

Gapless superconductivity and its real-space topology in quasicrystals

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We study superconductivity in Ammann-Beenker quasicrystals under magnetic field. By assuming an intrinsic s -wave pairing interaction and solving for mean-field equations self-consistently, we find gapless superconductivity in the quasicrystals at and near half filling. We show that gapless superconductivity results from the combination of broken translational symmetry and confined states that is characteristic of the quasicrystals. When Rashba spin-orbit coupling is present, the quasicrystalline gapless superconductor can be topologically nontrivial and characterized by a nonzero pseudospectrum invariant given by a spectral localizer. The gapless topological superconducting phase exhibits edge states with near-zero energy. These findings suggest that quasicrystals can be a unique platform for realizing gapless superconductivity with nontrivial topology.

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Introduction. Quasicrystals (QCs) are materials with aperiodic structure that has a long-range order [1,2]. Since translational symmetry is absent, one cannot employ a momentum-space picture to understand quantum phenomena in QCs. One such example is superconductivity, as the concept of Fermi surfaces in momentum space cannot be applied directly to QCs [3–6]. Nevertheless, superconductivity has recently been discovered in some QCs experimentally [7–9].

There has recently been theoretical works showing that not only conventional superconductivity, but also topologically nontrivial superconductivity can occur in QCs [10–22]. While unconventional pairing symmetry such as p wave is typically required to realize topological superconductivity, even with s -wave pairing, spin-orbit-coupled superconductors can be topologically nontrivial under strong magnetic field [23,24]. It has been shown theoretically by some of the present authors that such topological superconductivity with broken time-reversal symmetry is realizable even in QCs when the chemical potential is near the top or bottom of the kinetic-energy band [16,21]. At the same time, however, there is no fundamental difference in the nature of topological superconductivity compared with that in periodic systems and thus, no new phenomenon that is specific to QCs has been found so far.

In the tight-binding model of some QCs, where electrons hop from site to site along the edge of a tile in the quasiperiodic tiling, local geometry of the quasicrystalline lattice can give rise to strictly localized states of electrons. Such localized or confined states are highly degenerate and result in a large peak at zero energy in the density of states (DOS) [3,25–36]. For example, Penrose [25–28], Ammann-Beenker [3,30,31],

and Socolar dodecagonal [33,34,37] QCs exhibit a flat band at zero kinetic energy, which arises from a large family of confined states. If the chemical potential is at or close to zero energy, a flat band of confined states—when combined with the underlying inhomogeneity of the quasicrystalline lattice—may lead to a superconducting phase that has no counterpart in periodic systems.

In this work, we investigate superconductivity in Ammann-Beenker QCs at and near half filling. We demonstrate that this system presents gapless superconductivity under magnetic field, where the bulk energy spectrum is gapless despite the finite superconducting order parameter at all sites. Gapless superconductivity was originally studied in conventional s -wave superconductors with magnetic impurities [38–40]. In such systems, a band is formed inside the superconducting energy gap, which is broadened as the impurity concentration is increased, leading up to gapless superconductivity [41–43]. In the presence of magnetic field, disordered superconductors can also exhibit gapless superconductivity [44–46]. In contrast to these previous studies, we find that quasicrystalline gapless superconductivity occurs without additional disorder or magnetic impurities, and only at and very close to half filling in Ammann-Beenker QCs. This phenomenon is unique to QCs in that it originates in the combination of two intrinsic features of QCs, namely, the lack of translational symmetry and the highly degenerate flat band of confined states. Furthermore, we show that gapless superconductivity can coexist with nontrivial topology induced by Rashba spin-orbit coupling. Gapless superconducting phases can be classified topologically in terms of a real-space topological invariant called the pseudospectrum invariant [47,48]. The topological characterization differs from conventional approaches that rely on the existence of an energy gap [49]. In the gapless topological superconducting phase, we find topologically protected edge modes.

Model. We start with studying superconductivity in Ammann-Beenker QCs at half filling. Structure of the QC

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is shown in the Supplemental Material [50]. The mean-field Hamiltonian for an s -wave superconductor with Rashba spin-orbit coupling and Zeeman field [23,24] can be written in the Bogoliubov de-Gennes (BdG) formalism as

$$H = \frac{1}{2} \sum_{ij\alpha\beta} (c_{i\alpha}^\dagger c_{i\alpha}) \mathcal{H}_{\text{BdG}} \begin{pmatrix} c_{j\beta} \\ c_{j\beta}^\dagger \end{pmatrix},$$

$$\mathcal{H}_{\text{BdG}} = \begin{pmatrix} \mathcal{H} & \Delta \\ \Delta^\dagger & -\mathcal{H}^* \end{pmatrix}, \quad (1)$$

with $c_{i\alpha}^\dagger$ the creation operator of an electron with spin α at site i , which corresponds to the i th vertex of the Ammann-Beenker tiling. The single-particle Hamiltonian $\mathcal{H}$ is given by [21,51]

$$\mathcal{H}_{i\alpha,j\beta} = [t_{ij}\sigma_0 - (\tilde{\mu} - V_{i\alpha}^{(H)} + \bar{V}_{\alpha}^{(H)})\delta_{ij}\sigma_0 - \tilde{h}\delta_{ij}\sigma_3 + i\lambda_R(\sigma_1 y_{ij} - \sigma_2 x_{ij})]_{\alpha\beta}, \quad \bar{\alpha} \neq \alpha, \quad (2)$$

where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices acting on spin space, σ_0 the 2×2 identity matrix, and i the imaginary unit $\sqrt{-1}$. $t_{ij} \equiv -t$ ($t > 0$) is the hopping amplitude from the j th to the i th site, λ_R is the Rashba spin-orbit coupling constant, x_{ij} (y_{ij}) is the x (y) component of the unit vector connecting site j to site i , and

$$V_{i\alpha}^{(H)} = U \langle c_{i\alpha}^\dagger c_{i\alpha} \rangle, \quad \bar{V}_{\alpha}^{(H)} = \frac{1}{N} \sum_i V_{i\alpha}^{(H)}, \quad (3)$$

where $U < 0$ is the coupling constant of an onsite pairing interaction and N is the number of lattice sites in the system. $\tilde{\mu} = \mu - (\bar{V}_{\uparrow}^{(H)} + \bar{V}_{\downarrow}^{(H)})/2$ and $\tilde{h} = h_z + (\bar{V}_{\uparrow}^{(H)} - \bar{V}_{\downarrow}^{(H)})/2$, with the chemical potential μ and the Zeeman energy h_z . The off-diagonal elements in Eq. (1) are given by

$$\Delta_{i\alpha,j\beta} = [i\Delta_i\delta_{ij}\sigma_2]_{\alpha\beta}, \quad \Delta_i = U \langle c_{i\downarrow} c_{i\uparrow} \rangle. \quad (4)$$

We solve the BdG equations self-consistently for the Hartree potential and the superconducting order parameter in Eqs. (3) and (4), respectively. Below we present results for $U = -1.5t$ and $h_z \leq 0$ at zero temperature. In the following, we use an approximant (a quasicrystal) with $N = 2786$ ($N = 2869$) with the periodic (open) boundary condition, unless N is specified otherwise.

Gapless superconductivity and confined states. Using the model for superconducting Ammann-Beenker QCs in Eq. (1), we examine the bulk energy spectrum with the periodic boundary condition. Figure 1(a) [1(b)] shows the magnitude of the converged order parameter in the absence (presence) of Rashba spin-orbit coupling as a function of $|h_z|/t$, along with the result for a larger system with $N = 16238$. The red, black, and blue data are, respectively, the maximum, mean, and minimum values of $\{|\Delta_i|\}$. The mean value is defined as $\bar{\Delta} = (1/N) \sum_i |\Delta_i|$. Superconductivity is destroyed at $h_z \simeq -0.195t$ ($-0.366t$) when $\lambda_R = 0$ ($0.5t$), with the order parameter vanishing at most (all) sites.

Figures 1(c) and 1(d) present the lowest absolute energy $\varepsilon_{\min}$ of quasiparticle excitation for $\lambda_R = 0$ and $0.5t$, respectively. In both cases, the bulk energy gap is suppressed as $|h_z|/t$ increases. The bulk gap closes at $h_z \simeq -0.17t$ ($-0.265t$) for $\lambda_R = 0$ ($0.5t$), while the superconducting order parameter remains finite at all sites. We have thus

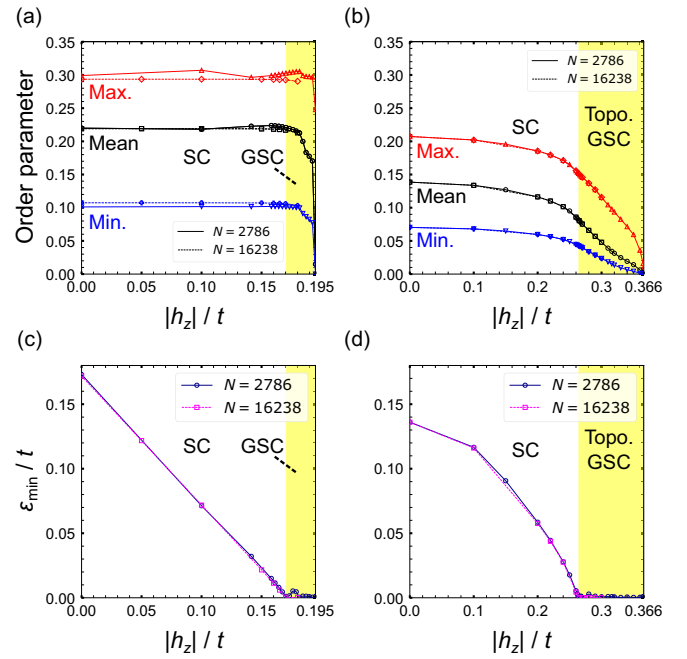


FIG. 1. (a), (b) Maximum, mean, and minimum absolute values of the superconducting order parameter scaled by t as a function of $|h_z|/t$ for $\lambda_R = 0$ and $0.5t$, respectively, in the Ammann-Beenker QC with the periodic boundary condition. SC (GSC) indicates gapful (gapless) superconducting phase. (c), (d) Lowest absolute excitation energy $\varepsilon_{\min}/t$ as a function of $|h_z|/t$ for $\lambda_R = 0$ and $0.5t$, respectively. The results for $N = 16238$ are also shown. In (b) and (d), the gapless superconducting phase is topologically nontrivial. The finite gap near $|h_z|/t \simeq 0.18t$ in (c) for $N = 2786$ is an artifact due to relatively small size.

found a stable gapless superconducting phase in Ammann-Beenker QCs at half filling under magnetic field. This quasicrystalline gapless superconductivity is qualitatively different from that in disordered systems, in which strong impurities make the order parameter vanish at some lattice sites [44,45,52]. We have also confirmed the occurrence of gapless superconductivity in the system with $N = 16238$ as can be seen in Figs. 1(c) and 1(d). Moreover, we show in the Supplemental Material [50] that gapless superconductivity appears even when the system is not exactly at half filling.

We examine the DOS of the superconducting Ammann-Beenker QC to understand the mechanism of gapless superconductivity. Presented in Figs. 2(a1)–2(a5) is the DOS for $\lambda_R = 0$ and varying h_z/t . In the absence of magnetic field [Fig. 2(a1)], the superconductor has $\bar{\Delta}_0 \equiv \bar{\Delta}(h_z = 0) \simeq 0.22t$ [Fig. 1(a)] and $\varepsilon_0 \equiv \varepsilon_{\min}(h_z = 0) \simeq 0.173t$ [Fig. 1(c)]. Without spin-orbit coupling, we need a Zeeman energy comparable to the bulk superconducting gap ε_0 for realizing gapless superconductivity [53]. In periodic crystals, such strong Zeeman field would destroy conventional superconductivity as $|h_z| \geq \varepsilon_0 = \bar{\Delta}_0$, which exceeds the Chandrasekhar-Clogston limit [54,55]. On the contrary, Ammann-Beenker QCs at half filling exhibit gapless superconductivity under magnetic field. With nonzero Zeeman field, the twofold degeneracy of each orbital state is lifted and quasiparticle states near the gap edges approach zero energy as $|h_z|$ increases, reducing the energy gap

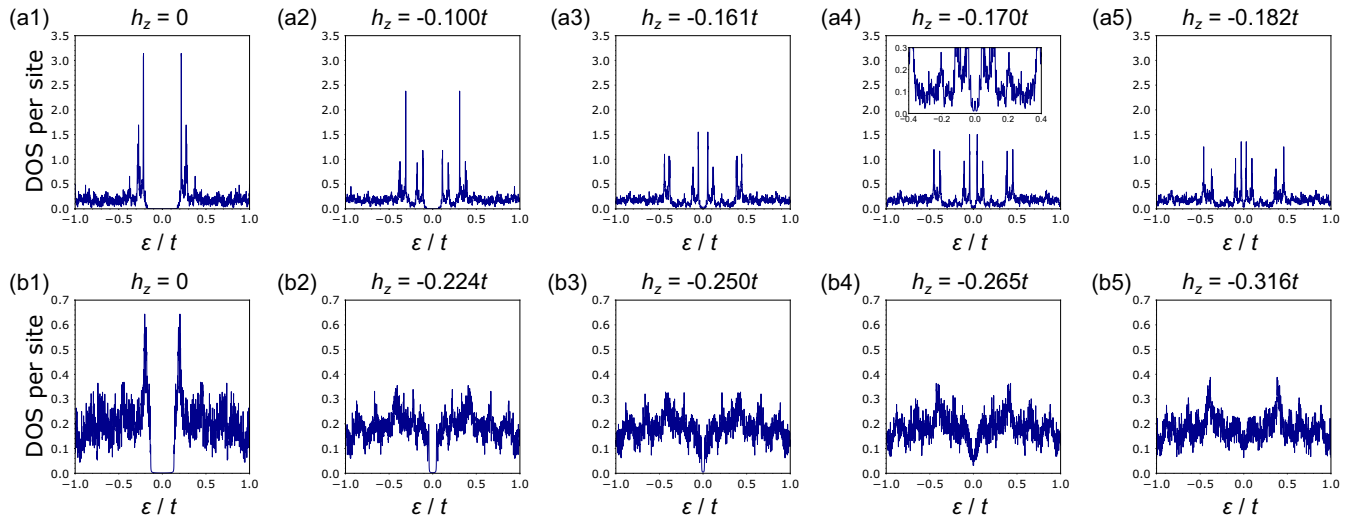


FIG. 2. Dependence of the DOS of the superconducting Ammann-Beenker QC on h_z/t for $\lambda_R = 0$ [(a1)–(a5)] and $0.5t$ [(b1)–(b5)] with the periodic boundary condition. The spectral gap closes at $h_z \simeq -0.170t$ and $-0.265t$ for $\lambda_R = 0$ and $0.5t$, respectively, and the DOS in (a4), (a5), (b4), and (b5) shows gapless superconductivity. A magnified view is given in the inset in (a4).

[Figs. 2(a2) and 2(a3)]. The gap closes at $|h_z| \simeq 0.17t \simeq \varepsilon_0$ [see the inset in Fig. 2(a4)] and a gapless superconducting phase emerges [Figs. 2(a4) and 2(a5)].

Gapless superconductivity can be realized in Ammann-Beenker QCs because a large number of confined states and broken translational symmetry allow the QC to satisfy $\varepsilon_0 < \bar{\Delta}_0$, a favorable condition for gapless superconductivity to occur. First, at or close to half filling, $\bar{\Delta}_0$ is strongly enhanced due to highly degenerate confined states in the normal state and is much larger than that at low or high filling. The strong degeneracy of confined states results in the sharp coherence peaks as can be seen in Fig. 2(a1). Second, as also visible in Fig. 2(a1), the gap-edge singularity is smeared by the underlying inhomogeneity of the quasicrystalline lattice, i.e., lowest-energy quasiparticle states spread toward zero energy in the DOS [40,52,56]. These two factors combine to yield $\varepsilon_0 < \bar{\Delta}_0$ in Ammann-Beenker QCs at half filling. This mechanism of gapless superconductivity originating in confined states and the lack of translational symmetry is further elucidated in the Supplemental Material [50], where it is also shown that gapless superconductivity disappears as one moves away from half filling when $\varepsilon_0 \simeq \bar{\Delta}_0$.

When Rashba spin-orbit coupling is present, confined states are all destroyed. It can be seen in Figs. 2(b1)–2(b5) for $\lambda_R = 0.5t$ (note the reduced scale for the DOS) that the coherence peaks are reduced drastically for $h_z = 0$ and almost nonexistent for $h_z \neq 0$. Even though $\varepsilon_0 \simeq 0.136t$ is smaller than that for $\lambda_R = 0$ and the bulk gap decreases quickly as $|h_z|$ increases [Figs. 2(b2) and 2(b3)], stronger Zeeman field is required for gapless superconductivity to occur at $|h_z| \simeq 0.265t$ [Figs. 2(b4) and 2(b5)].

Topology of gapless superconductivity. Spin-orbit coupling can lead to various topological phases in crystals that can host topological boundary modes [57,58]. Motivated by this, we investigate whether Rashba spin-orbit coupling can induce nontrivial topology in gapless superconductivity in Ammann-Beenker QCs—despite the lack of a bulk energy gap as well as translational symmetry. To explore topological properties,

we utilize a spectral localizer [47,48,59]. Topological classification based on the spectral localizer is applicable to gapless and aperiodic systems. The model in Eq. (1) belongs to class D in the Altland-Zirnbauer classification since it describes a superconductor with broken time-reversal symmetry due to the Zeeman field [60]. For class D in two dimensions, the spectral localizer $L(X, Y, \mathcal{H}_{\text{BdG}})$ is defined as [11,47,48]

$$L_{(x,y)}(X, Y, \mathcal{H}_{\text{BdG}}) = \begin{pmatrix} \kappa(X - xI) & \kappa(Y - yI) - i\mathcal{H}_{\text{BdG}} \\ \kappa(Y - yI) + i\mathcal{H}_{\text{BdG}} & -\kappa(X - xI) \end{pmatrix}, \quad (5)$$

where (x, y) is the coordinate, X (Y) is the position operator for the x (y) component of a lattice site, and I is the identity matrix. Here, $\kappa > 0$ is a scaling coefficient to compensate the different dimensions of the position operators and the Hamiltonian. In terms of the spectral localizer in Eq. (5), a topological invariant called a pseudospectrum invariant $C_L(x, y)$ is defined by [47,48,59]

$$C_L(x, y) = \frac{1}{2} \text{sig}[L_{(x,y)}(X, Y, \mathcal{H}_{\text{BdG}})], \quad (6)$$

where $\text{sig}[L_{(x,y)}]$ denotes the difference between the numbers of positive and negative eigenvalues of the spectral localizer $L_{(x,y)}$. The system is topologically nontrivial if the pseudospectrum invariant is nonzero in some areas of the system in real space.

We have calculated the eigenvalue spectrum $\sigma(L_{(x,y)})$ of the spectral localizer in the Ammann-Beenker QC presenting gapless superconductivity, with the open boundary condition. Figure 3(a) shows the spectrum $\sigma(L_{(x,y)})$ in units of t as a function of x along the red-dotted path ($y = 0$) in Fig. 3(c) for $\lambda_R = 0$ and $h_z = -0.182t$. $\kappa = 0.02t$ was used in Eq. (5) for this calculation. We find that without Rashba spin-orbit coupling, the superconducting system has no area where the pseudospectrum invariant is nonzero, irrespective of the magnetic field. Namely, the system is topologically trivial before and after the bulk gap closing.

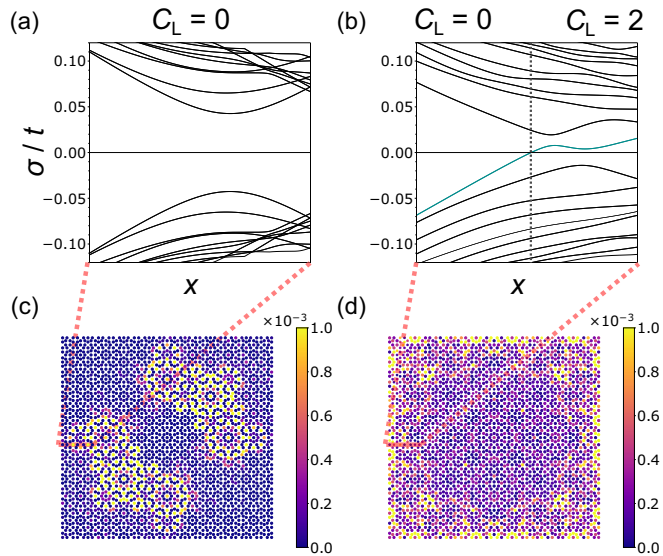


FIG. 3. Eigenvalue Spectra $\sigma(L_{x,y})$ of the spectral localizer as a function of x along the red-dotted path ($y = 0$) in (c) and (d), respectively, for (a) $\lambda_R = 0$ and $h_z = -0.182t$ and (b) $\lambda_R = 0.5t$ and $h_z = -0.316t$. In (b), the eigenvalue that crosses zero is highlighted in cyan. (c), (d) Probability distribution of a bulk mode and an edge mode, respectively, in the Ammann-Beenker QC with the open boundary condition.

In contrast, when spin-orbit coupling is present, gapless superconductivity can be topologically nontrivial. The nontrivial topology can be confirmed by $C_L(x, y)$ in Eq. (6) being nonzero. In Fig. 3(b) $\sigma(L_{x,y})/t$ is plotted as a function of x along the red-dotted path in Fig. 3(d) [the same path as for Fig. 3(a)] for $\lambda_R = 0.5t$ and $h_z = -0.316t$. As x varies from the edge toward the bulk, $C_L(x, 0)$ changes from 0 to 2 as a doubly degenerate eigenvalue of the spectral localizer crosses zero [Fig. 3(b)]. Therefore, this gapless phase is topologically nontrivial. In the Supplemental Material [50], we discuss topological phase transitions for gapless superconductivity in detail. We note that the same model in Eq. (1) on a square lattice does not exhibit gapless topological superconductivity in the presence of Rashba spin-orbit coupling [23,24].

The gap of the spectral localizer closing indicates the existence of edge modes associated with a nonzero pseudospectrum invariant in the bulk, regardless of whether they

are hybridized with bulk states [48]. In a gapless system with continuous spectrum across zero energy, edge modes are expected to hybridize with bulk states, with their energy lifted from zero as a result. In the absence of Rashba spin-orbit coupling, we have not found any edge state in the gapless superconducting phase. The probability distribution of a bulk state with energy $\varepsilon \simeq 9 \times 10^{-5}t$ is shown for $h_z = -0.182t$ in Fig. 3(c). With nontrivial topology induced by Rashba spin-orbit coupling, edge states do appear with near-zero energy. This is illustrated for $\lambda_R = 0.5t$ and $h_z = -0.316t$ in Fig. 3(d), where the probability distribution of an edge state with energy $\varepsilon \simeq 3.2 \times 10^{-3}t$ is shown.

The pseudospectrum invariant is equivalent to the Chern number in systems with a bulk energy gap [61,62]. In a topological gapless superconductor, if the bulk spectrum can be made to have a gap about zero energy while $C_L(x, y)$ remains nonzero in the bulk, the edge modes turn out to be Majorana zero modes. Therefore, the edge state presented in Fig. 3(d) can be regarded as a Majorana mode buried in the bulk spectrum.

Conclusion. In this Letter, we have demonstrated gapless superconductivity in Ammann-Beenker quasicrystals at half filling under magnetic field. We have revealed that gapless superconductivity stems from interplay of broken translational symmetry and highly degenerate confined states, which are both characteristic of quasicrystals. In the presence of Rashba spin-orbit coupling, the gapless superconducting phase can be topologically nontrivial. The nontrivial topology can be identified by a nonzero pseudospectrum invariant given by a spectral localizer applicable to gapless and aperiodic systems. The quasicrystalline gapless superconducting phase hosts edge modes protected by a nonzero pseudospectrum invariant in the bulk. Our findings on Ammann-Beenker quasicrystals are expected to hold in other quasicrystals that exhibit a flat band of confined states [25–28,33,34,37]. Quasicrystals can be a unique platform for realizing gapless superconductivity with nontrivial topology in the bulk, presenting topological edge modes.

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