

Boson-anyon-fermion mapping and anyon construction in one dimension

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(Received 9 November 2024; revised 27 December 2024; accepted 27 May 2025; published 25 June 2025)

We establish an exact mapping between identical particles in one dimension with arbitrary exchange statistics, including bosons, anyons, and fermions, provided they share the same scattering length. This boson-anyon-fermion mapping facilitates the construction of anyons from a linear superposition of spatially symmetric and antisymmetric states. This scheme is general and has been demonstrated in a spin-1/2 Fermi gas, where both s- and p-wave bound states can be supported by manipulating spin channels. With a suitable symmetry-breaking field, these bound states are hybridized to form a fractional-wave molecule. The condensation of these molecules in a many-body system leads to anyonic superfluidity, characterized by fractional statistics upon spin exchange within a Cooper pair. These anyonic states can be detected through asymmetric momentum distributions for each spin with a chiral k^{-3} tail. Our results demonstrate the inadequacy of the contact interaction model for anyons in continuum and lattices, and have also suggested a convenient route for engineering fractional phases on the platform of ultracold atoms.

DOI: [10.1103/np63-xnh8](https://doi.org/10.1103/np63-xnh8)

The statistics and interactions of quantum particles are often closely related, in that different exchange symmetries determine distinct scattering properties. For instance, identical bosons and fermions are dominated by s- and p-wave scatterings, respectively, at low energy, resulting in completely different interaction forms. In one dimension (1D), the correlation between statistics and interaction is even more pronounced because particles can only exchange via collision. This leads to many unique properties of 1D systems, one of which is an exact mapping between identical bosons and fermions with short-range interactions. The boson-fermion mapping was initially established for hard-core bosons and noninteracting fermions [1], and later extended to general couplings with reversed role of coupling strengths [2]. Importantly, it reveals the equivalence of bosons and fermions in 1D under certain interaction conditions, providing a powerful tool for understanding universal quantum physics in two seemingly distinct systems.

Beyond bosons and fermions, quantum particles can also manifest as anyons with fractional statistics, characterized by a fractional phase α that interpolates between 0 (boson) and π (fermion) upon their position exchange. Over the past decades, anyon physics has generated significant interest ranging from fractional quantum Hall effects in electron materials [3,4] to non-Abelian braiding for topological quantum computations [5]. Recently, anyonic statistics have been ex-

perimentally realized in lattice systems using electric circuits [6] or ultracold atoms with density-dependent hoppings [7]. Theoretically, static and dynamical properties of 1D interacting anyons have been extensively studied in both continuum [8–15] and lattices [15–24].

Despite these developments, the exploration of anyon physics still encounters fundamental challenges. First, most existing studies have assumed a contact (δ -function) interaction for anyons based on Kundu's model [8], which is identical to that for bosons. Given the distinct statistics between bosons and anyons, this assumption is questionable. For example, fermions experience a completely different (p-wave) interaction compared with (s-wave) bosons due to their different statistics. As of now, it remains unclear how to properly describe the interaction of anyons, which should cover both bosons and fermions by varying phase α . With a proper description, one might be able to identify a more general relation between 1D systems with arbitrary statistics not limited to bosons and fermions [1,2]. Second, the experimentally realized anyons in lattices turn out to experience the same on-site interaction and on-site permutation relation as bosons [7]. Such an anyon-Hubbard model, sharing the same spirit as Kundu's model, fails to capture the crucial effect of fractional statistics on short-range collisions when two anyons approach each other. Therefore, on the experimental side, it is imperative to engineer anyon physics without lattices, such as in continuum or a single trap, where the interplay of fractional statistics and short-range interaction can be more transparently visualized. Unfortunately, a feasible scheme to engineer anyon without lattices is still lacking at the moment.

This work was designed to address these problems. We have identified a unified description of short-range interactions for all 1D systems with different statistics by utilizing the short-range boundary condition for pairwise collision on

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one side. Such a boundary condition is fully parametrized by scattering length (l), while the statistics (α) just determine the asymptotic wave function on the other side of the collision. Based on this description, we have established an exact mapping between 1D systems with arbitrary statistics, including bosons, anyons, and fermions, termed the boson-anyon-fermion (BAF) mapping. It shows that all these systems can be mapped to each other in both energy and real-space wave function as long as they have the same l . The BAF mapping suggests a convenient new way to construct anyons, that is, by directly superposing spatially symmetric (boson-like) and antisymmetric (fermion-like) states at the basic two-body level. The scheme is general and has been demonstrated in a spin-1/2 Fermi gas, which can support both s- and p-wave bound states in different spin channels. Under a proper symmetry-breaking field, these bound states are hybridized to form fractional-wave molecules. The condensation of these molecules further leads to anyonic superfluidity, characterized by fractional statistics upon spin exchange in a Cooper pair. These anyonic states can be detected through asymmetric momentum distributions for each spin with a chiral k^{-3} tail. Finally, we discuss possible ultracold systems to realize our scheme.

We start by considering two identical particles (x_1, x_2) moving in 1D with relative and center-of-mass (CoM) coordinates $x = x_2 - x_1$ and $R = (x_1 + x_2)/2$. Their wave function obeys

$$\Psi(R, -x) = e^{i\alpha} \Psi(R, x), \quad ; (x > 0), \quad (1)$$

where α determines the quantum statistics and has the period 2π . In this work we take half of a period $\alpha \in [0, \pi]$, since α and $-\alpha$ are just related by the reflection symmetry $x \leftrightarrow -x$. Boson, fermion, and anyon then, respectively, correspond to $\alpha = 0, \pi$, and $\in (0, \pi)$.

When we turn on short-range interactions, Ψ and/or its derivatives will develop certain discontinuities when two particles approach each other ($x \rightarrow 0^\pm$). At $x \rightarrow 0^+$, Ψ asymptotically behaves as

$$\Psi(R, x) \rightarrow f(R)(x - l), \quad ; (x \rightarrow 0^+), \quad (2)$$

where l is defined as the scattering length [25]. According to Eq. (1), Ψ at $x \rightarrow 0^-$ automatically follows

$$\Psi(R, x) \rightarrow f(R)e^{i\alpha}(-x - l), \quad ; (x \rightarrow 0^-). \quad (3)$$

Obviously, different statistics (α) lead to different behaviors of Ψ at short range, as shown in Fig. 1. For bosons ($\alpha = 0$), Ψ is continuous and well-defined at $x \rightarrow 0^\pm$ [Fig. 1(a)], and therefore the interaction of bosons simply follows a δ -function potential in s-wave channel:

$$U_s = g_s \delta(x). \quad (4)$$

For fermions ($\alpha = \pi$), Ψ is no longer continuous at $x \rightarrow 0^\pm$ but $\Psi' \equiv \partial\Psi/\partial x$ is [Fig. 1(c)], so their interaction can only occur in a p-wave channel:

$$U_p = g_p \partial_x \delta(x) \partial_x. \quad (5)$$

For anyons, however, neither Ψ nor Ψ' is well-defined at $x = 0^\pm$ [Fig. 1(c)], and therefore their mutual interaction cannot solely follow U_s or U_p . This is why the anyon model with

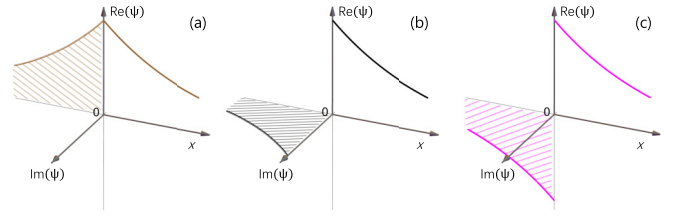


FIG. 1. Typical two-body wave functions Ψ as functions of relative coordinate $x \equiv x_2 - x_1$ in 1D for (a) bosons, (b) anyons with $\alpha = \pi/2$, and (c) fermions. For all cases, Ψ at $x > 0$ are taken to be real and share the same function. The shaded area highlights Ψ at $x < 0$.

a δ -function potential (identical to U_s) in previous studies is problematic.

As shown in Fig. 1, the interaction effect has actually been fully incorporated in the asymptotic behavior of Ψ on one side of the collision ($x > 0$), while Ψ on the other side, ($x < 0$) is just given by exchange statistics (α). It follows that one can simply utilize the short-range boundary condition on one side ($x > 0$) to describe the interaction effect. For bosons, this boundary condition is equivalent to applying U_s [Eq. (4)], and for fermions, it can also be related to U_p [Eq. (5)], given that the bare coupling g_p is renormalized [26]. Here we generalize this approach to arbitrary statistics: for a general 1D system of N particles with wave function $\Psi(x_1, \dots, x_N)$, we write the short-range boundary condition as

$$\lim_{x \equiv x_j - x_i \rightarrow 0^+} \left(\frac{1}{l} + \partial_x \right) \Psi(x_1, x_2, \dots, x_N) = 0. \quad (6)$$

The exchange statistics for a given α requires

$$\Psi(x_1, \dots, x_j, \dots, x_i, \dots, x_N) = e^{i\alpha w} \Psi(x_1, \dots, x_i, \dots, x_j, \dots, x_N), \quad (7)$$

where $w = \sum_{k=i+1}^j \epsilon(x_k - x_i) - \sum_{k=i+1}^{j-1} \epsilon(x_k - x_j)$, with $\epsilon(x) = 1$ for $x > 0$ and -1 for $x < 0$. Note that $\epsilon(x)$ is not defined at $x = 0$ due to the discontinuity of Ψ at this singular point [Fig. 1(b)]. This is different from previous studies of interacting anyons based on Kundu's model [8].

Equation (6) provides a unified description of short-range interaction for 1D systems with arbitrary statistics, where the scattering length l serves as the unique physical quantity to characterize interaction strength. Based on this description, we demonstrate here an exact mapping between 1D systems with different statistics (α), termed the *BAF mapping*: all α -systems can be mapped to each other in both energy and real-space wave function as long as they share the same l . When applied to two limiting cases $\alpha = 0, \pi$, it reproduces the boson-fermion mapping [1,2], where the reversed role of coupling strength is equivalent to requiring the same l for both systems.

A rigorous way to prove BAF mapping in the homogeneous case is through the Bethe ansatz solutions. As shown in the supplementary material [27], all α -systems share the same Bethe ansatz equation at a given l , thereby ensuring the same quasimomentum distribution and the same energy. This is in contrast to previous studies of anyons with δ -function potential, where the coupling constant in the Bethe ansatz equation is modified from that of bosons by a factor related to

α [8–12]. This modification can be attributed to the artificial definition of anyon wave function at singular point $x_i = x_j$. In comparison, here, we apply Eq. (6) to pairs of anyons coming close to each other but not exactly staying at the same site.

The BAF mapping can also apply to inhomogeneous systems with external potentials. A general proof is as follows: First, let us denote $\psi(x_1, x_2, \dots, x_N)$ as the wave function of N particles aligning in the region $x_1 < x_2 < \dots < x_N$. We can prove that all α -systems share the same ψ (up to a global phase) and the same energy E . This is because for all $x_i \neq x_{i+1}$, the system is simply governed by the noninteracting Hamiltonian, which is the same for all α -systems; whenever two neighboring particles come close ($x_{i+1} \rightarrow x_i + 0^+$), all systems satisfy the same boundary condition [Eq. (6)] with the same l . Therefore, all these systems have the same ψ and E , regardless of the value of α . Second, for any other region, the wave function can be related to ψ via Eq. (7), where α just determines their relative phase but does not change the eigen-energy of the system.

By revealing the intimate relation between anyonic and bosonic/fermionic 1D systems, the BAF mapping provides an important insight for constructing anyons. At the basic two-body level, this can be done by directly superposing the spatially symmetric (s-wave, boson-like) and antisymmetric (p-wave, fermion-like) wave functions under the same l , denoted by $\phi_s(x)$ and $\phi_p(x) = \epsilon(x)\phi_s(x)$, respectively, through

$$\phi_{\text{any}}(x) = \phi_s(x) - i \tan \frac{\alpha}{2} \phi_p(x). \quad (8)$$

By such superposition, ϕ_{any} satisfies anyonic statistics $\phi_{\text{any}}(-x) = e^{i\alpha} \phi_{\text{any}}(x)$ ($x > 0$). Eq. (8) implies that the interaction of anyons is also the superposition of U_s and U_p , where U_s (U_p) just takes effect on ϕ_s (ϕ_p) but not the other part. This is drastically different from Kundu's model with only contact (U_s) interaction, which is equivalent to a bosonic model with complicated double- δ potentials [8,15]. Generalized to lattice systems, the anyon interaction should involve both on-site (s-wave) and nearest-neighbor (p-wave) interactions, rather than simply the on-site one as in the anyon-Hubbard model [7,15,17–21,23]. Next, we show that anyons can be engineered in physical systems with realistic two-body potentials. The key challenge is to find both s- and p-wave states in a single system and meanwhile hybridize them with a pure imaginary coefficient ($\sim i \tan(\alpha/2)$).

Here we propose the spin-1/2 ($\uparrow, \downarrow$) system, where both spatially symmetric and antisymmetric states can be supported by manipulating the associated spin channels. Take spin-1/2 fermions for example, the s-wave interaction between $\uparrow$ and $\downarrow$ can support a spatially symmetric state in the spin-singlet channel:

$$\Psi_s(R, x) = \Phi_0(R) \phi_s(x) |\uparrow_1 \downarrow_2 - \downarrow_1 \uparrow_2\rangle / \sqrt{2}, \quad (9)$$

and the p-wave interaction supports a spatially antisymmetric state in spin-triplet channel:

$$\Psi_p(R, x) = \Phi_0(R) \phi_p(x) |\uparrow_1 \downarrow_2 + \downarrow_1 \uparrow_2\rangle / \sqrt{2}, \quad (10)$$

with Φ_0 the ground state of CoM motion. Under a suitable symmetry-breaking field, these two states can hybridize to result in anyonic states with fractional statistics, as schematically shown in Fig. 2(a).

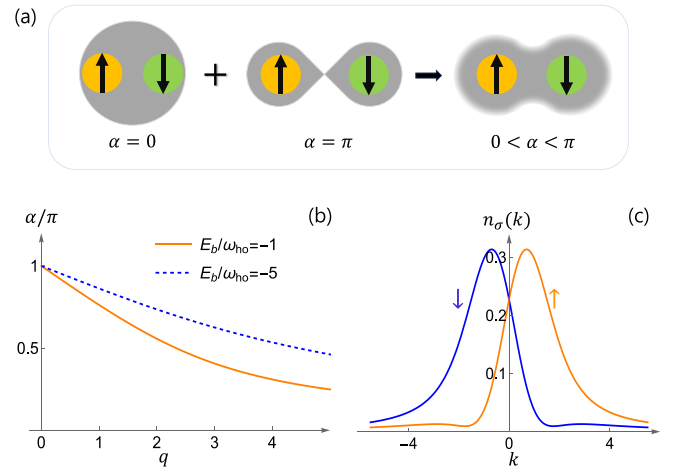


FIG. 2. (a) Schematics of the hybridization between spatially symmetric ($\alpha = 0$) and antisymmetric ($\alpha = \pi$) two-body states to achieve a fractional-wave molecule with fractional statistics ($0 < \alpha < \pi$). (b) Fractional phase α as a function of q (in unit of $1/l_{ho}$) for two fermions in a harmonic trap with equal s- and p-wave binding energies $E_b/\omega_{ho} = -5, -1$. (c) Momentum distribution of $\uparrow$ and $\downarrow$ spins for a fractional-wave molecule at $\alpha = \pi/2$ and $E_b/\omega_{ho} = -1$. Here n_σ and k are, respectively, in units of l_{ho} and $1/l_{ho}$.

To be concrete, we consider two fermions ($\uparrow, \downarrow$) in a harmonic trap, with frequency ω_{ho} and typical length $l_{ho} = (m\omega_{ho})^{-1/2}$. Further, we take the symmetry-breaking field as a weak spin-orbit coupling [28]:

$$V_{\text{soc}} = \Omega \sum_i [e^{iqx_i} \sigma_i^+ + e^{-iqx_i} \sigma_i^-], \quad (11)$$

where q is the transferred momentum and $\sigma^\pm = \sigma_x \pm i\sigma_y$ ($\sigma_{x,y,z}$ are Pauli matrices). Given the same positive s- and p-wave scattering length $l_s = l_p \equiv l > 0$, both s- and p-wave molecules can be supported with the same binding energy E_b . Based on the second-order perturbation theory, V_{soc} induces virtual excitations of these molecules (Ψ_s and Ψ_p) to other spin states ($|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$) at higher levels of relative and CoM motions [27]. This leads to an effective Hamiltonian $H_{\text{eff}} = v\mathcal{M}v^T$ with $v = (\Psi_s, \Psi_p)$ and

$$\mathcal{M} = -\frac{\Omega^2}{\omega_{ho}} \begin{pmatrix} A & iC \\ -iC & B \end{pmatrix}. \quad (12)$$

Here A, B, C are all real numbers that solely depend on E_b/ω_{ho} (or l/l_{ho}) and ql_{ho} . Crucially, V_{soc} gives rise to a purely imaginary off-diagonal coupling $\sim \pm iC$, which is necessary for anyon construction in Eq. (8). Finally, we arrive at the ground state wave function:

$$\Psi_G = \Phi_0(R) \left[\left(\phi_s - i \tan \frac{\alpha}{2} \phi_p \right) |\uparrow_1 \downarrow_2\rangle - \left(\phi_s + i \tan \frac{\alpha}{2} \phi_p \right) |\downarrow_1 \uparrow_2\rangle \right], \quad (13)$$

with

$$\tan \frac{\alpha}{2} \equiv \frac{1}{C} \left(\sqrt{\left(\frac{A-B}{2} \right)^2 + C^2} - \frac{A-B}{2} \right). \quad (14)$$

Remarkably, Ψ_G describes a fractional-wave molecule, in that the position exchange of $\uparrow$ and $\downarrow$ exactly gives rise to a fractional phase α [27].

Figure 2(b) shows how α varies with ql_{ho} for different molecule binding energies. Take $E_b/\omega_{ho} = -1$, for example, we can see a highly tunable α from π to $\sim\pi/4$ as increasing ql_{ho} from 0 to 5. Figure 2(c) further shows the spin-dependent momentum distribution $n_\sigma(k)$ for a fractional-wave molecule at $\alpha = \pi/2$ and $E_b/\omega_{ho} = -1$. These distributions exhibit strong asymmetry, that is, $n_\sigma(k) \neq n_\sigma(-k)$, and we have checked that such asymmetry is preserved for all fractional $\alpha \in (0, \pi)$ but not for purely s-wave ($\alpha = 0$) and p-wave ($\alpha = \pi$) cases. Similar asymmetries in the momentum distribution or transport dynamics of anyons have also been previously revealed [7, 11, 12, 14, 16, 19, 20, 23, 24]. In our case, despite the loss of symmetry for each spin, a spin-momentum combined symmetry is preserved as $n_\uparrow(k) = n_\downarrow(-k)$. Such combined symmetry is protected by the reflection symmetry of CoM motion, that is, $\Phi_0(R) = \Phi_0(-R)$ [27].

The physical origin of asymmetric $n_\sigma(k)$ can be seen more clearly from the free space case, where the wave functions of s- and p-wave molecules simply follow $\phi_s = e^{-\kappa|x|}$ and $\phi_p = \epsilon(x)e^{-\kappa|x|}$ ($\kappa = (m|E_b|)^{-1/2}$), and their Fourier transformations are (up to a common factor)

$$\phi_s(k) = \frac{\kappa}{k^2 + \kappa^2}, \quad \phi_p(k) = \frac{-ik}{k^2 + \kappa^2}. \quad (15)$$

The fractional-wave molecule in Eq. (13) can be created by

$$d^\dagger = \sum_k \left[\phi_s(k) - i \tan \frac{\alpha}{2} \phi_p(k) \right] c_{-k\uparrow}^\dagger c_{k\downarrow}^\dagger, \quad (16)$$

where $c_{k\sigma}^\dagger$ is to create a spin- σ fermion at momentum k . From Eqs. (15) and (16), we can obtain

$$n_\sigma(k) \sim \left(\frac{\kappa + \epsilon_\sigma \tan(\alpha/2)k}{k^2 + \kappa^2} \right)^2, \quad ; (\epsilon_\uparrow = 1, \epsilon_\downarrow = -1). \quad (17)$$

Now it is clear that asymmetric momentum distributions of anyons exactly come from the superposition of s- and p-wave states (with imaginary superposition coefficient), and therefore such asymmetry can be seen as a characteristic feature of fractional statistics. At high k , such asymmetry is reflected in the tail $\sim k^{-3}$, which has an opposite sign for different spins and thus is chiral. Note that this should be distinguished from $\sim k^{-3}$ tails in previous studies that are either due to CoM excitations [29] or appear in off-diagonal distributions [30].

In a many-body system, these fractional-wave molecules can condense, giving rise to superfluidity. Given $d^\dagger$ in Eq. (16) to create one such molecule [31], we write down a coherent state $\sim e^{\lambda d^\dagger}$ for its condensation, which is equivalent to a BCS-type wave function

$$|\Psi\rangle_{\text{ASF}} = \prod_k \left(1 + \lambda \left[\phi_s(k) - i \tan \frac{\alpha}{2} \phi_p(k) \right] c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger \right). \quad (18)$$

Remarkably, Eq. (18) represents an anyonic superfluidity (ASF) featuring fractional statistics upon spin exchange within a Cooper pair. Similar to a fractional-wave molecule, ASF exhibits asymmetric momentum distributions with chiral k^{-3} tail and spin-momentum combined symmetry.

The anyonic states could be realized in two-component Fermi gases with coexistent s- and p-wave interactions. For instance, a ^6Li Fermi gas with hyperfine states $|F = 1/2, m_F = 1/2\rangle$ and $|F = 3/2, m_F = -3/2\rangle$ undergoes a broad p-wave Feshbach resonance with low inelastic collision loss near $B \sim 225\text{G}$ [32], and meanwhile it has a large s-wave scattering length $a_s \sim 1000a_0$ (a_0 is the Bohr radius). Under confinement-induced resonances [33, 34], this system may achieve the same s- and p-wave scattering length in quasi-1D. Note that this condition is relaxed to the same E_b in the presence of an effective range. In such a quasi-1D geometry, ASF could be supported at low temperature when the phase fluctuations are suppressed [35]. Another potential system is alkali-earth atoms with ground state $^1\text{S}_0$ and long-lived $^3\text{P}_0$ states, which can experience s- and p-wave interactions in different orbital and spin channels, and the interaction strengths can be conveniently tuned by magnetic fields or confinements [36]. The application of spin-orbit coupling has also been very successful in ultracold gases of both alkali [28] and alkali-earth [37] atoms.

We emphasize that the anyon construction based on Eq. (8) is very general and not limited to the specific case of fermions with positive l . For instance, it can also be implemented in two-species bosons, or in the hard-core limit with $l_s = l_p = 0$. For the latter case, all spin states (singlet or triplet) have the same energy due to spin-charge separation, giving a natural situation for the coexistence of s- and p-wave states in Eq. (8). More details on constructing anyons in this case can be found in Ref. [27]. Moreover, spin-orbit coupling in Eq. (11) can also be replaced by other fields, as long as they produce a purely imaginary off-diagonal coupling as in Eq. (12).

In summary, we established a general mapping between short-range interacting 1D systems with arbitrary statistics, encompassing bosons, anyons, and fermions. This BAF mapping highlights the scattering length as the unique physical quantity to characterize the interaction effect. Inspired by this mapping, we can construct anyons without resorting to lattices, that is, by a proper superposition of boson- and fermion-like states. For demonstration, we successfully constructed a fractional-wave molecule and anyonic superfluidity in a spin-1/2 Fermi gas. These states feature fractional statistics upon spin exchange within a basic two-body unit (molecule or Cooper pair), and such statistics can be visualized by asymmetric spin distributions in momentum space. These results can be observed in ultracold experiments involving alkali or alkali-earth atoms.

Finally, we remark that the BAF mapping and its resultant anyon construction are unique to 1D systems. For higher dimensions, such as 2D or 3D, s- and p-wave states no longer have a good mapping as their short-range singularities are qualitatively different. For instance, in 3D, the relative wave function in the l -wave channel follows $\psi_l(r) \sim r^{-(2l+1)}$ at short distance r , and therefore it is impossible to achieve anyons by superposing different partial waves. In this regard, the 1D geometry provides a rare and surprisingly simple situation to simulate intriguing quantum physics with fractional statistics.

We thank Jiaming Li and Peng Zhang for useful discussions on potential experimental setups. This work is

supported by the National Natural Science Foundation of China (12074419, 12134015, 92476104), and the Strategic

Priority Research Program of the Chinese Academy of Sciences (XDB33000000).

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