

Statistics of the random matrix spectral form factor

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The spectral form factor of random matrix theory plays a key role in the description of disordered and chaotic quantum systems. While its moments are known to be approximately Gaussian, corrections subleading in the matrix dimension, D , have recently come to researchers' attention, with conflicting results in the literature. In this work, we investigate these departures from Gaussianity for both circular and Gaussian ensembles. Using two independent approaches—sine-kernel techniques and supersymmetric field theory—we identify the form factor statistics to next leading order in a D^{-1} expansion. Our sine-kernel analysis highlights inconsistencies with previous studies, while the supersymmetric approach backs these findings and suggests an understanding of the statistics from a complementary perspective. Our findings fully agree with numerics. They are presented in a pedagogical way, highlighting pathways (and pitfalls) in the study of statistical signatures at next leading order, which are increasingly becoming important in applications.

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I. INTRODUCTION

The spectral form factor

$$\text{SFF} \equiv |\text{tr} U_t|^2 \quad (1)$$

is one of the most important observables in the characterization of disordered or chaotic quantum systems [1–4]. It owes its significance to a combination of factors: The SFF probes the dynamics of the time evolution operator U_t and its Fourier transform energy-dependent correlations in spectra, both key signatures of a system. The time dependence of numerous observables, such as correlation functions between operators [5–7], out of time order correlations (OTOC) functions [8,9], or return probabilities [10] approximately or even exactly reduce to the SFF. Finally, the SFF has come to prominence in the context of many-body physics [11–15] and holography [16–21], where it features in the effective boundary description of two-dimensional gravitational theories.

The spectral form factor is commonly discussed with reference to three temporal regimes: at times $t < t_{\text{Th}}$ smaller than an effective *Thouless time*, it probes physical relaxation processes preceding the entry into an ergodic regime at times $t > t_{\text{Th}}$. (Our focus here will be on systems with late-time ergodic behavior.) Once ergodic, it typically increases in a linear or near-linear manner—the so-called “ramp regime”—up to a time $t_{\text{H}} = 2\pi/\Delta$, the *Heisenberg time* defined as the inverse

of a system's characteristic level spacing Δ . At later times, the form factor approaches constant behavior, the “plateau regime.” Our focus in this paper will be on ergodic late-time physics, ramp, and plateau.

Even in the ergodic regime, the spectral form factor remains a statistically distributed quantity and is subject to three different sources of fluctuations:

(1) The SFF of generic chaotic quantum systems typically shows deviations from that of random matrix theory (RMT), even deep in the ergodic regime at times $t \simeq t_{\text{H}}$. These deviations are due to large sample-to-sample fluctuations of the spectral density, which in many-body systems generally exceed those of Gaussian matrix ensembles. The deviations from the RMT form factor caused by these fluctuations have become a subject of study in their own right [22–26].

(2) The form factor makes reference to physical time so that, by energy/time uncertainty, it includes an integral over energies [cf. Eqs. (2) and (3) below]. The nonconstancy of the spectral density of a quantum system affects the form factor and its statistical moments via this convolution mechanism [27].

(3) Even within RMT, the spectral form factor is a famously non-self-averaging quantity whose statistical fluctuations are of about the same order as the average, and not diminishing in the large system size limit [28–30].

In generic interacting quantum systems, these three fluctuation channels act in combination, leading to intricate statistics. Their importance is highlighted by the fact that the quantum systems relevant to present day applications are typically large, $D \gg 1$, but not thermodynamically large. Some of the fluctuation channels mentioned above diminish in the large system size limit, $D \rightarrow \infty$, but not all of them do.

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Finally, statistical data on the form factor are typically collected by sampling over a finite number of runs N_{sim} , either on a computer or in experiments. This opens a fourth fluctuation channel reflecting the statistics of a finite data set. In view of this level of complexity, it is important to understand the statistics of the random matrix SFF as a "null-model" before comparing it to realistic quantum systems.

The statistics of the form factor in RMT has been the subject of a number of previous studies [18,31–35], a key finding being that it is approximately Gaussian, which at the second order implies $\overline{\text{SFF}^2} \simeq 2(\overline{\text{SFF}})^2$, where $\overline{}$ is the ensemble average. Deviations from Gaussianity are subleading in D but otherwise universal signatures of the quantum ergodic phase. They, too, have come under scrutiny, with a plethora of not entirely consistent results: Ref. [35] rigorously computes $\overline{\text{SFF}}$ and $\text{Var}(\text{SFF})$ for Wigner matrices, but only for short times $t \ll N^{5/17}$, where it is dominated by the edge spectrum. Reference [31] agrees with Ref. [35] when such an edge effect is neglected, but it also predicts the emergence of non-Gaussian statistics linearly increasing in time, t , at $t = t_H/2$. The sudden (nonanalytic) onset of this correction is indicative of a nonperturbative effect, similar to the celebrated "ramp-plateau" transition in the average SFF. Finally, Refs. [18,32] predict a more complicated time dependence containing a linear term throughout the semiclassical regime $t < t_H/2$, and contributions of next leading order in the $1/D$ expansion. (We note that for [18] this equation is ancillary to the paper's main results.)

In the comparison of these results, it is important to remember that the SFF is subject to energy averaging, the second channel of fluctuations mentioned above: The SFF contains physical time in the dimensionless combination $\tau \equiv t\Delta/2\pi$, where $\Delta \sim D^{-1}$ is the characteristic level spacing. Being defined in terms of an unconstrained trace $\text{SFF} = |\text{tr}(U_t)|^2$ the scale $\Delta = \Delta(\epsilon)$ enters averaged over the entire spectrum blurring otherwise sharply defined structures over scales $\sim \Delta^{-1}$ [27]. This mechanism is absent in *circular ensembles*, defined in terms of Haar-distributed unitary operators with statistically uniform eigenphase spacing, $\Delta = 2\pi/D$, making them attractive candidates for the study of the more interesting "intrinsic" statistical fluctuations, channel 3.

In this paper we analyze the statistics of the SFF in random matrix theory and discuss the departures from Gaussianity, which are nonperturbative and universal. We apply a combination of different methods—random matrix techniques, field theory, diagrammatic perturbation theory, and direct diagonalization—to analyze the statistics, emphasis put on the variance. Our numerical analysis focuses on models with relatively small D , where computing power is channeled into runs over many realizations and the deviations from Gaussian statistics are clearly resolved, as shown in Fig. 1, where we exemplify our results.

As summarized below, we analytically confirm and numerically validate the results of Ref. [31] for circular ensembles. Indeed, the results of that reference follow by Fourier transformation from mathematically rigorous results for CUE spectral determinants [36] and on this basis are exact. We demonstrate that a conflicting formula for $\overline{\text{SFF}^2}$ in Refs. [18,32] results from computational mistakes. Referring to the mathematical

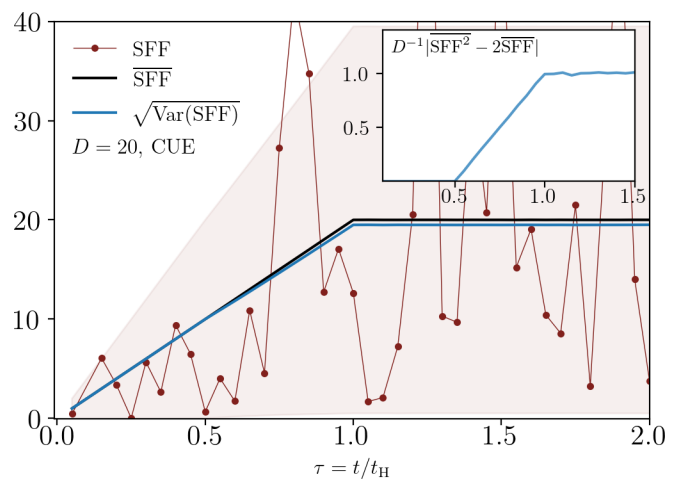


FIG. 1. Numerical summary of the paper. The spectral form factor (SFF) of a single realization (red dots) fluctuates over time around its average $\overline{\text{SFF}}$ (black line), with a variance $\text{Var}(\text{SFF}) = \overline{\text{SFF}^2} - \overline{\text{SFF}}^2$ proportional to the average in the large- D limit (blue line), i.e., the Gaussian distribution. The amplitude of these fluctuations is indicated by the shaded red region. The data show deviations from the Gaussian behavior for $\tau = t/t_H > 1/2$, as emphasized in the inset, which shows $D^{-1}|\overline{\text{SFF}^2} - 2\overline{\text{SFF}}^2|$. This work addresses precisely such deviations from Gaussianity, providing two analytical frameworks to account for them. Numerical simulation from a CUE for $D = 20$, average with $N_{\text{sim}} = 2 \times 10^7$ samples.

Ref. [35], which critically reviewed common analytical approaches to random Hermitian matrices, we qualitatively discuss our understanding of their status in the analysis of $1/D$ -subleading signatures to the SFF statistics.

In the next section, we provide a brief summary of our results. The rest of the paper is organized by technique: in Sec. II we use the sine-kernel approach of random matrix theory. After a review of the method and notations, we compute the second moment of the spectral form factor, emphasizing the differences with the formula for the second moment in Refs. [18,32]. In Sec. III we use the supersymmetric method, employing a twofold replicated nonlinear sigma model.

A. Summary of results

In this paper we study the statistical moments of the spectral form factor $\text{SFF} = |\text{tr} U_t|^2$ where U_t is a $D \times D$ matrix sampled from a random matrix theory ensemble. It is well known that in the large D limit, the connected part of the spectral form factor,

$$\text{SFF}_c = |\text{tr} U_t|^2 - |\overline{\text{tr} U_t}|^2,$$

contains universal properties about the spectral correlations of the level density $\rho(\lambda) = \sum_{i=1}^D \delta(\lambda - \lambda_i)$ where $\{\lambda_i\}_{i=1,\dots,D}$ are the eigenphases of the unitary generating the dynamics U_t . For definiteness, we focus on the unitary case and consider (1) evolution generated by the circular unitary ensemble (CUE), i.e., the case $U_t = U^t$, where $t \in \mathbb{N}$ is discrete-time and U is sampled uniformly from the unitary group and (2) $U_t = e^{-iHt}$,

where H is sampled according to the Gaussian unitary ensemble (GUE).

The statistical properties of the SFF in the ergodic regime can be summarized via its moments as follows:

$$\overline{(\text{SFF}_c)^n} = D^n \left[n! z(t)^n + \frac{1}{D} \text{Conn}_n(t) + \mathcal{O}\left(\frac{z(t)^n}{\sqrt{N_{\text{sim}}}}\right) \right]. \quad (2)$$

Here the leading term is given by the Gaussian contribution, expressed in terms of the average of the SFF, denoted $z(t)$, which is given by the Fourier integral

$$z(t) = \frac{1}{D} \int d\lambda_1 d\lambda_2 \bar{\rho}_c(\lambda_1, \lambda_2) e^{i(\lambda_1 - \lambda_2)t}, \quad (3)$$

where $\bar{\rho}_c(\lambda_1, \lambda_2) = \overline{\rho(\lambda_1)\rho(\lambda_2)} - \bar{\rho}(\lambda_1)\bar{\rho}(\lambda_2)$ is the connected part of the two-point correlation function of the spectral density. Specifically, the SFF of the CUE with its constant density of states shows the celebrated ramp-plateau behavior [31]:

$$z(t) = \begin{cases} t/t_H & \text{for } t < t_H \\ 1 & \text{for } t \geq t_H \end{cases}. \quad (4)$$

For the GUE, this result holds only approximately. Taking the nonconstant spectral density into account leads to the modified result, also valid for finite D , that for $t < \frac{4}{\pi}t_H$ reads [27]

$$z(t) = \frac{1}{2} \frac{t}{t_H} \sqrt{1 - \left(\frac{\pi}{4} \frac{t}{t_H}\right)^2} + \frac{2}{\pi} \arcsin\left(\frac{\pi}{4} \frac{t}{t_H}\right), \quad (5)$$

while it is one otherwise.

Second, the term $\mathcal{O}(\cdot)$ indicates the error resulting from an average over a finite number of simulations N_{sim} . It is of the order of $z(t)^n$, since SFF is approximately Gaussian (the variance of SFF^n scales as the square of its average), and it is suppressed as $N_{\text{sim}}^{-1/2}$ due to the central limit theorem.

Finally, $\text{Conn}_n(t)$, denotes the subleading connected contribution to the higher moments of the spectral form factor (it vanishes for $n = 1$). Its computation for $n = 2$ is the main result of the present study.

The rest of the paper is summarized as follows:

(1) We review how, to leading order in D , the SFF shows approximately Gaussian statistics, i.e., $\overline{\text{SFF}^n} = n! z(t)^n$. This also determines the size of the oscillatory error $\mathcal{O}(\cdot)$, due to the sample-to-sample fluctuations in numerical averages. We show how this asymptotic Gaussian result can be understood intuitively and in terms of straightforward perturbation theory.

(2) We compute the subleading corrections at finite D by focusing on $n = 2$. They read

$$\text{Conn}_2(t) = z(2t) - 2z(t)^2. \quad (6)$$

When considering the CUE, this result reduces to a linear regime between $t_H/2$ and t_H [31], thus reproducing the data in the inset of Fig. 1. Equation (6) will follow from the application of two complementary analytical approaches:

(a) Random matrix theory provides an explicit result for higher-order spectral correlations in terms of a determinant of the so-called sine-kernel, which allows an explicit computation of the second moment of the spectral form factor, and in particular the subleading connected correction in Eq. (6). This approach has been previously pursued

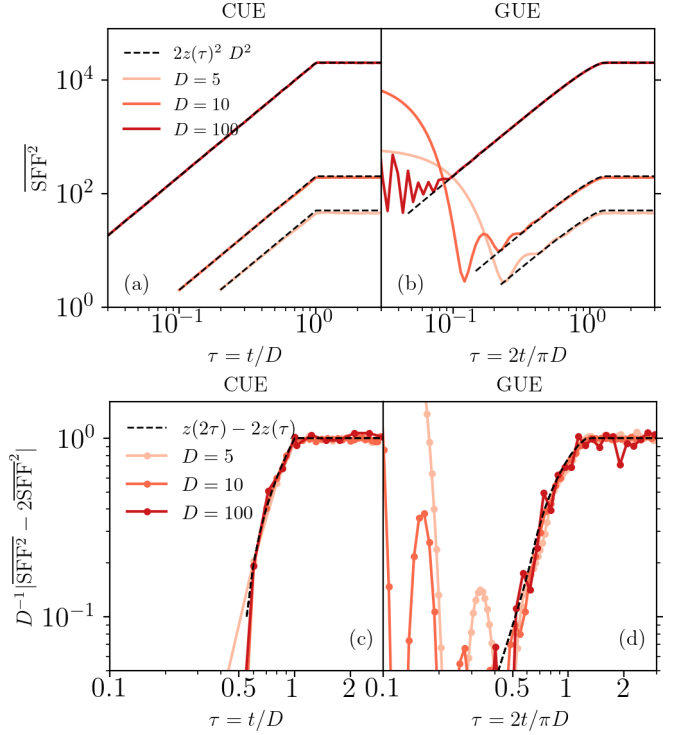


FIG. 2. Leading (a, b) and subleading (c, d) behavior of the second moment of the spectral form factor. Results were sampled from the CUE (left) and for the evolution of a GUE Hamiltonian (right). The numerical data for $D = 5, 10, 100$ (red lines with increasing color intensity) are compared with the analytical prediction [$2z(\tau)^2$ (a, b) and $2z(\tau) - z(2\tau)$ (c, d)] (dashed black) using Eq. (4) and Eq. (5) in the two panels. The empirical average is obtained with $N_{\text{sim}} = (5 \times 10^4, 5 \times 10^5, 2 \times 10^6)$ with $D = 5, 10, 100$, respectively.

in Refs. [18,32] with, however, some computational errors affecting the formula for $\overline{\text{SFF}^2}$.

(b) We apply the *supersymmetric sigma model*, fourfold replicated to compute the fourth moment of the density of states. The integration manifold featuring in this framework contains a discrete number of saddle points, which are not accessible in perturbation theory and are responsible for the non-Gaussian contributions.

(3) These findings are confirmed numerically, as shown in Fig. 2 for the leading and subleading behavior, in the two ensembles CUE and GUE. The plots show numerical averages obtained for large values of N_{sim} , such that the sampling fluctuations are already suppressed, as discussed in Sec. II A 1.

II. RANDOM MATRIX THEORY

In this section we review the statistics of the SFF [cf. Eq. (2)] as described by techniques of Random Matrix Theory. Specifically, we will consider evolution generated by two standard ensembles:

(1) The CUE, where

$$U_t = U^t \quad \text{with} \quad U \sim \text{uniform} \quad t \in \mathbb{N}, \quad (7)$$

U being sampled uniformly from the unitary group. In this case, the eigenphases of U , λ_i , are uniformly distributed over

the unit circle, leading to a constant density of states $\bar{\rho}(\lambda) = \frac{D}{2\pi}$, such that $t_H = 2\pi\bar{\rho} = D$.

(2) The GUE, where

$$U_t = e^{-iHt} \text{ with } P(H) \propto \exp\left[-\frac{D}{2}\text{Tr}(H^2)\right]. \quad (8)$$

The average density of states here is $\bar{\rho}(\lambda) = \frac{D}{2\pi}\sqrt{4-\lambda^2}$, so one has D eigenvalues in an interval of size 4, and the mean level density can be estimated as $\bar{\rho} = D/4$ and $t_H = D\frac{\pi}{2}$ [37].

We begin in Sec. II A with a discussion of the "asymptotically Gaussian" statistics of the SFF, in the $D \rightarrow \infty$ limit. On this basis, we will discuss the size of the fluctuations in empirical, i.e., numerical averages, explaining the first and third terms in Eq. (2). In Sec. II B, we will then employ the sine-kernel formalism to derive the subleading corrections to the Gaussian result, namely, the second term in Eq. (2).

A. Gaussian statistics of the SFF for $D \rightarrow \infty$

We begin by formulating a simple argument for the approximately Gaussian behavior. Let us define the random variable

$$u_t = \text{tr}(U_t) = \sum_i e^{i\lambda_i t} \quad (9)$$

and represent it as the Fourier transform of the density of states

$$u_t = \int d\lambda \rho(\lambda) e^{i\lambda t}. \quad (10)$$

In the CUE, the constancy of $\bar{\rho}$ implies the vanishing of its average $\bar{u}_t$ for any $t > 0$, namely,

$$\bar{u}_t = \int d\lambda \bar{\rho}(\lambda) e^{i\lambda t} = \frac{D}{2\pi} \delta(t). \quad (11)$$

By contrast, the Fourier transform of the semi-circular spectral density of the GUE is given by

$$\bar{u}_t = \int d\lambda \bar{\rho}(\lambda) e^{i\lambda t} = D \frac{J_1(2t)}{t}, \quad (12)$$

where $J_1(t)$ is the Bessel function of the first kind. For times $t \gtrsim 1$ exceeding the microscopically short inverse of the band width, this function, too, is approximately vanishing. As discussed in the introduction, we refer at the time regime where $\bar{u}_t$ is suppressed as the "ergodic regime."

Hence, the fluctuation part

$$u_f(t) = u_t - \bar{u}_t \quad (13)$$

is the Fourier transform of $\rho_f(\lambda) = \rho(\lambda) - \bar{\rho}(\lambda)$. In the ensembles under consideration, in the large D limit, the spectral density becomes approximately Gaussian distributed [31,35] with variance

$$\bar{\rho}_c(\lambda_1, \lambda_2) \equiv \overline{\rho(\lambda_1)\rho(\lambda_2)} - \bar{\rho}(\lambda_1)\bar{\rho}(\lambda_2).$$

This statistics is inherited by u_t , implying that in the large D limit $u_f(t)$ is a Gaussian random variable with zero average and variance

$$\overline{|u_f|^2} = D z(t), \quad (14)$$

where $z(t)$ —defined in Eq. (3)—encodes universal properties of the spectrum and can be computed via RMT techniques as

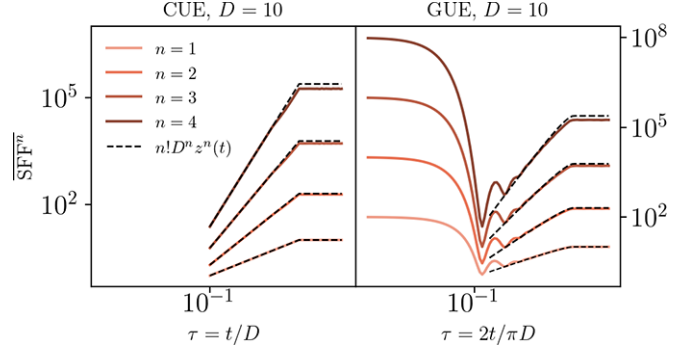


FIG. 3. Higher-order moments of the spectral form factor, $\overline{\text{SFF}^n}$ for $n = 1, 2, 3, 4$ (orange lines with increasing intensity) compared with the Gaussian prediction, cf. Eq. (15) (black dashed line). Results are sampled from the CUE (left panel) and for the evolution of a GUE Hamiltonian (right panel) using $z(t)$ from Eq. (4) and Eq. (5) in the two panels. The numerical data at fixed $D = 10$ averaged over $N_{\text{sim}} = 5 \times 10^5$ samples.

discussed below in Sec. II B. On this basis, the moments of the SFF, which are the moments of $u_f(t)$, are those of a Gaussian random variable [38]:

$$\lim_{D \rightarrow \infty} \overline{\text{SFF}_c^n} = \lim_{D \rightarrow \infty} \overline{|u_f(t)|^{2n}} = n! [D z(t)]^n. \quad (15)$$

This behavior is confirmed by numerically evaluating the moments of the spectral form factor up to $n = 4$, as reported in Fig. 3 in the two ensembles of interest. As the plot shows, the numerical averages of $\overline{\text{SFF}^n}$ follow the Gaussian prediction $n! D^n z^n(t)$, with some deviations, which are visible here for a small matrix of size $D = 10$.

1. Variances and empirical averages

Unlike many other observables characterizing random systems, the SFF is not self-averaging in the $D \rightarrow \infty$ limit [28,29]. Rather, its approximate Gaussianity in the large D limit implies fluctuations of about the same size as the average:

$$\text{Var}(\text{SFF}) = \overline{\text{SFF}^2} - \overline{\text{SFF}}^2 = [D z(t)]^2 = \overline{\text{SFF}}^2, \quad (16)$$

as numerically verified in Fig. 1. This property is true at every order n : via Eq. (15), one has

$$\begin{aligned} \text{Var}(\text{SFF}^n) &= \overline{\text{SFF}^{2n}} - \overline{\text{SFF}^n}^2 \\ &= ((2n)! - (n!)^2) [D z(t)]^{2n} \propto \overline{\text{SFF}^n}^2, \end{aligned} \quad (17)$$

showing that the variance of the n th moment scales as the square of its average.

This property has implications for the empirical (i.e., numerical) computation of expectation values $[\cdot]_{N_{\text{sim}}}$ averaged over N_{sim} samples. According to the central limit theorem, the fluctuations are suppressed with $N_{\text{sim}}^{-1/2}$ with an amplitude proportional to the standard deviation,

$$\begin{aligned} \lim_{D \rightarrow \infty} [\overline{\text{SFF}^n}]_{N_{\text{sim}}} &= \overline{\text{SFF}^n} + \mathcal{O}[\sqrt{\text{Var}(\text{SFF}^n)} N_{\text{sim}}^{-1/2}] \\ &= D^n (n! z^n(t) + \mathcal{O}[z^n(t) N_{\text{sim}}^{-1/2}]), \end{aligned} \quad (18)$$

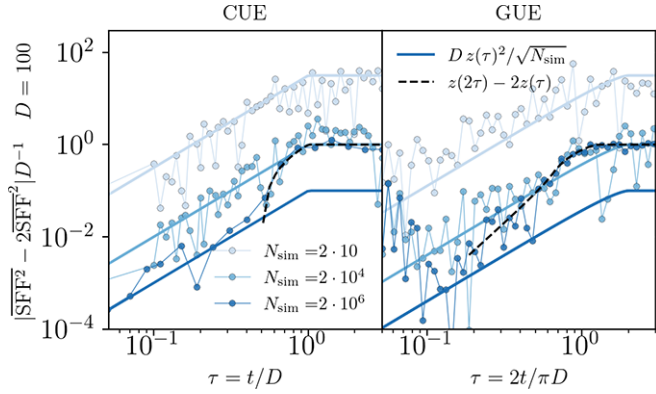


FIG. 4. Oscillations in the subleading correction to the second moment of the spectral form factor for different empirical averages over N_{sim} simulations. Results are sampled from the CUE (left panel) and for the evolution of a GUE Hamiltonian (right panel). The numerical data at fixed $D = 100$ averaged over $N_{\text{sim}} = 20, 2 \times 10^4, 2 \times 10^6$ (blue lines with increasing color intensity) are compared with the prediction in Eq. (2), with full blue lines we indicate the amplitude of the oscillatory term $O[Dz^2(\tau)/\sqrt{N_{\text{sim}}}]$ and with a dashed black line the subleading correction in Eq. (6), using Eq. (4) and Eq. (5) in the two panels.

where in the second line, we have used Eq. (17) and Eq. (15). We thus have recovered the first and third terms in Eq. (2). This shows that the sampling fluctuations vanish as the number of samples N_{sim} goes to infinity, making the average well defined. Note, however, that these oscillations may be relevant when addressing numerically subleading contributions $\propto D^{-1}$ and, in those cases, one shall choose N_{sim} adequately large.

Figure 4 shows data for $\frac{1}{D}|\text{SFF}^2 - 2\text{SFF}|^2$ with $D = 100$. For few simulation runs, the distribution of data points reflects the fluctuations due to the finiteness of N_{sim} , and it is centered around the estimate functions $Dz^2(t)/N_{\text{sim}}^{-1/2}$, indicated by solid blue lines. However, for large values $N_{\text{sim}} \gg D^2$, the fluctuations converge to a different function, namely, the genuine deviation from Gaussianity $z(2t) - 2z(t)$ defined in Eqs. (2) and (6) and represented by a dashed black line in the figure. The goal of the next section is to understand the origin of this subleading term using the tools of RMT.

B. Sine-Kernel predictions

In this section we review the random matrix theory framework for analyzing moments of the SFF. We closely follow Ref. [32], where explicit calculations are presented at a high level of detail. However, we also correct a number of inconsistencies which ultimately led to incorrect final results in that reference.

Before delving into the analysis of the SFF, let us review a few elements of RMT spectral statistics. Random matrix theory describes the energy spectra in terms of correlation functions defined as

$$R_n(\lambda_1, \dots, \lambda_n) = \sum_{i_1 \neq i_2 \neq \dots \neq i_n} \overline{\delta(\lambda_1 - \lambda_{i_1}) \cdots \delta(\lambda_n - \lambda_{i_n})}, \quad (19)$$

where the overbar denotes averaging over the statistical ensemble of the eigenvalues of either of Gaussian or Haar

distributed random operators (the results summarized in this section apply to both cases unless explicitly noted). These functions are commonly defined to exclude the autocorrelation of identical levels, as indicated by the condition $i_1 \neq i_2 \dots$ beneath the sum. This exclusion leads to modifications relative to the unconditioned sum, which are small in D^{-1} , but become important when it comes to higher order cumulative correlations—we need to keep an eye on them [39].

The two lowest-order correlation functions of this type are

$$R_1(\lambda) = \overline{\rho}(\lambda), \quad (20)$$

defining the average of the spectral density

$$\rho(\lambda) = \sum_i \delta(\lambda - \lambda_i), \quad (21)$$

normalized as $\int d\lambda R_1(\lambda) = \int d\lambda \overline{\rho}(\lambda) = D$, and

$$R_2(\lambda_1, \lambda_2) = \overline{\rho(\lambda_1)\rho(\lambda_2)} - \delta(\lambda_1 - \lambda_2)\overline{\rho}(\lambda_1), \quad (22)$$

where the subtraction removes the autocorrelation part entering the first term.

A crucial result of RMT expresses the correlations between n levels as the determinant of an n -dimensional matrix,

$$R_n(\lambda_1, \dots, \lambda_n) = \det[K_D(\lambda_i, \lambda_j)]_{i,j=1}^n, \quad (23)$$

where the specific form of the kernel $K_D(\lambda, \lambda')$ depends on the symmetries of the matrix ensemble, the magnitude of the spacings $|\lambda - \lambda'|$, and the distinction unitary vs Hermitian matrix realization. However, in the case $|\lambda - \lambda'| \sim D^{-1}$, where eigenvalues are close on the scale of the level spacing, a limit of high universality is realized. The kernel $K_D(\lambda, \lambda') = K_D(\lambda - \lambda')$ then depends only on eigenvalue differences, with a single-argument function $K_D(\lambda)$ solely determined by symmetries. Specifically, for class A (aka GUE/CUE) describing Hermitian or unitary random operators, excluding further symmetry constraints, K_D is the celebrated sine kernel defined as

$$\lim_{D \rightarrow \infty} K_D(\lambda_i, \lambda_j) = \begin{cases} \overline{\rho} \frac{\sin[\pi \overline{\rho}(\lambda_i - \lambda_j)]}{\pi \overline{\rho}(\lambda_i - \lambda_j)} & \text{for } i \neq j, \\ \overline{\rho} & \text{for } i = j, \end{cases} \quad (24)$$

where $\overline{\rho} \approx \overline{\rho}(\lambda_i) \approx \overline{\rho}(\lambda_j)$ is the characteristic spectral density at the point where the spectrum is probed.

On this basis, we now turn to the analysis of the spectral form factor, whose k th moment is defined as

$$\begin{aligned} \overline{\text{SFF}^k} &= \overline{[\text{Tr}(e^{-iHt})\text{Tr}(e^{iHt})]^k} \\ &= \overline{\sum_{i_1 \dots i_k, j_1 \dots j_k} e^{i(\lambda_{i_1} + \dots + \lambda_{i_k} - \lambda_{j_1} - \dots - \lambda_{j_k})t}} \\ &= \int_{\lambda} \overline{\rho(\lambda_1) \dots \rho(\lambda_{2k})} e^{i(\lambda_1 + \dots + \lambda_k - \lambda_{k+1} - \dots - \lambda_{2k})t} \end{aligned} \quad (25)$$

in terms of eigenvalues, where here and throughout $\int_{\lambda}$ is shorthand notation for the integration over all λ variables. In the following, we use this representation as a starting point to obtain explicit representations in terms of random matrix correlation functions for the cases $k = 1$ and 2.

1. First moment of the SFF

As a warmup, consider Eq. (25) for $k = 1$,

$$\overline{\text{SFF}} = \int_{\lambda} \overline{\rho(\lambda_1)\rho(\lambda_2)} e^{i(\lambda_1 - \lambda_2)t}. \quad (26)$$

Using the definition of R_2 in Eq. (22) and its expression given by determinant formula in Eq. (23),

$$R_2(\lambda_1, \lambda_2) = \bar{\rho}(\lambda_1)\bar{\rho}(\lambda_2) - K(\lambda_1, \lambda_2)^2,$$

we obtain

$$\begin{aligned} \overline{\text{SFF}} &= D + \int_{\lambda} R_2(\lambda_1, \lambda_2) e^{i(\lambda_1 - \lambda_2)t} \\ &= D + |\bar{u}_t|^2 - \int_{\lambda} K_D(\lambda_1, \lambda_2)^2 e^{i(\lambda_1 - \lambda_2)t}, \end{aligned} \quad (27)$$

where $\bar{u}_t$ is the Fourier transform of the average density of states; cf. Eqs. (10), (11), and (12). Accordingly, the connected part of the SFF [cf. Eq. (2)], $\overline{\text{SFF}}_c = \overline{\text{SFF}} - |\bar{u}_t|^2$, is determined by the sine kernel as

$$\overline{\text{SFF}}_c = Dz(t),$$

with

$$z(t) = 1 - \frac{1}{D} \int_{\lambda} K_D(\lambda_1, \lambda_2)^2 e^{i(\lambda_1 - \lambda_2)t}. \quad (28)$$

Assuming a constant average density of states $\bar{\rho} \equiv t_H/2\pi$, the Fourier transform of Eq. (24) then leads to the celebrated result

$$\overline{\text{SFF}}_c = D \begin{cases} \tau & \text{for } \tau < 1, \\ 1 & \text{for } \tau > 1, \end{cases} \quad \tau \equiv \frac{t}{t_H}, \quad (29)$$

showing the characteristic ramp and plateau structure with linear growth followed by instantaneous saturation at $t = t_H$.

For the GUE, the nonconstancy of the average spectral density leads to modifications of this result. A detailed analysis of the integral in Eq. (28) with account for an energy-dependent $\bar{\rho}(\lambda)$ in Eq. (24) yields Eq. (5) [27], which provides an improved basis for the comparison with numerical simulations for values of D where these modifications become sizable.

2. Second moment of the SFF

Turning to the second moment, $k = 2$ in Eq. (25), we start from the formal eigenvalue sum

$$\overline{\text{SFF}}_2 = \sum_{i_1, i_2, j_1, j_2} \overline{e^{i(\lambda_{i_1} + \lambda_{i_2} - \lambda_{j_1} - \lambda_{j_2})t}}. \quad (30)$$

The first task now is to organize this sum into contributions with a definite number of pairwise equal indices. These terms can then be individually represented in terms of the correlation functions $R_{1,\dots,4}$, Eq. (19), leading to

$$\begin{aligned} \overline{\text{SFF}}^2 &= \int_{\lambda} R_4(\lambda_1, \lambda_2, \lambda_3, \lambda_4) e^{i(\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4)t} \\ &\quad + 2 \int_{\lambda} R_3(\lambda_1, \lambda_2, \lambda_3) e^{i(2\lambda_1 - \lambda_2 - \lambda_3)t} \\ &\quad + \int_{\lambda} R_2(\lambda_1, \lambda_2) e^{2i(\lambda_1 - \lambda_2)t} \end{aligned}$$

$$\begin{aligned} &+ 4(D-1) \int_{\lambda} R_2(\lambda_1, \lambda_2) e^{i(\lambda_1 - \lambda_2)t} \\ &+ 2D^2 - D. \end{aligned} \quad (31)$$

Here the third and fourth lines have already been computed in terms of $z(t)$ in Eq. (28) as

$$\int_{\lambda} R_2(\lambda_1, \lambda_2) e^{i(\lambda_1 - \lambda_2)t} = D[z(t) - 1] + |\bar{u}_t|^2.$$

Turning to the first and second line, we use Eq. (23) to represent them through determinants of rank 4 and 3 matrices, and then do the integral over energy arguments. Referring to Appendix A for details, this leads to

$$\begin{aligned} &\int_{\lambda} R_3(\lambda_1, \lambda_2, \lambda_3) e^{i(2\lambda_1 - \lambda_2 - \lambda_3)t} \\ &= 2D[1 - z(2t)] + \text{Re}(\bar{u}_{2t} \bar{u}_{-t}^2), \end{aligned} \quad (32a)$$

$$\begin{aligned} &\int_{\lambda} R_4(\lambda_1, \lambda_2, \lambda_3, \lambda_4) e^{i(\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4)t} \\ &= 2D^2[1 - z(t)]^2 - 2D[3 - z(t) - 2z(2t)] \\ &\quad + |\bar{u}_t|^4 - 4D|\bar{u}_t|^2[1 - z(t)]. \end{aligned} \quad (32b)$$

Summing over all the contributions, the second moment of the SFF reads

$$\begin{aligned} \overline{\text{SFF}}^2 &= |\bar{u}_t|^4 + [|\bar{u}_{2t}|^2 - 4|\bar{u}_t|^2] \\ &\quad + 2[\text{Re}(\bar{u}_{2t} \bar{u}_{-t}^2) + 2|\bar{u}_t|^2 Dz(t)] \\ &\quad + 2D^2 z(t)^2 + D[z(2t) - 2z(t)]. \end{aligned} \quad (33)$$

Thus, neglecting the rapidly vanishing factors $\bar{u}_t$, this equation simplifies to

$$\overline{\text{SFF}}^2 \simeq 2D^2 z(t)^2 + D[z(2t) - 2z(t)], \quad (34)$$

where the first term corresponds to the asymptotic Gaussian result discussed in Sec. II A. However, the second term defines the subleading non-Gaussian contribution in Eq. (2) as

$$\text{Conn}_2(t) = z(2t) - 2z(t). \quad (35)$$

For the CUE, where $z(t)$ is given by Eq. (4), it assumes the form

$$\text{Conn}_2(t) = \begin{cases} 0 & \text{for } t < \frac{t_H}{2}, \\ 1 - 2t/t_H & \text{for } \frac{t_H}{2} < t < t_H, \\ -1 & \text{for } t > t_H. \end{cases} \quad (36)$$

Adding the Gaussian contribution, we obtain the final result

$$\overline{\text{SFF}}^2 = D^2 \begin{cases} 2\tau^2 & \text{for } \tau < \frac{1}{2} \\ 2\tau^2 + (1 - 2\tau)/D & \text{for } \frac{1}{2} < \tau < 1, \\ 2 - 1/D & \text{for } \tau > 1 \end{cases} \quad (37)$$

first obtained in [31] by a different method operating entirely in the time domain. Equation (37) describes the second moment of the form factor, distinguishing between three time domains. The quadratic growth for early times ($t < t_H/2$) represents purely Gaussian statistics, i.e., $\overline{\text{SFF}}^2 = (\overline{\text{SFF}})^2 + \text{var}(\text{SFF})$, with $\text{var}(\text{SFF}) = \overline{\text{SFF}}^2 = \tau^2$. This is a regime accessible to perturbation theory, as discussed in Sec. III A. The constancy for late times, $\tau > 1$, can be heuristically understood from Eq. (30). In this regime, only contributions without

oscillatory phases survive. Elementary combinatorics shows that there are $2D^2 - D$ of these, explaining the constant. If the perturbative result $\sim \tau^2$ would extend up to $\tau = 1$, we would observe a mismatch between $2D^2$ (perturbative) and $2D^2 - D$ (plateau). The form factor avoids this problem by sandwiching in a third regime between $\tau = 1/2$ and $\tau = 1$. The sudden (nonanalytic) change at $\tau = 1/2$ hints at the nonperturbative nature of this intermediate regime.

Finally, for the GUE ensemble, the substitution of Eq. (5) into Eq. (35) leads to a refined representation which we do not display explicitly but use in our comparison with simulations.

This result is in agreement with numerics, as shown in Fig. 2 above for both the CUE and GUE.

The sine-kernel approach naturally generalizes to the computation of higher-order moments SFF^k . The method remains structurally similar, with the k th moment involving $2k$ -point spectral correlation functions, expressible as determinants of $2k \times 2k$ sine-kernel matrices. While the analytic complexity increases with k , the structure remains governed by universal kernel identities. As an illustration, for $k = 3$, the dominant Gaussian contribution is $\text{SFF}^3 \approx 6D^3 z(t)^3$, while the leading subleading correction arises from connected contractions in the six-point correlation function R_6 and is expected to involve terms such as $z(3t)$, $z(2t)z(t)$, and their combinations. Although we do not compute this correction explicitly, the method's extensibility is evident, and such terms can be systematically extracted by extending the combinatorics developed similarly to Eq. (31) and computing the various integrals, such as $\int_{\lambda} R_k(\lambda_1, \lambda_2, \dots, \lambda_k) e^{i\lambda_1 t_1 + i\lambda_2 t_2 + \dots + i\lambda_k t_k}$.

In the next section we show how to retrieve the same behavior using field theory computations, reinforcing the validity of our corrected approach.

III. FIELD-THEORY CALCULATIONS

In this section we will discuss the computation of SFF statistics by field theory techniques. However, before turning to the discussion of the SFF in terms of such methods in Sec. III B, it is worthwhile taking a look at the situation within the framework of perturbation theory in D^{-1} . The advantage of this approach, reviewed in Sec. III A, is that it keeps transparent track of this expansion parameter at all stages of the computation and provides an intuitive explanation for the Gaussian correlation of the SFF discussed in Sec. II A. Readers primarily interested in final results may skip the next section and proceed directly to Sec. III B.

A. Asymptotic Gaussian SFF from perturbation theory

For definiteness, we consider a GUE ensemble of Hamiltonians $H = \{H_{\mu\nu}\}$ distributed according to Eq. (8) with second moment $|H_{\mu\nu}|^2 = D^{-1}$. The perturbative approach is best formulated in energy-Fourier space, where the retarded Green function, or resolvent,

$$G^+(\lambda) \equiv -i \int_0^\infty dt U_t e^{i(\lambda+i0)t} = (\lambda^+ - H)^{-1} \quad (38)$$

and its advanced sibling are center stage, where we use the short-hand notations $\lambda^\pm = \lambda \pm i0$. An expansion of these objects in H yields formal series, which we need to analyze

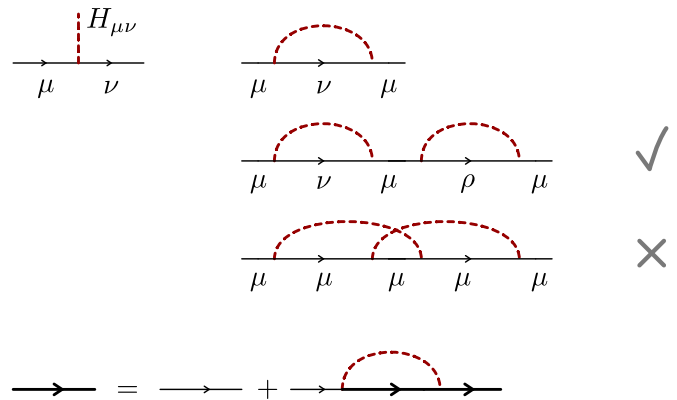


FIG. 5. Top left: Graphical representation of the expansion of the resolvent in Eq. (38) to first order in H . Top right: The averaging of a second-order term leads to index correlations as indicated. Right center: To leading order in the D^{-1} expansion, each additional order in H^2 must be accompanied by a free running index summation. The diagram shown satisfies this criterion. Right bottom: A suppressed diagram, which we disregard. Bottom: The recursive summation of all diagrams without crossing lines defines the SCBA.

to leading order in a D^{-1} expansion. This analysis is best organized in a two-stage process, where one first computes $\bar{G}^\pm \equiv G^\pm$, before turning to the computation of correlations between different Green functions.

1. Average Green function

The way in which the parameter $D \gg 1$ facilitates the computation of the Green function is illustrated in Fig. 5. At each additional order of the expansion of the resolvent in Eq. (38) in even powers of H , we must pick up a free running summation over μ to compensate for the D^{-1} in the GUE variance. In the diagrammatic code used in the figure, this amounts to the exclusion of the diagrams with intersecting lines. The survivor class of noncrossing diagrams can be summed up, and leads to the equation

$$\bar{G}^\pm(\lambda) = \frac{1}{\lambda^\pm - \bar{G}^\pm(\lambda)}.$$

Alluding to the resemblance of the denominator to that of a quantum state subject to a "self-energy" $\Sigma = G$, this equation is known as the *self-consistent Born approximation* (SCBA), or Pastur equation in the context of Random Matrix Theory [40]. The straightforward solution of the quadratic equation leads to

$$\bar{G}_{\mu\nu}^\pm(\lambda) = \frac{\delta_{\mu\nu}}{\frac{\lambda}{2} \pm i\Sigma(\lambda)} = \delta_{\mu\nu} \left(\frac{\lambda}{2} \mp i\Sigma(\lambda) \right), \quad (39)$$

with

$$\Sigma(\lambda) = \left(1 - \left(\frac{\lambda}{2} \right)^2 \right)^{1/2}.$$

From here, the average spectral density is seen to realize the celebrated semi-circular profile,

$$\bar{\rho}(\lambda) = -\frac{1}{2\pi i} \text{tr}[\bar{G}^+(\lambda) - \bar{G}^-(\lambda)] = \frac{D}{\pi} \Sigma(\lambda).$$

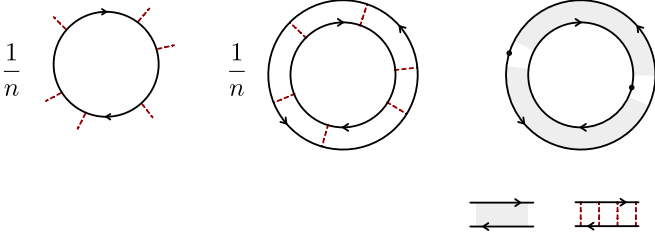


FIG. 6. Left: Perturbative expansion of the Green function generating function. The logarithmic form of the generating function leads to a factor $1/n$ at n th order. Center: Average of two generating functions, where the direction of the arrows distinguishes between retarded and advanced resolvent. Notice the thickened lines indicating that the individual resolvents are SCBA dressed. Right: Diagrammatic interpretation of the correlation function between two closed loops (traces) of Green functions. The shaded regions represent ladder diagrams, i.e., geometric series of noncrossing scattering processes. This can be interpreted as a heuristic version of the resolvent method in Ref. [35].

2. Average SFF

With these results in place, we can readily turn to the computation of the average SFF. Working in the energy representation, we represent the spectral density as the difference between retarded and advanced Green functions to obtain the connected correlation function $\bar{\rho}_c(\lambda_1, \lambda_2)$ featuring in Eq. (3) as

$$\bar{\rho}_c(\lambda_1, \lambda_2) = -\frac{1}{(2\pi)^2} \sum_{s_1, s_2 = \pm} s_1 s_2 X^{s_1 s_2},$$

$$X^{s_1 s_2} \equiv \overline{\text{tr}[G^{s_1}(\lambda_1)] \text{tr}[G^{s_2}(\lambda_2)]}.$$

Notice that X here and in the following only accounts for the connected part of the correlation function. Let us first focus on the building block X^{+-} . An economical way to compute it in perturbation theory represents the traces as derivatives of "generating functions": $\text{tr}[G^\pm(\lambda)] = \partial_{\lambda^\pm} \text{tr} \ln(\lambda^\pm - H)$. Now, consider the representation

$$X^{+-}(\lambda_1, \lambda_2) = \partial_{\lambda_1 \lambda_2}^2 \overline{\text{tr} \ln(\lambda_1^+ - H) \text{tr} \ln(\lambda_2^- - H)}$$

expanded in a formal series in H . To leading order in the D^{-1} expansion, the subsequent average over H will then select all diagrams excluding intersections, and the corresponding contraction lines fall into two categories, lines connecting individual logarithmic factors with themselves, and lines between the two logarithms. The summation over the former amounts to a replacement of the resolvent operators featuring as arguments of the $\text{tr} \ln$ by the dressed resolvents discussed in the foregoing section. For a visualization of the ladder structures connecting the two logarithms, we refer to Fig. 6. Focusing on a region where $|\lambda_1 - \lambda_2| \sim D^{-1}$, one can expand the density of state around the average energy $\lambda \equiv (\lambda_1 + \lambda_2)/2$, and the resummation of the resulting diagrams leads to the result

$$\begin{aligned} X^{+-}(\lambda_1, \lambda_2) &\simeq \partial_{\lambda_1 \lambda_2}^2 \ln \left(1 - \frac{1}{D} \text{tr}[\bar{G}^+(\lambda_1) \bar{G}^-(\lambda_2)] \right) \\ &\simeq \partial_{\lambda_1 \lambda_2}^2 \ln \left(\frac{i\pi \bar{\rho}(\lambda)}{2D} (\lambda_1 - \lambda_2) \right) \\ &\simeq -\frac{1}{(\lambda_1 - \lambda_2)^2}, \end{aligned}$$

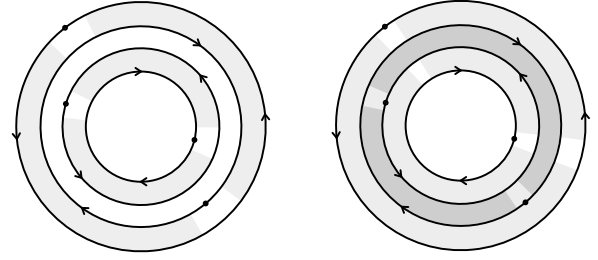


FIG. 7. To leading order in D^{-1} , i.e., to an order satisfying the one-summation-per-scattering-process rule, products of $2n$ generating functions statistically decouple, as indicated here for $n = 2$. The sum of the two diagrams on the right-hand side defines the second moment of the SFF. Notice that the individual propagators (distinguished by shading for improved visibility) correspond to energy denominators $1/(\lambda_{1+} - \lambda_{1-})^2 (\lambda_{2+} - \lambda_{2-})^2$ and $1/(\lambda_{1+} - \lambda_{2-})^2 (\lambda_{2+} - \lambda_{1-})^2$, respectively.

where the " $\simeq$ " are equalities up to corrections of subleading order in D^{-1} . For an interpretation of the twofold derivative in the language of diagrams, we again refer to Fig. 6. Turning to the remaining terms $X^{ss'}$, X^{+-} is symmetric in λ_1 and λ_2 , implying that $X^{+-}(\lambda_1, \lambda_2) = X^{+-}(\lambda_2, \lambda_1)$. However, the uncausal terms X^{++} and X^{--} contain the essential logarithm in the form $\ln(1 - \frac{1}{D} \text{tr}(\bar{G}^s \bar{G}^s))$. Since $\bar{G}^s \bar{G}^s \approx -1$, we obtain no singularity $\lambda_1 - \lambda_2$. Upon Fourier transformation back to time, these terms generate short-lived contributions $\sim \delta(t)$, where the δ function is smeared over scales $\sim \lambda^{-1}$. We neglect them as inessential in the finite time form factor. Adding the contributions $X^{+-} + X^{-+}$, we obtain the perturbative result

$$\bar{\rho}_c(\lambda_1, \lambda_2) \approx -\frac{1}{2} \left(\frac{1}{\pi(\lambda_1 - \lambda_2)} \right)^2, \quad (40)$$

which, when substituted back into Eq. (3) and subsequently Fourier transformed, leads to the linear ramp of the SFF.

3. Second moment of SFF

The framework above affords straightforward generalization to the n th moment of the SFF. All we need to do is consider $2n$ generating function factors with appropriately multiplied energy arguments. It is now straightforward to show that to leading order in the D^{-1} expansion there are no diagrams coupling more than two factors together. In other words, the average decouples into n factors, as indicated in Fig. 7 for the case $n = 2$. The combinatorial factor $n!$ counting the different choices of pairings explains the approximately Gaussian statistics of the SFF (again, to leading order in D^{-1}).

B. Subleading corrections via the supersymmetric method

In order to access the nonperturbative contributions to the statistics of the SFF, we employ a supersymmetric approach [41]. As our focus will be on universal properties of the deviations from Gaussianity, we consider the CUE ensemble, where the density of states is constant. We will start from a representation of SFF moments in terms of ratios of determinants, which are then expressed as integrals over bosonic and fermionic "ghost" fields.

The subsequent averaging over Haar distributed randomness is performed by a variant of the Hubbard-Stratonovich

known as color-flavor transformation [42]. This transformation expresses moments of the SFF in terms of an integral over auxiliary variables, which can be performed exactly by saddle point integration, due to a feature known as "semiclassical exactness." The results we obtain in this manner are therefore likewise exact. However, before turning to the execution of this program, some comments are in order. First, the result for the SFF discussed below is implicit in Ref. [36], which obtained it by representation theoretic methods. We here rederive this result in a framework perhaps more accessible to most physicists. Second, and unlike the method of Ref. [36], the CFT is known to run into convergence issues if the number of colors becomes of $\mathcal{O}(1)$ [43]. At the same time, numerics for random matrices of the smallest nontrivial dimensions $D = 2, 3$ turn out to be in perfect agreement with the analytical results obtained by CFT, indicating that the saddle point program works even beyond its range of mathematical backing. Finally, semiclassical exactness is a precious feature

of a few symmetry classes, including class A discussed in this paper. For instance, the SFF statistics of Haar distributed orthogonal matrices require a much more elaborate analysis, beyond the scope of this paper.

1. Generating function and nonlinear sigma model

The supersymmetry approach to the SFF is formulated in a phase representation, Fourier conjugate to discrete time. We thus start from the Fourier transform

$$\begin{aligned}\mathrm{tr}(U^t) &= \int \frac{d\phi}{2\pi} e^{-i\phi t} \mathrm{tr} \left(\frac{1}{1 - e^{i\phi^+} U} \right), \\ \mathrm{tr}([U^\dagger]^t) &= \int \frac{d\phi}{2\pi} e^{i\phi t} \mathrm{tr} \left(\frac{1}{1 - e^{-i\phi^-} U^\dagger} \right),\end{aligned}$$

with $t \in \mathbb{N}$ the discrete time of evolution and $\phi_s^\pm = \phi_s \pm i0$ slightly shifted into the upper/lower complex plane.

This motivates us to introduce

$$\mathcal{Z}(\alpha, \phi) = \left\langle \frac{\det(1 - e^{i(\alpha_1 + \phi_1^+)} U^\dagger) \det(1 - e^{i(\alpha_2 + \phi_2^+)} U^\dagger) \det(1 - e^{-i(\alpha_3 + \phi_3^-)} U) \det(1 - e^{-i(\alpha_4 + \phi_4^-)} U)}{\det(1 - e^{i\phi_1^+} U^\dagger) \det(1 - e^{i\phi_2^+} U^\dagger) \det(1 - e^{-i\phi_3^-} U) \det(1 - e^{-i\phi_4^-} U)} \right\rangle_U, \quad (41)$$

as a generating function for the second moments of the form factor. Indeed, it can be verified that

$$\overline{\mathrm{SFF}^2} = \frac{1}{(2\pi)^4} \int d\phi \mathcal{F}(\phi) e^{-it(\phi_1 + \phi_2 - \phi_3 - \phi_4)} \quad (42)$$

$$\mathcal{F}(\phi) = \partial_{\alpha_1} \partial_{\alpha_2} \partial_{\alpha_3} \partial_{\alpha_4} \mathcal{Z}(\alpha, \phi)|_{\alpha=0}, \quad (43)$$

where $\int d\phi$ denotes integration over all ϕ variables. The four determinants in the numerator and denominator of Eq. (41) then allow for integral-representations involving four complex ("bosonic") and four Grassmann ("fermionic") variables, respectively. Collecting the eight integration variables into the "supervector" ψ , we obtain the representation

$$\mathcal{Z} = \int_{\mathrm{U}(D)} dU \int d(\bar{\psi}, \psi) e^{\bar{\psi}(\mathbf{1}_{8D} - EU)\psi}, \quad (44)$$

where the integration of bosonic variables is along the imaginary axis, and $\mathbf{1}_{8D}$ is the identity matrix operating in the eight-fold replicated Hilbert space of graded ("bosonic/fermionic"), causal ("retarded/advanced"), and replicated ("1/2") variables. Here we upgraded the time-evolution operator

$$\mathcal{U} = \begin{pmatrix} U^\dagger & \\ & U \end{pmatrix}_{ra} \otimes \mathbf{1}_2 \otimes \mathbf{1}_{bf}, \quad (45)$$

to act on the replicated Hilbert space, where "flavor" indices "ra," "2," and "bf" denote the subspace of causal, replica, and graded indices. Furthermore, we introduced

$$E = \exp[i\Lambda(\phi + \alpha)], \quad (46)$$

involving sources and phases organized into matrices

$$\alpha = \mathrm{diag}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \otimes \pi_f, \quad (47)$$

$$\phi = \mathrm{diag}(\phi_1^+, \phi_2^+, \phi_3^-, \phi_4^-) \otimes \mathbf{1}_{bf}, \quad (48)$$

where $\pi_f = \frac{1}{2}(1 - \sigma_3^{bf})$ is a projection onto Grassmann components and we omitted the trivial Hilbert-space structure $\mathbf{1}_D$.

Finally, the matrix

$$\Lambda = \sigma_3^{ra} \otimes \mathbf{1}_2 \otimes \mathbf{1}_{bf} \quad (49)$$

follows the causal structure of time-evolution Eq. (45) and plays an important role in the following.

To perform the average over the Haar ensemble we employ the color-flavor transformation [42,44,45], which exchanges the integral over unitaries U , acting on the "color" indices of Hilbert space, and structureless in flavor space, for an integral over matrices $Z, \tilde{Z}$. These latter, on the other hand, act nontrivially on "flavor" indices but are structureless in "color" space. Referring for details to Appendix B, we notice that the transformation followed by ψ integration leads to

$$\mathcal{Z} = \int d(Z, \tilde{Z}) e^{D(\mathrm{Str} \ln(1 - \tilde{Z}Z) - \mathrm{Str} \ln(1 - E_+ Z E_- \tilde{Z}))}, \quad (50)$$

where "Str" is the graded trace and $E_\pm$ refers to the retarded and advanced sectors of the matrix E .

We have thus arrived at a representation of the generating function in terms of a "flavor" matrix integral involving supermatrices $Z, \tilde{Z}$. The Haar average has erased all microscopic details. Remaining structures are singlet in Hilbert space, explaining the overall factor D in the exponent of Eq. (50). This large factor suggests a semiclassical evaluation of the matrix integral, which turns out to be exact for the unitary class.

Before delving into details, let us notice that Eq. (50) can be brought to the standard form of a nonlinear sigma model interpreting $Z, \tilde{Z}$ as the rational coordinates of

the matrix [45]

$$Q(Z, \tilde{Z}) = \begin{pmatrix} 1 & Z \\ \tilde{Z} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & Z \\ \tilde{Z} & 1 \end{pmatrix}^{-1} = T \Lambda T^{-1}, \quad (51)$$

fulfilling the nonlinear constraint $Q^2 = \mathbf{1}_8$. Expressed in the variable Q , the matrix integral allows for the compact representation

$$\mathcal{Z} = \int d(Z, \tilde{Z}) e^{-D \text{Str} \ln \left(1 + \frac{1-E}{1+E} Q \Lambda \right)}. \quad (52)$$

Anticipating that the correlation function of interest only depends on phase differences and that the Fourier transform fixes phases to values $\phi_1 \simeq \phi_2 \simeq \phi_3 \simeq \phi_4 \sim \mathcal{O}(1/D)$, we can linearize Eq. (52) in sources and phases to obtain

$$\mathcal{Z} = \int dQ e^{\frac{iD}{2} \text{Str}[(\alpha + \phi)Q]}. \quad (53)$$

We note that the linearization $e^{i\phi} - 1 \approx i\phi$ is a matter of convenience but not essential. In time space, this amounts to an approximation of a discrete time translation operator by a derivative. In line with this observation, the somewhat more elaborate stationary phase analysis of the unapproximated integral (52) yields identical results for the SFF, but then as a function of discrete time. In other words, Eqs. (56) and (57) below are exact, provided $\tau = t/D$ is considered for $t \in \mathbb{N}$. This finding is also confirmed by our numerical experiments.

Proceeding with the linearized formalism, we note that Eq. (53) is the nonlinear sigma model for random matrix correlations [41,44]. In this representation, the matrix degree of freedom Eq. (51) is parametrized as a rotation around the matrix Λ , defined in Eq. (49), which is known as the "standard saddle" in the present context. A Gaussian expansion in generators of the rotations T reproduces the diagrammatic perturbation theory of the previous subsection. Importantly, the model has additional "nonperturbative" saddle points. As we discuss next, the latter describes correlations at longer timescales set by the Heisenberg time.

2. Semiclassical correlation function

As mentioned above, the matrix-integral generating (moments of) the SFF in symmetry class A enjoys "semiclassical exactness"[44–47]: Its evaluation in terms of "saddle points plus Gaussian fluctuations" provides the exact result. We thus need to understand the structure of saddle points in addition to the standard saddle Λ .

Appendix B provides further details, while here we notice that causality fixes the bosonic sector of the saddles to take the form $Q_{bb} = \sigma_3^{ra} \otimes \mathbf{1}_2$. In the fermionic sector, on the other hand, no such restriction applies, and any of the 16 matrices $Q_{ff} = \text{diag}(\pm 1, \pm 1, \pm 1, \pm 1)$ provides a legitimate saddle of the sigma model. In case of the CUE, however, only the traceless matrices, sharing the signature of the standard saddle point, can be accessed [48]. There are thus $\binom{4}{2} = 6$ relevant saddle points for our calculation. They can be parametrized as permutations of the standard saddle Λ and classified in terms of the number of involved transpositions T ; see Table I.

TABLE I. Six permutations (σ) giving rise to standard (first entry) and nonstandard saddle points (remaining five entries). The last entry corresponds to the Andreev-Altshuler saddle.

σ	1	2	3	4
I	1	2	3	4
T_{13}	3	2	1	4
T_{14}	4	2	3	1
T_{23}	1	3	2	4
T_{24}	1	4	3	2
$T_{14}T_{23}$	4	3	2	1

A straightforward, if tedious, semiclassical evaluation using the Harish-Chandra-Itzykson-Zuber (HCIZ) formula [46,49,50] then leads to

$$\mathcal{Z} = \sum_{\sigma} \prod_{i=1,2} \prod_{k=3,4} \frac{(\phi_i - \phi_{\sigma(k)} - \alpha_{\sigma(k)})(\phi_k - \phi_{\sigma(i)} - \alpha_{\sigma(i)})}{(\phi_i - \phi_k)(\phi_{\sigma(i)} - \phi_{\sigma(k)} + \alpha_{\sigma(i)} - \alpha_{\sigma(k)})} \times e^{\frac{iD}{2} (\sum_{i=1,2} - \sum_{i=3,4}) (\phi_i - \phi_{\sigma(i)} - \alpha_{\sigma(i)})}, \quad (54)$$

and similar results are obtained for the GUE.

Next, we compute the contribution of each saddle to the second moment of the spectral form factor. Introducing the compactified notation $\Delta_{ij} = \phi_i - \phi_j$ and performing the four-fold derivative, we get

$$\begin{aligned} \mathcal{F}_I(\phi) &= \left(\frac{D}{2}\right)^4 - \left(\frac{D}{2}\right)^2 \left(\frac{1}{\Delta_{13}^2} + \frac{1}{\Delta_{23}^2} + \frac{1}{\Delta_{14}^2} + \frac{1}{\Delta_{24}^2}\right) \\ &\quad + \frac{1}{\Delta_{23}^2 \Delta_{14}^2} + \frac{1}{\Delta_{13}^2 \Delta_{24}^2}, \\ \mathcal{F}_{T_{13}}(\phi) &= \left[\frac{D^2}{4\Delta_{24}^2} + \frac{iD}{2\Delta_{13}^2} \left(\frac{1}{\Delta_{12}} + \frac{1}{\Delta_{23}} + \frac{1}{\Delta_{43}} + \frac{1}{\Delta_{14}} \right) \right. \\ &\quad \left. - \frac{1}{\Delta_{13}^2 \Delta_{24}^2} - \frac{1}{\Delta_{12} \Delta_{23} \Delta_{34} \Delta_{41}} \right] e^{iD\Delta_{13}}, \\ \mathcal{F}_{TT}(\phi) &= \frac{\Delta_{12}^2 \Delta_{34}^2}{\Delta_{13}^2 \Delta_{32}^2 \Delta_{24}^2 \Delta_{41}^2} e^{iD(\Delta_{13} + \Delta_{24})}, \end{aligned} \quad (55)$$

the second single transposition following from symmetry.

Finally, we Fourier transform. Since $\mathcal{Z}$ depends only on phase differences, one of the ϕ_i will be trivially integrated out. To fix the redundancy related to the poles with identical causal structure ($\phi_1^+ - \phi_2^+$ and $\phi_3^- - \phi_4^-$), we introduce $\phi_s^\pm = \phi_s \pm i\delta_s$ and regulate by enforcing $0 < \delta_1 < \delta_2$ and $0 < \delta_3 < \delta_4$ (notice that the result is independent of the chosen order). Extending the remaining integrals over ϕ_i to infinity, and neglecting contributions involving δ functions in time, we arrive at a disconnected contribution

$$\overline{\text{SFF}}_{\text{dis}}^2 = 2D^2 \begin{cases} \tau^2 & \text{for } \tau < 1 \\ 1 & \text{for } \tau \geq 1 \end{cases} \quad (56)$$

and connected contribution

$$D\text{Conn}_2(t) = -D \begin{cases} 0 & \text{for } \tau < 1/2 \\ 2\tau - 1 & \text{for } 1 < \tau \leq 1/2, \\ 1 & \text{for } \tau \geq 1 \end{cases} \quad (57)$$

where $\tau = t/D$ is the total time of evolution in units of the Heisenberg time $t_H = D$ in the CUE. We thus recover the sine-kernel result (36).

Finally, we comment on the possibility of computing higher moments through the supersymmetric approach. For the k th moment of the SFF, we need to consider a k -fold replicated theory, which contains a total of $\binom{2k}{k}$ relevant saddles. The same semiclassical evaluation can be carried out, yielding a straightforward generalization of Eq. (54). After taking $2k$ α derivatives, one obtains $\mathcal{F}_k(\phi)$, which can then be Fourier transformed to yield SFF^k .

To assess the extent of the difficulty of the calculation for higher orders, we look at the example of $k = 3$. In this case the generating function becomes a 12-dimensional integral over bosonic and fermionic fields, and the saddle point structure includes $\binom{6}{3} = 20$ relevant configurations corresponding to traceless sign permutations of the eigenvalue phases. These saddle points generalize those shown in Table I for $k = 2$. As with $k = 2$, each nonstandard saddle contributes to the connected part $\text{Conn}_3(t)$, and these contributions scale as one order down in $1/D$ compared to the leading Gaussian contribution $\sim 6D^3 z(t)^3$. While the number of relevant saddle point contributions increases combinatorially, their structure is systematically organized by the replica formalism, and the semiclassical exactness of the color-flavor transformation ensures that all such terms are captured. This provides a clear and controlled route to studying the full distribution of the SFF beyond the variance.

Summarizing, we have obtained the non-Gaussian statistics as a term of next-leading order in D^{-1} relative to the leading order term; cf. the relative factor of D between Eqs. (56) and (57). As pointed out above, these results are exact for the CUE, and as such, in agreement with the results of Ref. [36]. They are also consistent with the findings of Ref. [31], which obtained the SFF statistics in a discrete-time dependent setting via the analysis of Toeplitz determinants. In this context, the authors of Ref. [31] further develop the Toeplitz determinant approach by representing the spectral form factor $|\tau_n^2| = |\text{tr} U^n|^2$ through determinants of Toeplitz matrices. By expanding the determinant of these matrices and their logarithm, and evaluating derivatives, they derive *exact* expressions for the mean and variance of SFF. Remarkably, for discrete times $n \leq N/2$, the moments conform to Gaussian statistics. The plots shown in Fig 8 provide numerical support for these statements. They show data for the average SFF and its second moment for ensembles of matrices of very small dimension, $D = 2, 3, 4$, in perfect agreement with the analytical results. (The dots represent discrete time steps, the Heisenberg time assuming the values $t_H = 2, 3, 4$, respectively.)

It is interesting to note that *all* contributions to Conn_2 are nonperturbative in nature. Even though they are smaller by just one power D compared to Eq. (56), they cannot be obtained from a perturbative, diagrammatic, or semiclassical expansion. For example, in the perturbative approach outlined in Sec. III A, connected contributions would assume the form of four "wheels," fully interconnected by ladder "spokes." Such diagrams exist, but they are of order D^{-3} relative to the leading order contributions, and hence negligibly small. The nonperturbative nature of the supersymmetry results is witnessed by the nonanalytic kinks in the time dependence of

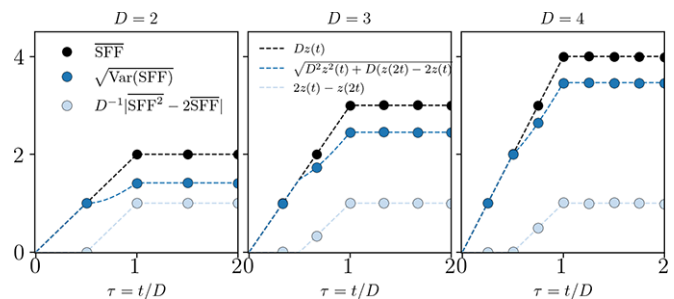


FIG. 8. Numerical vs analytical statistics of the SFF statistics of the CUE for very small D . Numerical averages (large dots) of $\overline{\text{SFF}}$ (black), the variance $\text{Var}(\text{SFF}) = (\overline{\text{SFF}^2} - \overline{\text{SFF}}^2)^{1/2}$ and the deviation from Gaussianity $D|\text{SFF}^2 - 2\overline{\text{SFF}}^2|$ are plotted against with the analytical prediction in Eq. (2) and Eq. (6) (dashed lines) for $D = 2, 3, 4$ in the three panels, respectively. Numerical simulations obtained with $N_{\text{sim}} = 2 \times 10^5$.

Conn_2 at $\tau = \frac{1}{2}, 1$. They have a status similar to the celebrated ramp-plateau kink in the average SFF which, likewise, evades perturbative approaches.

Finally, the situation with the GUE is different. There the background of a nonconstant average spectral density gives rise to a plethora of correction terms of subleading order in $1/D$. However, these do not affect the finding that Conn_2 (35) remains the leading non-Gaussian contribution to the variance of the SFF.

IV. CONCLUSION

In this paper we studied the statistical moments of the spectral form factor of circular and Gaussian matrix ensembles. To leading order in the matrix dimension, D , the SFF is Gaussian with deviations appearing at next leading order. Employing two complementary approaches, sine-kernel techniques and supersymmetric matrix integration, we identified a universal subleading correction to the second moment, consistent with previous findings for the circular ensemble [31]. Our sine-kernel analysis clarified discrepancies with previous studies, while the field-theoretic analysis revealed new nonperturbative saddle points that extend beyond the standard ones in the nonlinear sigma model. These findings were further corroborated by numerical simulations, which confirmed the theoretical predictions.

Our results open several avenues for further exploration. Alongside the generalization to other symmetry classes, one is the extension of our approach to the computation of higher-order subleading corrections to the SFF or even its probability distribution. Another interesting question is to what degree the subleading corrections to the SFF are universal across different chaotic systems beyond Random Matrix Theory. Our focus has been on RMT ensembles as a controlled setting to explore subleading corrections to the spectral form factor; however, it is natural to ask whether such effects are observable in concrete quantum many-body systems. While these corrections may be subtle and challenging to detect directly in physical observables, their universality makes them valuable as benchmarks for diagnosing quantum chaoticity in many-body systems. In particular, moments of the spectral

form factor naturally arise in the analysis of time-dependent observables such as correlation functions (including out-of-time-order ones) and return probabilities. As such, the random matrix theory predictions provide a baseline upon which one can analyze the influence of the specific many-body spectrum in realistic systems. One promising direction is provided by the Sachdev-Ye-Kitaev (SYK) model, where finite- N corrections lead to subleading deviations from random matrix predictions, including in the statistics of the SFF.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [51].

APPENDIX A: SINE-KERNEL PREDICTIONS FOR GAUSSIAN UNITARY ENSEMBLE

In this appendix we seek to implement the sine-kernel formalism outlined in Sec. II B to compute the second moment of the spectral form factor of the GUE ensemble. We adhere closely to the framework established in [32] while rectifying critical inconsistencies, which lead to a different result for the second moment of the SFF.

Equation (23) demonstrates how all higher-order correlations can be expressed as powers of the sine kernel. To transition from this expression to the spectral form factor in Eq. (25), it is necessary to apply the appropriate Fourier transform. For this purpose, the following *convolution formula* [cf. Eq. (6.2.18) in [52]], valid in the large D limit, proves particularly useful:

$$\int \prod_{i=1}^m d\lambda_i K(\lambda_1, \lambda_2) K(\lambda_2, \lambda_3) \cdots K(\lambda_m, \lambda_1) e^{i \sum_{i=1}^m k_i \lambda_i} = \delta\left(\sum_{i=1}^m k_i\right) \int dk g(k) \prod_{i=1}^{m-1} g\left(k + \sum_{l=1}^i \frac{k_l}{2D}\right), \quad (\text{A1})$$

where the function $g(k)$ is defined as

$$g(k) = \begin{cases} 1 & \text{for } |k| < \frac{1}{2}, \\ 0 & \text{for } |k| > \frac{1}{2}. \end{cases} \quad (\text{A2})$$

We note that the omission of the sum over k_l in Eq. (2.36) of [32] is partially responsible for the errors that propagate through the subsequent computations.

Let us here clarify the differences with the notations of Ref. [32], which define the moments of the SFF in terms of some functions $r_k(t)$ with $k = 1, 2, 3$. These are related to our notations by $r_2(t) = 1 - z(t)$ and $Dr_1(t) = \bar{u}_t = \int_{\lambda} K(\lambda_i, \lambda_i) e^{i\lambda_i t}$. Finally, we neglect $r_3(t)$, corresponding to the finite size correction of the delta function in Eq. (A1).

To evaluate $\overline{\text{SFF}^2}$ for GUE, here we proceed by computing the first and second terms of Eq. (31) in the following.

(a) Third-order term: The term $\int_{\lambda} R_3(\lambda_1, \lambda_2, \lambda_3) e^{i(2\lambda_1 - \lambda_2 - \lambda_3)t}$, expressed in terms of the determinant of the sine kernel using Eq. (23) results in a multiple number of terms that need to be computed through Eq. (24) and Eq. (A1). Due to the large number and tedious nature of the computation of these terms, stating the calculation of each term is beyond the scope of this discussion. For those, we refer the reader to Ref. [32], while here we only explicitly adjust the inconsistencies in Eq. (2.68) (the so-called three-type contribution), namely,

$$2\text{Re} \int_{\lambda} K(\lambda_1, \lambda_2) K(\lambda_2, \lambda_3) K(\lambda_3, \lambda_1) e^{i(2\lambda_1 - \lambda_2 - \lambda_3)t} = 2D[1 - z(2t)]. \quad (\text{A3})$$

Considering this correction we arrive at Eq. (32a).

(b) Fourth-order term: A similar approach is taken for the integral $\int_{\lambda} R_4(\lambda_1, \lambda_2, \lambda_3, \lambda_4) e^{i(\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4)t}$, which is rewritten in terms of the sine-kernel determinant. Following Ref. [32] and correcting Eq. (2.56) (the so-called four-type contribution), we derive

$$\int_{\lambda} K(\lambda_1, \lambda_3) K(\lambda_3, \lambda_2) K(\lambda_2, \lambda_4) K(\lambda_4, \lambda_1) e^{i(\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4)t} = D[1 - z(t)], \quad (\text{A4})$$

$$\int_{\lambda} K(\lambda_1, \lambda_2) K(\lambda_2, \lambda_3) K(\lambda_3, \lambda_4) K(\lambda_4, \lambda_1) e^{i(\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4)t} = D[1 - z(2t)], \quad (\text{A5})$$

$$\int_{\lambda} K(\lambda_1, \lambda_2) K(\lambda_2, \lambda_4) K(\lambda_4, \lambda_3) K(\lambda_3, \lambda_1) e^{i(\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4)t} = D[1 - z(2t)]. \quad (\text{A6})$$

By incorporating this correction, we obtain Eq. (32b).

c. The final result: By aggregating the previously derived results along with the two-point function contributions from the third and fourth terms of Eq. (31), we arrive at Eq. (33). To obtain the corresponding expression for the GUE, we substitute $z(t)$ with Eq. (5).

APPENDIX B: FIELD-THEORY DETAILS

1. Matrix action

In this appendix, we provide additional details on the derivation of the effective field theory discussed in Sec. III. Our starting point is the generating function

$$\mathcal{Z}(\alpha, \phi) = \left\langle \frac{\det(1 - e^{i(\alpha_1 + \phi_1^+)} U^\dagger) \det(1 - e^{i(\alpha_2 + \phi_2^+)} U^\dagger) \det(1 - e^{-i(\alpha_3 + \phi_3^-)} U) \det(1 - e^{-i(\alpha_4 + \phi_4^-)} U)}{\det(1 - e^{+i\phi_1^+} U^\dagger) \det(1 - e^{+i\phi_2^+} U^\dagger) \det(1 - e^{-i\phi_3^-} U) \det(1 - e^{-i\phi_4^-} U)} \right\rangle_U, \quad (\text{B1})$$

involving Haar average of the ratio of eight spectral determinants. The latter can be expressed in terms of integrals over $4D$ fermionic variables χ_a^i and $4D$ bosonic variables S_a^i , where $i = 1, \dots, D$ and $a = 1, 2, 3, 4$. To compactify notation, it is convenient to introduce the $8D$ -component supervector ψ ,

$$\psi_a^i = \begin{pmatrix} \chi_a^i \\ S_a^i \end{pmatrix}, \quad (\text{B2})$$

comprising of eight internal degrees of freedom, namely, the two "graded" fermionic/bosonic components (b/f), the two "causal" retarded/advanced components (+/-), and the two replica components (1/2). The ensemble average of the generating function can then be expressed as

$$\mathcal{Z} = \int_{U(D)} dU \int d(\psi, \bar{\psi}) \exp \times (\bar{\psi}_+(1 - E_+ U^\dagger) \psi_+ + \bar{\psi}_-(1 - E_- U) \psi_-), \quad (\text{B3})$$

where $E_\pm$ denote the retarded/advanced sectors of E , defined in Eq. (46).

The next step is to perform the integral over the Haar ensemble employing the color-flavor transformation identity [44,45]

$$\begin{aligned} & \int_{U(D)} dU \exp(\bar{\psi}_1 U \psi_{2'} + \bar{\psi}_2 U^\dagger \psi_{1'}) \\ &= \int d(\tilde{Z}, Z) \text{Sdet}(1 - \tilde{Z}Z)^D \exp(\bar{\psi}_1 \tilde{Z} \psi_{1'} + \bar{\psi}_2 Z \psi_{2'}), \end{aligned} \quad (\text{B4})$$

where "Sdet" denotes the superdeterminant [44]. We are then left with a quadratic action in the superfield ψ that can be integrated out, resulting in

$$\begin{aligned} \mathcal{Z} &= \int d(Z, \tilde{Z}) d(\psi, \bar{\psi}) \text{Sdet}(1 - \tilde{Z}Z)^D \exp(\bar{\psi} M \psi) \\ &= \int d(Z, \tilde{Z}) \text{Sdet}(1 - \tilde{Z}Z)^D \text{Sdet}(M)^{-D}, \end{aligned}$$

where $M = \begin{pmatrix} 1 & E_+ \tilde{Z} \\ E_- Z & 1 \end{pmatrix}$ is the structure in causal indices.

Evaluating the determinant in causal indices and using that $\ln \text{Sdet} = \text{Str} \ln$, we arrive at Eq. (50). To rewrite this in the standard form of a sigma model, we follow [45] and introduce

$$Q(Z, \tilde{Z}) = T \Lambda T^{-1}, \quad T = \begin{pmatrix} 1 & Z \\ \tilde{Z} & 1 \end{pmatrix}, \quad (\text{B5})$$

and in terms of which

$$S[Q] = D \text{Str} \ln(1 + \frac{1-E}{1+E} Q \Lambda), \quad (\text{B6})$$

with $\text{Str} A = \text{tr} A_{BB} - \text{tr} A_{FF}$. Notice that $Q^2 = \mathbf{1}_8$, so Eq. (B5) indeed corresponds to a nonlinear sigma model.

2. Semiclassical evaluation

For the semiclassical evaluation, it is convenient to start out from an alternative matrix action, involving a quadratic contribution $\frac{D}{2} \text{Str}(Q^2)$ which enforces the nonlinear constraint as a result of the saddle approximation. Specifically, the sigma model in the main text is semiclassically equivalent to the matrix integral

$$\mathcal{Z} = \int dQ \exp \left[\frac{D}{2} \text{Str}(Q^2) - D \text{Str} \ln \left(Q + \frac{i}{2} (\alpha + \phi) \right) \right], \quad (\text{B7})$$

where the integration now is not restricted to the manifold $Q^2 = \mathbf{1}_8$.

Now, we are allowed to shift $Q \rightarrow Q - \frac{i}{2} (\alpha + \phi)$, making the logarithm rotationally invariant. This then motivates the introduction of angle and radial coordinates

$$A = V R V^{-1}, \quad R = \text{diag}(z_1, z_2, z_3, z_4, w_1, w_2, w_3, w_4), \quad (\text{B8})$$

where rotations V contain the angles and the integration contour of radial coordinates is over the real axis for the w variables (fermionic sector) and over the imaginary axis for the z variables (bosonic sector). The change of integration variables introduces a Berezinian $B = \frac{\Delta^2(z, w)}{\Delta^2(z) \Delta^2(w)}$ given by the Vandermonde determinants

$$\Delta(x) = \prod_{i < j} (x_i - x_j), \quad \Delta(x, y) = \prod_{i, k} (x_i - y_k). \quad (\text{B9})$$

The integration over angles V is nontrivial but can be carried out employing the Harish-Chandra-Itzykson-Zuber (HCIZ) integration formula [49,50], leading to

$$\begin{aligned} \mathcal{Z} &= \frac{\Delta(\phi, \phi + \alpha)}{\Delta(\phi) \Delta(\phi + \alpha)} \int dZ dW \frac{\Delta(z) \Delta(w)}{\Delta(z, w)} \exp \\ &\times \left[\frac{D}{2} \text{Str} \left(R - \frac{i}{2} (\alpha + \phi) \right)^2 - D \text{Str} \ln R \right]. \end{aligned} \quad (\text{B10})$$

Shifting back $R \rightarrow R + \frac{i}{2} (\alpha + \phi)$ and performing the saddle point analysis, one finds that the integration contours over the bosonic block variables can only be deformed to reach the saddle points $z_{1,2} = +1$, $z_{3,4} = -1$, while the contours over

the fermionic block variables are allowed to reach any of the saddles $w_i = \pm 1$. While in the sigma model Eq. (51) $\text{tr} Q = 0$ and the signature of Q is fixed, this is not the case in Eq. (B7), and all 16 saddles must be considered. However, fluctuations around saddles with finite trace are suppressed by additional factors $1/D$ and are thus subleading [44]. In any case, only six saddles are relevant for our calculation, so we arrive at

$$\mathcal{Z} = \frac{\Delta(\phi, \phi + \alpha)}{\Delta(\phi)\Delta(\phi + \alpha)} \sum_P \frac{\Delta(z - i\phi/2)\Delta(w - i\phi/2 - i\alpha/2)}{\Delta(z - i\phi/2, w - i\phi/2 - i\alpha/2)} \times e^{i\frac{D}{2}\text{Str}(\alpha + \phi)}, \quad (\text{B11})$$

with $\sum_P$, the sum over the six saddle points. As discussed in the main text, they can be expressed as permutations of the standard saddle $P_0 \equiv \Lambda$ and classified according to the number of transpositions T (see Table I) resulting in

$$\mathcal{Z} = \sum_{\sigma} \prod_{i=1,2} \prod_{k=3,4} \frac{(\phi_i - \phi_{\sigma(k)} - \alpha_{\sigma(k)})(\phi_k - \phi_{\sigma(i)} - \alpha_{\sigma(i)})}{(\phi_i - \phi_k)(\phi_{\sigma(i)} - \phi_{\sigma(k)} + \alpha_{\sigma(i)} - \alpha_{\sigma(k)})} \times e^{i\frac{D}{2}(\sum_{i=1,2} - \sum_{i=3,4})(\phi_i - \phi_{\sigma(i)} - \alpha_{\sigma(i)})}. \quad (\text{B12})$$

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- $$\rho^{(n)}(\lambda_1, \dots, \lambda_n) = \frac{(D-n)!}{D!} R_n(\lambda_1, \dots, \lambda_n) \quad (\text{B13})$$
- defined such that the average is normalized to one, $\int d\lambda \rho^{(1)}(\lambda) = 1$.
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