21) for $\mathcal{M}$.

2. Expression for probability amplitude b

To derive an expression for b we recall Eq. (1) which in the complex scaling, Eq. (A4), reads

$$i\frac{d}{dz}a(z) = i\frac{z}{2}a(z) - \frac{e^{-i\frac{\pi}{4}}}{\sqrt{2\epsilon}}b(z), \quad (\text{A25})$$

and cast it into the form

$$b(z) = \sqrt{2\epsilon}e^{-i\frac{\pi}{4}} \left[\frac{d}{dz}a(z) - \frac{z}{2}a(z) \right]. \quad (\text{A26})$$

Together with Eqs. (A6) and (A10), for a and its derivative, we find

$$b(z) = \sqrt{2\epsilon}e^{-i\frac{\pi}{4}} \left[\mathcal{A}_+ \left(\dot{\mathcal{D}}_v(z) - \frac{z}{2}\mathcal{D}_v(z) \right) - \mathcal{A}_- \left(\dot{\mathcal{D}}_v(-z) + \frac{z}{2}\mathcal{D}_v(-z) \right) \right], \quad (\text{A27})$$

which with the help of the relation, Eq. (A20), for the derivative of the PCF, reduces to

$$b(z) = -\sqrt{2\epsilon}e^{-i\frac{\pi}{4}} [\mathcal{A}_+\mathcal{D}_{v+1}(z) - \mathcal{A}_-\mathcal{D}_{v+1}(-z)]. \quad (\text{A28})$$

We now return to the initial conditions, Eq. (A8), for which the constants $\mathcal{A}_\pm$ are given by Eq. (A23). Thus, we arrive at the expression

$$b(z) = -\sqrt{2\epsilon}e^{-i\frac{\pi}{4}} \frac{\mathcal{D}_{v+1}(z_0)\mathcal{D}_{v+1}(z) - \mathcal{D}_{v+1}(-z_0)\mathcal{D}_{v+1}(-z)}{\mathcal{D}_v(-z_0)\mathcal{D}_{v+1}(z_0) + \mathcal{D}_v(z_0)\mathcal{D}_{v+1}(-z_0)} \quad (\text{A29})$$

for the probability amplitude b which satisfies the initial condition $b(-z_0) = 0$, in complete agreement with Eq. (4). Indeed, our analysis brings out clearly that the asymptotic value of b relies on the behavior of the PCF on both sides of the Stokes lines [43,66]. In contrast, the derivation of a only requires this knowledge on one side.

3. Expression for solution η of the Riccati equation

In Sec. III, we introduced the ansatz Eq. (8) for the probability amplitude a , which defines the function η and leads directly to the first-order nonlinear Riccati equation Eq. (13). Here, we derive an explicit representation of η in terms of PCFs.

For this purpose, we first recall Eq. (11) which in the complex scaling provides us with the expression

$$\eta(z) = -\frac{b(z)}{a(z)} \quad (\text{A30})$$

for the function η . Next, we insert the representations, Eqs. (A24) and (A29) of the probability amplitudes a and b into Eq. (A30) and arrive at

$$\eta(z) = \sqrt{2\epsilon}e^{-i\frac{\pi}{4}} \frac{\mathcal{D}_{v+1}(z_0)\mathcal{D}_{v+1}(z) - \mathcal{D}_{v+1}(-z_0)\mathcal{D}_{v+1}(-z)}{\mathcal{D}_{v+1}(z_0)\mathcal{D}_v(z) + \mathcal{D}_{v+1}(-z_0)\mathcal{D}_v(-z)}, \quad (\text{A31})$$

which satisfies the initial condition Eq. (14).

4. Asymptotics of a , b and η

We now evaluate the asymptotic forms of a , b , and η in the limit $\tau_0 \rightarrow \infty$, based on the asymptotic expansions of

the PCFs. For clarity, we begin by collecting the relevant expressions before applying them to the results obtained in the preceding sections.

The asymptotic behavior of a follows directly from the elementary expansion of the PCF and is characterized by a logarithmic phase singularity. In contrast, obtaining the asymptotic form of b requires accounting for both dominant and subdominant contributions, rendering its derivation more intricate. Since η is defined as the ratio of a and b , its asymptotic behavior is primarily determined by that of b .

a. Asymptotic expressions for parabolic cylinder functions

It is essential to distinguish the behavior of the PCF $\mathcal{D}_\nu$ at negative times from that at positive times, as these correspond to different phases of the argument z . To highlight this distinction, we set $z = z_0$ in Eqs. (A24), (A29), and (A31) and adopt the scaling

$$z_0 \equiv \sqrt{2\epsilon}\tau_0 e^{i\frac{\pi}{4}} \quad (\text{A32})$$

as motivated by Eq. (A4), corresponding to positive values of τ_0 . For negative times we use the representation

$$-z_0 \equiv e^{-i\pi} z_0 = \sqrt{2\epsilon}\tau_0 e^{-i\frac{3\pi}{4}}, \quad (\text{A33})$$

where we have included the minus sign as a phase using the relation $-1 = \exp(-i\pi)$. In the domain $|\arg[z]| < 3\pi/4$ of the complex plane the dominant contribution to the asymptotic expansion of the PCF reads [36,37]

$$\mathcal{D}_\rho(z) \cong e^{-\frac{z^2}{4}} z^\rho, \quad (\text{A34})$$

which for $\rho = \nu$ and the definition, Eq. (A7), of ν , takes the form

$$\mathcal{D}_\nu(z) \cong e^{-\frac{z^2}{4}} z^{-1-\frac{i}{2\epsilon}} \quad (\text{A35})$$

and in the limit $\tau_0 \rightarrow \infty$ we arrive at

$$\mathcal{D}_\nu(z) \rightarrow 0. \quad (\text{A36})$$

From Eq. (A34) we find

$$\mathcal{D}_{\rho+1}(z) \cong e^{-\frac{z^2}{4}} z^{\rho+1}, \quad (\text{A37})$$

which for $\rho = \nu$ leads us to the expression

$$\mathcal{D}_{\nu+1}(z) = e^{-\frac{z^2}{4}} z^{-\frac{i}{2\epsilon}} = e^{-\frac{z^2}{4}} e^{-\frac{i}{2\epsilon} \ln(z)}. \quad (\text{A38})$$

The two PCFs $\mathcal{D}_\nu$ and $\mathcal{D}_{\nu+1}$ differ by an extra power of z which due to the special form of ν causes $\mathcal{D}_\nu$ to vanish whereas $\mathcal{D}_{\nu+1}$ contains a logarithmic phase. For negative times, that is for the domain $-\pi/4 > \arg[z] > -5\pi/4$, the PCF consists [36,37] to lowest order in $1/z$ of the superposition

$$\mathcal{D}_\rho(z) \cong e^{-\frac{z^2}{4}} z^\rho + \frac{\sqrt{2\pi}}{\Gamma(-\rho)} e^{-i(\rho+1)\pi} e^{\frac{z^2}{4}} z^{-\rho-1}. \quad (\text{A39})$$

Hence, for $\rho = \nu$ only the second term in Eq. (A39) survives in the asymptotic limit leading us to the expression

$$\mathcal{D}_\nu(z) \cong \frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{-i(\nu+1)\pi} e^{\frac{z^2}{4}} z^{-\nu-1}. \quad (\text{A40})$$

In contrast, for $\rho = \nu + 1$ the first contribution in Eq. (A39) is the dominant one, and we arrive at

$$\mathcal{D}_{\nu+1}(z) \cong e^{-\frac{z^2}{4}} z^{\nu+1} = e^{-\frac{z^2}{4}} z^{-\frac{i}{2\epsilon}}. \quad (\text{A41})$$

We emphasize that for negative times the contributions for ν and $\nu + 1$ are fundamentally different as expressed by the opposite signs of the Gaussians. Moreover, Eq. (A40) involves the Γ function which is crucial for the asymptotics of b .

b. Probability amplitude a

We now employ the asymptotic expressions derived in the preceding section to obtain the familiar Landau-Zener result for the probability amplitude a .

For this purpose, we set in the formula, Eq. (A24) for a , $z = z_0$, and recall the asymptotic expression, Eq. (A36) for $\mathcal{D}_\nu(z_0)$ and find

$$\lim_{\tau_0 \rightarrow \infty} a(\tau_0) = \lim_{\tau_0 \rightarrow \infty} \frac{\mathcal{D}_{\nu+1}(-z_0)}{\mathcal{D}_{\nu+1}(z_0)}, \quad (\text{A42})$$

which with Eqs. (A38) and (A41) reduces to

$$\lim_{\tau_0 \rightarrow \infty} a(\tau_0) = \lim_{\tau_0 \rightarrow \infty} \frac{(-z_0)^{-\frac{i}{2\epsilon}}}{z_0^{-\frac{i}{2\epsilon}}}. \quad (\text{A43})$$

The identity $(-1)^{-\frac{i}{2\epsilon}} = e^{-\frac{\pi}{2\epsilon}}$ leads us to the familiar Landau-Zener result

$$\lim_{\tau_0 \rightarrow \infty} a(\tau_0) = e^{-\frac{\pi}{2\epsilon}}. \quad (\text{A44})$$

This derivation highlights three key features of the asymptotic amplitude a : (i) it arises solely from the ratio $(-z_0)^{-i/(2\epsilon)}/z_0^{-i/(2\epsilon)}$, (ii) since z_0 cancels, the limit $\tau_0 \rightarrow \infty$ is unnecessary, and (iii) it originates from a logarithmic phase singularity, $\exp[-i/(2\epsilon) \ln(-1)]$ where $-1 = \exp(-i\pi)$. Notably, we do not rely on the asymptotic expansion of $\mathcal{D}_\nu$ involving the Γ function, making this derivation especially direct.

c. Probability amplitude b

Next, we derive the asymptotic expression for the probability amplitude b . This analysis is more involved, since in contrast to a which according to Eq. (A42) only involves the PCF $\mathcal{D}_{\nu+1}$ at positive and negative times, we now have to use the PCFs $\mathcal{D}_\nu$ and $\mathcal{D}_{\nu+1}$ for positive and negative times. Hence, we have to employ all four asymptotic expansions obtained in Appendix A 4 a.

From Eq. (A29) we find with the help of the asymptotic formula, Eq. (A36), the expression

$$\lim_{\tau_0 \rightarrow \infty} b(\tau_0) = \lim_{\tau_0 \rightarrow \infty} \frac{-\sqrt{2\epsilon} e^{-i\frac{\pi}{4}}}{\mathcal{D}_\nu(-z_0)} \left[\mathcal{D}_{\nu+1}(z_0) - \frac{\mathcal{D}_{\nu+1}^2(-z_0)}{\mathcal{D}_{\nu+1}(z_0)} \right], \quad (\text{A45})$$

which due to Eqs. (A38), (A40), and (A41) takes the form

$$\begin{aligned} \lim_{\tau_0 \rightarrow \infty} b(\tau_0) &= \lim_{\tau_0 \rightarrow \infty} \sqrt{\frac{\epsilon}{\pi}} \Gamma(-\nu) e^{-i\frac{5\pi}{4}} \frac{e^{-\frac{z_0^2}{4}} z_0^{2(\nu+1)}}{e^{-i(\nu+1)\pi} (-1)^{-\nu-1}} \\ &\quad \times [1 - (-1)^{2(\nu+1)}]. \end{aligned} \quad (\text{A46})$$

With the identities $\exp(-z_0^2/2) = \exp(-i\epsilon\tau_0^2)$ and

$$z_0^{2(\nu+1)} = e^{-\frac{i}{\epsilon} \ln(z_0)} = e^{-\frac{i}{\epsilon} \ln(\sqrt{2\epsilon}\tau_0)} e^{\frac{\pi}{4\epsilon}} \quad (\text{A47})$$

following from the scaling, Eq. (A32), of τ_0 together with the relation $\exp(-i(\nu+1)\pi)(-1)^{-\nu-1} = 1$ we arrive at the expression

$$b(\tau_0) = \sqrt{\frac{\epsilon}{\pi}} \Gamma(-\nu) e^{i\xi} e^{\frac{\pi}{4\epsilon}} [1 - e^{-\frac{\pi}{\epsilon}}], \quad (\text{A48})$$

where we have introduced the phase

$$\xi \equiv -\frac{5\pi}{4} - \epsilon\tau_0^2 - \frac{1}{\epsilon} \ln(\sqrt{2\epsilon}\tau_0). \quad (\text{A49})$$

Next, we recall the definition, Eq. (A7), of ν and use the decomposition

$$\Gamma(-\nu) = \left| \Gamma\left(1 + \frac{i}{2\epsilon}\right) \right| e^{i \arg[\Gamma(1 + \frac{i}{2\epsilon})]} \quad (\text{A50})$$

into amplitude and phase, together with the functional equation $\Gamma(1 + \frac{i}{2\epsilon}) = \frac{i}{2\epsilon} \Gamma(\frac{i}{2\epsilon})$ and the identity [36,37]

$$\left| \Gamma\left(\frac{i}{2\epsilon}\right) \right| = \sqrt{\frac{\pi}{\frac{1}{2\epsilon} \sinh(\frac{\pi}{2\epsilon})}} = 2\sqrt{\pi\epsilon} \frac{e^{-\frac{\pi}{4\epsilon}}}{\sqrt{1 - e^{-\frac{\pi}{\epsilon}}}} \quad (\text{A51})$$

of the Γ function to further simplify the expression, Eq. (A48), for b .

After minor algebra we arrive at the final expression

$$b(\tau_0) = \sqrt{1 - e^{-\frac{\pi}{\epsilon}}} e^{i\phi}, \quad (\text{A52})$$

where we have introduced the phase

$$\phi \equiv \xi + \arg\left[\Gamma\left(1 + \frac{i}{2\epsilon}\right)\right]. \quad (\text{A53})$$

It is interesting to identify the origins of the amplitude of b and the individual contributions to the phase ϕ . Indeed, the square root in Eq. (A52) is a consequence of the difference of the PCFs in Eq. (A45) and the appearance of the Γ function in Eq. (A40) whose absolute value reduces with the help of Eq. (A51) the expression in the square brackets in Eq. (A48) to the square root. According to Eqs. (A48) and (A50) the phase of the Γ function is one contribution to phase ϕ . The others arise from the complex-valued Gaussian, which appears in the asymptotic expansions, Eqs. (A38) and (A40), of $\mathcal{D}_{\nu+1}(z_0)$ and $\mathcal{D}_\nu(-z_0)$ with opposite signs leading to a factor of two. The logarithmic phase emerges in Eq. (A47), since the PCFs $\mathcal{D}_{\nu+1}(z_0)$ and $\mathcal{D}_{\nu+1}(-z_0)$ given by Eqs. (A38) and (A41) are proportional to $z^{\nu+1}$ whereas, according to Eq. (A40), $\mathcal{D}_\nu(-z_0)$ is proportional to $z^{-\nu-1}$. The constant phase of $-5\pi/4$ is a consequence of the minus sign in the definition, Eq. (A29), of b and the transformation, Eq. (A4), from τ to z .

d. Solution η of the Riccati equation

Finally, we address the asymptotic behavior of η . With the help of the asymptotic expressions, Eqs. (A44) and (A52), for the probability amplitudes a and b , we find from Eq. (A30) the result

$$\eta(\tau_0) = \frac{\sqrt{1 - e^{-\frac{\pi}{\epsilon}}}}{e^{-\frac{\pi}{2\epsilon}}} e^{i\phi_\eta(\tau_0)}, \quad (\text{A54})$$

where we have defined the phase

$$\phi_\eta \equiv \pi + \xi + \arg\left[\Gamma\left(1 + \frac{i}{2\epsilon}\right)\right] \quad (\text{A55})$$

of η . Needless to say, the same result follows from Eq. (A31) when we apply the asymptotic expressions, Eqs. (A36), (A38), (A40), and (A41).

APPENDIX B: INTERACTION PICTURE

In a recent article [48], we have analyzed the Landau-Zener problem in the interaction picture given by the coupled differential equations

$$i\dot{\tilde{a}} = e^{-i\epsilon\tau^2} \tilde{b}, \quad (\text{B1})$$

$$i\dot{\tilde{b}} = e^{i\epsilon\tau^2} \tilde{a}, \quad (\text{B2})$$

with the corresponding initial conditions

$$\tilde{a}(-\tau_0) = e^{-i\epsilon\tau_0^2/2} \quad \text{and} \quad \tilde{b}(-\tau_0) = 0, \quad (\text{B3})$$

where the *tilde* indicates the probability amplitudes in the interaction picture. Equations (1) and (2) are connected to Eq. (B1) and Eq. (B2) through the unitary transformation $U \equiv \text{diag}(\exp[-i\epsilon\tau^2/2], \exp[i\epsilon\tau^2/2])$.

At this point, we emphasize two key differences compared to the corresponding equations in Ref. [48]: (i) The quadratic phase factors in front of $\tilde{a}$ and $\tilde{b}$ differ by a factor of two. This factor was lost in Ref. [48] during the transition from the Hamiltonian [Eq. (1)] to the interaction picture [Eqs. (3) and (4)]. Replacing $\alpha \rightarrow 2\alpha$ in those and subsequent equations restores consistency with the original Hamiltonian. (ii) In Ref. [48], the initial condition $\tilde{a}(-\tau_0) = 1$ was used, in contrast to Eq. (B3), which follows from Eq. (4). To reproduce the same probability amplitudes as in Ref. [48], we must choose $\tau_0 \equiv \sqrt{2\pi T_0/\epsilon}$ where T_0 is a large positive integer.

Next, we briefly highlight the difference between the Markov approximation in the Schrödinger picture and the interaction picture used in Ref. [48]. We already have derived the integro-differential equation, Eq. (23), for a which reads

$$\dot{a}(\tau) = -e^{-i\epsilon\tau^2} \int_{-\tau_0}^{\tau} d\tau' e^{i\epsilon\tau'^2} \tilde{a}(\tau') \quad (\text{B4})$$

in the interaction picture. Due to the transformation the Markov approximation takes the form $\tilde{a}(\tau') \approx \tilde{a}(\tau)$ and the approximated integral differential equation reads

$$\dot{\tilde{a}}_M(\tau) = i\eta_M(\tau) \tilde{a}_M(\tau), \quad (\text{B5})$$

with the solution

$$\tilde{a}_M(\tau) = e^{-i\epsilon\tau_0^2/2} e^{iH_M(\tau)} \quad (\text{B6})$$

satisfying the initial condition Eq. (B3).

We conclude by noting that in Ref. [48], the initial condition was expressed in the interaction picture rather than in the Schrödinger picture. As a result, the quadratic phase factor in τ_0 does not appear in the expression for $\tilde{a}_M$ in Ref [48]. Moreover, we have already performed the limit $\tau_0 \rightarrow \infty$ in the lower boundary of the integral in Eq. (B6).

APPENDIX C: NONLINEAR INTEGRAL EQUATIONS

In Sec. III, we derived the Riccati equation Eq. (13) for η , linking the amplitudes a and b . Here, we reformulate it as a nonlinear integral equation, which admits two useful approximations: (i) a linearized (Markov) solution by neglecting nonlinearity, and (ii) an improved solution from a single iteration starting from the Markov result.

1. Exponential representation

In contrast to the derivation of the expression Eq. (11) for b , we equivalently find a solution Eq. (22) by a formal integration using the initial condition Eq. (4). On first sight, Eq. (22) is very different from the representation of b given by Eq. (11) since now the probability amplitude a appears under the integral, requiring knowledge of a for all times prior to τ . In contrast, in Eq. (11), a is evaluated at τ . However, it is possible to cast, Eq. (22), into the form of Eq. (11). For this purpose, we multiply and divide the right-hand side of Eq. (22) by $a(\tau)$ yielding

$$b(\tau) = -ia(\tau)e^{-i\epsilon\tau^2/2} \int_{-\tau_0}^{\tau} d\tau' e^{i\epsilon\tau'^2/2} \frac{a(\tau')}{a(\tau)}. \quad (C1)$$

We note the relation

$$\frac{a(\tau')}{a(\tau)} = e^{i\epsilon(\tau^2 - \tau'^2)/2} e^{i\mathcal{H}(\tau', \tau)} \quad (C2)$$

following from the exponential ansatz, Eq. (8), for a . Here, we have introduced the function

$$\mathcal{H}(\tau', \tau) \equiv H(\tau') - H(\tau) = \int_{\tau}^{\tau'} d\tau'' \eta(\tau''), \quad (C3)$$

which leads us to the representation

$$b(\tau) = -ia(\tau)e^{-i\epsilon\tau^2} \int_{-\tau_0}^{\tau} d\tau' e^{i\epsilon\tau'^2} e^{i\mathcal{H}(\tau', \tau)}. \quad (C4)$$

A comparison with Eq. (11) yields the nonlinear integral equation

$$\eta(\tau) = ie^{-i\epsilon\tau^2} \int_{-\tau_0}^{\tau} d\tau' e^{i\epsilon\tau'^2} e^{i\mathcal{H}(\tau', \tau)}. \quad (C5)$$

We can easily verify the identity, Eq. (C5), by differentiation which yields

$$\dot{\eta} = -2i\epsilon\tau\eta + i - i\eta^2, \quad (C6)$$

in complete agreement with the Riccati equation, Eq. (13). Moreover, the integral, Eq. (C5), vanishes for $\tau = -\tau_0$ satisfying the initial condition, Eq. (14), for η .

2. Approximations

It is interesting that η^2 appears in Eq. (C6) due to the presence of η in the exponent of Eq. (C5). When we neglect η in this exponent, that is the nonlinearity in the Riccati equation, Eq. (C6), we arrive at the linear differential equation

$$\dot{\eta}_M = -2i\epsilon\tau\eta_M + i, \quad (C7)$$

with the solution η_M given by Eq. (26).

Moreover, since Eq. (C5) is an integral equation we are inclined to iterate the equation using the Markov solution,

Eq. (26), as a starting point. By direct differentiation, we verify that the iterated approximate solution

$$\eta_I(\tau) \cong ie^{-i\epsilon\tau^2} \int_{-\tau_0}^{\tau} d\tau' e^{i\epsilon\tau'^2} e^{i\mathcal{H}_M(\tau', \tau)} \quad (C8)$$

satisfies the differential equation

$$\dot{\eta}_I \cong -2i\epsilon\tau\eta_I + i - i\eta_I\eta_M, \quad (C9)$$

where we have introduced the definition $\mathcal{H}_M$ as the Markov variant of Eq. (C3) and the subscript I to indicate that the solution emerged from an iteration.

When we compare the differential equations, Eq. (C6) and Eq. (C9), for η and η_I , we note that the quadratic nonlinearity of the Riccati equation is replaced by the product of η_I and η_M . Hence, for $\eta_I \approx \eta_M$, we find $\eta_I\eta_M \approx \eta_I^2$ and η_I is then an approximate solution of the Riccati equation, Eq. (C6).

3. Implicit integral equation for η

Using the relation between the probability amplitudes a and b , Eq. (11), together with the normalization condition Eq. (3), we can obtain an expression for η . Its modulus $|\eta|$ is determined by the amplitude A of a , as defined in Eq. (17). We now reformulate the nonlinear integral equation Eq. (C5) to derive $|\eta|$ independently of the normalization condition.

With the help of the relation Eq. (C3) we find from Eq. (C5) the alternative nonlinear integral equation

$$\eta(\tau) = ie^{-i\epsilon\tau^2} e^{-iH(\tau)} \int_{-\tau_0}^{\tau} d\tau' e^{i\epsilon\tau'^2} e^{iH(\tau')}. \quad (C10)$$

Next, we decompose η into its real and imaginary part following Eq. (15), and recall the relation, Eq. (17), for A and Eq. (18) which yields the formula

$$\eta(\tau) = iA^{-1}(\tau) e^{i\lambda(\tau)} \int_{-\tau_0}^{\tau} d\tau' e^{-i\lambda(\tau')} A(\tau'), \quad (C11)$$

where we have introduced the abbreviation

$$\lambda(\tau) \equiv -\epsilon\tau^2 - \int_{-\tau_0}^{\tau} d\tau' \eta^{(R)}(\tau'). \quad (C12)$$

4. Integral identity

On first sight, Eq. (C11) seems to be an explicit equation for η . However, this impression is misleading since A involves the integral of $\eta^{(I)}$, and Φ contains, besides the quadratic time dependence, the integral of $\eta^{(R)}$.

Hence, Eq. (C11) is a rather complicated integral equation for η . Nevertheless, we can gain some insight.

Indeed, we now use the nonlinear integral equation, Eq. (C11), to establish the identity

$$|\mathcal{I}(\tau)| = \sqrt{1 - A^2(\tau)}, \quad (C13)$$

where we have introduced the integral

$$\mathcal{I}(\tau) \equiv \int_{-\tau_0}^{\tau} d\tau' e^{-i\lambda(\tau')} A(\tau'). \quad (C14)$$

From the derivative

$$\frac{d}{d\tau} |\mathcal{I}|^2 = \frac{d}{d\tau} (\mathcal{I}\mathcal{I}^*) = \dot{\mathcal{I}}\mathcal{I}^* + \mathcal{I}\dot{\mathcal{I}}^* \quad (C15)$$

of the absolute value squared of $\mathcal{I}$, we find with the help of Eq. (C14) the identity

$$\frac{d}{d\tau} |\mathcal{I}(\tau)|^2 = 2A^2(\tau) \text{Re}[A^{-1}(\tau) e^{-i\lambda(\tau)} \mathcal{I}(\tau)], \quad (\text{C16})$$

which with Eq. (C11) reduces to

$$\frac{d}{d\tau} |\mathcal{I}(\tau)|^2 = 2A^2(\tau) \eta^{(I)}(\tau). \quad (\text{C17})$$

By differentiation we find the relation

$$\frac{d}{d\tau} [1 - A^2(\tau)] = -2A(\tau) \dot{A}(\tau) = 2A^2(\tau) \eta^{(I)}(\tau), \quad (\text{C18})$$

where in the last step we have recalled the definition, Eq. (17), of A .

After we have equated Eqs. (C17) and (C18) we arrive at the relation

$$\frac{d}{d\tau} |\mathcal{I}(\tau)|^2 = \frac{d}{d\tau} [1 - A^2(\tau)]. \quad (\text{C19})$$

Next, we integrate both sides of Eq. (C19) from $-\tau_0$ to τ and note the initial conditions $\mathcal{I}(-\tau_0) = 0$ following from the definition of the integral $\mathcal{I}$ given by Eq. (C14), and the condition $A(-\tau_0) = 1$ determined from Eq. (17).

Thus, we arrive indeed at the identity Eq. (C13). We emphasize that in this derivation we have not employed the normalization condition, Eq. (3).

5. Summary

We are now in the position to make the connection between the nonlinear integral equation, Eq. (C11), for η and the representation in amplitude $|\eta|$ and phase ϕ_η Eq. (15).

Indeed, the identity, Eq. (C13), leads to the representation

$$e^{i\lambda(\tau)} \int_{-\tau_0}^{\tau} d\tau' e^{-i\lambda(\tau')} A(\tau') = \sqrt{1 - A(\tau)^2} e^{i\phi_\eta(\tau)}, \quad (\text{C20})$$

where the phase $\phi_\eta(\tau)$ is determined by the argument of the left-hand side of this equation, that is

$$\phi_\eta = \lambda(\tau) + \arg \left[\int_{-\tau_0}^{\tau} d\tau' e^{-i\lambda(\tau')} A(\tau') \right]. \quad (\text{C21})$$

This result demonstrates that the asymptotic phase ϕ of b given by Eq. (7) follows from an asymptotic evaluation of Eq. (C21) for $\tau = \tau_0$.

APPENDIX D: APPROXIMATE EXPRESSIONS FOR THE MARKOV SOLUTION

In this appendix, we use elementary properties of integrals to derive a connection formula for η_M , which connects positive and negative times, and brings out most clearly the origin of the Stueckelberg oscillations [5]. Moreover, this connection formula allows us to obtain the exact asymptotic value of a . However, it fails to predict correctly b due to the lack of the nonlinearity.

1. Large negative times

We begin with the case of a vanishing initial condition, Eq. (14), at a finite but large negative time $\tau = -\tau_0$. Since η starts at zero, it remains small during the initial evolution. This

permits a linearization of the Riccati equation by neglecting its nonlinear term, leading to the Markov approximation η_M , defined in Eq. (26).

Next, we use the asymptotic identity

$$\int_{-\tau_0}^{-|\tau|} d\tau' e^{i\epsilon\tau'^2} \cong \frac{1}{2i\epsilon} \left[\frac{e^{i\epsilon\tau_0^2}}{\tau_0} - \frac{e^{i\epsilon\tau^2}}{|\tau|} \right], \quad (\text{D1})$$

valid for large τ_0 and in the regime $-\tau_0 \lesssim \tau$. This result follows by substituting $\bar{\tau} \equiv -\tau'$, then $t \equiv \bar{\tau}^2$, and applying partial integration. Using this identity in the Markov solution, Eq. (26), we obtain the approximation

$$\eta(-|\tau|) \cong \eta_M(-|\tau|) \cong -\frac{1}{2\epsilon} \left[\frac{1}{|\tau|} - \frac{1}{\tau_0} e^{i\epsilon(\tau_0^2 - \tau^2)} \right], \quad (\text{D2})$$

which satisfies the initial condition, Eq. (14). Hence, two distinct features characterize the early-time dynamics of η_M : (i) a growth proportional to $1/|\tau|$ as $\tau \rightarrow 0$, and (ii) oscillations with a quadratic phase. Both are reflected in the time dependence of the probability amplitude

$$a_M(-|\tau|) = e^{i\epsilon(\tau^2 - \tau_0^2)/2} e^{iH_M(-|\tau|)} \quad (\text{D3})$$

as given by Eq. (28).

Indeed, from the the definition, Eq. (29), of H_M we find from the approximate expression, Eq. (D2), of η_M the formula

$$H_M(-|\tau|) = \frac{1}{2\epsilon} \ln \left(\frac{|\tau|}{\tau_0} \right) + \frac{e^{i\epsilon\tau_0^2}}{2\epsilon\tau_0} \int_{-\tau_0}^{-|\tau|} d\tau' e^{-i\epsilon\tau'^2}, \quad (\text{D4})$$

which with the help of the asymptotic formula, Eq. (D1), yields

$$H_M(-|\tau|) \cong \frac{1}{2\epsilon} \ln \left(\frac{|\tau|}{\tau_0} \right) + \frac{i}{4\epsilon^2\tau_0^2} - \frac{ie^{i\epsilon(\tau_0^2 - \tau^2)}}{4\epsilon^2\tau_0|\tau|}. \quad (\text{D5})$$

Hence, for large negative times we find with Eqs. (19) and (D2) the asymptotic phase velocity

$$\dot{\Phi}_M(-|\tau|) = -\epsilon|\tau| - \eta_M^{(R)}(-|\tau|). \quad (\text{D6})$$

Therefore, the first term in Eq. (D2) leads to a negative phase velocity which increases as $\tau \rightarrow 0$. In addition, this growth is modulated by an oscillation with an increasing frequency. The amplitude of this oscillation decreases linearly with increasing τ_0 . Only when $\tau_0 = \infty$ do these oscillations cease to exist. The quadratic phase factor in Eq. (D2) not only affects the phase velocity but also the amplitude. Indeed, we find that the amplitude

$$A_M(-|\tau|) = \exp \left(- \int_{-\tau_0}^{-|\tau|} d\tau' \eta_M^{(I)}(\tau') \right) \quad (\text{D7})$$

also displays oscillations.

Deeper insight into the oscillations arises by differentiating η_M in time, as given in Eq. (D2). The $1/\tau$ term yields a negligible $1/\tau^2$ contribution. In contrast, the oscillatory term from the quadratic chirp produces a linear τ dependence, resulting in an amplitude of order $\tau/\tau_0 \sim 1$. Thus, the oscillations originate from a nonvanishing time derivative.

In contrast, the solution of the Riccati equation with initial condition $\eta(\tau_0 = -\infty) = 0$ has a time derivative that scales

purely as $1/\tau^2$. To demonstrate this fact, we start from Eq. (13) in the form

$$\eta(\tau) = \frac{1}{2\epsilon\tau} + \frac{\eta(\tau)^2}{2\epsilon\tau} - \frac{\dot{\eta}(\tau)}{2i\epsilon\tau} \quad (\text{D8})$$

and solve it by neglecting the contributions from η^2 and $\dot{\eta}$, which yields

$$\eta(\tau) \cong \frac{1}{2\epsilon\tau}. \quad (\text{D9})$$

This approximation satisfies the initial condition $\eta(\tau_0 = -\infty) = 0$ and exhibits no oscillations. Indeed, differentiating Eq. (D9) yields a $1/\tau^2$ term, which can be neglected. Hence, to this order, it is consistent to drop the time derivative $\dot{\eta}$ in the Riccati equation Eq. (13). Furthermore, in the limit $\tau_0 \rightarrow \infty$, Eqs. (D2) and (D9) coincide.

2. At the time of the transition

In the neighborhood of $\tau = 0$ we perform a Taylor expansion

$$\eta_M(\tau) = \eta_M(0) + \dot{\eta}_M(0)\tau + \frac{1}{2}\ddot{\eta}_M(0)\tau^2 + \dots \quad (\text{D10})$$

of η_M around $\tau = 0$ and use derivatives of the corresponding differential equation, Eq. (30), to evaluate the expansion coefficients.

Indeed, we find immediately

$$\dot{\eta}_M(0) = i. \quad (\text{D11})$$

Differentiating Eq. (30) once more yields the identity

$$\ddot{\eta}_M = -2i\epsilon\eta_M - 2i\epsilon\tau\dot{\eta}_M, \quad (\text{D12})$$

which reduces for $\tau = 0$ to

$$\ddot{\eta}_M(0) = -2i\epsilon\eta_M(0). \quad (\text{D13})$$

Since η_M is given explicitly, its value at $\tau = 0$ follows directly from Eq. (26) and we arrive at

$$\eta_M(0) = i\mathcal{F}(0). \quad (\text{D14})$$

When we substitute Eqs. (D11), (D13), and (D14) into Eq. (D10) we arrive at the approximate expression

$$\eta_M(\tau) = i\mathcal{F}(0)(1 - i\epsilon\tau^2) + i\tau + \dots \quad (\text{D15})$$

for η_M in the neighborhood of $\tau = 0$

3. Large positive times

We now address the behavior of η_M for large positive times. To do so, we first derive a relation that connects the behavior of η_M at negative and positive times. This identity is then used to obtain an expression for a_M . From the definition, Eq. (26), of η_M we find with the new integration variable $\bar{\tau} \equiv -\tau'$ the expression

$$\eta_M(-\tau) = ie^{-i\epsilon\tau^2} \int_{\tau}^{\tau_0} d\bar{\tau} e^{i\epsilon\bar{\tau}^2}. \quad (\text{D16})$$

Next, we subtract and add the integral $\mathcal{F}(\tau)$ to obtain the relation

$$\eta_M(-\tau) = ie^{-i\epsilon\tau^2} \left[\mathcal{F}(\tau) - \mathcal{F}(\tau) + \int_{\tau}^{\tau_0} d\bar{\tau} e^{i\epsilon\bar{\tau}^2} \right]. \quad (\text{D17})$$

Combining the definition of $\mathcal{F}$, Eq. (27), with the integral and using the definition of η_M from Eq. (26), we obtain the identity

$$\eta_M(-\tau) = -\eta_M(\tau) + ie^{-i\epsilon\tau^2} \mathcal{F}(\tau_0). \quad (\text{D18})$$

However, the connection formula allows us to derive an expression for the Markov solution a_M for large positive times. For this purpose, we start from the identity

$$H_M(\tau) = H_M(0) + \int_0^{\tau} d\tau' \eta_M(\tau') \quad (\text{D19})$$

for the integral H_M defined in Eq. (29). Substituting the connection formula Eq. (D18) into the integrand and changing variables via $\bar{\tau} = -\tau'$ in the integral over $\eta_M(-\tau)$, we obtain

$$H_M(\tau) = H_M(-|\tau|) + i\mathcal{F}(\tau_0) \int_0^{\tau} d\tau' e^{-i\epsilon\tau'^2}. \quad (\text{D20})$$

Next, we combine the Markov solution Eq. (28) with Eq. (D20), yielding

$$a_M(|\tau|) = a_M(-|\tau|) \exp \left(-\mathcal{F}(\tau_0) \int_0^{\tau} d\tau' e^{-i\epsilon\tau'^2} \right), \quad (\text{D21})$$

where we have used the expression for $a_M(-|\tau|)$ from Eq. (37).

APPENDIX E: ORIGIN OF OSCILLATIONS AT EARLY TIMES

In Sec. V, we identified oscillations in the approximate analytical expression for η_M , valid at large negative times. This Appendix demonstrates that these oscillations arise from the superposition of the two instantaneous eigenstates of the time-dependent Hamiltonian [24,29,67,68]. To bring out this connection, we first review the essential elements of the adiabatic basis approach and then use it to derive an expression for the amplitude a in the limit of large negative times. Finally, we compare the resulting approximation with Eq. (38), as obtained from the Markov solution Eq. (37).

1. Instantaneous eigenstates

We start by recalling the expressions of the instantaneous eigenvalues and eigenstates. For this purpose, we cast the equations, Eqs. (1) and (2) for the probability amplitudes a and b into the matrix form

$$i \frac{d}{d\tau} \begin{pmatrix} a(\tau) \\ b(\tau) \end{pmatrix} = \begin{pmatrix} -\epsilon\tau & 1 \\ 1 & \epsilon\tau \end{pmatrix} \begin{pmatrix} a(\tau) \\ b(\tau) \end{pmatrix}, \quad (\text{E1})$$

which leads us to the instantaneous eigenvalues

$$\theta_{\pm}(\tau) \equiv \pm\theta(\tau) \equiv \pm\sqrt{(\epsilon\tau)^2 + 1}, \quad (\text{E2})$$

with the corresponding normalized eigenstates

$$u_{\pm}(\tau) \equiv N_{\pm}(\tau) \begin{pmatrix} v_{\pm}(\tau) \\ 1 \end{pmatrix}. \quad (\text{E3})$$

Here, we have introduced the abbreviation

$$v_{\pm}(\tau) \equiv -\epsilon\tau \pm \theta \quad (\text{E4})$$

and the normalization constant

$$N_{\pm}(\tau) \equiv \frac{1}{\sqrt{1 + v_{\pm}^2(\tau)}}. \quad (\text{E5})$$

To connect the diabatic with the adiabatic basis we consider the limit of $\tau \rightarrow \infty$. From the definitions, Eqs. (E2) and (E4), we find

$$v_{\pm}(\tau) \cong \pm(2\epsilon|\tau|)^{\mp \text{sgn}(\tau)}, \quad (\text{E6})$$

where $\text{sgn}(\tau) \equiv \tau/|\tau|$ is the sign function. This leads to the asymptotic eigenstates

$$u_{-} \cong \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad u_{+} \cong \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (\text{E7})$$

for large *negative* times, whereas for large *positive* times we find

$$u_{-} \cong \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad \text{and} \quad u_{+} \cong \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (\text{E8})$$

Hence, u_{+} and u_{-} correspond asymptotically to the diabatic states a and b .

2. Diagonalization

Next, we diagonalize the Hamiltonian derived from Eqs. (1) and (2) using the instantaneous eigenstates introduced in Eq. (E3). This procedure is applicable under specific conditions. In particular, we demonstrate its validity in the regime of large negative times, which is the focus of the present discussion.

For the diagonalization we find the transformation matrix

$$G \equiv \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} \frac{v_{+}}{\sqrt{1+v_{+}^2}} & \frac{v_{-}}{\sqrt{1+v_{+}^2}} \\ \frac{1}{\sqrt{1+v_{+}^2}} & \frac{1}{\sqrt{1+v_{-}^2}} \end{pmatrix} \quad (\text{E9})$$

from the diabatic to the adiabatic basis, and the corresponding inverse matrix

$$G^{-1} \equiv \begin{pmatrix} G_{11}^{-1} & G_{12}^{-1} \\ G_{21}^{-1} & G_{22}^{-1} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{1+v_{+}^2}}{v_{+}-v_{-}} & -\frac{\sqrt{1+v_{+}^2}v_{-}}{v_{+}-v_{-}} \\ -\frac{\sqrt{1+v_{-}^2}}{v_{+}-v_{-}} & \frac{\sqrt{1+v_{-}^2}v_{+}}{v_{+}-v_{-}} \end{pmatrix}, \quad (\text{E10})$$

which is the transformation from the adiabatic to the diabatic basis, that is

$$\begin{pmatrix} a(\tau) \\ b(\tau) \end{pmatrix} = G(\tau) \begin{pmatrix} \tilde{a}(\tau) \\ \tilde{b}(\tau) \end{pmatrix}. \quad (\text{E11})$$

The matrices G and G^{-1} diagonalize the Hamiltonian $\mathcal{H}$ and the Schrödinger equation, Eq. (E1), takes the form

$$i \frac{d}{d\tau} \begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} = D \begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} - iG^{-1} \left(\frac{d}{d\tau} G \right) \begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix}, \quad (\text{E12})$$

where

$$D \equiv G^{-1} \begin{pmatrix} -\epsilon\tau & 1 \\ 1 & \epsilon\tau \end{pmatrix} G. \quad (\text{E13})$$

With the help of the explicit expressions, Eqs. (E9) and (E10), for G and G^{-1} , it is easy to verify that D is indeed the diagonal matrix $D = \text{diag}(\theta, -\theta)$ where θ is defined by Eq. (E2).

3. Solution for the diagonalized system

We emphasize that we can solve Eq. (E12) only when we can neglect the time derivative of G governing the last term. Since in the limit of large negative times the matrix

$$G(-|\tau|) \cong \begin{pmatrix} 1 & -\frac{1}{2\epsilon|\tau|} \\ \frac{1}{2\epsilon|\tau|} & 1 \end{pmatrix} \quad (\text{E14})$$

is almost constant we can indeed neglect it. Therefore, we solve the equation

$$i \frac{d}{d\tau} \begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} = D \begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} \quad (\text{E15})$$

obtained from Eq. (E12) in the limit of large negative times, and find the solutions

$$\tilde{a}(-|\tau|) \cong \tilde{a}(-\tau_0) e^{-i\chi(-|\tau|)}, \quad (\text{E16})$$

$$\tilde{b}(-|\tau|) \cong \tilde{b}(-\tau_0) e^{i\chi(-|\tau|)}. \quad (\text{E17})$$

Here, we have introduced the time-dependent function

$$\chi(\tau) \equiv \int_{-\tau_0}^{\tau} d\tau' \theta(\tau'). \quad (\text{E18})$$

The initial conditions $\tilde{a}(-\tau_0)$ and $\tilde{b}(-\tau_0)$ in Eqs. (E16) and (E17) are connected to the initial conditions $a(-\tau_0)$ and $b(-\tau_0)$, Eq. (4), by the matrix G^{-1} which leads to

$$\tilde{a}(-\tau_0) = G_{11}^{-1}(-\tau_0) \quad \text{and} \quad \tilde{b}(-\tau_0) = G_{21}^{-1}(-\tau_0). \quad (\text{E19})$$

Now we are in the position to obtain an expression for the probability amplitude a in the diabatic basis. For this purpose, we substitute Eq. (E9) into Eq. (E11) and find

$$a(-|\tau|) = G_{11}(-|\tau|) \tilde{a}(-|\tau|) + G_{12}(-|\tau|) \tilde{b}(-|\tau|), \quad (\text{E20})$$

which with the help of the solutions, Eqs. (E16) and (E17), in the adiabatic basis, and the initial conditions, Eq. (E19), takes the form

$$a(-|\tau|) = G_{11}(-|\tau|) G^{-1}(-\tau_0) e^{-i\chi(-|\tau|)} + G_{12}(-|\tau|) G_{21}^{-1}(-\tau_0) e^{i\chi(-|\tau|)}. \quad (\text{E21})$$

We emphasize apart from the assumption that G is constant for large negative times, we have made no further approximations.

4. Asymptotic behavior for large negative times

Since we are interested in an approximate expression for a in the neighborhood of $\tau = -\tau_0$ we expand θ and $v_{\pm}$ for large negative times, and find the expression

$$\theta_{\pm}(-|\tau|) \cong \pm \left(\epsilon|\tau| + \frac{1}{2\epsilon|\tau|} \right) \quad (\text{E22})$$

as well as from Eq. (E6) the terms $v_{+}(-|\tau|) \cong 2\epsilon|\tau|$ and $v_{-}(-|\tau|) \cong -1/(2\epsilon|\tau|)$. Hence, the difference of v_{+} and v_{-} is given by $v_{+} - v_{-} = 2\epsilon|\tau| + 1/(2\epsilon|\tau|)$. We are now in the position to approximate each matrix element in Eq. (E21), we find the expressions

$$G_{11}(-|\tau|) \cong 1 \quad \text{and} \quad G_{12}(-|\tau|) \cong -\frac{1}{2\epsilon|\tau|}. \quad (\text{E23})$$

Moreover, we find

$$G_{11}^{-1}(-\tau_0) \cong 1 - \frac{1}{4\epsilon^2\tau_0^2} \quad \text{and} \quad G_{21}^{-1}(-\tau_0) \cong -\frac{1}{2\epsilon|\tau_0|} \quad (\text{E24})$$

for the matrix elements of the inverse matrix G^{-1} , dependent on the initial time $-\tau_0$.

Next, we insert Eqs. (E23) and (E24) into Eq. (E21), which yields

$$a(-|\tau|) = \left(1 - \frac{1}{4\epsilon^2\tau_0^2}\right) e^{-i\chi(-|\tau|)} + \frac{1}{4\epsilon^2\tau_0|\tau|} e^{i\chi(-|\tau|)}, \quad (\text{E25})$$

where we do not yet use the asymptotic expression for θ .

In the next step, we factor out the first exponential and obtain the expression

$$a(-|\tau|) = e^{-i\chi(-|\tau|)} \left(1 - \frac{1}{4\epsilon^2\tau_0^2} + \frac{e^{2i\chi(-|\tau|)}}{4\epsilon^2\tau_0|\tau|}\right), \quad (\text{E26})$$

which with the help of the Taylor expansion $\exp(X) \cong 1 + X$ of the exponential function yields

$$a(-|\tau|) \cong e^{-i\chi(-|\tau|)} \exp\left(-\frac{1}{4\epsilon^2\tau_0^2} + \frac{e^{2i\chi(-|\tau|)}}{4\epsilon^2\tau_0|\tau|}\right). \quad (\text{E27})$$

Next, we evaluate the function χ , Eq. (E18), for large negative times by inserting the asymptotic behavior of θ , Eq. (E22), which leads us to

$$\chi(-|\tau|) \cong \frac{\epsilon}{2}(\tau_0^2 - \tau^2) - \frac{1}{2\epsilon} \ln\left(\frac{|\tau|}{\tau_0}\right). \quad (\text{E28})$$

Thus, we arrive at the expression

$$a_M(-|\tau|) = e^{-i\epsilon(\tau_0^2 - \tau^2)/2} e^{-i\tilde{H}_M(-|\tau|)}, \quad (\text{E29})$$

where

$$\tilde{H}_M(-|\tau|) \equiv \frac{1}{2\epsilon} \ln\left(\frac{|\tau|}{\tau_0}\right) + \frac{i}{4\epsilon^2\tau_0^2} - \frac{ie^{2i\chi(-|\tau|)}}{4\epsilon^2\tau_0|\tau|}. \quad (\text{E30})$$

Here we have used Eqs. (E28) and (E27).

On first sight, this expression seems to be different from Eq. (37) due to the presence of the logarithmic phase singularity in the exponential function. However, the asymptotic expansion

$$\ln\left(\frac{|\tau|}{\tau_0}\right) = \ln\left[1 + \frac{|\tau| - \tau_0}{\tau_0}\right] \cong \frac{|\tau| - \tau_0}{\tau_0} \quad (\text{E31})$$

of the logarithm shows that the correction scales like $1/\tau_0$ and leads to a higher order correction. For this reason, we find

$$H_M(-|\tau|) \cong \tilde{H}_M(-|\tau|), \quad (\text{E32})$$

and Eq. (E30) reduces indeed to Eq. (38).

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