

Optimal work control for an activity-driven odd tracer in a viscoelastic bath

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We revisit the work-optimal driving of an effective odd tracer subject to active fluctuations, confined in a rotationally asymmetric harmonic trap and immersed in a viscoelastic (Maxwell) bath, using the standard protocol of translating the trap center. The odd mobility of an odd tracer need not be an intrinsic single-particle property, but can instead emerge from coarse graining over particles coupled to a viscoelastic bath. Two control schemes are considered: open loop (without measuring the active force) and closed loop (conditioned on the initial value of the active force). In the open-loop case, we show that viscoelastic memory increases the energetic cost and breaks time-reversal symmetry in the individual components of the optimal protocol, while the sum of the two protocol components can remain time symmetric. In closed-loop control, an initial measurement of the active force determines the strategy adopted by the optimal protocol. We identify the regimes of force configuration, the timescales associated with the persistence of the active force and bath relaxation, and the tracer's odd mobility that govern optimal performance—maximizing work extraction or, equivalently, minimizing the control input. In particular, for odd tracers with specific orientations of the initial force, the interplay between the persistence time and the bath relaxation time becomes crucial, selecting an intermediate persistence that optimizes the work cost. In strongly viscoelastic regimes, we find that the cost generally increases with the bath relaxation time, reflecting the energetic penalty associated with long memory. Our results offer practical design principles for optimizing active micromachines in complex biomolecular environments, illustrating how measurement, memory, and controlled tuning of persistence can be harnessed for efficient energy conversion.

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I. INTRODUCTION

Designing efficient machines—and improving their efficiency—remains a long-standing challenge across scales [1–5]. Over recent decades, advances in micro- and nanofabrication have drawn attention to mesoscopic devices, with biomolecular motors providing concrete prototypes [6–8]. At this scale, thermal fluctuations and viscous dissipation constrain extractable work, motivating an optimal-control formulation: drive the system between two thermodynamic states with time-dependent controls $\lambda(t)$ and choose protocols that optimize a task-specific cost functional (e.g., net work, process duration, or total entropy production) [9–17]. Prior studies explored several protocol strategies by tuning control parameters—dragging the trap center or varying trap stiffness, modifying the potential landscape (including geometric asymmetry), and adjusting

the strength of external or initial forces [18–32]. In open-loop control—where the protocol is prespecified without state measurements, typically from an equilibrium or steady-state distribution—the optimal schedule is fixed in advance and often features boundary jumps associated with minimal-dissipation principles [9]. In contrast, closed-loop (feedback) control conditions the protocol on measurements of the system state; when information costs are accounted for, such feedback can substantially improve performance [33–37].

These considerations are especially relevant for active matter, where energy is injected by the environment or by the constituents themselves (e.g., ATP-driven motors, bacterial baths, and self-propelled tracers), producing persistent nonequilibrium fluctuations [38–45]. While open-loop schemes cannot exploit instantaneous active fluctuations, feedback that steers the direction or magnitude of self-propulsion can enhance work extraction and reduce dissipation [16,34,46–48]. Activity breaks time-reversal symmetry (TRS) and sustains steady probability currents [49–63]. Other symmetry-breaking mechanisms are common in biomatter, such as self-rotating particles that break parity symmetry [64–69]. Concrete realizations include charged tracers in magnetic fields, spinning objects subject to Coriolis- or Magnus-type forces [70–73], and tracers in chiral active baths [74,75]; in a broad sense, these systems fall under odd dynamics [64,65]. For such odd-diffusive systems, a

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convenient two-dimensional parametrization introduces an antisymmetric (odd) contribution to the diffusivity, $D_{ij} = D_0(\delta_{ij} + \kappa \epsilon_{ij})$, with dimensionless oddness parameter κ , Kronecker delta δ_{ij} , and Levi-Civita tensor ϵ_{ij} ($\epsilon_{12} = +1$, $\epsilon_{21} = -1$, otherwise 0) [65]. The antisymmetric part generates solenoidal (circulating) probability currents and a transverse bias in response—i.e., motion perpendicular to applied forces or density gradients—analogue to the Hall effect [71,76]. Consequently, pairwise interactions (collisions) between odd tracers induce mutual rolling (orbiting), which enhances self-diffusion and yields density-enhanced self-diffusion in many-body settings [67,69].

Most stochastic-control studies of tracers—passive, active, and more recently odd—have been developed for Newtonian fluids, where friction is instantaneous and the noise is white [9,32,77]. In Newtonian media, bath degrees of freedom relax on timescales much shorter than the tracer dynamics, so memory generated by coarse graining is often neglected, and the Langevin description is effectively Markovian [78]. In contrast, most biological environments are viscoelastic and crowded [79,80]. For example, the cytoplasm comprises an entangled, cross-linked network of actin, microtubules, and intermediate filaments, with motor proteins (e.g., myosin, kinesin, and dynein) generating active kicks; analogous viscoelastic scaffolds exist in the nucleus (chromatin and the nuclear lamina) [81,82]. Other viscoelastic media include blood plasma (protein crowding), polymeric hydrogels, dense bacterial suspensions, and micellar solutions [83–87]. In such settings, coarse graining over bath degrees of freedom produces temporal correlations that render the dynamics non-Markovian, and an explicit memory kernel becomes essential [87,88]. A minimal, analytically tractable representation is a Maxwell medium, in which the generalized Langevin equation acquires a convolutional friction $\int_0^t \mathcal{K}(t-t') \dot{\mathbf{x}}(t') dt'$, with memory kernel $\mathcal{K}(t) = D_s e^{-t/\tau_s} / \tau_s$, characterized by the bath relaxation time τ_s [89–92]. In the frequency domain, $\tilde{\mathcal{K}}(\Omega)$ corresponds to the complex modulus $G^*(\Omega) = i\Omega \tilde{\mathcal{K}}(\Omega) = G'(\Omega) + iG''(\Omega)$, with storage and loss moduli, $G'(\Omega) = D_s \Omega^2 \tau_s / (1 + \Omega^2 \tau_s^2)$ and $G''(\Omega) = D_s \Omega / (1 + \Omega^2 \tau_s^2)$, respectively, while the Newtonian limit is recovered smoothly as $\tau_s \rightarrow 0$, restoring Stokes friction and δ -correlated thermal noise [93,94].

Motivated by the above, we investigate optimal control of work extraction in a minimal setting: an activity-driven odd tracer confined by a time-dependent harmonic trap and embedded in a Maxwell medium. We consider both open-loop protocols (preassigned schedules initiated from a stationary state) and closed-loop (measurement-based) protocols, where the control is conditioned on the measured initial state. The setup may admit several experimental realizations, with the physical origin of odd mobility differing from case to case. A first class of realizations involves a charged tracer confined to two dimensions inside a dense bacterial bath and subjected to a magnetic field perpendicular to the plane. In this case, the Lorentz force deflects the particle velocity transverse to the applied forces, giving rise to nonreciprocal response and steady rotational probability currents [66,76,95].

A second possibility for realizing an effective odd tracer arises when the trapped probe itself is a chiral microswimmer, such as *Escherichia coli*, spermatozoa, or starfish embryos

confined by optical tweezers and suspended in viscoelastic environments such as micellar solutions, polymeric gels, or the cytoplasm [74,96]. The intrinsic rotation of these swimmers generates persistent circular (or, in three dimensions, helical) trajectories. Upon coarse graining over the orientation dynamics, this behavior can be interpreted as an effective odd mobility of the particles, and we focus on a representative particle as the probe [68,97]. Artificial analogs include magnetically driven rotating colloids, spinning microrobots, and asymmetrically shaped particles such as L-shaped Janus colloids [73,98,99]. A third possibility is to use passive tracers held by optical tweezers and immersed in suspensions of chiral active particles within viscoelastic media, e.g., micellar solutions. In this scenario, the odd response arises not from the tracer itself but from the surrounding nonequilibrium medium: Odd stresses generated by the chiral active bath can, after coarse graining, produce an effective antisymmetric mobility for the confined tracer [100,101].

In what follows, we present a unified minimal framework that incorporates odd mobility into the overdamped dynamics of a prototypical tracer, capturing in a simplified manner the range of physical scenarios outlined above. We analyze how the oddness parameter κ , the initial orientation and magnitude of the active force $\mathbf{F}_0 \equiv (f_{10}, f_{20})^T$, and the Maxwell relaxation time of the bath τ_s govern optimal work extraction under externally applied protocols. Furthermore, we delineate regimes in which feedback control offers a distinct advantage.

II. MODEL

Let $X_i(t)$ and $Y_i(t)$ denote the tracer and bath coordinates, respectively, with $i = 1, 2$ labeling the two spatial directions. In a Maxwell viscoelastic model, the tracer is linearly coupled by springs to auxiliary bath variables Y_i [57,89,102]. The bath variables Y_i evolve according to

$$\begin{aligned}\tau_s \dot{Y}_1(t) &= -Y_1(t) + X_1(t) + \eta_{y1}(t), \\ \tau_s \dot{Y}_2(t) &= -Y_2(t) + X_2(t) + \eta_{y2}(t),\end{aligned}\quad (1)$$

where τ_s is the relaxation time of the medium and $\eta_{yi}(t)$ are zero-mean white Gaussian noises, with $\langle \eta_{yi}(t) \rangle = 0$ and $\langle \eta_{yi}(t) \eta_{yj}(t') \rangle = 2D_s \delta_{ij} \delta(t-t')$. For an odd tracer, the orthogonal components are coupled anisotropically (and nonreciprocally), captured by a friction tensor $\boldsymbol{\Gamma}_t = \begin{pmatrix} 1 & -\kappa \\ \kappa & 1 \end{pmatrix}$, with κ being the oddness parameter, so that the dissipative force on the tracer is $\boldsymbol{\Gamma}_t \dot{\mathbf{X}}(t)$ [67,76]. Before driving, the tracer is confined by a harmonic potential of stiffness k_0 with a rotational asymmetry set by the parameter γ . In addition, the tracer is linearly coupled to the Maxwell bath variables $\mathbf{Y} = (Y_1, Y_2)^T$; i.e., each pair (X_i, Y_i) interacts through a harmonic potential of stiffness $k_I \propto \tau_s^{-1}$. This coupling transmits the bath's relaxation dynamics and generates the viscoelastic memory experienced by the tracer. Under a time-dependent protocol $\boldsymbol{\lambda}(t) = (\lambda_1, \lambda_2)^T$, the potential reads

$$\begin{aligned}V(\mathbf{X}, \mathbf{Y}; \boldsymbol{\lambda}) &= \frac{k_0}{2} ((X_1 - \lambda_1)^2 + (X_2 - \lambda_2)^2 + 2\gamma X_1 X_2) \\ &\quad + \frac{k_I}{2} ((X_1 - Y_1)^2 + (X_2 - Y_2)^2).\end{aligned}\quad (2)$$

Apart from the thermal noise arising from collisions with solvent molecules, the tracer—described by the degrees of freedom $\mathbf{X}(t)$ —is also driven by an additional stochastic contribution, hereafter referred to as the active force. This force may represent either the self-propulsion of an active particle or athermal random kicks generated by the surrounding active bath. In the overdamped limit (or low-Reynolds-number regime), the dynamics of the tracer are thus governed by the Langevin equation of the form

$$\tau_c \Gamma_t \dot{\mathbf{X}}(t) = -\nabla_{\mathbf{X}} V(\mathbf{X}, \mathbf{Y}; \lambda) + \mathbf{F}(t) + \boldsymbol{\eta}(t), \quad (3)$$

where τ_c is the characteristic viscous (or relaxation) timescale of the tracer, $\mathbf{F}(t) = (F_1, F_2)^T$ is the active force, and $\boldsymbol{\eta}(t) = (\eta_1, \eta_2)^T$ represents the thermal noise. The latter is modeled as zero-mean Gaussian noise, satisfying $\langle \eta_i(t) \rangle = 0$ and $\langle \eta_i(t) \eta_j(t') \rangle = 2D_c \delta_{ij} \delta(t - t')$. As an analytically tractable model for active force, we adopt the active Ornstein-Uhlenbeck process (AOUP) [55,103],

$$\dot{F}_i(t) = -\frac{1}{\tau_A} F_i(t) + \sqrt{\frac{2f_A^2}{\tau_A}} \xi_{A,i}(t), \quad (4)$$

where τ_A is the persistence time and $\xi_{A,i}(t)$ are independent unit white noises, with $\langle \xi_{A,i}(t) \rangle = 0$ and $\langle \xi_{A,i}(t) \xi_{A,j}(t') \rangle = \delta_{ij} \delta(t - t')$. With this normalization, the stationary two-time correlation is

$$\langle F_i(t) F_j(t') \rangle = \delta_{ij} f_A^2 e^{-|t-t'|/\tau_A}. \quad (5)$$

Defining $\alpha_k = k_0/(k_l(1 + \kappa^2))$, $k_r^{-1} = k_0/k_l$, and $\alpha = 1/(1 + \kappa^2)$, and writing Eq. (3) componentwise using Eq. (2) yields

$$\begin{aligned} \tau_c \dot{X}_1(t) = & -\alpha_k (X_1(t) - \lambda_1(t) + \gamma X_2(t)) - \alpha (X_1(t) - Y_1(t)) \\ & - \kappa \alpha_k (X_2(t) - \lambda_2(t) + \gamma X_1(t)) \\ & - \kappa \alpha (X_2(t) - Y_2(t)) + \alpha (\eta_1(t) + \kappa \eta_2(t)) \\ & + \alpha (F_1(t) + \kappa F_2(t)), \end{aligned} \quad (6)$$

$$\begin{aligned} \tau_c \dot{X}_2(t) = & \kappa \alpha_k (X_1(t) - \lambda_1(t) + \gamma X_2(t)) + \kappa \alpha (X_1(t) - Y_1(t)) \\ & - \alpha_k (X_2(t) - \lambda_2(t) + \gamma X_1(t)) - \alpha (X_2(t) - Y_2(t)) \\ & - \alpha (\kappa \eta_1(t) - \eta_2(t)) - \alpha (\kappa F_1(t) - F_2(t)). \end{aligned} \quad (7)$$

Eliminating $\mathbf{Y}$ using Eq. (1) yields a single-exponential memory kernel for the tracer, consistent with the Maxwell model.

The particle is subjected to an externally controlled protocol $\lambda(t) = (\lambda_1, \lambda_2)^T$, with the initial conditions $\lambda_1(0) = \lambda_2(0) = 0$, and final values $\lambda_1(t_f) = L_1$ and $\lambda_2(t_f) = L_2$ at the end of the protocol of duration t_f . Following Sekimoto's convention [104], the external work performed on the tracer during the process is expressed as

$$W = \int_0^{t_f} dt \dot{\lambda}(t) \frac{\partial V}{\partial \lambda}. \quad (8)$$

Here, contributions from active forces are excluded from W ; being nonconservative and generally not directly measurable, they may be assigned to housekeeping (medium) heat [49]. Our aim is to determine optimal protocols $\lambda^*(t)$ that minimize the mean work $\omega \equiv \langle W \rangle$, i.e., $\omega^* = \min_{\lambda} \langle W \rangle$, where the average is taken over the ensemble of trajectories generated by the thermal and active noises. Here, $\omega^* \geq 0$

denotes work done on the tracer by the controller, whereas extracted work corresponds to $\omega^* < 0$. We apply the calculus of variations—using the Euler-Lagrange equations (see Appendix A for details)—to obtain the optimality and boundary jump conditions [105]. We consider two settings: (1) open loop, a protocol with no initial measurements; and (2) closed loop, a measurement-conditioned protocol in which an initial observation of the active force $\mathbf{F}(0) = (f_{10}, f_{20})^T$ is used to select the control. In the following, we present results for both strategies.

III. OPEN-LOOP CONTROL

In the open-loop setting—no initial information and protocols fixed *a priori*—the controller cannot utilize the instantaneous active force. Consequently, within this linear framework, the active force averages to zero and does not modify the mean trajectories; it therefore does not influence either the optimal protocol or the mean work. One can obtain the optimal protocol $\lambda^*(t)$ by substituting the optimal trajectories $\mathbf{x}^*(t)$, $\mathbf{y}^*(t)$ (with $f_{10} = f_{20} = 0$) from Eq. (A14) into Eqs. (A6) and (A7). Representative results are shown in Fig. 1; see also Fig. S1 in the Supplemental Material [106]. As in the Newtonian fluid [77], $\lambda_i^*(t)$ exhibits asymmetric boundary jumps at $t = 0$ and $t = t_f^-$; in general, for $\kappa \neq 0$, $L_i - \lambda_i^*(t_f^-) \neq \lambda_i^*(0^+)$. At $t = 0$, increasing κ drives the initial jump in λ_1^* further backward relative to the target L_1 , while simultaneously enhancing the forward jump in λ_2^* . For larger values of κ , viscoelastic damping moderates both the initial and final jumps, reducing their magnitudes as τ_s increases. Unlike in a Newtonian fluid ($\tau_s = 0$), the continuous path traced by $\lambda^*(t)$ in the (λ_1, λ_2) plane becomes intrinsically nonlinear at intermediate times in a viscoelastic medium, as illustrated in Fig. 1(c). This nonlinearity develops over a characteristic timescale $q_s^{-1} \tau_c \tau_s = \tau_s / \sqrt{1 + \tau_s/\tau_c}$, with q_s in Eq. (A19), and is further amplified by increasing oddness κ and longer bath relaxation times τ_s . This timescale sets the characteristic time associated with the hyperbolic terms in Eqs. (A15)–(A18) and is generally different from t_f .

In multidimensional control problems, the optimal protocol is generally symmetric in time about the midpoint. Each component of $\lambda^*(t)$ obeys time-reversal symmetry, which reads [16,107]

$$\dot{\lambda}_i^*(t) = \dot{\lambda}_i^*(t_f - t) \iff \lambda_i^*(t) + \lambda_i^*(t_f - t) = 2\lambda_i^*\left(\frac{t_f}{2}\right), \quad (9)$$

where $i = 1, 2$. To quantify deviations from TRS, one can also define an asymmetry measure

$$\mathcal{A}_{\lambda_i} = \frac{2}{t_f} \int_0^{t_f/2} \left[\lambda_i(t) + \lambda_i(t_f - t) - 2\lambda_i\left(\frac{t_f}{2}\right) \right]^2 dt, \quad (10)$$

which satisfies $\mathcal{A}_{\lambda_i} = 0$ if and only if Eq. (9) holds. In a Newtonian solvent ($\tau_s \rightarrow 0$), $\lambda_i^*(t)$ satisfies TRS for any κ [77]. In a Maxwell medium ($\tau_s > 0$) with a normal tracer ($\kappa = 0$), TRS continues to hold, consistent with Ref. [16]. In contrast, for an odd tracer with $\kappa \neq 0$ and $\tau_s > 0$, each component λ_i^* separately breaks the TRS ($\mathcal{A}_{\lambda_i} > 0$) (see Fig. 2). In fact, in a viscoelastic bath with fixed relaxation time τ_s , increasing

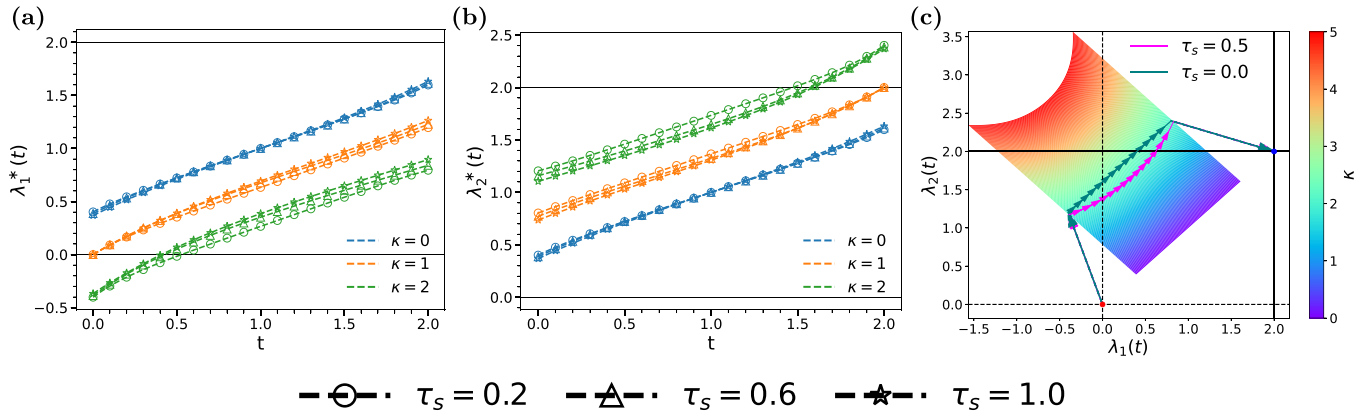


FIG. 1. Optimal open-loop protocols ($f_{10} = f_{20} = 0$). (a), (b) $\lambda_1^*(t)$ and $\lambda_2^*(t)$ shown for three values of oddness parameter κ ; line styles encode the timescale τ_s associated with bath particles. (c) Parametric plot $(\lambda_1^*(t), \lambda_2^*(t))$ for several κ values (colors as in the legend). A representative trajectory of the protocol $\Lambda(t)$ is shown for both a Newtonian fluid ($\tau_s = 0$) and a Maxwell bath ($\tau_s = 0.5$) at $\kappa = 2.5$. Arrows indicate the initial jump from $(0, 0)$, the subsequent continuous evolution, and the final jump toward the targets at $\lambda_1(t_f) = L_1$ and $\lambda_2(t_f) = L_2$, where $L_1 = L_2 = 2.0$. Other parameters are $\tau_c = 1.0$, $\tau_A = 1.0$, $k_r = 1.0$ (with $k_0 = k_f$), $\gamma = 0.5$, and $t_f = 2.0$.

oddness κ systematically strengthens TRS violations in both protocol components λ_1^* and λ_2^* [panels (d) and (e)]. Likewise, for fixed κ , slower bath relaxation (larger τ_s) produces the same trend. In particular, the optimal trajectories of the tracer $x_i^*(t)$ remain time symmetric, i.e., $\dot{x}_i^*(t) = \dot{x}_i^*(t_f - t)$ for $i = 1, 2$, while the bath variables $y_i^*(t)$ generally do not. It suggests that componentwise TRS is broken (local asymmetry, $\mathcal{A}_{\lambda_i} > 0$) by coupling between auxiliary (hidden) bath degrees of freedom and the observed coordinates with bro-

ken parity symmetry. Interestingly, for $L_1 = L_2$, the combined protocol $\Lambda^*(t) \equiv \lambda_1^*(t) + \lambda_2^*(t)$ follows $\dot{\Lambda}^*(t) = \dot{\Lambda}^*(t_f - t)$, and the TRS measure $\mathcal{A}_\Lambda$ for $\Lambda^*(t)$ remains near zero across parameters, indicating that the combined protocol preserves a global TRS [panel (f)].

The work W defined in Eq. (8) is obtained by averaging over all realizations of the thermal and active noise trajectories, yielding the optimal mean work $\omega^*(t_f)$ for a protocol of duration t_f . Specifically, we evaluate W along the optimal

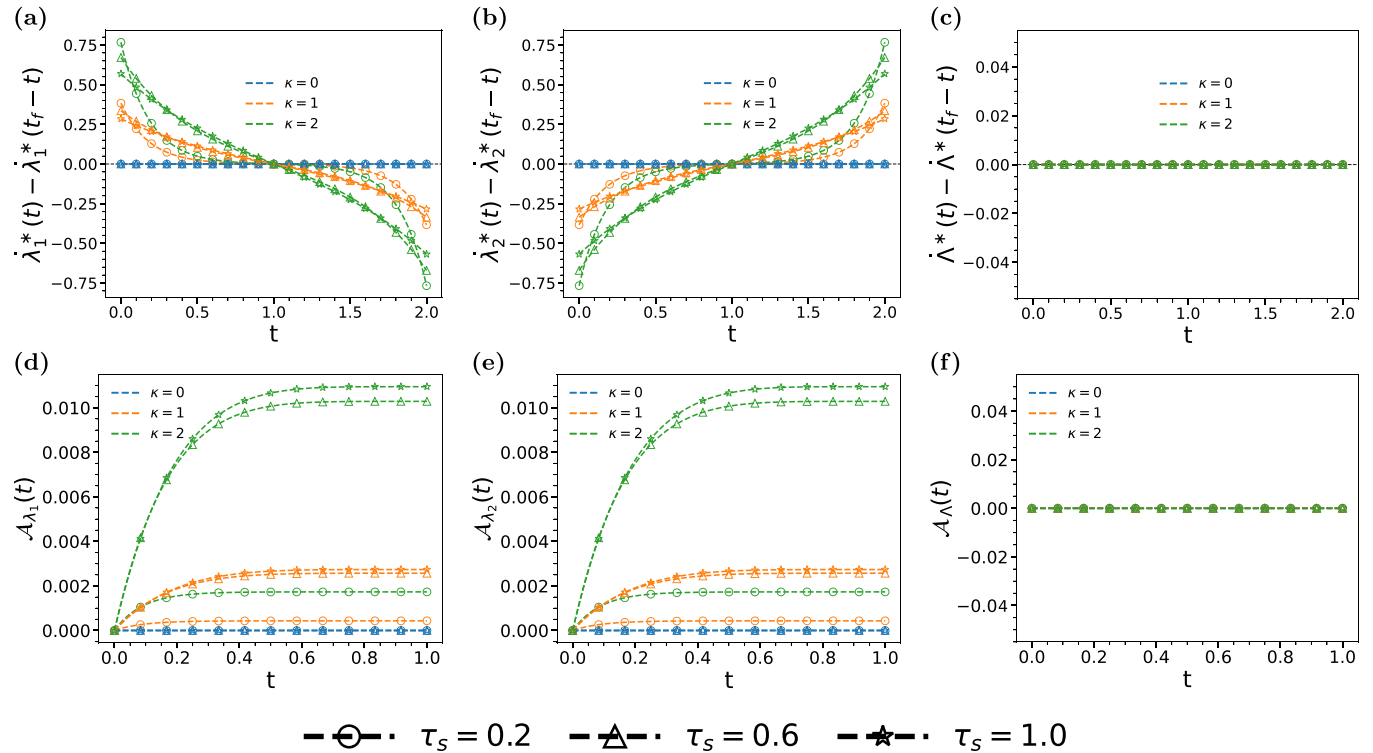


FIG. 2. Time-reversal symmetry tests for optimal protocols in open-loop control. (a)–(c) Deviations from TRS shown as $\dot{\lambda}_i^*(t) - \dot{\lambda}_i^*(t_f - t)$ for λ_1^* and λ_2^* , and as $\dot{\Lambda}^*(t) - \dot{\Lambda}^*(t_f - t)$ for $\Lambda^* = \lambda_1^* + \lambda_2^*$. Curves are shown for $\kappa \in \{0, 1, 2\}$; for each κ , three values of τ_s are considered. Color encodes κ ; line style encodes τ_s . (d)–(f) Corresponding asymmetry measures $\mathcal{A}_{\lambda_i}$ [Eq. (10)] for λ_1^* , λ_2^* , and Λ^* . All other parameters are as in Fig. 1.

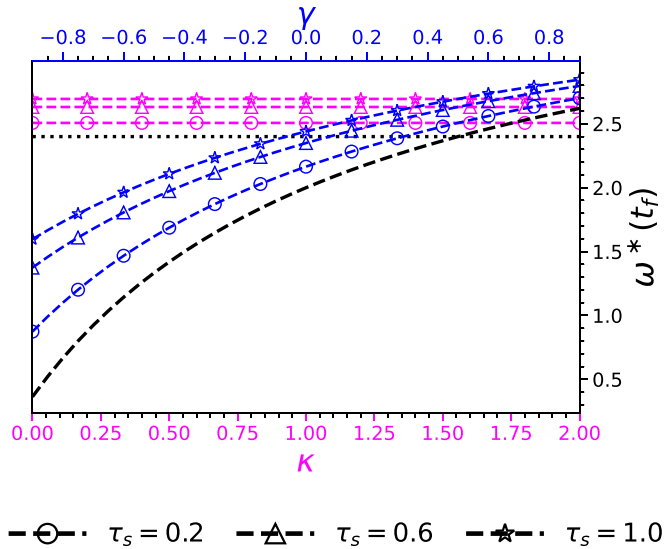


FIG. 3. Open-loop optimal mean work $\omega^*(t_f)$ as a function of the oddness parameter κ (bottom axis, with $\gamma = 0.5$ fixed) and the trap asymmetry γ (top axis, with $\kappa = 1.0$ fixed). Different colors distinguish the κ and γ scans; for each scan, results are shown for three values of the bath relaxation time τ_s (see legend). In the Newtonian limit ($\tau_s = 0$), $\omega^*(t_f)$ is shown by the black dashed line (without symbols) as a function of γ , while the dotted line (without symbols) corresponds to the case with fixed $\gamma = 0.5$, and is independent of κ . These curves are included for comparison with the viscoelastic cases. All other parameters are the same as in Fig. 1.

trajectories $\mathbf{x}^*(t)$, $\mathbf{y}^*(t)$ [Eq. (A14)] driven by the optimal protocols $\lambda^*(t)$ [Eqs. (A6) and (A7)]. The resulting $\omega^*(t_f)$ is shown in Fig. 3. As in a Newtonian fluid [77], $\omega^*(t_f)$ is invariant with respect to κ for fixed τ_s and γ . In contrast, tuning γ —which sets the rotational asymmetry of the potential—modulates the optimal work; in particular, decreasing γ reduces the work performed by the controller. In a viscoelastic bath, increasing the relaxation time τ_s raises the controller's work input, reflecting stronger memory—slower stress relaxation and a larger elastic (storage) contribution on the protocol timescale (see Fig. 3). Notably, the rise of $\omega^*(t_f)$ with τ_s parallels the increase of the componentwise time-reversal asymmetry $\mathcal{A}_{\lambda_i}$, linking viscoelastic memory, broken TRS, and elevated work cost.

The cumulative work $\omega(t)$ up to time $t \in [0, t_f]$, evaluated along protocols $\lambda(t)$, provides additional insight into the temporal evolution of the system [77]. Analogous to Eq. (8), $\omega(t)$ is obtained by replacing the upper integration limit t_f with $t \in (0, t_f)$ and averaging over all stochastic trajectories; the results are reported in Figs. S1 and S2 of the Supplemental Material [106]. For comparison with the optimal case, we consider a smooth linear protocol without jumps, $\lambda_i(t) = L_i t/t_f$ (which satisfies TRS). In this case, $\omega(t)$ increases continuously from $t = 0$ as the trap is translated gradually (see Fig. S2 of the Supplemental Material [106]). Since the tracer cannot respond instantaneously to the moving trap, a persistent lag develops, leading to continuous dissipation. For the optimal protocol, the presence of boundary jumps produces a discontinuity in $\omega^*(t)$ at $t = 0^+$, corresponding to an instantaneous change in potential energy without dis-

sipative loss. The initial jump positions the tracer such that its subsequent evolution closely follows its relaxation dynamics under the protocol. This reduces the effective lag between the tracer and the moving trap during the course of the driving, thereby minimizing dissipation. A final jump then brings the system to the target configuration, and for specific asymmetries set by γ , the work input can be further reduced when the final value of the potential becomes significantly lower than the initial one. In a viscoelastic bath, the deformation of the medium does not relax instantaneously as the trap is driven. This leads to a history-dependent restoring force that enhances the effective lag between the tracer and the protocol. Consequently, the optimal protocol $\lambda^*(t)$ acquires a nonlinear time dependence, and $\omega^*(t)$ typically increases at later times, particularly for $t \gtrsim q_s^{-1} \tau_c \tau_s$. Furthermore, for an odd tracer, the two spatial directions are not equivalent: Driving along one direction can induce a transverse response associated with circulating probability currents, leading to asymmetric jump magnitudes in the two components of $\lambda^*(t)$. Overall, the optimal protocol does not eliminate lag uniformly in time, but redistributes it in a manner that minimizes the total energetic cost. Experimentally, the trap position is typically controlled using a piezoelectric stage or acousto-optic deflectors, which possess finite response times and inertia; therefore, strictly instantaneous jumps cannot be realized. Instead, the jumps are implemented as rapid but continuous changes, with switching times chosen to be much shorter than the relaxation times of both the tracer and the bath. Under these conditions, the protocol provides a good approximation to the ideal discontinuous control [16].

IV. CLOSED-LOOP CONTROL

In the closed-loop protocol, the initial active force $\mathbf{f}_0 = (f_{10}, f_{20})$ is measured. This measurement fixes the conditional means of the coordinates, (x_{10}, x_{20}) and (y_{10}, y_{20}) , via linear relations of the form $z_{i0} = z_{ii} f_{i0} + z_{ij} f_{j0}$, where $z \in \{x, y\}$, $i, j \in \{1, 2\}$, $i \neq j$; see Eqs. (B5)–(B8). As in the Newtonian case [77], the optimal protocol $\lambda_i^*(t)$ in direction i selects one of two strategies depending on whether the measured f_{i0} lies above or below a threshold f_{i0}^* . The threshold f_{i0}^* is determined by $\lambda_i^*(0^+) = 0$; at $f_{i0} = f_{i0}^*$, the protocol $\lambda_i^*(t)$ starts with the slowest possible variation—zero initial rate of change. If $f_{i0} > f_{i0}^*$, λ_i^* follows Strategy I; if $f_{i0} < f_{i0}^*$, it follows Strategy II (with $f_{i0} = f_{i0}^*$ marking the boundary). See Fig. 4 for the time evolution of the optimal protocols $\lambda_i^*(t)$ ($i = 1, 2$); see also Figs. S3 and S4 in the Supplemental Material [106]. *Strategy I* ($\dot{\lambda}_i^*(0^+) > 0$): After a forward or reverse jump at $t = 0^+$, $\lambda_i^*(t)$ either increases monotonically toward L_i or reaches an interior maximum before the final jump to L_i [Fig. 5(a)]. *Strategy II* ($\dot{\lambda}_i^*(0^+) < 0$): After the initial jump, $\lambda_i^*(t)$ first decreases to an intermediate minimum and then increases to the terminal jump at L_i [Fig. 5(a)]. These conditions partition the (f_{10}, f_{20}) plane into four regions, each labeled by the strategy pair $\{s_1, s_2\}$ selected by $(\lambda_1^*, \lambda_2^*)$ [Fig. 5(b)], with $s_1, s_2 \in \{I, II\}$. In all regions, the optimal protocols exhibit an initial boundary jump at $t = 0^+$, whose sign (forward or reverse) and magnitude depend on κ and τ_s . At fixed τ_s , increasing κ generally lengthens the initial jump in a given direction. However, for comparatively

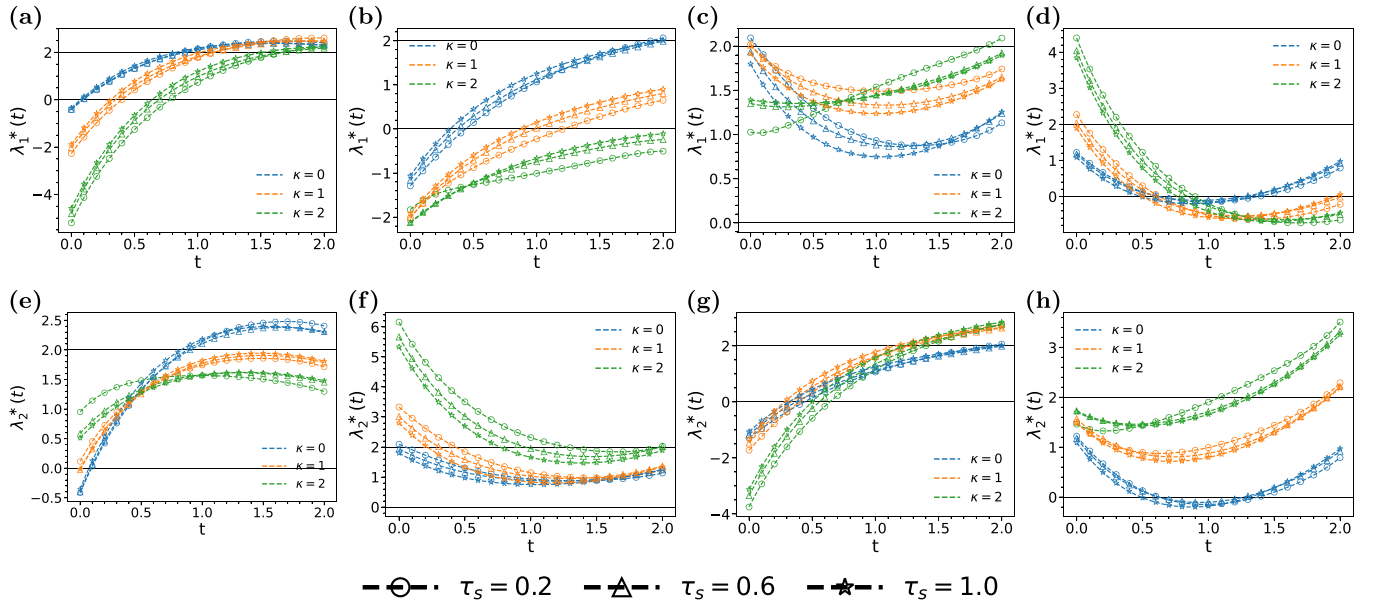


FIG. 4. Closed-loop optimal protocols. Time evolution of $\lambda_1^*(t)$ for four representative initial-force pairs (f_{10}, f_{20}) , respectively: (a) $f_{10} = 5.0$, $f_{20} = 5.0$; (b) $f_{10} = 5.0$, $f_{20} = -5.0$; (c) $f_{10} = -5.0$, $f_{20} = 5.0$; and (d) $f_{10} = -5.0$, $f_{20} = -5.0$. (e)–(h) Corresponding $\lambda_2^*(t)$ for the same set of initial-force pairs in the same order. Different colors are used for different oddness parameters κ ; line styles indicate the bath relaxation time τ_s (see legend). The final values of protocol is given by $\lambda_1^*(t_f) = L_1$ and $\lambda_2^*(t_f) = L_2$, where $L_1 = L_2 = 2.0$. Other parameters are $\tau_c = 1.0$, $\tau_A = 1.0$, $k_r = 1.0$ (so $k_0 = k_I$), $\gamma = 0.5$, and $t_f = 2.0$.

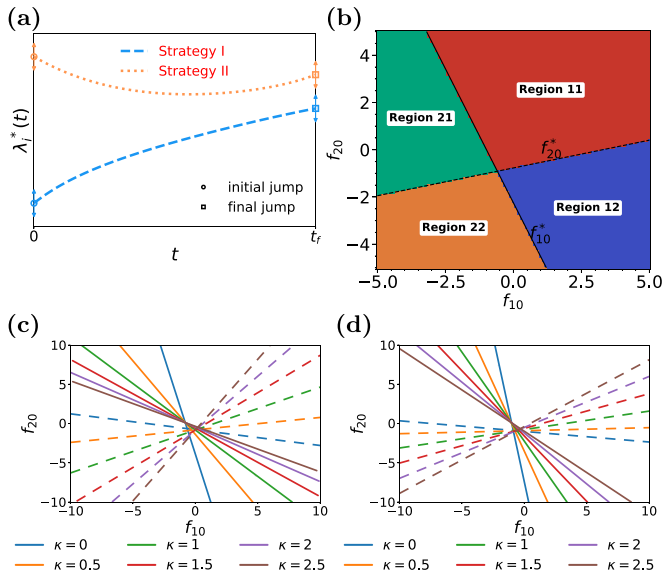


FIG. 5. (a) Schematic of the two strategies (Strategy I and Strategy II) adopted by the optimal protocols—each involving either a forward or a reverse jump at $t = 0$ and at $t = t_f$, as described. (b) Partition of the (f_{10}, f_{20}) plane into four regions determined by whether f_{10} lies above or below its threshold f_{10}^* ($i = 1, 2$); the boundaries are oblique (tilted) lines $f_{10} = f_{10}^*(f_{20})$ and $f_{20} = f_{20}^*(f_{10})$. Regions 11, 12, 22, and 21 correspond to the four force-pair regimes in which the strategy combinations {I,I}, {I,II}, {II,II}, and {II,I} are adopted by $(\lambda_1^*, \lambda_2^*)$, respectively. Panels (c) and (d) show the threshold curves f_{10}^* (solid lines) and f_{20}^* (dashed lines) in the (f_{10}, f_{20}) plane for a Newtonian bath ($\tau_s = 0.0$) and a Maxwell bath ($\tau_s = 1.0$), respectively, at different values of the oddness parameter κ .

small τ_s (here $\tau_s = 0.2$), notable exceptions appear at large κ , as seen in panels (b), (c), and (h) of Fig. 4 (regions 12, 21, and 22), where either $\lambda_1^*(t)$ [panels (b) and (c)] or $\lambda_2^*(t)$ [panel (h)] exhibits a shorter forward jump for $\kappa = 2$; in contrast, in panel (e) (region 11), the jump reverses sign with increasing κ . At fixed κ , smaller τ_s typically produces larger initial jumps—reflecting faster bath relaxation and consistent with open-loop results—though slight deviations appear for large κ (here $\kappa = 2$) and very small τ_s , as in panels (b), (c), and (h) of Fig. 4. Interestingly, the threshold forces f_{i0}^* also depend sensitively on both κ and τ_s . Compared with the Newtonian limit ($\tau_s \rightarrow 0$), f_{i0}^* in the viscoelastic case varies more slowly with f_{j0} ($i \neq j$) for a given κ [see Figs. 5(c) and 5(d)]. The overall trend follows that of the Newtonian case [77], $f_{10}^* \sim -\text{sign}(f_{20})|f_{10}|$ and $f_{20}^* \sim \text{sign}(f_{10})|f_{20}|$ for $\kappa \neq 0$. However, for a fixed f_{j0} , the corresponding f_{i0}^* values at nearby κ remain closer in the Maxwell bath than in the Newtonian limit. In other words, a force pair located well within a given region in Fig. 5(b) is less likely to switch to a different region when κ is varied slightly under viscoelastic conditions. For instance, the pair $(f_{10}, f_{20}) = (10, 0)$, which belongs to region 11 at $\kappa = 0$, remains within the same region at $\kappa = 0.5$ in the Maxwell bath [Fig. 5(d)], whereas in the Newtonian case it shifts from region 11 to region 12 for the same change in κ [Fig. 5(c)]. The variation of the optimal work $\omega^*(t_f)$ with system parameters is summarized in Figs. 6–9 (see also Figs. S5 and S6 in the Supplemental Material [106]). Unlike open-loop control, here $\omega^*(t_f)$ depends on the oddness parameter κ and on the initial-force pair (f_{10}, f_{20}) denoting different regions (11, 12, 21, and 22), for other fixed parameters. In region 11—where both initial-force components are aligned with the targets—net work can be extracted ($\omega^* < 0$), but the

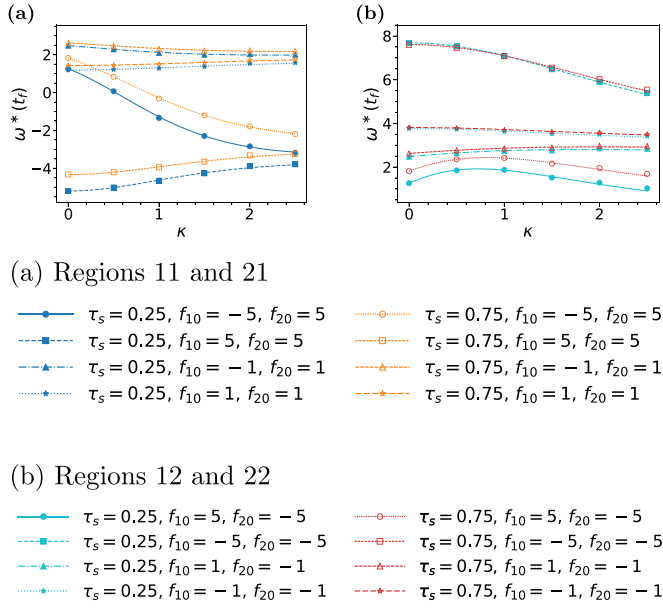


FIG. 6. Closed-loop optimal mean work $\omega^*(t_f)$ vs κ for two sets of regions; the corresponding legends are shown below. All other parameters are as in Fig. 4.

magnitude of extraction decreases with increasing κ [see Fig. 6(a)]. In contrast, in region 21—where f_{10} is oppositely aligned with the target L_1 —the controller’s work input is reduced with κ ; for sufficiently large κ , the sign of ω^* reverses, enabling net extraction. Moreover, a larger magnitude of the initial active force enhances the extent of extractable work [Fig. 6(a)]. In region 12, the dependence on κ is nonmonotonic: The input work is maximized at an intermediate κ [see Fig. 6(b)]. In region 22—where both f_{10} and f_{20} are initially misaligned with the targets—the controller’s input work decreases with κ ; nonetheless, $\omega^*(t_f) > 0$ over the explored range, so no net extraction is achieved.

For fixed κ , the minimum of $\omega^*(t_f)$ —and hence the maximum extractable work—occurs at a characteristic persistence time $\tau_A = \tau_A^*(\kappa, \tau_s)$ (see Fig. 7; see also Fig. S6 in the Supplemental Material [106]). Unlike in a Newtonian medium, here the interplay between the two timescales τ_A and τ_s dictates the location of τ_A^* . In this study, we focus primarily on the intermediate viscoelastic regime ($\tau_s \lesssim \tau_c$), where several distinct behaviors emerge. In region 22, where work extraction is not feasible, the minimum of ω^* occurs in the passive limit ($\tau_A \rightarrow 0$), as shown in Fig. 7(d). In region 11, increasing τ_s shifts τ_A^* to higher values within the range $\tau_A \leq t_f$ considered here, reflecting that longer persistence facilitates work extraction when the activity and driving are cooperatively oriented [see Fig. 7(a)]. In regions 12 and 21, τ_A^* assumes an intermediate value of order $O(\tau_c)$. For an even tracer ($\kappa = 0$), both directions remain equivalent, and τ_A^* decreases gradually with growing τ_s . For an odd tracer ($\kappa \neq 0$), this trend persists in region 12, whereas in region 21 the dependence of τ_A^* on τ_s becomes nonmonotonic, exhibiting a clear maximum at a characteristic relaxation time $\tau_s^*(\kappa)$ (see Fig. 8 and Fig. S7 in the Supplemental Material [106]). The trend in region 12 indicates that stronger viscoelastic memory penalizes highly persistent forcing and favors shorter

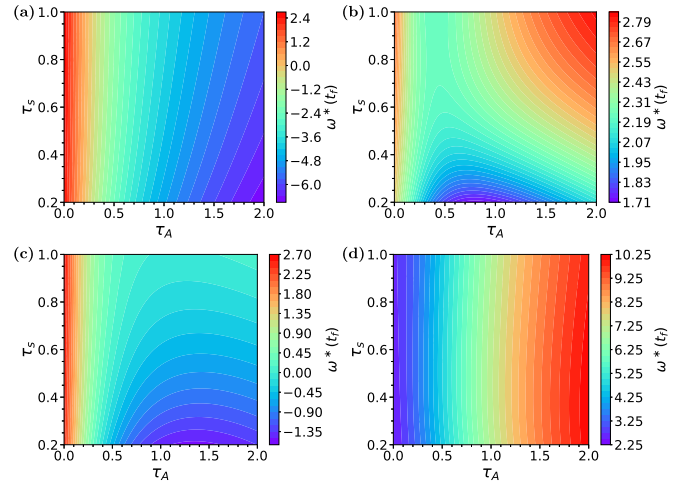


FIG. 7. Contour maps of optimal mean work $\omega^*(t_f)$ as functions of the persistence time of active force τ_A and the bath relaxation time τ_s . Results are shown for four representative initial-force pairs (f_{10}, f_{20}) : (a) $f_{10} = 5.0, f_{20} = 5.0$; (b) $f_{10} = 5.0, f_{20} = -5.0$; (c) $f_{10} = -5.0, f_{20} = 5.0$; and (d) $f_{10} = -5.0, f_{20} = -5.0$, corresponding to regions 11, 12, 21, and 22, respectively. Here, $\kappa = 1.0$. Other parameters are as in Fig. 4.

persistence for optimal extraction—consistent with one component of the initial active force being misaligned with the target. In contrast, in region 21, where one force component is also oppositely oriented, a clear competition emerges between the active persistence and bath relaxation times. At small τ_s , activity dominates and longer persistence enhances work extraction or reduces work cost, whereas beyond τ_s^* , viscoelastic memory governs the response, rendering highly persistent motion detrimental to optimal performance. For larger κ , the crossover τ_s^* shifts to higher values, implying that odd tracers require relatively longer persistence to reach their optimal regime—a consequence of the delayed relaxation induced by antisymmetric mobility, which generates nondissipative circular drift and slows the approach to steady state. As shown in Fig. 9, when τ_A is comparable to τ_c , increasing τ_s generally elevates $\omega^*(t_f)$, consistent with the prior

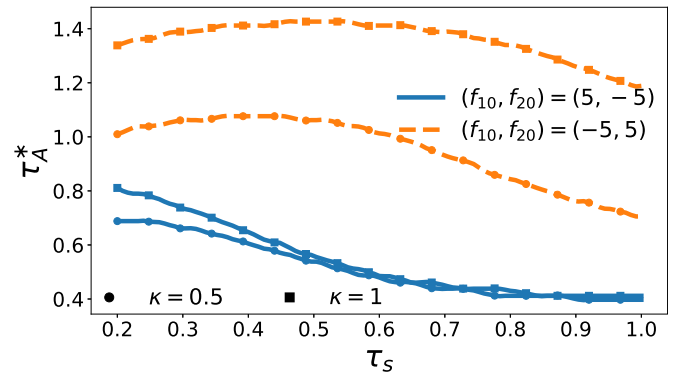


FIG. 8. The intermediate persistence time τ_A^* , at which the optimal mean work $\omega^*(t_f)$ attains its minimum [as shown in Figs. 7(b) and 7(c)], plotted as a function of the bath relaxation time τ_s for two different oddness parameters κ and two representative initial-force pairs corresponding to regions 12 and 21.

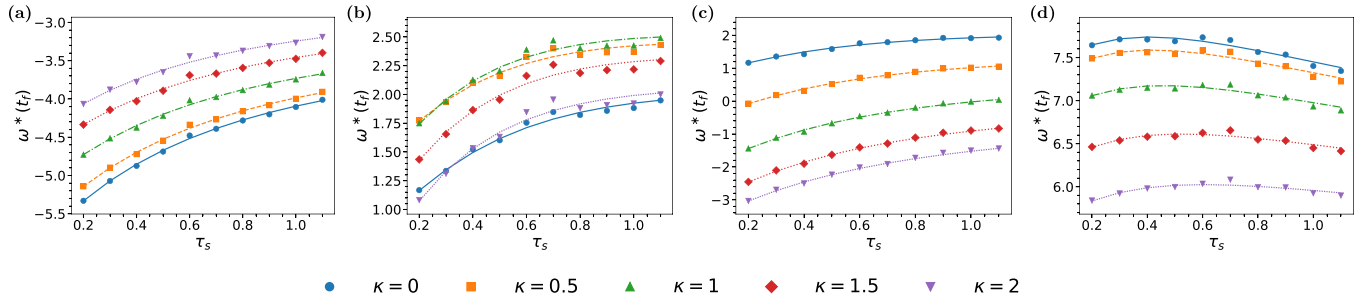


FIG. 9. Closed-loop optimal mean work $\omega^*(t_f)$ vs bath relaxation time τ_s for four initial-force pairs (f_{10}, f_{20}) . Results are shown for four representative initial-force pairs (f_{10}, f_{20}) : (a) $f_{10} = 5.0$, $f_{20} = 5.0$; (b) $f_{10} = 5.0$, $f_{20} = -5.0$; (c) $f_{10} = -5.0$, $f_{20} = 5.0$; and (d) $f_{10} = -5.0$, $f_{20} = -5.0$. Curves correspond to five values of the oddness parameter κ (see shared legend below). Other parameters are as in Fig. 4.

analysis. However, in region 22, the input work exhibits a nonmonotonic dependence on τ_s , reaching a maximum at an intermediate relaxation time of the bath.

To analyze how initial information about the active force (f_{10}, f_{20}) can be exploited during the driving, we compute the cumulative work $\omega^*(t)$, as shown in Fig. S5 of the Supplemental Material [106]. As in the open-loop case, $\omega^*(t)$ exhibits an initial jump at $t = 0^+$ due to the instantaneous change in the potential associated with the protocol, which effectively places the tracer in a favorable configuration. During the subsequent evolution, the role of the active force depends on its alignment with the target L_i . When the active force is aligned with L_i , self-propulsion assists the motion toward the predefined target (Strategy I). In contrast, when the force is oppositely aligned, the protocol follows Strategy II: The active force initially drives the system away from L_i (or toward it only if the initial jump overshoots the target) over a timescale $\sim \tau_A$, allowing work to be extracted during this interval, after which additional work must be supplied to reach the final configuration. This behavior, shown in Fig. S5 of the Supplemental Material [106], illustrates how measurement-based control can be designed so that activity assists the protocol, thereby reducing the energetic cost and enabling transient work extraction.

V. CONCLUSIONS

We investigated the optimal work control of an odd tracer confined in a rotationally asymmetric harmonic trap and embedded in a viscoelastic environment modeled by a Maxwell medium, focusing on the regime where the bath relaxation time τ_s is comparable to the tracer relaxation time τ_c . In open-loop control, as in the Newtonian case $\tau_s \rightarrow 0$, the optimal work ω^* is independent of the oddness parameter κ , which primarily influences the evolution of the individual protocol components $\lambda_i^*(t)$. However, in contrast to the Newtonian limit, the optimal protocol exhibits a nonlinear time dependence during the driving due to the finite bath relaxation time. In the Newtonian case, the protocol components satisfy TRS for both normal tracers ($\kappa = 0$) and odd tracers ($\kappa > 0$) [77]. In a viscoelastic bath ($\tau_s > 0$), while TRS is preserved for a normal tracer [16], we find that it is broken at the level of the individual protocol components $\lambda_i^*(t)$ for an odd tracer, even though the combined protocol $\Lambda^*(t) = \lambda_1^*(t) + \lambda_2^*(t)$ remains

time symmetric. This deviation from TRS is accompanied by an increase in the work input with increasing τ_s , reflecting the additional energetic cost associated with viscoelastic memory. Other trends persist from the Newtonian limit: Reducing the rotational asymmetry (i.e., smaller cross-coupling γ) consistently lowers the input work [77].

In closed-loop control—conditioned on the measured initial active force $\mathbf{f}_0 = (f_{10}, f_{20})$ —the optimal work is governed by the choice of strategy. When both components of $\mathbf{f}_0$ are aligned with the targets and exceed their respective thresholds (region 11), the system follows Strategy I, enabling net work extraction, although the magnitude of extraction decreases with increasing κ , consistent with the Newtonian case [77]. A key feature in the viscoelastic setting is the role of the bath relaxation timescale τ_s . As in the open-loop case, the extractable work (work input) generally decreases (increases) with increasing τ_s ; however, in region 22—where both components of the initial force are misaligned—the work cost is maximized at intermediate τ_s . Furthermore, while an optimal persistence time τ_A^* minimizing ω^* also exists in the Newtonian limit [77], in a viscoelastic bath its location depends sensitively on τ_s and the force configuration. For instance, in region 12, increasing viscoelasticity favors smaller persistence for optimal extraction, whereas in region 21, an intermediate τ_s promotes larger persistence, with shorter persistence becoming favorable again in both low and high τ_s limits.

Our results extend previous studies on optimal control in Newtonian fluids [9,32,77] to nonequilibrium baths with finite memory. Recent experiments on silica particles driven through wormlike micellar networks [16] already demonstrate comparable dynamics for normal tracers and offer an immediate testing ground for our predictions. Further experimental validation could be achieved using magnetically driven charged particles in dense bacterial baths, chiral Janus particles, or spinning microrobots confined by optical tweezers in micellar or polymeric solutions, or by tracking passive colloids immersed in suspensions of chiral active swimmers suspended in micellar environments [97,99,101,108]. More broadly, the principles uncovered here inform the design of microscale engines and soft robotic systems operating in cellular or biomolecular environments, where activity, broken symmetries, and memory jointly determine performance. Extensions to non-Maxwellian rheology—incorporating complex media such as gel networks, porous media, or

dynamically rearranging crowded environments—represent promising directions for future research [86,99,109].

ACKNOWLEDGMENTS

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DATA AVAILABILITY

The data that support the findings of this article are provided in the Supplemental Material [106], which contains supporting figures and details of the numerical optimization method discussed in Refs. [110,111]. The data generated using this method are openly available in Ref. [112].

APPENDIX A: DETAILED DERIVATION OF OPTIMAL VARIABLES AND PROTOCOLS

We denote ensemble averages by $\langle \cdot \rangle$ and define

$$x_i(t) = \langle X_i(t) \rangle, \quad y_i(t) = \langle Y_i(t) \rangle, \quad f_i(t) = \langle F_i(t) \rangle. \quad (\text{A1})$$

Using Eqs. (6) and (7) together with Eqs. (1) and (4), the dynamical equations for the average quantities can be expressed as

$$\begin{aligned} \tau_c \dot{x}_1(t) = & -\alpha_k[x_1(t) - \lambda_1(t) + \gamma x_2(t)] - \alpha[x_1(t) - y_1(t)] - \alpha_k[\kappa x_2(t) - \kappa \lambda_2(t) + \kappa \gamma x_1(t)] \\ & - \alpha[\kappa x_2(t) - \kappa y_2(t)] + \alpha[f_1(t) + \kappa f_2(t)], \end{aligned} \quad (\text{A2})$$

$$\begin{aligned} \tau_c \dot{x}_2(t) = & \alpha_k[\kappa x_1(t) - \kappa \lambda_1(t) + \kappa \gamma x_2(t)] + \alpha[\kappa x_1(t) - \kappa y_1(t)] - \alpha_k[x_2(t) - \lambda_2(t) + \gamma x_1(t)] \\ & - \alpha[x_2(t) - y_2(t)] - \alpha[\kappa f_1(t) - f_2(t)], \end{aligned} \quad (\text{A3})$$

$$\tau_s \dot{y}_1(t) = -y_1(t) + x_1(t), \quad \tau_s \dot{y}_2(t) = -y_2(t) + x_2(t). \quad (\text{A4})$$

If the initial components of the active force are measured as f_{10} and f_{20} at $t = 0$, then the average force evolves as

$$f_1(t) = f_{10}e^{-t/\tau_A}, \quad f_2(t) = f_{20}e^{-t/\tau_A}. \quad (\text{A5})$$

Combining Eqs. (A2) and (A3) and rearranging terms, the driving protocol $\lambda(t) = (\lambda_1, \lambda_2)^T$ is obtained from the resulting closed set of coupled equations,

$$\lambda_1(t) = k_r \tau_c [\dot{x}_1(t) - \kappa \dot{x}_2(t)] + (1 + k_r)x_1(t) + \gamma x_2(t) - k_r y_1(t) - k_r f_1(t), \quad (\text{A6})$$

$$\lambda_2(t) = k_r \tau_c [\kappa \dot{x}_1(t) + \dot{x}_2(t)] + \gamma x_1(t) + (1 + k_r)x_2(t) - k_r y_2(t) - k_r f_2(t). \quad (\text{A7})$$

Using Eq. (8), the mean work is then written as

$$\begin{aligned} \omega(t_f) = & k_0 \int_0^{t_f} dt' \dot{\lambda}_1(t') [\lambda_1(t') - x_1(t')] + k_0 \int_0^{t_f} dt' \dot{\lambda}_2(t') [\lambda_2(t') - x_2(t')] \\ = & \Delta \tilde{V} + k_I \int_0^{t_f} dt' [\tau_c \dot{x}_1(t')^2 + \tau_c \dot{x}_2(t')^2 - \dot{x}_1(t')y_1(t') - \dot{x}_2(t')y_2(t') - \dot{x}_1(t')f_1(t') - \dot{x}_2(t')f_2(t')], \end{aligned} \quad (\text{A8})$$

where

$$\begin{aligned} \tilde{V}(x_1, x_2, t_f) = & \frac{k_0}{2}(L_1 - x_1(t_f))^2 + \frac{k_0}{2}(L_2 - x_2(t_f))^2 + k_0 \gamma x_1(t_f)x_2(t_f) + \frac{k_I}{2}x_1(t_f)^2 + \frac{k_I}{2}x_2(t_f)^2, \\ \tilde{V}(x_1, x_2, 0) = & \frac{k_0}{2}x_{10}^2 + \frac{k_0}{2}x_{20}^2 + k_0 \gamma x_{10}x_{20} + \frac{k_I}{2}x_{10}^2 + \frac{k_I}{2}x_{20}^2, \quad \Delta \tilde{V} = \tilde{V}(x_1, x_2, t_f) - \tilde{V}(x_1, x_2, 0). \end{aligned} \quad (\text{A9})$$

Thus, we can define the Lagrangian as

$$\begin{aligned} \mathcal{L}(x_i, \dot{x}_i, y_i, \dot{y}_i, \Theta_i) = & \tau_c \dot{x}_1(t')^2 + \tau_c \dot{x}_2(t')^2 - \dot{x}_1(t')y_1(t') - \dot{x}_2(t')y_2(t') - \dot{x}_1(t')f_1(t') - \dot{x}_2(t')f_2(t') \\ & + \Theta_1(t)[\tau_s \dot{y}_1(t) + y_1(t) - x_1(t)] + \Theta_2(t)[\tau_s \dot{y}_2(t) + y_2(t) - x_2(t)], \end{aligned} \quad (\text{A10})$$

and apply the Euler-Lagrange equation [105]

$$\frac{d}{dt'} \left(\frac{\partial \mathcal{L}}{\partial \dot{z}_i} \right) - \frac{\partial \mathcal{L}}{\partial z_i} = 0,$$

where we collect the variables as $\mathbf{z}_i \equiv (x_i, y_i, \Theta_i)$, with Θ_i the corresponding Lagrange multipliers enforcing the constraints given by Eq. (A4). This yields the following equations:

$$2\tau_c \ddot{x}_1(t) - \dot{y}_1(t) + \frac{f_{10}}{\tau_A} e^{-\frac{t}{\tau_A}} + \Theta_1(t) = 0, \quad 2\tau_c \ddot{x}_2(t) - \dot{y}_2(t) + \frac{f_{20}}{\tau_A} e^{-\frac{t}{\tau_A}} + \Theta_2(t) = 0, \quad (\text{A11})$$

$$\dot{x}_1(t) + \tau_s \dot{\Theta}_1(t) - \Theta_1(t) = 0, \quad \dot{x}_2(t) + \tau_s \dot{\Theta}_2(t) - \Theta_2(t) = 0, \quad (\text{A12})$$

$$\tau_s \dot{y}_1(t) + y_1(t) - x_1(t) = 0, \quad \tau_s \dot{y}_2(t) + y_2(t) - x_2(t) = 0. \quad (\text{A13})$$

We impose the following boundary and initialization conditions:

$$x_1(0) = x_{10}, \quad x_2(0) = x_{20}, \quad y_1(0) = y_{10}, \quad y_2(0) = y_{20}, \quad \dot{x}_1(0) = c_1, \quad \dot{x}_2(0) = c_2, \quad \Theta_1(0) = c_3, \quad \Theta_2(0) = c_4.$$

The protocol endpoints are fixed, $\lambda_1(t_f) = L_1$ and $\lambda_2(t_f) = L_2$, while nonzero boundary jumps at $t = 0$ and t_f are allowed to admit nontrivial optimal trajectories.

With the above boundary conditions, the solutions of Eqs. (A11)–(A13) take the form

$$x_1^*(t) = \frac{x_{n1}(t)}{x_d}, \quad x_2^*(t) = \frac{x_{n2}(t)}{x_d}, \quad y_1^*(t) = \frac{y_{n1}(t)}{x_d}, \quad y_2^*(t) = \frac{y_{n2}(t)}{x_d}, \quad (\text{A14})$$

where

$$\begin{aligned} x_{n1}(t) = & 2\tau_c q_s Q_s \left((x_{10} - y_{10} + 2c_1 \tau_c + c_3 \tau_s)t - f_{10}t + f_{10}\tau_A + 2x_{10}\tau_c + (x_{10} + y_{10})\tau_s + c_3 \tau_s^2 \right) + 2\tau_c e^{-\frac{t}{\tau_A}} f_{10} \tau_A (\tau_c + \tau_s) q_s \\ & \times (\tau_s^2 - \tau_A^2) + 2\tau_s \tau_c q_s \cosh\left(\frac{q_s t}{\tau_c \tau_s}\right) [x_{10} Q_s - y_{10} Q_s - c_3 \tau_s Q_s - f_{10} \tau_s^2 \tau_A] + 2\tau_s \tau_c^2 \sinh\left(\frac{q_s t}{\tau_c \tau_s}\right) \\ & \times [-x_{10} Q_s + y_{10} Q_s + 2c_1 \tau_s Q_s - c_3 \tau_s Q_s + f_{10} \tau_s^3], \end{aligned} \quad (\text{A15})$$

$$\begin{aligned} x_{n2}(t) = & 2\tau_c q_s Q_s \left((x_{20} - y_{20} + 2c_2 \tau_c + c_4 \tau_s)t - f_{20}t + f_{20}\tau_A + 2x_{20}\tau_c + (x_{20} + y_{20})\tau_s + c_4 \tau_s^2 \right) + 2\tau_c e^{-\frac{t}{\tau_A}} f_{20} \tau_A (\tau_c + \tau_s) q_s \\ & \times (\tau_s^2 - \tau_A^2) + 2\tau_s \tau_c q_s \cosh\left(\frac{q_s t}{\tau_c \tau_s}\right) [x_{20} Q_s - y_{20} Q_s - c_4 \tau_s Q_s - f_{20} \tau_s^2 \tau_A] + 2\tau_s \tau_c^2 \sinh\left(\frac{q_s t}{\tau_c \tau_s}\right) \\ & \times [-x_{20} Q_s + y_{20} Q_s + 2c_2 \tau_s Q_s - c_4 \tau_s Q_s + f_{20} \tau_s^3], \end{aligned} \quad (\text{A16})$$

$$\begin{aligned} y_{n1}(t) = & 2\tau_c q_s Q_s \left((x_{10} - y_{10} + 2c_1 \tau_c + c_3 \tau_s)t - f_{10}t + f_{10}(\tau_A + \tau_s) + 2(x_{10}\tau_c + y_{10}\tau_s - c_1 \tau_c \tau_s) \right) - 2\tau_c e^{-\frac{t}{\tau_A}} f_{10} \tau_A^2 (\tau_A + \tau_s) \\ & \times (\tau_c + \tau_s) q_s + 2 \cosh\left(\frac{q_s t}{\tau_c \tau_s}\right) \tau_c^2 [-2q_s x_{10} Q_s + 2q_s y_{10} Q_s + 2c_1 \tau_s Q_s q_s + f_{10} \tau_s^2 q_s (\tau_A + \tau_s)] \\ & + 2 \sinh\left(\frac{q_s t}{\tau_c \tau_s}\right) \tau_c^2 [x_{10} Q_s (2\tau_c + \tau_s) - y_{10} Q_s (2\tau_c + \tau_s) - 2c_1 Q_s \tau_c \tau_s - \tau_s^2 c_3 Q_s - \tau_s^2 f_{10} [\tau_c \tau_s + \tau_A (\tau_c + \tau_s)]], \end{aligned} \quad (\text{A17})$$

$$\begin{aligned} y_{n2}(t) = & 2\tau_c q_s Q_s \left((x_{20} - y_{20} + 2c_2 \tau_c + c_4 \tau_s)t - f_{20}t + f_{20}(\tau_A + \tau_s) + 2(x_{20}\tau_c + y_{20}\tau_s - c_2 \tau_c \tau_s) \right) - 2\tau_c e^{-\frac{t}{\tau_A}} f_{20} \tau_A^2 (\tau_A + \tau_s) \\ & \times (\tau_c + \tau_s) q_s + 2 \cosh\left(\frac{q_s t}{\tau_c \tau_s}\right) \tau_c^2 [-2q_s x_{20} Q_s + 2q_s y_{20} Q_s + 2c_2 \tau_s Q_s q_s + f_{20} \tau_s^2 q_s (\tau_A + \tau_s)] \\ & + 2 \sinh\left(\frac{q_s t}{\tau_c \tau_s}\right) \tau_c^2 [x_{20} Q_s (2\tau_c + \tau_s) - y_{20} Q_s (2\tau_c + \tau_s) - 2c_2 Q_s \tau_c \tau_s - \tau_s^2 c_4 Q_s - \tau_s^2 f_{20} [\tau_c \tau_s + \tau_A (\tau_c + \tau_s)]], \end{aligned} \quad (\text{A18})$$

and

$$\tau_A^2 (\tau_c + \tau_s) - \tau_c \tau_s^2 = Q_s, \quad \sqrt{\tau_c (\tau_c + \tau_s)} = q_s, \quad x_d = 4 q_s^3 Q_s. \quad (\text{A19})$$

Substituting the expressions for x_1, x_2, y_1 , and y_2 from Eq. (A14) into the work expression given by Eq. (A8), we can compute the total work in terms of the undetermined coefficients c_1, c_2, c_3 , and c_4 . The optimal coefficients are then obtained by minimizing the work, i.e., by setting $\partial \omega / \partial c_i = 0$. Substituting these optimal coefficients back into Eq. (A14) yields the optimal trajectories x_1^*, x_2^*, y_1^* , and y_2^* , along with the corresponding optimal work $\omega^*(t_f)$ from Eq. (A8). Finally, the optimal protocols $\lambda_1^*(t)$ and $\lambda_2^*(t)$ are obtained using Eqs. (A6) and (A7).

APPENDIX B: CONDITIONAL INITIALIZATION OF $\mathbf{x}_0$ AND $\mathbf{y}_0$ IN CLOSED-LOOP CONTROL

If the initial active force is measured as $\mathbf{F}(0) = \mathbf{f}_0 = (f_{10}, f_{20})^T$, the initial means of the slow variables are fixed by Gaussian conditioning [113]. Denote the (unconditional) means by $\mu_{x0} = \langle \mathbf{X}(0) \rangle$, $\mu_{y0} = \langle \mathbf{Y}(0) \rangle$, $\mu_{f0} = \langle \mathbf{F}(0) \rangle$, and the

cross-covariances by

$$\mathbf{C}_{xf} = \langle \mathbf{X}(0) \mathbf{F}(0)^\top \rangle, \quad \mathbf{C}_{yf} = \langle \mathbf{Y}(0) \mathbf{F}(0)^\top \rangle, \quad \mathbf{C}_{ff} = \langle \mathbf{F}(0) \mathbf{F}(0)^\top \rangle.$$

Then, the conditional means are determined as [113]

$$\langle \mathbf{X}(0) | \mathbf{F}(0) = \mathbf{f}_0 \rangle = \boldsymbol{\mu}_{x0} + \mathbf{C}_{xf} \mathbf{C}_{ff}^{-1} (\mathbf{f}_0 - \boldsymbol{\mu}_{f0}), \quad \langle \mathbf{Y}(0) | \mathbf{F}(0) = \mathbf{f}_0 \rangle = \boldsymbol{\mu}_{y0} + \mathbf{C}_{yf} \mathbf{C}_{ff}^{-1} (\mathbf{f}_0 - \boldsymbol{\mu}_{f0}). \quad (\text{B1})$$

In the common zero-mean stationary case ($\boldsymbol{\mu}_{x0} = \boldsymbol{\mu}_{y0} = \boldsymbol{\mu}_{f0} = \mathbf{0}$),

$$\mathbf{x}_0 \equiv \langle \mathbf{X}(0) | \mathbf{f}_0 \rangle = \mathbf{C}_{xf} \mathbf{C}_{ff}^{-1} \mathbf{f}_0 = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}, \quad \mathbf{y}_0 \equiv \langle \mathbf{Y}(0) | \mathbf{f}_0 \rangle = \mathbf{C}_{yf} \mathbf{C}_{ff}^{-1} \mathbf{f}_0 = \begin{bmatrix} y_{10} \\ y_{20} \end{bmatrix}. \quad (\text{B2})$$

For an AOUP with $\mathbf{C}_{ff} = f_A^2 \mathbf{I}$ [cf. Eq. (5)], Eq. (B2) reduces to $\mathbf{x}_0 = (\mathbf{C}_{xf}/f_A^2) \mathbf{f}_0$ and $\mathbf{y}_0 = (\mathbf{C}_{yf}/f_A^2) \mathbf{f}_0$.

We work in the Laplace domain, $\tilde{g}(s) \equiv \int_0^\infty e^{-st} g(t) dt$. From Eqs. (A2) and (A3), one obtains the steady-state cross-covariances

$$\langle \tilde{x}_1(s) \tilde{f}_1(s) \rangle_{ss} = f_A^2 \frac{\tilde{S}_{11}}{\tilde{S}_0}, \quad \langle \tilde{x}_1(s) \tilde{f}_2(s) \rangle_{ss} = f_A^2 \frac{\tilde{S}_{12}}{\tilde{S}_0}, \quad \langle \tilde{x}_2(s) \tilde{f}_1(s) \rangle_{ss} = f_A^2 \frac{\tilde{S}_{21}}{\tilde{S}_0}, \quad \langle \tilde{x}_2(s) \tilde{f}_2(s) \rangle_{ss} = f_A^2 \frac{\tilde{S}_{22}}{\tilde{S}_0}, \quad (\text{B3})$$

where

$$\begin{aligned} \tilde{S}_{11} &= \tilde{S}_{22} = \alpha [\tau_s s + 1] ([\tau_s s + 1] [s \tau_c + 1 + k_r^{-1}] - 1), \\ \tilde{S}_{12} &= \alpha [\tau_s s + 1]^2 [\kappa \tau_c s - \gamma k_r^{-1}], \quad \tilde{S}_{21} = -\alpha [\tau_s s + 1]^2 [\kappa \tau_c s + \gamma k_r^{-1}], \\ \tilde{S}_0 &= \alpha [\tau_s s + 1]^2 ([s \tau_c + 1 + k_r^{-1}]^2 + [\kappa^2 (\tau_c s)^2 - \gamma^2 k_r^{-2}]) - 2\alpha [\tau_s s + 1] [s \tau_c + 1 + k_r^{-1}] + \alpha. \end{aligned} \quad (\text{B4})$$

Evaluating the cross-covariances at $s = 1/\tau_A$ and using Eq. (B2), the closed-loop initial means (x_{10}, x_{20}) conditioned on $\mathbf{f}_0 = (f_{10}, f_{20})^\top$ are given by

$$x_{10} = \frac{x_{11}}{T_0} f_{10} + \frac{x_{12}}{T_0} f_{20}, \quad x_{20} = \frac{x_{21}}{T_0} f_{10} + \frac{x_{22}}{T_0} f_{20}, \quad (\text{B5})$$

with coefficients (at $s = 1/\tau_A$)

$$x_{11} = x_{22} = k_r \tau_A [\tau_s + \tau_A] T_d, \quad x_{12} = k_r \tau_A [\tau_s + \tau_A]^2 [\kappa k_r \tau_c - \gamma \tau_A], \quad x_{21} = k_r \tau_A [\tau_s + \tau_A]^2 [-\kappa k_r \tau_c - \gamma \tau_A], \quad (\text{B6})$$

and

$$T_d = \tau_A^2 + \tau_A [\tau_s + k_r (\tau_s + \tau_c)] + k_r \tau_c \tau_s, \quad T_0 = T_d^2 + [\tau_s + \tau_A]^2 (k_r^2 \kappa^2 \tau_c^2 - \gamma^2 \tau_A^2). \quad (\text{B7})$$

By the same conditioning procedure applied to $\mathbf{y}$, the closed-loop initial means $\mathbf{y}_0 = (y_{10}, y_{20})^\top$ are

$$y_{10} = \frac{y_{11}}{T_0} f_{10} + \frac{y_{12}}{T_0} f_{20}, \quad y_{20} = \frac{y_{21}}{T_0} f_{10} + \frac{y_{22}}{T_0} f_{20}, \quad (\text{B8})$$

with coefficients

$$y_{11} = y_{22} = k_r \tau_A^2 T_d, \quad y_{12} = k_r \tau_A^2 [\tau_s + \tau_A] [\kappa k_r \tau_c - \gamma \tau_A], \quad y_{21} = k_r \tau_A^2 [\tau_s + \tau_A] [-\kappa k_r \tau_c - \gamma \tau_A]. \quad (\text{B9})$$

As a consistency check, setting $\tau_s = 0$ recovers the Newtonian limit. In this case, Eq. (B5) reduces to expressions consistent with those reported in Ref. [77]. One also obtains $x_{10} = y_{10}$ and $x_{20} = y_{20}$, as expected from the dynamical equations in Eq. (A4).

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