

Universality and weak-ergodicity breaking in quantum quenches

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Understanding equilibration and universality after sudden quantum quenches remains a central challenge in isolated many-body systems. Persistent oscillations and anomalous scaling reported in lattice models appear to challenge the standard picture based on integrability and quantum-to-classical correspondence. Focusing on the quantum $O(n)$ model in the large- n limit, we show that these apparent anomalies originate from lattice effects and an underlying integrable structure. In particular, we map the postquench dynamics onto an integrable Neumann system and we derive an exact action-angle description valid in the thermodynamic limit. This framework explains both the convergence of long-time averages to the generalized Gibbs ensemble and the emergence of persistent oscillations associated with an isolated mode at the upper edge of the phonon spectrum. We further show that the expected thermal universality class is recovered in the appropriate scaling regime, while the oscillations disappear in the quantum-field-theory limit.

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I. INTRODUCTION

The approach to equilibrium in isolated quantum systems is a central problem in modern statistical mechanics [1]. While thermalization in classical systems is typically attributed to dynamical chaos [2,3], quantum systems evolve unitarily and do not admit a direct classical notion of chaos. Nevertheless, a statistical description of equilibration has been established through the eigenstate thermalization hypothesis [4–7], whose broad validity in interacting systems has been extensively investigated [8–15].

Quantum quenches provide a particularly powerful protocol to probe equilibration and universality in isolated many-body systems [16,17]. Beyond asymptotic thermalization, quenches often display universal dynamical scaling [18,19] and universal energy statistics [20,21], reflecting the universality class of the associated equilibrium critical point via the quantum-to-classical correspondence [22]. Integrable systems play a crucial role in this context: Although they do not thermalize due to the presence of infinitely many conserved quantities, they can equilibrate through dephasing—often termed kinematical chaos [23–25]—leading to relaxation toward a generalized Gibbs ensemble (GGE) [26–31]. The kinematical chaos hypothesis has been shown to be violated in systems with strong long-range interactions [32–34].

Other reports of persistent oscillations in the long-time dynamics, referred to as weak ergodicity breaking, have arisen in local lattice systems after quantum quenches [35–42]. In parallel, scaling behavior apparently incompatible with thermal universality has been observed in the large- n $O(n)$ model [43,44], while the applicability of the GGE in this setting has been questioned [45]. These findings would suggest the emergence of unconventional dynamical universality classes, intermediate between integrability and thermalization, or genuinely nonthermal critical behavior.

In this work, we show that these seemingly anomalous features admit a coherent and unified interpretation once lattice effects and the full dynamical structure of the theory are properly accounted for. Focusing on the $O(n)$ symmetric model in the large- n limit—a paradigmatic and exactly solvable setting for nonequilibrium field theory [22,46,47]—we demonstrate that the model is integrable and exhibits weak equilibration to a GGE—meaning that long-time averages converge to the GGE predictions—resolving the puzzle raised in Ref. [45]. We further show that the expected thermal universality class [48] is recovered in the appropriate dynamical scaling regime, while lattice-induced crossover effects account for the deviations reported in Refs. [43,44].

Most importantly, we uncover an action-angle description of the postquench dynamics in the thermodynamic limit: This structure naturally explains both the weak equilibration to the GGE and the presence of persistent oscillations, the latter being associated with a resonance at the upper edge of the phonon spectrum. This mechanism provides a physically transparent explanation of weak-ergodicity breaking in lattice systems and clarifies why such oscillations disappear in the quantum-field-theory limit, offering insights expected

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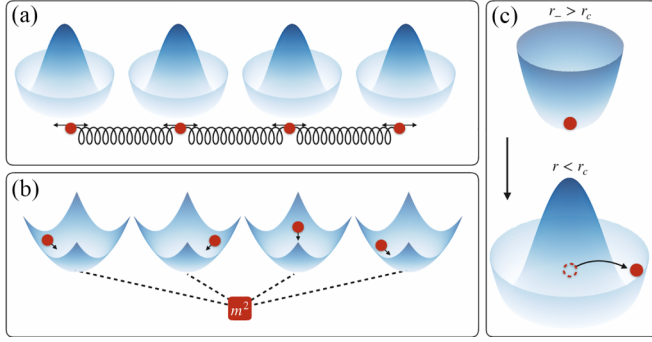


FIG. 1. (a) Sketch of the quantum $O(n)$ model for $r < 0$: The model consists of a chain of locally coupled bosons kept in place by a quartic potential. (b) Sketch of the large- n limit of the model: In this regime, the model can be described as a gas of phonons scattering through the time-dependent, self-consistent mass. (c) Sketch of the quench dynamics: Starting from the ground state, the system is driven out of equilibrium by a sudden quench of the bare mass $r^- \rightarrow r$ (here $r < 0$).

to remain relevant beyond the specific integrable model considered.

II. THE $O(n)$ QUANTUM ROTORS

Let us consider a d -dimensional (cubic) lattice with lattice spacing a and $N = (L/a)^d$ sites: For each site $\mathbf{j}$, we introduce an n -component vector variable $\Phi_{\mathbf{j}}$ and its conjugate momentum $\Pi_{\mathbf{j}}$, such that $[\Phi_{\mathbf{j}}^\alpha, \Pi_{\mathbf{j}'}^{\alpha'}] = i \delta_{\alpha, \alpha'} \delta_{\mathbf{j}, \mathbf{j}'}$, with $\alpha, \alpha' = 1, \dots, n$. The interaction is described by the $O(n)$ -symmetric quartic nearest-neighbor Hamiltonian

$$H = \sum_{\mathbf{j}} \frac{1}{2} \Pi_{\mathbf{j}}^2 + U(\Phi_{\mathbf{j}}^2) + \frac{1}{2a^2} \sum_{\langle \mathbf{j}, \mathbf{j}' \rangle} (\Phi_{\mathbf{j}} - \Phi_{\mathbf{j}'}^2), \quad (1)$$

where $U(\Phi^2) = r \Phi^2/2 + \lambda(\Phi^2)^2/(4n)$, with $\lambda > 0$ and r representing the bare mass of the theory [see Fig. 1(a)]. The Hamiltonian (1) represents a lattice discretization of the usual $O(n)$ bosonic field theory, which is recovered as $a \rightarrow 0$, $L = N^{1/d} a \rightarrow \infty$: In particular, for $r < 0$, depending on the dimensionality, the ground state of the model may undergo a transition between a massive ($\langle \Phi \rangle = 0$) and a massless ($\langle \Phi \rangle \neq 0$) phase [49], $\langle \cdot \rangle$ denoting the quantum expectation value.

The dynamics of the model for a ground-state quench is known to become tractable in the limit $n \rightarrow \infty$, due to two facts: (1) the ground state of the initial Hamiltonian is homogeneous and factorized over the component indices; (2) such a factorization is maintained at any finite time t [43,45,47], so that it is possible to replace $\Phi_{\mathbf{j}}^2 \Phi_{\mathbf{j}'}^\alpha \rightarrow n \langle \Phi_\alpha^2 \rangle \Phi_{\mathbf{j}}^\alpha$ in the Heisenberg equations of motion (e.o.m.) coming from Eq. (1).

As a result, in this limit, the system can be naturally described in terms of n identical phononic chains, in which each bosonic field α interacts with each other through a collective degree of freedom, which plays the role of an effective, dressed mass [see Fig. 1(b)]. In particular, in terms of the

Fourier-transformed bosons

$$\Phi_{\mathbf{k}} = \frac{1}{\sqrt{N}} \sum_{\mathbf{j}} e^{i\mathbf{k} \cdot \mathbf{j}} \Phi_{\mathbf{j}},$$

$$\Pi_{\mathbf{k}} = \frac{1}{\sqrt{N}} \sum_{\mathbf{j}} e^{-i\mathbf{k} \cdot \mathbf{j}} \Pi_{\mathbf{j}} \quad (2)$$

($[\Phi_{\mathbf{k}}, \Pi_{\mathbf{k}'}] = i \delta_{\mathbf{k}, \mathbf{k}'}$), one has

$$\dot{\Phi}_{\mathbf{k}} = \Pi_{\mathbf{k}}, \quad \dot{\Pi}_{\mathbf{k}} = -(m^2(t) + \omega_{\mathbf{k}}^2) \Phi_{\mathbf{k}}. \quad (3)$$

As the evolution of each component is decoupled, the index α has been dropped. Here, $\omega_{\mathbf{k}}^2 = 4a^{-2} \sum_{v=1}^d \sin^2(k_v a/2)$ is the lattice dispersion relation, while the effective mass is determined self-consistently by $m^2(t) = r + \lambda \langle \Phi^2 \rangle$ (see Fig. 1). In the large- N limit, $\mathbf{k}$ will become continuous in the interval $[-\pi/a, \pi/a]$, and so will $\omega_{\mathbf{k}}^2$ in the interval $[0, \Lambda]$, with $\Lambda \equiv 4da^{-2}$. It is worth noting that $m^2(t)$ need not be positive at all times, while $m^2(t) \geq 0$ for any stationary state.

In the ground state, $|\eta_{\mathbf{k}}^2|_{\text{gs}} = 1/2(m_{\text{gs}}^2 + \omega_{\mathbf{k}}^2)^{-1/2}$ and the renormalized mass is given by the self-consistent relation [47]

$$m_{\text{gs}}^2 = r_- + \frac{\lambda}{2N} \sum_{\mathbf{k}} \frac{1}{\sqrt{m_{\text{gs}}^2 + \omega_{\mathbf{k}}^2}} \quad (4)$$

(see also Appendix C). This allows us to link the quantum critical properties of the model to the infrared behavior of the sum in Eq. (4), for $m_{\text{gs}} = 0$ in the $N \rightarrow \infty$ limit. If such sum converges, then a (negative) critical value r_{gs}^c of the bare mass r exists such that $m_{\text{gs}} \rightarrow 0$ as $r \rightarrow r_{\text{gs}}^c$. This marks the onset of Bose-Einstein condensation in the $\mathbf{k} = 0$ mode. The critical scaling behavior can be characterized by the scaling of the renormalized mass, $m_{\text{gs}} \sim (r - r_{\text{gs}}^c)^\nu$. For $d \geq 3$, the system belongs to the mean-field universality class, with a correlation length exponent $\nu = 1/2$. As the dimension is lowered within the range $1 < d < 3$, the exponent acquires a dimensional dependence, $\nu = 1/(d-1)$, reflecting the increasing influence of fluctuations. In the limit $d \rightarrow 1$, the transition disappears as $r_{\text{gs}}^c \rightarrow 0$ [48].

We will consider the following quench protocol: For $t < 0$, we choose $r^- > 0$ and assume the system to be in the ground state of the Hamiltonian in Eq. (1) with $r = r^-$; then, at $t = 0$, the bare mass is quenched to its (possibly negative) final value r [see Fig. 1(c)]. For $t > 0$, the evolution of the system is described by Eq. (3). In turn, these can be simplified by parametrizing the second-order moments as $|\eta_{\mathbf{k}}|^2 \equiv \langle \Phi_{\mathbf{k}}^\dagger \Phi_{\mathbf{k}} \rangle$, $2\Re(\eta_{\mathbf{k}}^* p_{\mathbf{k}}) \equiv \langle \Pi_{\mathbf{k}} \Phi_{\mathbf{k}} + \text{H.c.} \rangle$, and $|p_{\mathbf{k}}|^2 \equiv \langle \Pi_{\mathbf{k}}^\dagger \Pi_{\mathbf{k}} \rangle$. Up to normalization, $\eta_{\mathbf{k}}$, $p_{\mathbf{k}}$ can also be seen as the components of the expansion of the fields $\Phi_{\mathbf{k}}$, $\Pi_{\mathbf{k}}$ in the basis of the ladder operators $a_{\mathbf{k}}$, $a_{-\mathbf{k}}$ (see, e.g., Ref. [45]). Then, the e.o.m. can be rewritten as

$$\dot{\eta}_{\mathbf{k}} = p_{\mathbf{k}}, \quad \dot{p}_{\mathbf{k}} = -(m^2 + \omega_{\mathbf{k}}^2) \eta_{\mathbf{k}}, \quad (5)$$

with

$$m^2 = r + \frac{\lambda}{N} \sum_{\mathbf{k}} |\eta_{\mathbf{k}}|^2. \quad (6)$$

Let us notice that, as in our protocol, both the e.o.m. and the initial condition only depend on $\omega_{\mathbf{k}}^2$, so that all the $\eta_{\mathbf{k}}$ corresponding to degenerate frequencies are in turn degenerate at

any t . The number of nondegenerate modes will be denoted by $\mathcal{N}$.

III. INTEGRABILITY

We now show that the semiclassical equations of motion arising in the large- n limit of the quantum $O(n)$ model can be mapped onto a classical integrable Neumann system. In particular, the e.o.m. in Eq. (5) describe the motion of the classical Hamiltonian

$$\mathcal{H} = \frac{1}{2} \sum_{\mathbf{k}} (|p_{\mathbf{k}}|^2 + (r + \omega_{\mathbf{k}}^2)|\eta_{\mathbf{k}}|^2) + \frac{\lambda}{4N} \left(\sum_{\mathbf{k}} |\eta_{\mathbf{k}}|^2 \right)^2. \quad (7)$$

As already noticed in Ref. [45], besides $\mathcal{H}$ itself, the $U(1)$ symmetry of the Hamiltonian results in the presence of a set of N -trivial integrals of motion, namely, $\ell_{\mathbf{k}} = \Im(\eta_{\mathbf{k}}^* p_{\mathbf{k}})$: Although these can be classically interpreted as angular momenta, at the quantum level their value is fixed by the energy of the quasiparticles in the initial state. Thus, the classical angular momentum cannot vanish but rather has to obey $\ell_{\mathbf{k}} \geq 1/2 \quad \forall \mathbf{k}$, due to the quantum zero-point energy [50]. As the number of (real) degrees of freedom is $2N$, these conserved quantities are not enough to make the dynamics integrable.

On the other hand, in the context of classical integrable systems, Hamiltonian (7) belongs to the Neumann class of integrable models [51–55], whose integrals of motion can be written in the form of dressed single-mode energies

$$\begin{aligned} \epsilon_{\mathbf{k}} = & \frac{1}{2} (|p_{\mathbf{k}}|^2 + (r + \omega_{\mathbf{k}}^2)|\eta_{\mathbf{k}}|^2) + \frac{\lambda}{4N} |\eta_{\mathbf{k}}|^2 \sum_{\mathbf{k}'} |\eta_{\mathbf{k}'}|^2 \\ & + \frac{\lambda}{4N} \sum_{\mathbf{k}'} \frac{|p_{\mathbf{k}}^2 \eta_{\mathbf{k}'}^2| + |p_{\mathbf{k}'}^2 \eta_{\mathbf{k}}^2| - 2\Re(\eta_{\mathbf{k}}^* p_{\mathbf{k}}) \Re(\eta_{\mathbf{k}'}^* p_{\mathbf{k}'})}{\omega_{\mathbf{k}'}^2 - \omega_{\mathbf{k}}^2}, \end{aligned} \quad (8)$$

where the primed sum symbol denotes the summation over all the modes nondegenerate with $\omega_{\mathbf{k}}$. Notice that $\mathcal{H} = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}}$ (see Appendix A for a proof). As shown in the following sections, this correspondence provides direct access to the thermodynamic limit of the integrable dynamics. In particular, it allows us to derive the properties of the generalized Gibbs ensemble and, more importantly, to construct an explicit action-angle description of the thermodynamic dynamics, from which we obtain the exact excitation spectrum. Such an action-angle description of the thermodynamic limit does not appear to have been previously derived in the literature on the classical Neumann model.

IV. GENERALIZED GIBBS ENSEMBLE

Since the quasiparticle spectrum $\omega_{\mathbf{k}}$ becomes continuous in the thermodynamic limit, it is natural to expect the long-time averages of observables such as $m^2(t)$, $|\eta_{\mathbf{k}}(t)|^2$, and $|p_{\mathbf{k}}(t)|^2$ to converge to the corresponding expectation values m_{gge}^2 , $|\eta_{\text{gge}}|^2$, and $|p_{\text{gge}}|^2$ predicted by the GGE. Within the GGE, these quantities can be related to the conserved charges $\epsilon_{\mathbf{k}}$ through the virial relation $|p_{\mathbf{k}}|_{\text{gge}}^2 = (m_{\text{gge}}^2 + \omega_{\mathbf{k}}^2)|\eta_{\text{gge}}|^2$, whose validity is proven in Appendix B. We will, however,

give here a physical interpretation: If we replace $m^2(t)$ with its (time-independent) GGE expectation value m_{gge}^2 , this relation corresponds to the virial theorem expected for a gas of noninteracting harmonic modes (phonons) with frequencies $\sqrt{\omega_{\mathbf{k}}^2 + m_{\text{gge}}^2}$. By exploiting this virial relation, we can thus rewrite $\epsilon_{\mathbf{k}}$ as

$$\frac{\epsilon_{\mathbf{k}}}{m_{\text{gge}}^2 + \omega_{\mathbf{k}}^2} = |\eta_{\mathbf{k}}|_{\text{gge}}^2 \left(1 + \frac{\lambda}{2N} \sum_{\mathbf{k}'}' \frac{|\eta_{\mathbf{k}'}|_{\text{gge}}^2}{\omega_{\mathbf{k}'}^2 - \omega_{\mathbf{k}}^2} \right) \quad (9)$$

where, again, the primed summation occurs over all the nondegenerate frequency modes. Equation (10) is valid to $O(N^{-1})$ terms, which are negligible in absence of macroscopic occupation of the finite k modes. By performing the sum of both sides over $\mathbf{k}$, the skew-symmetric sum in the right-hand side of Eq. (9) vanishes, so that one finds the self-consistent condition for m_{gge}^2 :

$$m_{\text{gge}}^2 = r + \frac{\lambda}{N} \sum_{\mathbf{k}} \frac{\epsilon_{\mathbf{k}}}{m_{\text{gge}}^2 + \omega_{\mathbf{k}}^2}, \quad (10)$$

in full agreement with the leading-order perturbative result in Refs. [47,56].

Assuming that $m_{\text{gge}}^2 \neq 0$, it is also possible to gain more information about the distribution of $|\eta_{\mathbf{k}}|_{\text{gge}}^2$ through the use of the generating function

$$\varphi_{\text{gge}}(u) = 1 - \frac{\lambda}{2N} \sum_{\mathbf{k}} \frac{|\eta_{\mathbf{k}}|_{\text{gge}}^2}{u - \omega_{\mathbf{k}}^2}. \quad (11)$$

Indeed, for any u at a finite distance from the spectral band $\{\omega_{\mathbf{k}}^2\}$ (i.e., $u \notin [0, \Lambda]$ for large N), one finds, up to $O(N^{-1})$ corrections,

$$\varphi_{\text{gge}}(u) = \sqrt{1 - \frac{\lambda}{N} \sum_{\mathbf{k}} \frac{\epsilon_{\mathbf{k}}}{m_{\text{gge}}^2 + \omega_{\mathbf{k}}^2} \frac{1}{u - \omega_{\mathbf{k}}^2}}, \quad (12)$$

which links the moment of the distribution of $|\eta_{\mathbf{k}}|_{\text{gge}}^2$ to those of the distribution of $\epsilon_{\mathbf{k}}$. In particular, as the spectrum becomes continuous in the thermodynamic limit, it is also possible to derive a closed expression for the distribution of $|\eta_{\mathbf{k}}|_{\text{gge}}^2$ (see Appendix B).

V. DYNAMICAL CRITICAL BEHAVIOR

When the sudden quench $r_- \rightarrow r$ from the ground state is performed, we expect the long-time averaged value $\overline{m^2} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T m^2(t) dt$ of the renormalized mass to coincide with its GGE value (10). It can be shown (see Appendix C) that for a ground-state quench,

$$r - r_- = (m_{\text{gge}}^2 - m_{\text{gs}}^2) \left(1 + \frac{\lambda}{4N} \sum_{\mathbf{k}} \frac{|\eta_{\mathbf{k}}|_{\text{gs}}^2}{(m_{\text{gge}}^2 + \omega_{\mathbf{k}}^2)} \right), \quad (13)$$

where $|\eta_{\mathbf{k}}|_{\text{gs}}^2 \equiv 1/2(m_{\text{gs}}^2 + \omega_{\mathbf{k}}^2)^{-1/2}$ are the ground-state amplitudes. It follows that $m_{\text{gge}} - m_{\text{gs}}$ always has the same sign as $r - r_-$, and, in particular, $m_{\text{gge}} = m_{\text{gs}}$ iff $r = r_- = r_{\text{gs}}^c$. Moreover, we find that the ground-state critical point ($r_- = r_{\text{gs}}^c$, $m_{\text{gs}} = 0$) is also the GGE critical point. By varying the postquench bare mass r in the region $r < r_-$, a critical point

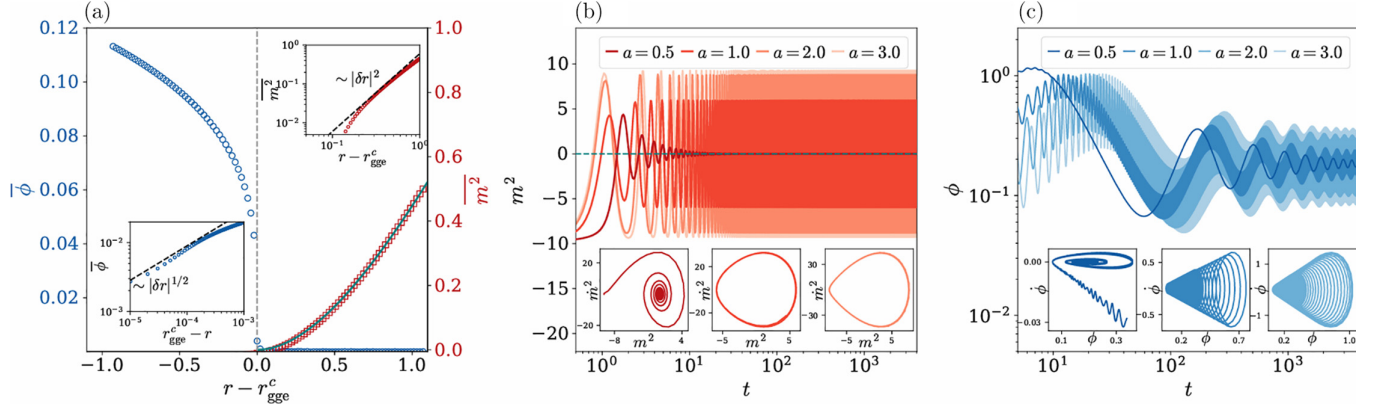


FIG. 2. (a) Out-of-equilibrium phase diagram. Blue dots represent the long-time average of the order parameter, $\phi = \eta_0/\sqrt{N}$ (right y axis), while red squares denote the squared mass, m^2 (left y axis), as functions of the final bare mass r . For $r < r_{\text{gge}}^c$, the system exhibits a finite order parameter ($\phi > 0$) and a vanishing mass ($m^2 = 0$), whereas for $r > r_{\text{gge}}^c$ the order parameter vanishes ($\phi = 0$) and the mass remains finite ($m^2 > 0$). The gray dashed line marks the GGE prediction for the dynamical critical point $r = r_{\text{gge}}^c$. The numerical results for the averaged squared mass (red squares) are also in excellent agreement with the GGE prediction (10) (green solid line). The insets display the scaling behavior of the averaged order parameter (blue dots, left inset) and of the averaged squared mass (red dots, right inset) as functions of $\delta r = |r - r_{\text{gge}}^c|$ in a log-log scale near the transition. The extracted critical exponents are consistent with the spherical-model universality class in $d = 3$: $\bar{m}^2 \sim |\delta r|^{2\nu}$, with $\nu = 1/(d-2) = 1$, and $\bar{\phi} \sim |\delta r|^\beta$, with $\beta = 1/2$. Deviations from the power-law scaling of the squared mass at very small δr arise from finite-size effects. The system size is $L = 5a \times 10^3$, the lattice spacing is $a = 3$, and the initial value of the bare mass is $r_- = 1 > r_{\text{gge}}^c$, $\lambda = 1$. (b) Time evolution of the squared mass $m^2(t)$ following a sudden quench from the disordered phase ($r_- = 1$) to the ordered phase ($r = -10$) in $d = 3$, for different lattice spacings a (curves in different shades). The system size is $L = 8a \times 10^3$, with $\lambda = 1$. In the field-theory limit ($a \rightarrow 0$), the dynamics converge to the GGE prediction (green dashed line). For any finite lattice spacing, $m^2(t)$ approaches persistent, undamped oscillations around the GGE prediction. For sufficiently small a , however, the asymptotic oscillation amplitude becomes extremely small and is barely visible on the scale of the figure. (c) Time evolution of the order parameter $\phi(t)$ for the same quench and system parameters as in panel (b). For finite lattice spacing, $\phi(t)$ displays fast, undamped oscillations superimposed on the standard oscillations around the potential minimum in the broken phase. The insets in panels (b) and (c) show the corresponding phase-space trajectories in $(m, \bar{m})$ and $(\phi, \bar{\phi})$ space, respectively.

r_{gge}^c emerges in the GGE equilibrium, such that the renormalized mass m_{gge} in Eq. (10) vanishes as $r \rightarrow r_{\text{gge}}^c$. For $r < r_{\text{gge}}^c$, the zero mode η_0 condenses, signaling the appearance of a spontaneous magnetization $\phi = \eta_0/\sqrt{N}$. Around the transition, the system exhibits dynamical scaling behavior, characterized by universal critical exponents $m_{\text{gge}} \sim (r - r_{\text{gge}}^c)^\nu$, $\phi_{\text{gge}} \sim (r_{\text{gge}}^c - r)^\beta$. From Eq. (10), we find that no symmetry-broken phase exists for dimensions $d \leq 2$, while $\nu = 1/(d-2)$ for $2 < d < 4$, which reaches the mean-field value $\nu = 1/2$ for $d \geq 4$. The scaling exponent of the order parameter remains dimension independent, with $\beta = 1/2$ [48,57].

Figure 2(a) presents the dynamical phase diagram of the model in $d = 3$ for initial states with $r_- > r_{\text{gs}}^c$. The solid green line shows the renormalized squared mass $\bar{m}_{\text{gge}}^2$, obtained by solving Eq. (10) as a function of the postquench bare mass r . The critical point r_{gge}^c is identified by the condition $\bar{m}_{\text{gge}}^2 = 0$. The empty squares represent numerical results from time evolution, where the long-time average of the squared mass is computed. These values collapse onto the GGE prediction, supporting the validity of the *kinematical chaos* hypothesis, which asserts that long-time averaged dynamics are governed by statistical ensembles.

The insets in Fig. 2(a) examine the scaling behavior of both the order parameter and the GGE mass. In the vicinity of the critical point, where $|r - r_{\text{gge}}^c| \ll 1$, the scaling exponents match those of the thermal universality class, consistent with

the dynamical quantum-to-classical correspondence [22]. For deeper quenches, a crossover to trivial scaling exponents $\nu = 1/2$ and $\beta = 1/4$ emerges, identified in Refs. [43,44]. This crossover can be attributed to the shrinking of the critical scaling window, which is bounded by the condition $m_{\text{gge}} \ll 3m_{\text{gs}}a^3/(8\pi^2)$ and becomes narrower for small values of a (see Appendix C for the details).

VI. WEAK-ERGODICITY BREAKING

Although the time-averaged squared mass $\bar{m}^2$ matches the GGE prediction, the full postquench dynamics from the disordered to the ordered phase depends on the lattice spacing a . For any finite a , $m(t)$ exhibits persistent, undamped oscillations around the GGE value, signaling weak-ergodicity breaking. As shown in Fig. 2(b), these oscillations become asymptotically periodic, with their amplitude vanishing as $a \rightarrow 0$. This behavior stems from a resonance at the edge of the quasiparticle spectrum and persists even in the thermodynamic limit ($N \rightarrow \infty$), challenging the expectations of the kinematical chaos hypothesis.

This resonance phenomenon is better understood in terms of Jacobi quasiparticles. The Jacobi coordinates are a set of separable coordinates [52] for the Hamiltonian (7), defined as the zeros of the generating function

$$\varphi(u) = 1 - \frac{\lambda}{2N} \sum_{\mathbf{k}} \frac{|\eta_{\mathbf{k}}|^2}{u - \omega_{\mathbf{k}}^2}, \quad (14)$$

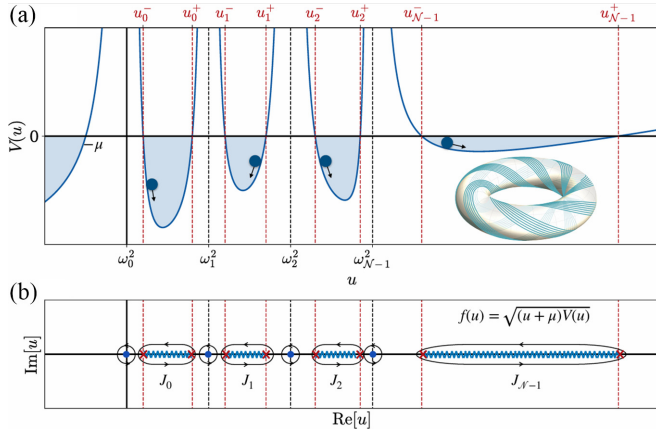


FIG. 3. (a) Jacobi quasiparticle picture: Each microscopic degree of freedom oscillates between the turning points $u_l^\pm$ of the potential $V(u)$ for $u > 0$. The negative zero $u = -\mu$ of $V(u)$ physically represents the effective GGE mass m_{gge}^2 . As $u_l^\pm$ are bounded by the bare frequencies ω_l^2 , as the spectrum becomes continuous for $N \rightarrow \infty$, the first $N-1$ Jacobi quasiparticles acquire a well-defined energy, while the oscillation of the N th quasiparticle beyond the edge of the spectral band has a finite support, leading to the emergence of a Higgs mode with persistent oscillations. (b) The set of action variables $\{J_l, \ell_l\}$ that constrain the dynamics on invariant tori (see inset) can be computed from the complex plane singularities of $f(u) = \sqrt{(u+\mu)V(u)}$, as integrals around its branch cuts and residues at the poles $\{\omega_l^2\}$, respectively. In the QFT limit ($a \rightarrow 0$), this allows the direct computation of the spectrum of the model and a proof of the equilibration to the GGE predictions.

analogous to that introduced in Eq. (11) for the GGE. Their peculiarity lies in the fact that u_l , together with the associated momenta P_l , form a set of separable coordinates of $\mathcal{H}$, as their motion is constrained to the zero-energy subspace of the effective Hamiltonian $\mathcal{H}_J = \lambda^2 P_l^2 / N^2 + V(u_l) = 0$. Here, $V(u)$ is the potential

$$V(u) = u + r - \frac{\lambda}{N} \sum_{\mathbf{k}} \frac{\epsilon_{\mathbf{k}}}{u - \omega_{\mathbf{k}}^2} + \frac{\lambda^2}{4N^2} \sum_{\mathbf{k}} \frac{\ell_{\mathbf{k}}^2}{(u - \omega_{\mathbf{k}}^2)^2} \quad (15)$$

(see Appendix D for a derivation of these properties). Since the motion is confined to the region where $V(u) < 0$ and $u > 0$, the positive zeros of $V(u)$ define the turning points of each coordinate u_l , denoted as $u_l^\pm$. The quantum condition $\ell_{\mathbf{k}} \geq 1/2$ ensures that these coordinates remain bounded. For $l \neq N-1$, the turning points satisfy $\omega_l^2 < u_l^- < u_l^+ < \omega_{l+1}^2$, while the final pair $u_{N-1}^\pm$ lies beyond the band edge and is only bounded from below by the ultraviolet cutoff $\omega_{N-1}^2 \equiv \Lambda$, as shown in Fig. 3(a). The potential also has a negative zero at $u = -\mu < 0$. In the thermodynamic limit ($N \rightarrow \infty$), the spectrum becomes continuous: $|\omega_l - \omega_{l+1}| \approx 1/(\rho(\omega_l^2)N)$ and the quasiparticles u_l for $l = 0, \dots, N-2$ localize around well-defined energies. However, the behavior of the final quasiparticle u_{N-1} depends on whether the limit is taken at finite or vanishing lattice spacing. In either of these limits, we can thus derive an integral expression for the potential $V(u)$. In particular, for $u \notin [0, \Lambda]$ (i.e., far from the spectral band) one has that $V(u)$ approaches the regular

Hilbert transform

$$V_{\text{reg}}(u) = u + r - \lambda \int_0^\Lambda \frac{du' \rho(u') \epsilon(u')}{u - u'}, \quad (16)$$

where $\rho(u')$ is the density of states of the phononic modes in the interval $u' \in [0, \Lambda]$. On the other hand, for $\omega_\ell < u < \omega_{\ell+1}$ we have to take into account the singular structure of the potential. It is thus convenient to parametrize $u_l = \omega_l^2 + \theta_l / (\pi \rho(\omega_l^2) N)$, with $\theta_l \in [0, \pi]$. In the new variable θ_ℓ , $V(u_l) \equiv V_l(\theta)$ takes the (local) universal form

$$V_l(\theta) = V_{\text{reg}}(\omega_l^2) - \frac{\lambda \pi \rho(\omega_l^2) \epsilon_l}{\tan \theta} + \frac{(\lambda \pi \rho(\omega_l^2) \ell_l)^2}{4 \sin^2 \theta} \quad (17)$$

(see Appendix E for a proof).

The Jacobi particle formalism allows us to derive explicit expressions for the excitation spectrum $\{\Omega_{\mathbf{k}}\}$ in the thermodynamic limit. Indeed, the spectrum of the model can be linked to the fundamental frequencies of the classical Hamiltonian $\mathcal{H}$. In turn, this can be obtained by expressing $\mathcal{H}$ in terms of the action variables

$$J_l = \oint \frac{du_l}{2\pi} P_l(u_l) = \frac{N}{\pi \lambda} \int_{u_l^-}^{u_l^+} du \sqrt{-V(u)}, \quad (18)$$

as $\Omega_l = \partial \mathcal{H} / \partial J_l$. In the next section, we will discuss the result both in the quantum-field-theoretical (QFT) limit ($a \rightarrow 0$, $N a^d \rightarrow \infty$) and on the lattice ($N \rightarrow \infty$, $a = \text{const}$).

A. Spectrum in the QFT limit

In the limit where the lattice spacing vanishes ($a \rightarrow 0$), the excitation spectrum $\{\Omega_{\mathbf{k}}\}$ takes a strikingly simple form, namely,

$$\Omega_{\mathbf{k}} = \sqrt{m_{\text{gge}}^2 + \omega_{\mathbf{k}}^2}, \quad (19)$$

where m_{gge}^2 is fixed by $\epsilon_{\mathbf{k}}$ through Eq. (10). Before examining the formal proof of Eq. (19), let us notice that the expression has a straightforward interpretation: The model can be seen as a relativistic gas of free bosons in which the mass m_{gge} is not an external parameter, but rather it is fixed by the initial conditions. Therefore, the QFT regime adheres strictly to the kinematical chaos hypothesis [23], which posits strong equilibration to the GGE.

Although the technical details will be given in Appendix F, let us now examine briefly the origin of Eq. (19). The main idea behind the argument is that in the QFT regime one can avoid dealing with the explicit expressions (16) and (17) of the potential $V(u)$ in the thermodynamic limit. Rather, one can exploit the relatively simple analytic structure for finite N and take the $N \rightarrow \infty$ limit only at the end.

To this aim, we introduce the auxiliary function $f(u) = \sqrt{(u + m_{\text{gge}}^2)V(u)}$ and study its singularities on the complex plane. Let us notice that for large N , $V(-m_{\text{gge}}^2) = 0$ as the last term in Eq. (15) is $O(N^{-1}) \forall u \notin [0, \Lambda]$. As a consequence, this choice ensures that $f(u)$ is analytic for $\Re u < 0$ half plane. On the other hand, the residues at the poles located at $u = \omega_l^2$ are linked to ℓ_l , while contour integrals around the branch cuts of $f(u)$ can be directly related to the classical action variables (18), by exploiting the fact that the support of each integral becomes of $O(N^{-1})$. Finally, the residue of $f(u)$ at $u = \infty$

can be linked to the Hamiltonian $\mathcal{H} = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}}$, thus allowing us to express the Hamiltonian in terms of the action variable. The schematic idea of the procedure is illustrated in Fig. 3(b).

B. Spectrum on the lattice

Also for finite a , the phonon dispersion ω_l^2 becomes continuous in the thermodynamic limit. Then, the width of each oscillation, $u_l^+ - u_l^-$ for $l = 0, \dots, \mathcal{N} - 2$, scales as $O(N^{-1})$. However, the final mode $u_{\mathcal{N}-1}$ lies beyond the band edge Λ and retains a finite support, resulting in an isolated frequency that persists in the continuum limit $N \rightarrow \infty$, thus hindering the idea behind the complex-analysis argument exposed before.

As a consequence, the final expression of $\{\Omega_l\}$ and its derivation become much more cumbersome and will be left entirely to Appendix G. Luckily, we can also provide here a transparent physical interpretation of our result: The isolated mode corresponding to $u_{\mathcal{N}-1}$ is indeed responsible for long-lived oscillations in the system, as its characteristic frequency $\Omega_{\mathcal{N}-1} = \partial\mathcal{H}/\partial J_{\mathcal{N}-1}$, remains isolated in the thermodynamic limit, thus surviving the dephasing process. As explicitly demonstrated in Ref. [58], similar synchronization phenomena arise due to singularities in the quasiparticle spectrum and are, therefore, absent when the momentum space integrals are smoothly regularized e.g. with a Gaussian or exponential regularization [59,60].

As $m^2(t)$ is thus periodic at large times, it acts as a Floquet driving in the equations of motion (5): The continuous part of the spectrum ($\{\Omega_l\}$ $l = 0, \dots, \mathcal{N} - 2$) can thus be seen as the Floquet spectrum arising from this driving.

VII. CONCLUSIONS

In this work, we identify a general dynamical mechanism underlying weak-ergodicity breaking after quantum quenches: a resonance at the upper edge of the quasiparticle band. Using the large- n limit of the $O(n)$ -symmetric model as a paradigmatic and exactly solvable setting, we show that persistent oscillations are a high-energy, lattice-induced phenomenon, which vanish as the lattice spacing a —or equivalently, the ultraviolet cutoff Λ —is removed. The relevance of this mechanism to ergodicity breaking observed in numerical simulations [35,37,40] can be tested by analyzing how the amplitude and frequency of oscillations depend on the lattice spacing.

Within the same framework, we provide an explicit demonstration of GGE thermalization in this model, resolving the puzzle posed in Ref. [45]. In the quantum-field-theory limit ($N, \Lambda \rightarrow \infty$), the relevant observables equilibrate to the GGE in a strong sense, while for finite lattice spacing they display persistent oscillations around the GGE averages [Figs. 2(b) and 2(c)]. We further identify the dynamical critical scaling regime and show that it belongs to the thermal universality class, in agreement with the quantum-to-classical correspondence [22].

An important open question concerns the fate of persistent oscillations beyond the integrable limit. While perturbative approaches suggest undamped oscillations even at long times [61–63], other studies predict a finite damping that increases

with quench strength [64]. In our case, oscillations are protected by integrability, but incorporating $1/n$ corrections is expected to clarify their stability and may enable the exploration of more general dynamical critical phenomena, such as symmetry rebreaking recently observed in ϕ^4 theories [65].

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DATA AVAILABILITY

The data that support the findings of this article are openly available [66].

APPENDIX A: INTEGRALS OF MOTION

We will now show that the quantities $\epsilon_{\mathbf{k}}$ defined in Eq. (8) are actually integrals of motion for the Hamiltonian evolution (5). First, it is convenient to rewrite $\epsilon_{\mathbf{k}}$ as

$$\epsilon_{\mathbf{k}} = \frac{1}{2} (p_{\mathbf{k}}^*, \eta_{\mathbf{k}}^*) \begin{pmatrix} U_{\mathbf{k}} & -Q_{\mathbf{k}} \\ -Q_{\mathbf{k}} & T_{\mathbf{k}} \end{pmatrix} \begin{pmatrix} p_{\mathbf{k}} \\ \eta_{\mathbf{k}} \end{pmatrix}, \quad (\text{A1})$$

where

$$\begin{aligned} U_{\mathbf{k}} &= 1 + \frac{\lambda}{2N} \sum_{\mathbf{k}'}' \frac{|\eta_{\mathbf{k}'}|^2}{\omega_{\mathbf{k}'}^2 - \omega_{\mathbf{k}}^2}, \\ T_{\mathbf{k}} &= r + \omega_{\mathbf{k}}^2 + \frac{\lambda}{2N} \sum_{\mathbf{k}'}' \frac{|p_{\mathbf{k}'}|^2}{\omega_{\mathbf{k}'}^2 - \omega_{\mathbf{k}}^2} + \frac{\lambda}{2N} \sum_{\mathbf{k}'} |\eta_{\mathbf{k}'}|^2, \\ Q_{\mathbf{k}} &= \frac{\lambda}{2N} \sum_{\mathbf{k}'}' \frac{\Re(\eta_{\mathbf{k}'}^* p_{\mathbf{k}'})}{\omega_{\mathbf{k}'}^2 - \omega_{\mathbf{k}}^2}. \end{aligned} \quad (\text{A2})$$

The evolution of $U_{\mathbf{k}}, Q_{\mathbf{k}}, T_{\mathbf{k}}$ is described by a similar set of closed ordinary differential equations, namely,

$$\begin{aligned} \dot{U}_{\mathbf{k}} &= 2Q_{\mathbf{k}}, \\ \dot{T}_{\mathbf{k}} &= -2(m^2 + \omega_{\mathbf{k}}^2)Q_{\mathbf{k}} + \frac{\lambda \nu_{\mathbf{k}}}{2N} \Re(\eta_{\mathbf{k}}^* p_{\mathbf{k}}), \\ \dot{Q}_{\mathbf{k}} &= T_{\mathbf{k}} - (m^2 + \omega_{\mathbf{k}}^2)U_{\mathbf{k}} + \frac{\lambda \nu_{\mathbf{k}}}{2N} |\eta_{\mathbf{k}}|^2, \end{aligned} \quad (\text{A3})$$

$\nu_{\mathbf{k}}$ being the degeneracy of the mode $\omega_{\mathbf{k}}$, so that

$$\dot{\epsilon}_{\mathbf{k}} = \frac{\lambda \nu_{\mathbf{k}}}{4N} (p_{\mathbf{k}}^*, \eta_{\mathbf{k}}^*) \begin{pmatrix} 0 & -|\eta_{\mathbf{k}}|^2 \\ -|\eta_{\mathbf{k}}|^2 & 2\Re(\eta_{\mathbf{k}}^* p_{\mathbf{k}}) \end{pmatrix} \begin{pmatrix} p_{\mathbf{k}} \\ \eta_{\mathbf{k}} \end{pmatrix} = 0. \quad (\text{A4})$$

APPENDIX B: GGE ENSEMBLE

In this Appendix, we will analyze the properties of the GGE equilibrium ensemble, namely,

$$P(p, \eta) = \frac{1}{Z} e^{-\sum_{\mathbf{k}} \beta_{\mathbf{k}} \epsilon_{\mathbf{k}}(p, \eta) + \varpi_{\mathbf{k}} \ell_{\mathbf{k}}(p, \eta)}, \quad (\text{B1})$$

where the Lagrange multipliers $\beta_{\mathbf{k}}$, $\varpi_{\mathbf{k}}$ are fixed by the values of the integrals of motion. In the following, we will use the notation $A(p, \eta)_{\text{gge}}$ to represent the average of the function $A(p, \eta)$ over the probability distribution (B1).

1. Virial relation

We will now show that the virial condition $|p_{\mathbf{k}}^2|_{\text{gge}} = (m_{\text{gge}}^2 + \omega_{\mathbf{k}}^2)|\eta_{\mathbf{k}}^2|_{\text{gge}}$ actually holds. We start by rescaling the momenta by a factor $p_{\mathbf{k}} = \sqrt{\mu + \omega_{\mathbf{k}}^2} \tilde{p}_{\mathbf{k}}$ for some $\mu > 0$. We find that

$$\begin{aligned} \ell_{\mathbf{k}}(\tilde{p}, \eta) &= \sqrt{\mu + \omega_{\mathbf{k}}^2} \Im(\eta_{\mathbf{k}}^* \tilde{p}_{\mathbf{k}}), \\ \epsilon_{\mathbf{k}}(\tilde{p}, \eta) &= \frac{1}{2}(\mu + \omega_{\mathbf{k}}^2) \left(|\tilde{p}_{\mathbf{k}}|^2 + |\eta_{\mathbf{k}}|^2 + \frac{\lambda}{2N} \right. \\ &\quad \times \sum_{\mathbf{k}'} \left. \frac{|\tilde{p}_{\mathbf{k}'}^2 \eta_{\mathbf{k}}^2| + |\tilde{p}_{\mathbf{k}}^2 \eta_{\mathbf{k}'}^2| - 2\alpha_{\mathbf{k}, \mathbf{k}'} \Re(\eta_{\mathbf{k}}^* \tilde{p}_{\mathbf{k}'} \eta_{\mathbf{k}'}^* \tilde{p}_{\mathbf{k}})}{\omega_{\mathbf{k}'}^2 - \omega_{\mathbf{k}}^2} \right) \\ &\quad + \frac{1}{2} \left(\mu - r - \frac{\lambda}{2N} \sum_{\mathbf{k}'} (|\eta_{\mathbf{k}'}|^2 + |\tilde{p}_{\mathbf{k}'}|^2) \right) |\eta_{\mathbf{k}}|^2, \end{aligned} \quad (\text{B2})$$

with $\alpha_{\mathbf{k}, \mathbf{k}'} = \sqrt{(\mu + \omega_{\mathbf{k}}^2)(\mu + \omega_{\mathbf{k}'}^2)}$. For $N \rightarrow \infty$, the fluctuations of the quantity within brackets in the last line are suppressed, so that we can replace it with its ensemble average. By choosing thus

$$\mu = r + \frac{\lambda}{2N} \sum_{\mathbf{k}'} (|\eta_{\mathbf{k}'}^2|_{\text{gge}} + |\tilde{p}_{\mathbf{k}'}^2|_{\text{gge}}), \quad (\text{B3})$$

we have that the probability measure becomes symmetric under the exchange $\eta_{\mathbf{k}} \leftrightarrow \tilde{p}_{\mathbf{k}}$. As a consequence, $|\tilde{p}_{\mathbf{k}}^2|_{\text{gge}} = |\eta_{\mathbf{k}}^2|_{\text{gge}}$ and

$$\mu = r + \frac{\lambda}{N} \sum_{\mathbf{k}'} |\eta_{\mathbf{k}'}^2|_{\text{gge}} = m_{\text{gge}}^2 \quad (\text{B4})$$

so that, finally, coming back to the original variables, one has the virial condition $|p_{\mathbf{k}}^2|_{\text{gge}} = (m_{\text{gge}}^2 + \omega_{\mathbf{k}}^2)|\eta_{\mathbf{k}}^2|_{\text{gge}}$.

Through the use of the virial relation we just proved, we can straightforwardly obtain Eqs. (9) and (10). Indeed, replacing the relation into the definition of the integrals of motion, we have

$$\epsilon_{\mathbf{k}} = (m_{\text{gge}}^2 + \omega_{\mathbf{k}}^2) |\eta_{\mathbf{k}}^2|_{\text{gge}} \left(1 + \frac{\lambda}{2N} \sum_{\mathbf{k}'} \frac{|\eta_{\mathbf{k}'}^2|_{\text{gge}}}{\omega_{\mathbf{k}'}^2 - \omega_{\mathbf{k}}^2} \right) - \frac{\lambda}{4N} |\eta_{\mathbf{k}}^4|_{\text{gge}}. \quad (\text{B5})$$

In the following, we will assume that $m_{\text{gge}} > 0$ so that $|\eta_{\mathbf{k}}^2|_{\text{gge}} = O(1)$ and the last term can be neglected for $N \rightarrow \infty$, so that we obtain Eq. (9).

2. Mode distribution

We want now to derive an expression for $|\eta_{\mathbf{k}}^2|_{\text{gge}}$ by inverting Eq. (12). Let us notice that Eq. (12) is defined for any u far from the spectral band $\{\omega_{\mathbf{k}}^2\}$ (i.e., $u \notin [0, \Lambda]$ for large N). In the $N \rightarrow \infty$ limit, the spectral band becomes a continuum $[0, \Lambda]$ so that the above expression can be written in integral form as

$$\begin{aligned} 1 - \frac{\lambda}{2} \int_0^\Lambda dE \frac{\rho(E) |\eta^2(E)|_{\text{gge}}}{u - E} \\ = \sqrt{1 - \lambda \int_0^\Lambda \frac{dE}{E + m_{\text{gge}}^2} \frac{\rho(E) \epsilon(E)}{u - E}}, \end{aligned} \quad (\text{B6})$$

$\rho(E)$ being the density of states of the phononic dispersion $\omega_{\mathbf{k}}^2$. Finally, by taking the analytic continuation of the equality above just above the branch cut $[0, \Lambda]$ and using the Plemelj formula, one finds

$$\frac{\lambda \pi \rho(E)}{2} |\eta^2(E)|_{\text{gge}} = \Im \sqrt{1 - \lambda \int_0^\Lambda \frac{dE'}{E' + m_{\text{gge}}^2} \frac{\rho(E') \epsilon(E')}{E - E' + i0}}. \quad (\text{B7})$$

APPENDIX C: GROUND-STATE QUENCH

For $t = 0$, the system is in its ground state with $r(t = 0) = r_-$. In this state, every mode is in the ground state of a two-dimensional harmonic oscillator of frequency $\Omega_{\mathbf{k}, -}^2 = m_{\text{gs}}^2 + \omega_{\mathbf{k}}^2$, m_{gs}^2 being the value of effective mass over the ground state, so that

$$\begin{aligned} |\eta_{\mathbf{k}}^2|_{\text{gs}} &= \langle \Phi_{\mathbf{k}} \Phi_{\mathbf{k}}^\dagger \rangle \equiv \frac{1}{2\sqrt{m_{\text{gs}}^2 + \omega_{\mathbf{k}}^2}}, \\ |p_{\mathbf{k}}^2|_{\text{gs}} &= \langle \Pi_{\mathbf{k}} \Pi_{\mathbf{k}}^\dagger \rangle \equiv \frac{1}{2}\sqrt{m_{\text{gs}}^2 + \omega_{\mathbf{k}}^2}, \\ \eta_{\mathbf{k}}^* p_{\mathbf{k}}|_{\text{gs}} &= \langle \Phi_{\mathbf{k}} \Pi_{\mathbf{k}} \rangle \equiv \frac{i}{2}, \end{aligned} \quad (\text{C1})$$

and m_{gs}^2 is determined by the self-consistency condition

$$m_{\text{gs}}^2 = r_- + \frac{\lambda}{2N} \sum_{\mathbf{k}} \frac{1}{\sqrt{m_{\text{gs}}^2 + \omega_{\mathbf{k}}^2}}. \quad (\text{C2})$$

so that the ground-state critical point takes place for $r_- = r_{\text{gs}}^c$, with

$$r_{\text{gs}}^c = -\frac{\lambda}{2N} \sum_{\mathbf{k}} \omega_{\mathbf{k}}^{-1}. \quad (\text{C3})$$

Let us now consider the quench $r_- \rightarrow r$: In this case, we have $\ell_{\mathbf{k}} = 1/2$, $\forall \mathbf{k}$, and

$$\begin{aligned} \epsilon_{\mathbf{k}} &= \frac{1}{2} \sqrt{m_{\text{gs}}^2 + \omega_{\mathbf{k}}^2} + \frac{r - r_-}{4\sqrt{m_{\text{gs}}^2 + \omega_{\mathbf{k}}^2}} \\ &\quad + \frac{\lambda}{8N} \sum_{\mathbf{k}'} \frac{1}{\omega_{\mathbf{k}'}^2 - \omega_{\mathbf{k}}^2} \sqrt{\frac{m_{\text{gs}}^2 + \omega_{\mathbf{k}}^2}{m_{\text{gs}}^2 + \omega_{\mathbf{k}'}^2}}. \end{aligned} \quad (\text{C4})$$

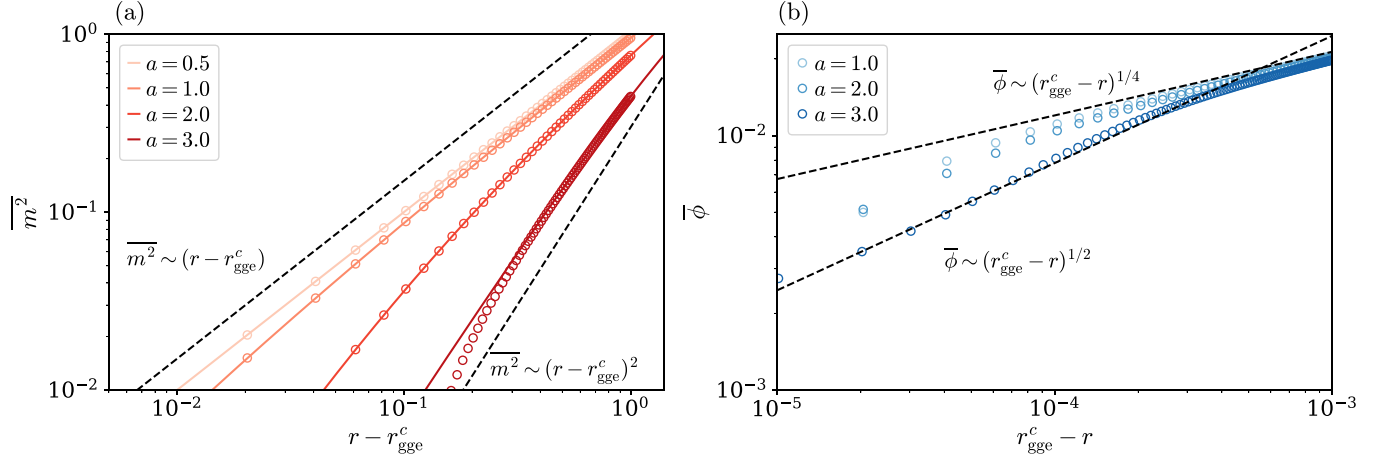


FIG. 4. Panels (a) and (b) show, respectively, the scaling of the long-time average of the squared mass m^2 and of the order parameter $\phi = \eta_0/\sqrt{N}$ as functions of $\delta r = |r - r_{\text{gge}}^c|$ close to the dynamical transition (log-log scale), for different values of the lattice spacing a . The numerical results for m^2 obtained from the equations of motion (red squares) are in excellent agreement with the GGE prediction (solid lines). The system size is $L = 5a \times 10^3$, $\lambda = 1$. The scaling behavior consistent with the spherical-model universality class in $d = 3$, $m^2 \sim (r - r_{\text{gge}}^c)^2$, $\bar{\phi} \sim (r_{\text{gge}}^c - r)^{1/2}$ emerges within the range of numerically accessible δr only for sufficiently large lattice spacings ($a > 1$). Conversely, for smaller lattice spacings ($a < 1$), the numerical data display an effective scaling $\bar{m}^2 \sim (r - r_{\text{gge}}^c)$, $\bar{\phi} \sim (r_{\text{gge}}^c - r)^{1/4}$ in agreement with previous findings in Refs. [43,44].

1. Critical properties for a ground-state quench

The GGE prediction for m_{gge} , given by Eq. (10),

$$m_{\text{gge}}^2 = r + \frac{\lambda}{N} \sum_{\mathbf{k}} \frac{\epsilon_{\mathbf{k}}}{m_{\text{gge}}^2 + \omega_{\mathbf{k}}^2}, \quad (\text{C5})$$

reads for a ground-state quench as

$$\begin{aligned} m_{\text{gge}}^2 = r + \frac{\lambda}{2N} \sum_{\mathbf{k}} \frac{|p_{\mathbf{k}}^2|_{\text{gs}}}{m_{\text{gge}}^2 + \omega_{\mathbf{k}}^2} \\ + (r - r_-) \frac{\lambda}{4N} \sum_{\mathbf{k}} \frac{|\eta_{\mathbf{k}}^2|_{\text{gs}}}{m_{\text{gge}}^2 + \omega_{\mathbf{k}}^2} \\ + (m_{\text{gs}}^2 - m_{\text{gge}}^2) \left(\frac{\lambda}{4N} \sum_{\mathbf{k}} \frac{|\eta_{\mathbf{k}}^2|_{\text{gs}}}{\sqrt{m_{\text{gs}}^2 + \omega_{\mathbf{k}}^2}} \right)^2. \end{aligned} \quad (\text{C6})$$

Together with Eq. (C2), which links r_- and m_{gs} , Eq. (C6) fixes the value of m_{gge} in terms of r_- (and thus m_{gs}). In particular, with the help of Eq. (C2), we find [Eq. (13)]

$$r - r_- = (m_{\text{gge}}^2 - m_{\text{gs}}^2) \left(1 + \frac{\lambda}{4N} \sum_{\mathbf{k}} \frac{|\eta_{\mathbf{k}}^2|_{\text{gs}}}{(m_{\text{gge}}^2 + \omega_{\mathbf{k}}^2)} \right). \quad (\text{C7})$$

From Eq. (13), setting $m_{\text{gge}} = 0$, we find

$$\begin{aligned} r_{\text{gge}}^c = r_- - m_{\text{gs}}^2 \left(1 + \frac{\lambda}{4N} \sum_{\mathbf{k}} \frac{1}{\omega_{\mathbf{k}}^2 \sqrt{m_{\text{gs}}^2 + \omega_{\mathbf{k}}^2}} \right) \\ = -\frac{\lambda}{2N} \sum_{\mathbf{k}} \frac{1}{\sqrt{m_{\text{gs}}^2 + \omega_{\mathbf{k}}^2}} \left(1 + \frac{m_{\text{gs}}^2}{2\omega_{\mathbf{k}}^2} \right). \end{aligned} \quad (\text{C8})$$

As the expression above is surprisingly linear in λ , the exact result coincides with the first-order perturbative expansion obtained in Refs. [47,56].

By expanding close to the critical point, we have

$$r - r_{\text{gge}}^c \sim m_{\text{gge}}^{d-2} + O(m_{\text{gge}}^2), \quad (\text{C9})$$

so that $\nu = (d-2)^{-1}$ for $2 < d < 4$. In particular, for $d = 3$ one can compute the series coefficients within the Debye approximation, finding

$$r - r_{\text{gge}}^c = \frac{3m_{\text{gs}}\pi}{8\Lambda^3} m_{\text{gge}} + \gamma_{\Lambda} m_{\text{gge}}^2, \quad (\text{C10})$$

where Λ is the ultraviolet cutoff and $\gamma_{\Lambda} \approx 1$.

This result shows that the GGE critical point coincides with the thermal one, giving $\nu = 1$ in $d = 3$. However, the critical scaling window in which such exponents can be observed is restricted by the condition $m_{\text{gge}} \ll 3m_{\text{gs}}a^3/(8\pi^2)$, which becomes increasingly narrow for smaller lattice spacings a . Consequently, as illustrated in Fig. 4, for a small but finite quench depth $\delta r = |r - r_{\text{gge}}^c|$, decreasing a induces a crossover toward the mean-field exponents $\nu = 1/2$ and $\beta = 1/4$. This accounts for the numerical scaling behavior reported in Refs. [43,44].

APPENDIX D: JACOBI COORDINATES

By setting $\eta_{\mathbf{k}} = \xi_{\mathbf{k}} e^{i\theta_{\mathbf{k}}}$, we have that $\theta_{\mathbf{k}}$, $\ell_{\mathbf{k}}$ are trivially a pair of separable canonical coordinates, as $\theta_{\mathbf{k}}$ is cyclic in $\mathcal{H}$. We will now prove that the Jacobi coordinates, defined by the relation

$$\phi(u) = 1 - \frac{\lambda}{2N} \sum_{\mathbf{k}} \frac{\xi_{\mathbf{k}}^2}{u - \omega_{\mathbf{k}}^2} = \prod_l \frac{u - u_l}{u - \omega_l}, \quad (\text{D1})$$

form a set of separable coordinates for the Hamiltonian $\mathcal{H}$ (7) (notice that in the last equality we chose to label the nondegenerate frequencies such that $\omega_{l+1} > \omega_l$). By differentiating

twice with respect to time and by using Eq. (5) together with the identity

$$\frac{1}{(u - u_l)(u - u_{l'})} = \frac{1}{u_l - u_{l'}} \left(\frac{1}{u - u_l} - \frac{1}{u - u_{l'}} \right) \quad (\text{D2})$$

(valid for $l \neq l'$), we find

$$V(u) = (u + m^2)\varphi^2 + \frac{1}{2}\varphi \partial_t^2 \varphi(u) - \frac{1}{4}(\partial_t \varphi)^2. \quad (\text{D3})$$

From this, we find that, in correspondence of $u = u_l$, one has

$$V(u_l) + \frac{1}{4}(\partial_t \varphi)^2|_{u=u_l} = 0. \quad (\text{D4})$$

To prove the separability, we have now to prove that the conjugate momentum to u_l is $P_l = N/(2\lambda) \partial_t \varphi|_{u=u_l}$. To see that, it is convenient to write explicitly ξ_l in terms of u_l , by expressing the residues of the poles of $\varphi(u)$ in terms of u_l , namely,

$$\frac{\lambda}{2N} \xi_l^2 = -(\omega_l^2 - u_l) \prod_{\omega_{l'} \neq \omega_l} \frac{\omega_l^2 - u_{l'}}{\omega_l^2 - \omega_{l'}^2}, \quad (\text{D5})$$

from which

$$\frac{1}{\xi_l} \frac{\partial \xi_l}{\partial u_{l'}} = \frac{1}{2} \frac{1}{\omega_l^2 - u_{l'}}. \quad (\text{D6})$$

We then find that the momentum P_l corresponding to u_l is given by

$$P_l = \sum_{l'} \frac{\partial \xi_{l'}}{\partial u_l} p_{l'} = \frac{1}{2} \sum_{l'} \frac{\xi_{l'} p_{l'}}{\omega_{l'}^2 - u_l} = -\frac{N}{2\lambda} \partial_t \varphi|_{u=u_l}, \quad (\text{D7})$$

where we used the equation of motion $\dot{\xi}_l = p_l$ for the original variables.

Let us notice that, more explicitly, one has

$$P_l = \frac{N}{2\lambda} \frac{\dot{u}_l}{u_l - \omega_l^2} \prod_{l' \neq l} \frac{u_l - u_{l'}}{u_l - \omega_{l'}^2}. \quad (\text{D8})$$

APPENDIX E: LOCAL FORM OF $V(u)$

Here, we derive the expression that the potential $V(u)$ of Eq. (15)

$$V(u) = u + r - \frac{\lambda}{N} \sum_{\mathbf{k}} \frac{\epsilon_{\mathbf{k}}}{u - \omega_{\mathbf{k}}^2} + \frac{\lambda^2}{4N^2} \sum_{\mathbf{k}} \frac{\ell_{\mathbf{k}}^2}{(u - \omega_{\mathbf{k}}^2)^2}, \quad (\text{E1})$$

takes in the thermodynamic limit. As noticed above, far from the singularities ($u \notin [0, \Lambda]$), one can replace the sum with integrals, obtaining

$$V_{\text{reg}}(u) = u + r - \lambda \int_0^\Lambda \frac{du' \rho(u') \epsilon(u')}{u - u'}, \quad (\text{E2})$$

$\rho(u)$ being the density of states of the phonon dispersion ω_l^2 . Notice that $V_{\text{reg}}(z)$ is analytic everywhere except the interval $z \in [0, \Lambda]$, where it has a branch cut. Moreover, for $u \in \mathbb{R} \setminus [0, \Lambda]$, we inherit the polynomial structure $V(u) < 0$ for $u < -\mu$ and $u_{N-1}^- < u < u_{N-1}^+$. Let us notice, however, that in this limit $u_{N-1}^- \rightarrow \Lambda$.

We want now to prove that for $u \in [\omega_l^2, \omega_{l+1}^2]$ in the $N \rightarrow \infty$ limit the potential is given by Eq. (17), namely,

$$V_l(\theta_l) = \text{p.v. } V_{\text{reg}}(\omega_l^2) - \frac{\lambda \pi \rho(\omega_l^2) \epsilon_l}{\tan \theta_l} + \frac{(\lambda \pi \rho(\omega_l^2) \ell_l)^2}{4 \sin^2 \theta_l}, \quad (\text{E3})$$

where θ_l is the rescaled variable

$$u_l = \omega_l^2 + \frac{\Delta \omega_l^2}{\pi} \theta_l, \quad (\text{E4})$$

with $\Delta \omega_l^2 = \omega_{l+1}^2 - \omega_l^2$ and $\theta_l \in [0, \pi]$. Notice that this definition coincides with the one of the paper, as one has $(N \Delta \omega_l^2)^{-1} = \rho(u)$ up to higher corrections in N^{-1} .

The idea behind the proof of Eq. (E3) is to separate $V(u)$ in Eq. (E1) into a slowly varying part, which is approximately constant over the interval (ω_l, ω_{l+1}) , plus a fast part, which contains the singularities around each ω_l^2 . The slowly varying part will be simply given by $V_{\text{reg}}(\omega_l^2)$. On the other hand, the singular part, coming from the poles in the vicinity of $u = \omega_l^2$, can be estimated as follows:

$$\frac{1}{N} \sum_{l'} \frac{\epsilon_{l'}}{u_l - \omega_{l'}^2} \rightarrow \sum_{l'} \frac{\epsilon_{l'}}{N(\omega_l^2 - \omega_{l'}^2) + N \Delta \omega_l^2 \theta / \pi}. \quad (\text{E5})$$

In the large- N limit, ω_l and ϵ_l vary slowly with l ; on the other hand, the contribution of the terms with $l' \gg l$ vanish, so that one can approximate $\epsilon_{l'} \sim \epsilon_l$, $N(\omega_l^2 - \omega_{l'}^2) \sim -N \Delta \omega_l^2 (l - l')$. We have that

$$\frac{1}{N} \sum_{l'} \frac{\epsilon_{l'}}{u_l - \omega_{l'}^2} = \frac{\epsilon_l}{N \Delta \omega_l^2} \sum_{l'} \frac{1}{l - l' + \theta / \pi} = \pi \rho(\omega_l^2) \epsilon_l \cot \theta. \quad (\text{E6})$$

Similarly,

$$\frac{1}{N^2} \sum_{l'} \frac{\ell_{l'}^2}{u_l - \omega_{l'}^2} \rightarrow \sum_{l'} \frac{\ell_{l'}^2}{[N(\omega_l^2 - \omega_{l'}^2) + N \Delta \omega_l^2 \theta / \pi]^2}, \quad (\text{E7})$$

which can be estimated analogously as before [or, equivalently, by deriving both sides of Eq. (E6) with respect to θ , and replacing ϵ_l with ℓ_l]. One has

$$\frac{1}{N^2} \sum_{l'} \frac{\ell_{l'}^2}{(u_l - \omega_{l'}^2)^2} = \pi^2 \rho^2(\omega_l^2) \frac{\ell_l^2}{\sin^2 \theta}, \quad (\text{E8})$$

which finally gives Eq. (17).

APPENDIX F: SPECTRUM IN THE QFT LIMIT

We will now present in detail the argument sketched above to compute the spectrum of the model in the QFT limit. In this limit, as the spectrum becomes continuous, and $u_l^+ - u_l^- \sim N^{-1} \forall l$, while the ultraviolet cutoff $\Lambda \sim a^{-2} \rightarrow \infty$. In this limit, the explicit form of the potential $V(u)$ in the thermodynamic limit is not needed, as the strategy is to exploit the finite N singularity structure of $V(u)$ in Eq. (E1) and take the thermodynamic limit only at the end.

In order to find the spectrum of the problem, we have to express $\mathcal{H} = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}}$ in terms of the action variables J_l . As

explained in the body of the paper, we introduce the auxiliary function

$$f(u) = \sqrt{(u + \mu)V(u)}. \quad (\text{F1})$$

This has $\mathcal{N}$ first-order poles for $u = \omega_l^2$ and $\mathcal{N}$ branch cuts for $u_l^- < u < u_l^+$. If we consider the contour at infinity C_∞ and we deform it on the real axis, we have

$$\oint_{C_\infty} dz f(z) = \sum_l 2\pi i \text{Res}_{\omega_l^2} f(z) + \sum_l \int_{C_l} dz f(z), \quad (\text{F2})$$

where C_l are counterclockwise contours that go around the $q_l^- < u < q_l^+$ branch cut (see also Fig. 3). It is crucial that, in the QFT limit, the factor $\sqrt{\mu + u}$ can be considered constant along each branch cut. While for $l \neq \mathcal{N} - 1$ this is simply a consequence of the fact that the spectral gap closes, for $l = \mathcal{N} - 1$, the integration interval is located beyond the edge of the spectrum Λ and thus unbounded. Yet our assumption becomes true in the field-theoretical limit in which $\Lambda \rightarrow \infty$: Indeed, as $u_{\mathcal{N}-1}^\pm = O(\Lambda)$, one has $\sqrt{u + \mu} = \sqrt{\Lambda^2 + \mu} + O(\Lambda^{-1})$. We have

$$\begin{aligned} \int_{C_l} dz f(z) &= -2i \int_{u_l^-}^{u_l^+} |f(u)| du \\ &\approx -2i \sqrt{\omega_l^2 + \mu} \int_{u_l^-}^{u_l^+} \sqrt{-V(u)} du \\ &= -\frac{2\pi i \lambda}{N} J_l \sqrt{\omega_l^2 + \mu}, \end{aligned} \quad (\text{F3})$$

while

$$\text{Res}_{\omega_l^2} f(z) = -\frac{\lambda}{2N} \ell_l \sqrt{\omega_l^2 + \mu}. \quad (\text{F4})$$

On the other hand, since $f(z) \sim z + (r + \mu) - [\lambda \mathcal{H}/(2N) + (\mu - r)^2/8]z^{-1} + O(z^{-2})$, one has

$$\oint_{C_\infty} dz f(z) = -2\pi i \left(\frac{\lambda}{2N} \mathcal{H} + \frac{(\mu - r)^2}{8} \right), \quad (\text{F5})$$

so that finally we get an expression for $\mathcal{H}$ in terms of J_l , ℓ_l and μ , namely,

$$\mathcal{H} = -N \frac{(\mu - r)^2}{4\lambda} + \sum_l \sqrt{\omega_l^2 + \mu} (2J_l + \ell_l). \quad (\text{F6})$$

We still have to express the zero μ in terms of J_l . To do so, we consider the function

$$g(z) = \sqrt{\frac{V(z)}{z + \mu}}, \quad (\text{F7})$$

which has the exact branch cut and pole structure of $f(z)$, but a different behavior at infinity, namely, $g(z) = 1 - (\mu - r)/2z^{-1} + O(z^{-2})$. If we integrate it on the same contour, we get

$$\mu - r = \frac{\lambda}{N} \sum_l \frac{2J_l + \ell_l}{\sqrt{\omega_l^2 + \mu}}, \quad (\text{F8})$$

which is an implicit equation for μ . Combining Eqs. (F6) and (F8), one has the form of the Hamiltonian $\mathcal{H}(J_l)$ in

terms of the action variables J_l , ℓ_l . Since the Hamiltonian depends only on $2J_l + \ell_l$, we have $\Omega_{l,2} = \Omega_{l,1}$, i.e., a set of doubly degenerate frequencies. The latter are given by

$$\begin{aligned} \Omega_{l,1} &= \frac{\partial \mathcal{H}}{\partial \ell_l} = \sqrt{\omega_l^2 + \mu} \\ &+ \frac{\partial \mu}{\partial \ell_l} \left(-\frac{\mu - r}{2\lambda} + \frac{1}{2N} \sum_l \frac{2J_l + \ell_l}{\sqrt{\omega_l^2 + \mu}} \right) \\ &= \sqrt{\omega_l^2 + \mu}. \end{aligned} \quad (\text{F9})$$

Notice that we can interpret μ as the asymptotic value of the (squared) effective mass. On the other hand, by exploiting the definition of μ , ($V(-\mu) = 0$), one gets the self-consistency equation in terms of ϵ_l , namely,

$$0 = -\mu + r + \frac{\lambda}{N} \sum_{\mathbf{k}} \frac{\epsilon_{\mathbf{k}}}{\mu + \omega_{\mathbf{k}}^2} + \frac{\lambda^2}{4N^2} \sum_{\mathbf{k}} \frac{\ell_{\mathbf{k}}^2}{(\mu + \omega_{\mathbf{k}}^2)^2}. \quad (\text{F10})$$

Unless $\mu \sim N^{-1}$, we see that the last term on the right-hand side is negligible for $N \rightarrow \infty$, so that one has

$$\mu = r + \frac{\lambda}{N} \sum_{\mathbf{k}} \frac{\epsilon_{\mathbf{k}}}{\mu + \omega_{\mathbf{k}}^2}, \quad (\text{F11})$$

from which $\mu = m_{\text{ge}}^2$.

APPENDIX G: LATTICE CASE

If we keep the lattice spacing a (and thus the momentum cutoff Λ) finite, we have to resort to the explicit expression for the potential $V(u)$ in the $N \rightarrow \infty$ limit derived in Appendix E, i.e., Eqs. (E2) and (E3).

In particular, as $u_{\mathcal{N}-1} \notin [0, \Lambda]$, we have that

$$J_{\mathcal{N}-1} = \frac{N}{\pi \lambda} \int_{\Lambda}^b du \sqrt{-V_{\text{reg}}(u)} \quad (\text{G1})$$

[where, as before, $b = u_{\mathcal{N}-1}^+$, the largest zero of $V_{\text{reg}}(u)$]. For $l = 0, \dots, \mathcal{N} - 2$, it is convenient to write the integral in θ :

$$\begin{aligned} J_l &= \frac{N \Delta \omega_l^2}{\pi^2 \lambda} \int_{\theta^-}^{\theta^+} d\theta \sqrt{-V_l(\theta)} \\ &= -\frac{\ell_l}{2} + \frac{1}{\pi \lambda \rho(\omega_l^2)} \Im \sqrt{\text{p.v.} V_{\text{reg}}(\omega_l^2) + i \lambda \pi \rho(\omega_l^2) \epsilon_l} \\ &= -\frac{\ell_l}{2} + \frac{1}{\pi \lambda \rho(\omega_l^2)} \Im \sqrt{V_{\text{reg}}(\omega_l^2 + i0)}, \end{aligned} \quad (\text{G2})$$

where we made use of the definition [Eq. (E2) of V_{reg}]. In the continuum, we can again introduce $E = \omega_l^2 \in [0, \Lambda]$ and $\mathcal{J}(E) = (J_l + \ell_l/2)$, so that

$$\pi \lambda \rho(u) \mathcal{J}(u) = \Im \sqrt{V_{\text{reg}}(u + i0)}. \quad (\text{G3})$$

By carefully handling the singularity structure of $\sqrt{V_{\text{reg}}(z)}$, this expression can be inverted with the use of the Cauchy-Plemelj formula. In particular, we have that, outside the interval $[0, \Lambda]$, $\sqrt{V(u)}$ has branch cuts for $u < -\mu$ and

$u \in [\Lambda, b]$, while at $z = \infty$ $\sqrt{V(z)} \sim \sqrt{z}$. We have thus

$$V(z) = \frac{(z + \mu)(z - \Lambda)}{z - b} \left(1 + \lambda \int_0^\Lambda \frac{du}{\sqrt{u + \mu}} \sqrt{\frac{b - u}{\Lambda - u}} \frac{\rho(u) \mathcal{J}(u)}{u - z} \right)^2, \quad (\text{G4})$$

which can alternatively be written by coming back to the discrete formulation as

$$V(z) = \frac{(z + \mu)(z - \Lambda)}{z - b} \left(1 + \frac{\lambda}{2N} \sum_{l=0}^{N-2} \frac{2J_l + \ell_l}{\omega_l^2 - z} \sqrt{\frac{b - \omega_l^2}{(\Lambda - \omega_l^2)(\mu + \omega_l^2)}} \right)^2. \quad (\text{G5})$$

By expanding both sides around $z = \infty$, and remembering that $V(z) = z + r - \lambda \mathcal{H}/Nz^{-1} + O(z^{-2})$, we find

$$\mu - r + b - \Lambda = \frac{\lambda}{N} \sum_{l=0}^{N-2} \frac{2J_l + \ell_l}{\sqrt{\omega_l^2 + \mu}} \sqrt{\frac{b - \omega_l^2}{\Lambda - \omega_l^2}} \quad (\text{G6})$$

and

$$\mathcal{H} = -\frac{N}{\lambda}(\Lambda + r)(b - \Lambda) - \frac{N}{4\lambda}(\mu - r + b - \Lambda)^2 + \sum_{l=0}^{N-2} \sqrt{\omega_l^2 + \mu} \sqrt{\frac{b - \omega_l^2}{\Lambda - \omega_l^2}} (2J_l + \ell_l); \quad (\text{G7})$$

which are the lattice analogs of the relations (F8) and (F6), respectively.

Yet, since on the lattice we still have the free parameter b , we ought to express this in terms of J_k . This can be done by exploiting the last action variable J_{N-1} :

$$J_{N-1} = \frac{N}{\pi\lambda} \int_\Lambda^b du \sqrt{\frac{(\mu + u)(\Lambda - u)}{u - b}} + \frac{1}{2\pi} \sum_{l=0}^{N-2} (2J_l + \ell_l) \sqrt{\frac{b - \omega_l^2}{(\Lambda - \omega_l^2)(\mu + \omega_l^2)}} \int_\Lambda^b \frac{du}{\omega_l^2 - u} \sqrt{\frac{(\mu + u)(\Lambda - u)}{u - b}}. \quad (\text{G8})$$

Equation (G7), together with conditions (G6) and (G8), allows us to compute directly the isolated frequency $\Omega = \partial \mathcal{H} / \partial J_{N-1}$ and the rest of the spectrum $\Omega_{l,1} = \partial \mathcal{H} / \partial J_l$, $\Omega_{l,2} = \partial \mathcal{H} / \partial \ell_l$.

APPENDIX H: NUMERICAL METHODS

The numerical results shown in the paper are obtained by numerically integrating the equations of motion in Eq. (5), which govern the evolution of the complex variables η_k and p_k , using a fourth-order Runge-Kutta algorithm with suitably small time steps.

In order to access larger system sizes in three dimensions ($d = 3$), we employ the Debye approximation [67,68], which consists in replacing the cubic Brillouin zone spanned by $\mathbf{k} = (k_x, k_y, k_z)$ with a sphere of radius Λ . This approximation preserves the low-momentum density of states while significantly reducing the computational cost. Within this scheme, the problem is effectively reduced to $2L/a$ degrees of freedom,

corresponding to the complex variables η_k and p_k , labeled by

$$k = 0, \frac{\pi}{L}, \frac{2\pi}{L}, \dots, \frac{\pi}{a}, \quad (\text{H1})$$

which evolve according to

$$p_k(t) = \dot{\eta}_k(t), \quad \dot{p}_k(t) = -(m^2(t) + \omega_k^2) \eta_k(t), \quad (\text{H2})$$

with the squared mass $m^2(t)$ computed self-consistently at each time step through

$$m^2(t) = r + \frac{ad}{L} \sum_{k \neq 0} \left(\frac{ak}{\pi} \right)^{(d-1)} |\eta_k(t)|^2 + |\eta_0(t)|^2. \quad (\text{H3})$$

The order parameter corresponds to the normalized zero mode, $\phi = \sqrt{L/a} \eta_0$.

The initial conditions are fixed by the ground-state conditions.

- [1] R. Kubo, M. Toda, and N. Hashitsume, *Statistical Physics II Nonequilibrium Statistical Mechanics* (Springer Science & Business Media, Berlin, Heidelberg, 2012).
- [2] H. G. Schuster and W. Just, *Deterministic Chaos: An Introduction* (Wiley Online Library, Hoboken, New Jersey, 1988), Vol. 2.
- [3] G. Gallavotti, *The Fermi-Pasta-Ulam Problem: A Status Report* (Springer, Berlin, Heidelberg, 2007), Vol. 728.

- [4] J. von Neumann, Proof of the ergodic theorem and the H-theorem in quantum mechanics: Translation of: Beweis des ergodensatzes und des H-theorems in der neuen mechanik, *Eur. Phys. J. H* **35**, 201 (2010).
- [5] M. Srednicki, Chaos and quantum thermalization, *Phys. Rev. E* **50**, 888 (1994).
- [6] M. Rigol and M. Srednicki, Alternatives to eigenstate thermalization, *Phys. Rev. Lett.* **108**, 110601 (2012).

- [7] S. Goldstein, J. L. Lebowitz, R. Tumulka, and N. Zanghì, Canonical typicality, *Phys. Rev. Lett.* **96**, 050403 (2006).
- [8] S. R. Manmana, S. Wessel, R. M. Noack, and A. Muramatsu, Strongly correlated fermions after a quantum quench, *Phys. Rev. Lett.* **98**, 210405 (2007).
- [9] M. Moeckel and S. Kehrein, Interaction quench in the Hubbard model, *Phys. Rev. Lett.* **100**, 175702 (2008).
- [10] G. Biroli, C. Kollath, and A. M. Läuchli, Effect of rare fluctuations on the thermalization of isolated quantum systems, *Phys. Rev. Lett.* **105**, 250401 (2010).
- [11] E. Khatami, G. Pupillo, M. Srednicki, and M. Rigol, Fluctuation-dissipation theorem in an isolated system of quantum dipolar bosons after a quench, *Phys. Rev. Lett.* **111**, 050403 (2013).
- [12] T. Yoshizawa, E. Iyoda, and T. Sagawa, Numerical large deviation analysis of the eigenstate thermalization hypothesis, *Phys. Rev. Lett.* **120**, 200604 (2018).
- [13] L. Foini and J. Kurchan, Eigenstate thermalization and rotational invariance in ergodic quantum systems, *Phys. Rev. Lett.* **123**, 260601 (2019).
- [14] S. Pappalardi, L. Foini, and J. Kurchan, Eigenstate thermalization hypothesis and free probability, *Phys. Rev. Lett.* **129**, 170603 (2022).
- [15] S. Pappalardi, F. Fritzsche, and T. Prosen, Full eigenstate thermalization via free cumulants in quantum lattice systems, *Phys. Rev. Lett.* **134**, 140404 (2025).
- [16] A. Mitra, Quantum quench dynamics, *Annu. Rev. Condens. Matter Phys.* **9**, 245 (2018).
- [17] A. Polkovnikov, K. Sengupta, A. Silva, and M. Vengalattore, *Colloquium: Nonequilibrium dynamics of closed interacting quantum systems*, *Rev. Mod. Phys.* **83**, 863 (2011).
- [18] P. Calabrese and J. Cardy, Time dependence of correlation functions following a quantum quench, *Phys. Rev. Lett.* **96**, 136801 (2006).
- [19] P. Calabrese and J. Cardy, Quantum quenches in extended systems, *J. Stat. Mech.* (2007) P06008.
- [20] A. Gambassi and A. Silva, Large deviations and universality in quantum quenches, *Phys. Rev. Lett.* **109**, 250602 (2012).
- [21] A. Solfanelli and N. Defenu, Universal work statistics in long-range interacting quantum systems, *Phys. Rev. Lett.* **134**, 030402 (2025).
- [22] A. Gambassi and P. Calabrese, Quantum quenches as classical critical films, *Europhys. Lett.* **95**, 66007 (2011).
- [23] G. Jona-Lasinio and C. Presilla, Chaotic properties of quantum many-body systems in the thermodynamic limit, *Phys. Rev. Lett.* **77**, 4322 (1996).
- [24] M. Lenci, Ergodic properties of the quantum ideal gas in the Maxwell-Boltzmann statistics, *J. Math. Phys.* **37**, 5136 (1996).
- [25] S. Graffi and A. Martinez, Ergodic properties of infinite quantum harmonic crystals: An analytic approach, *J. Math. Phys.* **37**, 5111 (1996).
- [26] M. Rigol, V. Dunjko, V. Yurovsky, and M. Olshanii, Relaxation in a completely integrable many-body quantum system: An *ab initio* study of the dynamics of the highly excited states of 1D lattice hard-core bosons, *Phys. Rev. Lett.* **98**, 050405 (2007).
- [27] M. Rigol, Breakdown of thermalization in finite one-dimensional systems, *Phys. Rev. Lett.* **103**, 100403 (2009).
- [28] P. Calabrese, F. H. L. Essler, and M. Fagotti, Quantum quench in the transverse-field Ising chain, *Phys. Rev. Lett.* **106**, 227203 (2011).
- [29] M. Rigol, Fundamental asymmetry in quenches between integrable and nonintegrable systems, *Phys. Rev. Lett.* **116**, 100601 (2016).
- [30] E. Ilievski, J. De Nardis, B. Wouters, J.-S. Caux, F. H. L. Essler, and T. Prosen, Complete generalized Gibbs ensembles in an interacting theory, *Phys. Rev. Lett.* **115**, 157201 (2015).
- [31] L. Biagetti, M. Łebek, M. Panfil, and J. De Nardis, Generalised BBGKY hierarchy for near-integrable dynamics, *Phys. Rev. X* (2026), doi:10.1103/sx1x-py11.
- [32] N. Defenu, Metastability and discrete spectrum of long-range systems, *Proc. Natl. Acad. Sci. USA* **118**, e2101785118 (2021).
- [33] G. Giachetti and N. Defenu, Entanglement propagation and dynamics in non-additive quantum systems, *Sci. Rep.* **13**, 12388 (2023).
- [34] G. Giachetti and N. Defenu, Conditions for quantum violent relaxation, *Phys. Rev. B* **111**, 214301 (2025).
- [35] M. C. Bañuls, J. I. Cirac, and M. B. Hastings, Strong and weak thermalization of infinite nonintegrable quantum systems, *Phys. Rev. Lett.* **106**, 050405 (2011).
- [36] G. Delfino and J. Viti, On the theory of quantum quenches in near-critical systems, *J. Phys. A: Math. Theor.* **50**, 084004 (2017).
- [37] T. Rakovszky, M. Mestyán, M. Collura, M. Kormos, and G. Takács, Hamiltonian truncation approach to quenches in the Ising field theory, *Nucl. Phys. B* **911**, 805 (2016).
- [38] M. Kormos, M. Collura, G. Takács, and P. Calabrese, Real-time confinement following a quantum quench to a non-integrable model, *Nat. Phys.* **13**, 246 (2017).
- [39] M. Collura, M. Kormos, and G. Takács, Dynamical manifestation of the Gibbs paradox after a quantum quench, *Phys. Rev. A* **98**, 053610 (2018).
- [40] N. J. Robinson, A. J. A. James, and R. M. Konik, Signatures of rare states and thermalization in a theory with confinement, *Phys. Rev. B* **99**, 195108 (2019).
- [41] O. A. Castro-Alvaredo, M. Lencsés, I. M. Szécsényi, and J. Viti, Entanglement oscillations near a quantum critical point, *Phys. Rev. Lett.* **124**, 230601 (2020).
- [42] S. Scopa, P. Calabrese, and A. Bastianello, Entanglement dynamics in confining spin chains, *Phys. Rev. B* **105**, 125413 (2022).
- [43] B. Sciolla and G. Biroli, Quantum quenches, dynamical transitions, and off-equilibrium quantum criticality, *Phys. Rev. B* **88**, 201110(R) (2013).
- [44] S. A. Weidinger, M. Heyl, A. Silva, and M. Knap, Dynamical quantum phase transitions in systems with continuous symmetry breaking, *Phys. Rev. B* **96**, 134313 (2017).
- [45] A. Chandran, A. Nandori, S. Gubser, and S. Sondhi, Equilibration and coarsening in the quantum $O(n)$ model at infinite n , *Phys. Rev. B* **88**, 024306 (2013).
- [46] M. Moshe and J. Zinn-Justin, Quantum field theory in the large N limit: A review, *Phys. Rep.* **385**, 69 (2003).
- [47] S. Sotiriadis and J. Cardy, Quantum quench in interacting field theory: A self-consistent approximation, *Phys. Rev. B* **81**, 134305 (2010).
- [48] T. Vojta, Quantum version of a spherical model: Crossover from quantum to classical critical behavior, *Phys. Rev. B* **53**, 710 (1996).

- [49] S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, 1999).
- [50] Notice that for $\mathbf{k} = 0$, the canonical commutation relations fix $\ell_0 = 1/2$.
- [51] D. Choodnovsky and G. Choodnovsky, Completely integrable class of mechanical systems connected with Korteweg-de Vries and multicomponent Schrödinger equations-I, Sémin. Équations Non Linéaires 1 (1978).
- [52] S. Wojciechowski, Integrability of one particle in a perturbed central quartic potential, *Phys. Scr.* **31**, 433 (1985).
- [53] C. Neumann, De problemate quodam mechanico, quod ad primam integralium ultraellipticorum classem revocatur, *Z. Angew. Math. (Crelle's J.)* 46 (1859).
- [54] C. Neumann, *De problemate quodam mechanico, quod ad primam integralium ultraellipticorum classem revocatur: Dissertatio inauguralis* (Dalkowski, 1856).
- [55] K. K. Uhlenbeck, Equivariant harmonic maps into spheres, in *Harmonic Maps: Proceedings of the NSF-CBMS Regional Conference, Held at Tulane University, New Orleans, 1980* (Springer, Berlin, Heidelberg, 1982), pp. 146–158.
- [56] P. Smacchia, M. Knap, E. Demler, and A. Silva, Exploring dynamical phase transitions and prethermalization with quantum noise of excitations, *Phys. Rev. B* **91**, 205136 (2015).
- [57] G. S. Joyce, Spherical model with long-range ferromagnetic interactions, *Phys. Rev.* **146**, 349 (1966).
- [58] H. P. O. Collado, N. Defenu, and J. Lorenzana, Engineering Higgs dynamics by spectral singularities, *Phys. Rev. Res.* **5**, 023011 (2023).
- [59] A. Maraga, A. Chiocchetta, A. Mitra, and A. Gambassi, Aging and coarsening in isolated quantum systems after a quench: Exact results for the quantum $O(n)$ model with $n \rightarrow \infty$, *Phys. Rev. E* **92**, 042151 (2015).
- [60] A. Chiocchetta, M. Tavora, A. Gambassi, and A. Mitra, Short-time universal scaling in an isolated quantum system after a quench, *Phys. Rev. B* **91**, 220302(R) (2015).
- [61] G. Delfino, Persistent oscillations after quantum quenches: The inhomogeneous case, *Nucl. Phys. B* **954**, 115002 (2020).
- [62] G. Delfino and M. Sorba, Persistent oscillations after quantum quenches in d dimensions, *Nucl. Phys. B* **974**, 115643 (2022).
- [63] M. Sorba, N. Defenu, and G. Delfino, Quantum quenches with long range interactions, *Nucl. Phys. B* **1019**, 117105 (2025).
- [64] J. H. Robertson, R. Senese, and F. H. L. Essler, Decay of long-lived oscillations after quantum quenches in gapped interacting quantum systems, *Phys. Rev. A* **109**, 032208 (2024).
- [65] F. Balducci, A. Chandran, and R. Moessner, Symmetry re-breaking in an effective theory of quantum coarsening, *Phys. Rev. Lett.* **136**, 020402 (2026).
- [66] A. Solfanelli, Universality and weak-ergodicity breaking in quantum quenches, Zenodo, 2026, <https://doi.org/10.5281/zenodo.22095019>.
- [67] P. Debye, Zur theorie der spezifischen wärmen, *Ann. Phys.* **344**, 789 (1912).
- [68] N. W. Ashcroft and N. D. Mermin, *Solid State Physics* (Holt, Rinehart and Winston, New York, 1976).