

Stochastic cloaking: Concealing a region from diffusive particles

Connor Roberts ^{1,*}, Ziluo Zhang ^{1,2}, Helder Rojas ³, Stefano Bo ⁴, Carlos Escudero ³,
Sébastien Guenneau ^{5,†} and Gunnar Pruessner ^{1,‡}

¹Department of Mathematics, Imperial College London, London SW7 2AZ, United Kingdom

²Wenzhou Institute, University of Chinese Academy of Sciences, Wenzhou, Zhejiang 325001, China

³Departamento de Matemáticas Fundamentales, Universidad Nacional de Educación a Distancia, Calle de Juan del Rosal 10, 28040 Madrid, Spain

⁴Department of Physics, King's College London, London WC2R 2LS, United Kingdom

⁵Department of Physics, Imperial College London, London SW7 2AZ, United Kingdom



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We introduce “stochastic cloaking,” where a region of space is concealed from an ensemble of diffusing particles whose individual trajectories are governed by a stochastic (Langevin) equation. Our simulations reveal how different interpretations of the Langevin equation affect the cloaking performance of an annular single-layer invisibility cloak of smoothly varying diffusivity in two dimensions. Near-perfect cloaking is achieved under the Itô convention, indicated by the cloak preventing particles from accessing an inner core without disturbing the particle density outside the cloak. The cloak’s performance can be further improved by regularizing its singular behavior. We believe our demonstration of stochastic cloaking is a significant milestone, comparable to earlier developments that extended cloaking from optics and acoustics to thermodynamics.

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Introduction. The advent of “metamaterials” has led to the proposal of various phenomena that were once considered the realm of science fiction [1–4]. Notably, by capitalizing on the form invariance of Maxwell’s equations under a coordinate transformation [5], Pendry suggested that tuning the magnetic permeability and electric permittivity of a metamaterial could be used to deviate electromagnetic radiation around an object in such a way that the radiation is undisturbed from its original trajectory [6]. The metamaterial acts as an invisibility cloak in this case, since any external observer would be unable to detect the presence of the concealed object and, crucially, the metamaterial itself. This application of “transformation optics” has inspired research into cloaking for other forms of radiation.

A major milestone is thermal cloaking, based on “transformation thermodynamics” [7–9], where a metamaterial of heterogeneous conductivity protects a region from heat flux. Thermal cloaking demonstrated that transformation-based cloaking applies not only to elliptic differential equations—describing optics and acoustics [10,11]—but also to parabolic differential equations. Yet, both are examples of *deterministic* differential equations.

Till now, no equivalent demonstration has been made for systems governed by *stochastic* differential equations, where behavior can be predicted only on a *statistical* (rather than *precise*) basis. Such a demonstration would be another significant milestone for cloaking, not least because of the inherent difficulties introduced by stochasticity. The major difficulty arises from the governing stochastic equation—the Langevin equation—being uniquely determined only after choosing a particular convention under which to integrate it [12]. This is often called the Itô vs Stratonovich dilemma [12–16], named after the first two conventions to be proposed [17,18]. Though both are over 60 years old, their physical interpretation remains of great interest today [19].

Switching conventions typically involves a simple transformation of the Langevin equation [20]. However, this requires a degree of regularity that turns out to be absent in our model, similarly to the one-dimensional cases of Refs. [14,21–23]. This negates the ability to readily switch conventions and thus makes the original choice even more critical. Additionally, we consider a multidimensional model, which poses further mathematical challenges as well as important implications for noise interpretation in physical systems [24].

Despite these difficulties, we demonstrate that an annular metamaterial of spatially varying diffusivity can cloak an inner core from stochastically diffusing particles when interpreted under the Itô convention. The properties of the metamaterial, or “cloak,” are determined from a coordinate transformation that maps the inner core to the surrounding cloak, as in Fig. 1.

In analogy to optical and thermal cloaking, “stochastic” cloaking prevents particles from accessing the core, while preserving the particle density outside the cloak. The latter

*Contact author: connor.roberts16@imperial.ac.uk

†Contact author: s.guenneau@imperial.ac.uk

‡Contact author: g.pruessner@imperial.ac.uk

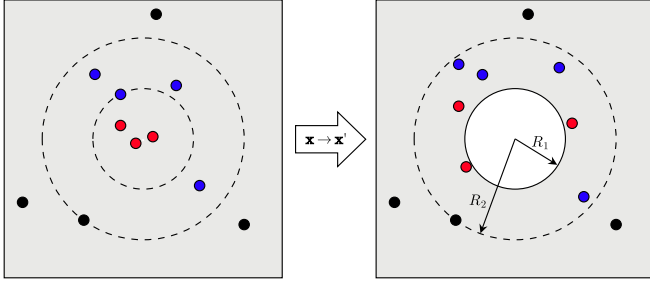


FIG. 1. Schematic representation of a coordinate transformation that tears a hole in the metric space of a two-dimensional plane (left) to produce a finite region, indicated in white, that particles are unable to enter (right). Particle positions at radii $r \geq R_2$ (black points) are invariant under the transformation, while the radii of particles at $r < R_2$ (red and blue points) increase under the transformation.

is quantified by analyzing the arrival distribution of particles downstream of the cloak.

Langevin equation. The time evolution of a particle's position $x(t)$ in one dimension is described by the stochastic Langevin equation,

$$\frac{m}{\gamma(x)}\ddot{x}(t) = -\dot{x}(t) + \sqrt{2D(x)}\xi(t), \quad (1)$$

where the dots above x denote time derivatives, m is the particle's mass, $\gamma(x)$ is the friction coefficient, $D(x)$ is the diffusivity, and $\xi(t)$ is Gaussian white noise with ensemble-averaged mean $\langle \xi(t) \rangle = 0$ and correlation $\langle \xi(t)\xi(t') \rangle = \delta(t - t')$. In the overdamped regime, where the typical time scale τ_o of interest satisfies $\tau_o \gg m/\gamma$, the Langevin equation simplifies to

$$\dot{x}(t) = \sqrt{2D(x)}\xi(t), \quad (2)$$

indicating the particle's dynamics are determined solely by $D(x)$.

Conventions. To simulate a particle trajectory, one can iteratively integrate Eq. (2) in small time steps Δt ,

$$x(t + \Delta t) = x(t) + \int_t^{t+\Delta t} ds \sqrt{2D(x(s))}\xi(s), \quad (3)$$

which presents a choice when it comes to evaluating the integral on the right-hand side of Eq. (3). Specifically, we must choose which value the random variable $x(t)$ takes in the interval $[x(t), x(t + \Delta t)]$ in order to evaluate the diffusivity. However, there is no uniquely determined point at which it *should* be evaluated. Instead, the choice is a matter of the adopted convention, which is parametrized by the continuous variable $\alpha \in [0, 1]$ in the following [20]:

$$x(t + \Delta t) = x(t) + \sqrt{2D[(1 - \alpha)x(t) + \alpha x(t + \Delta t)]} \times \int_t^{t+\Delta t} ds \xi(s), \quad (4)$$

where the integral $\int_t^{t+\Delta t} ds \xi(s)$ is numerically evaluated by drawing from a Gaussian distribution of variance Δt .

Three common conventions are Itô ($\alpha = 0$) [17], Stratonovich ($\alpha = 1/2$) [18], and isothermal/

Hänggi-Klimontovich ($\alpha = 1$) [25,26], the latter of which we will refer to as “anti-Itô.” Each convention has its merits [14,20,27]. However, integration under any $\alpha > 0$ convention invalidates the use of the standard Euler-Maruyama method of Itô calculus because $\alpha > 0$ requires evaluating the diffusivity at a future time step. The simplest way to simulate $\alpha > 0$ is thus to reformulate the dynamics in terms of the nonanticipatory Itô convention, $\alpha = 0$, via a correction term (if sufficiently regular) [20] or an auxiliary step [28,29]—see Supplemental Material [30], Sec. SII. Importantly, the chosen convention affects the statistics of the particle's dynamics, as elucidated by the Fokker-Planck equation describing the time evolution of the particle density,

$$\frac{\partial P(x, t)}{\partial t} = \frac{\partial}{\partial x} D^\alpha(x) \frac{\partial}{\partial x} D^{1-\alpha}(x) P(x, t). \quad (5)$$

Though similar to a heat equation, the Fokker-Planck equation differs by having an associated Langevin equation that directly governs the motion of each individual particle. However, due to their similarity, we will leverage methods from transformation thermodynamics [7,8] to inform our derivation of the spatially varying diffusivity that achieves stochastic particle cloaking in two dimensions. This diffusivity will then be used to simulate an ensemble of diffusive particles via the two-dimensional Langevin equation,

$$x_i(t + \Delta t) = x_i(t) + g_{ij}[(1 - \alpha)\mathbf{x}(t) + \alpha\mathbf{x}(t + \Delta t)] \times \int_t^{t+\Delta t} ds \xi_j(s), \quad (6)$$

where $i, j \in \{1, 2\}$ index the components of vectors [such as the position $\mathbf{x} = (x_1, x_2)$ and noise $\xi = (\xi_1, \xi_2)$] and matrices [such as the square root of the diffusivity/diffusion tensor $\mathbf{g} = \sqrt{2\mathbf{D}} = ((g_{11}, g_{21})^T, (g_{12}, g_{22})^T)$]. Above and throughout, repeated indices imply summation. We discuss the feasibility of generalizing our results to a three-dimensional spherical cloak in Sec. SIIC [30].

Transformed Fokker-Planck equation. To derive the cloak's spatially varying diffusivity, we consider a coordinate transformation that maps the protected circular core of radius R_1 to a surrounding annulus $R_1 \leq r' \leq R_2$, see Fig. 1, where r' is the radial distance from the center of the core in the new coordinates. This annular region acts as the cloak, with its diffusivity determined through the specific mapping.

First, we consider the effect of a general coordinate transformation $\mathbf{x} \rightarrow \mathbf{x}'$ on the Fokker-Planck equation in two dimensions. Starting from the Fokker-Planck equation describing homogeneous diffusion D_0 ,

$$\frac{\partial P(\mathbf{x}, t)}{\partial t} = \frac{\partial}{\partial x_i} D_0 \frac{\partial P(\mathbf{x}, t)}{\partial x_i}, \quad (7)$$

we seek the transformed diffusion tensor $\mathbf{D}'$ in the new coordinates that renders the Fokker-Planck equation form invariant. In other words, by manipulating the transformed Fokker-Planck equation into the same form as Eq. (7), identification of $\mathbf{D}'$ in place of D_0 readily follows. Leaving the details to Sec. SI [30], we have for the Fokker-Planck equation after a

general coordinate transformation [7]

$$\frac{\partial P(\mathbf{x}', t)}{\partial t} = \frac{1}{\det(\mathbf{J})} \frac{\partial}{\partial x'_j} J_{kj}^{-1} D_{0kl} J_{kl}^{-1} \det(\mathbf{J}) \frac{\partial P(\mathbf{x}', t)}{\partial x'_l}, \quad (8)$$

where the Jacobian matrix $\mathbf{J}$ has components $J_{ij} = \partial_{x'_j} x_i$. Equation (8) poses the dynamics on an effective manifold [31], whereas we desire the coordinate transformation to leave the Fokker-Planck equation in an invariant “Euclidean” form analogous to Eq. (7). However, this is prevented by the space-dependent factor $1/\det(\mathbf{J})$ preceding the spatial derivatives on the right-hand side of Eq. (8). For the heat equation in transformation thermodynamics, this factor is merely absorbed into the specific heat capacity [7,8]. However, there is no analogous trick for the case of stochastic particle diffusion here, since it depends solely on the diffusivity $\mathbf{D}'(\mathbf{x})$. Hence, unlike the heat equation, the Fokker-Planck equation is generally not form invariant under a coordinate transformation, as Eq. (8) is the furthest one can simplify before considering specific transformations.

Nonlinear transformation. Equation (8) suggests a transformation with constant $\det(\mathbf{J})$ would render the Fokker-Planck equation invariant. The most-studied transformation is the linear mapping $r' = R_1 + r(R_2 - R_1)/R_2$ for $r \leq R_2$ [6,32,33]. This has nonconstant $\det(\mathbf{J}) = (R_2/(R_2 - R_1))^2 (r' - R_1)/r'$ for $r \leq R_2$, which motivates us to consider alternative transformations. As it turns out, a transformation that simultaneously maps the protected region, $r' < R_1$, to the annulus, $R_1 \leq r' \leq R_2$, while maintaining constant $\det(\mathbf{J})$

is difficult to find. *Close* to satisfying both properties is the nonlinear transformation [34,35]

$$r' = \begin{cases} \sqrt{\beta r^2 + R_1^2}, & r \leq R_2, \\ r, & r > R_2, \end{cases} \quad (9)$$

where $\beta = (R_2^2 - R_1^2)/R_2^2$ and the azimuthal coordinate $\varphi' = \varphi$ is invariant. In Cartesian coordinates $\mathbf{x}' = (x', y')$, the Jacobian for this nonlinear transformation is

$$\begin{aligned} \mathbf{J} &= \frac{\partial(x, y)}{\partial(r, \varphi)} \frac{\partial(r, \varphi)}{\partial(r', \varphi')} \frac{\partial(r', \varphi')}{\partial(x', y')} \\ &= \begin{cases} \mathbf{R}(\varphi') \text{diag}\left(\frac{r'}{r\beta}, \frac{r}{r'}\right) \mathbf{R}(-\varphi'), & R_1 < r' \leq R_2, \\ \mathbb{1}, & r' > R_2, \end{cases} \end{aligned} \quad (10)$$

where $\mathbf{R}(\varphi)$ and $\mathbb{1}$ are the 2×2 rotation and identity matrices, respectively. The Jacobian (10) has a piecewise-constant determinant,

$$\det(\mathbf{J}) = \begin{cases} \frac{1}{\beta}, & R_1 < r' \leq R_2, \\ 1, & r' > R_2, \end{cases} \quad (11)$$

suggesting Eq. (8) can be brought into the desired form (7) on a piecewise basis under the transformation (9) above. On this piecewise basis, the factors of $\det(\mathbf{J})$ in Eq. (8) cancel, allowing us to approximate the Fokker-Planck equation under this transformation by

$$\frac{\partial P(\mathbf{x}', t)}{\partial t} \approx \frac{\partial}{\partial x'_j} \mathbf{D}'_{jk} \frac{\partial P(\mathbf{x}', t)}{\partial x'_k}, \quad (12)$$

where we have identified the piecewise diffusion tensor,

$$\mathbf{D}' = D_0 \mathbf{J}^{-T} \mathbf{J}^{-1} = \begin{cases} D_0 \beta \mathbf{R}(\varphi') \text{diag}\left(\frac{r'^2 - R_1^2}{r'^2}, \frac{r'^2}{r'^2 - R_1^2}\right) \mathbf{R}(-\varphi'), & R_1 < r' \leq R_2, \\ D_0 \mathbb{1}, & r' > R_2, \end{cases} \quad (13)$$

from which we can recover homogeneous diffusion $\mathbf{D}' = D_0 \mathbb{1}$, as in Eq. (7), by setting $R_1 = 0$. Strictly speaking, Eq. (12) is an equality everywhere except at $r' = R_2$, where there is an additional divergent contribution arising from the sharp jump in $\det(\mathbf{J})$, Eq. (11). By neglecting this term, we naïvely expect the diffusion tensor in Eq. (13) to yield far-from-perfect cloaking. Furthermore, Eq. (12) has the appearance of a two-dimensional analog to the one-dimensional Fokker-Planck equation (5) with $\alpha = 1$, naïvely suggesting the anti-Itô convention [25,26] of the Langevin equation (6) would yield the best cloaking performance. However, such a prediction is based on the misguided presumption that the identification of α from Eq. (12) equally applies to Fokker-Planck equations other than just the one-dimensional case [20]. In fact, in higher dimensions, the correction term that arises when treating anti-Itô as a perturbation to Itô is generally not of the divergence form necessary to write the Fokker-Planck equation as a two-dimensional analog of Eq. (5) [14]. As a result, we also trialed the diffusion tensor $\mathbf{D}'$, Eq. (13), in the Langevin equation (6) for conventions other than just anti-Itô. This was further justified *a posteriori*, since we found the Itô convention [17] resulted in the best cloaking performance. This is likely due to Itô requiring less

regularity than the other conventions [14,21–23], which have singular correction terms in this case. This singular behavior means the usual relation between the Langevin and Fokker-Planck equations is not guaranteed. Thus, the manipulations above can be considered only formal steps at first, becoming effectively valid (or not) only after appealing to simulations.

Simulation setup. We performed simulations to determine how well the cloak, with diffusivity given by Eq. (13), conceals the inner core. The core of radius R_1 is centered in a square box of linear size L . The box has periodic boundary conditions at the sides, $x' = -L/2$ and $x' = L/2$, a reflecting boundary at the top, $y' = L/2$, and an absorbing boundary at the bottom, $y' = -L/2$, through which all particles eventually leave the system. A uniform line of N particles is initialized at the top, $y' = L/2$, at time $t = 0$, i.e., $P(x', y'; t = 0) = \delta(y' - L/2) \sum_{i=1}^N \delta(x' + L/2 - iL/N)/N$. This setup is notably different from that of thermal cloaking, where the temperatures at the top and bottom of the box are kept fixed. Here, the setup is analogous to a single pulse of radiation emitted from the line $y' = L/2$ at $t = 0$. Strictly speaking, each particle contributes a Dirac delta function to the overall particle density $P(x', y'; t)$. However, for finite N , we rather view the particle density in a coarse-grained sense, i.e.,

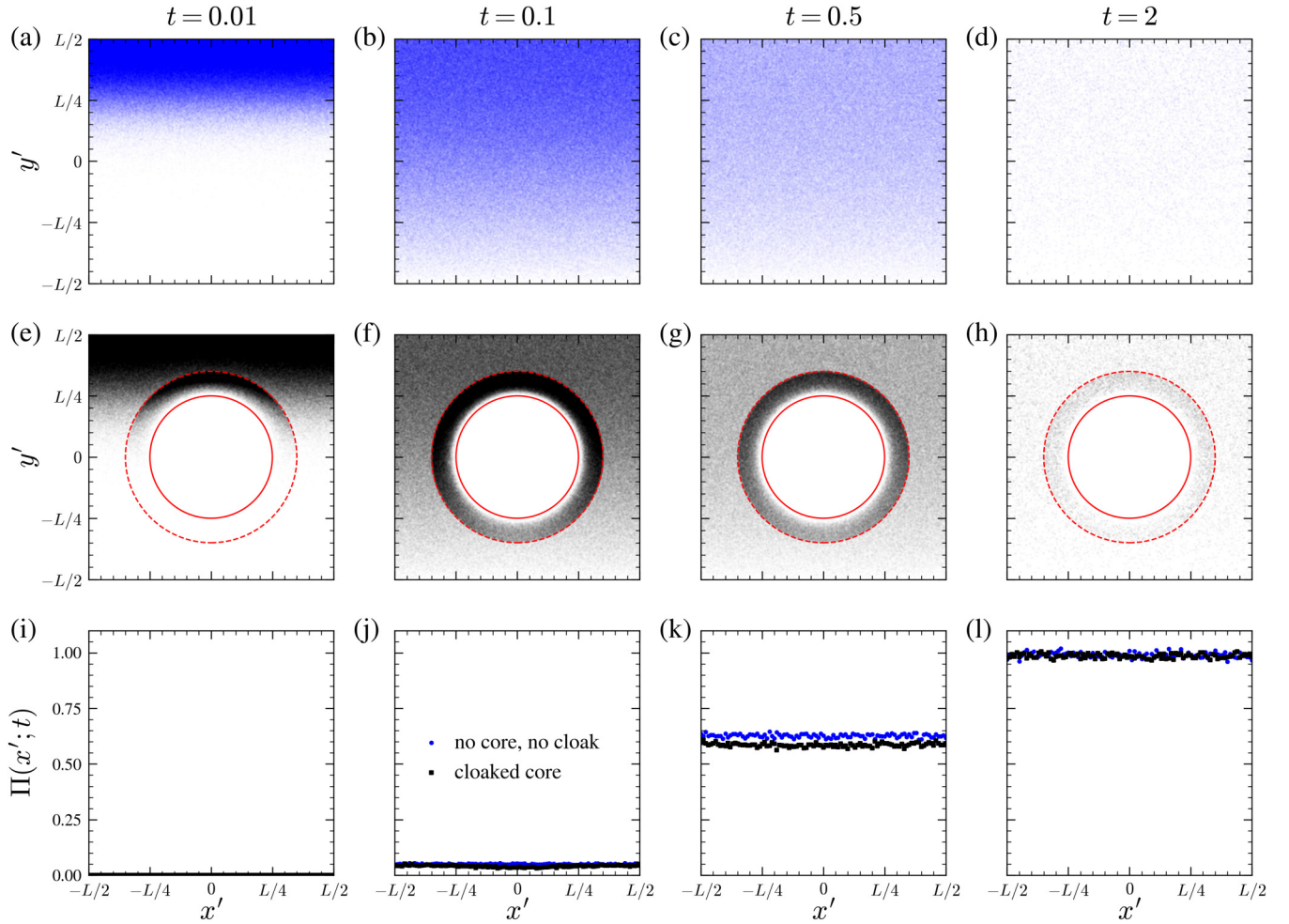


FIG. 2. Comparison between simulations of no core (a)–(d), and a cloaked core (e)–(h), with their corresponding arrival distributions $\Pi(x'; t)$, Eq. (14), (i)–(l) at times $t = 0.01$ in (a), (e), and (i); $t = 0.1$ in (b), (f), and (j); $t = 0.5$ in (c), (g), and (k); and $t = 2$ in (d), (h), and (l). Each point in (a)–(h) represents the position of one of $N = 10^6$ particles. The simulation parameters used in all subfigures were $\Delta t = 10^{-5}$, $L = 1$, and $D_0 = 1$. In (e)–(h), the inner core has radius $R_1 = 0.25$ (solid red line) and is surrounded by an annular cloak of outer radius $R_2 = 0.35$ (dashed red line). Particles are prevented from penetrating the core by the cloak in (e)–(h), and the arrival distributions $\Pi(x'; t)$ in (i)–(l) show that of the cloaked core closely matches that of no core for all times t , suggesting the diffusion tensor, Eq. (13), used in the cloaking simulations produces near-perfect cloaking.

envisage the simulation box as being divided into small cells of linear length ℓ , such that the density in a particular cell is well approximated by $n/(N\ell^2)$, where n is the number of particles contained in that cell. In this sense, the true particle density $P(x', y'; t)$ is recovered in the thermodynamic limit $N \rightarrow \infty$, while also taking $\ell \rightarrow 0$.

After initialization, particles diffuse according to Eq. (6) with diffusivity given by Eq. (13)—see Sec. SII for full details [30]. In case particles penetrated the core $r' \leq R_1$, we chose it to have the same homogeneous diffusivity D_0 as the medium surrounding the cloak. Perfect cloaking would prevent particles from penetrating the core, $r' \leq R_1$, while also ensuring the density outside the cloak, $r' > R_2$, matches that of a simulation containing no core at all times t . In other words, any observer measuring the density at a radius $r' > R_2$ would be unable to detect the presence of the core and the cloak because the density would be the same as in their absence. How closely these densities match gives a measure of the cloak's performance. To quantify cloaking performance,

we introduced the cumulative arrival distribution $\Pi(x, t_f)$ of particles at the absorbing boundary, which is analogous to a “splitting probability” in a first-passage problem [13],

$$\Pi(x'; t_f) = \int_0^{t_f} dt D_0 \frac{\partial P(x', y'; t)}{\partial y'} \Big|_{y'=-L/2}, \quad (14)$$

where $D_0 \partial_y P(x', y'; t)$ is the particle current at $\mathbf{x}' = (x', y')$ in the negative- y' direction. Equation (14) is the density of particles that have arrived at position x' on the absorbing boundary by time t_f . The spatial average of $\Pi(x'; t_f)$ asymptotically approaches unity as $t_f \rightarrow \infty$, i.e., $\lim_{t_f \rightarrow \infty} \int_{-L/2}^{L/2} dx' \Pi(x'; t_f)/L = 1$, signifying the absorption of all particles. For the cloaking to be considered effective, the arrival distribution in the simulation of the cloaked core must closely match that of a simulation of no core (and no cloak).

Simulation results. We performed the simulations described above for the Itô ($\alpha = 0$) [17], Stratonovich ($\alpha = 1/2$) [18], and anti-Itô ($\alpha = 1$) [25,26] conventions of interpreting

the Langevin equation (6). The Stratonovich and anti-Itô conventions resulted in poorer cloaking than that of Itô, signified by particles penetrating the core; spurious large jumps near $r' = R_1$; and noticeable differences in the arrival distributions, Eq. (14), compared to that of no core. This is not surprising given the application of these conventions to systems involving “problematic” boundaries—like the discontinuity at $r' = R_2$ and singularity at $r' = R_1$ in the diffusivity, Eq. (13)—can lead to ill-posed results, as mentioned in Refs. [14,21–23]. Given their relatively poor performance, we omit further discussion of the Stratonovich and anti-Itô conventions.

Our key result is that, remarkably, the Itô convention exhibited near-perfect cloaking signified by no particle penetration of the core and an arrival distribution, Eq. (14), that closely matched that of no core for all times t , see Fig. 2. “Near perfect” is to caveat a small discrepancy in the number of particles that have been absorbed up to time t between the simulations of the cloaked core and no core. However, this is similar to time lags typically seen in thermal cloaking [36]. Moreover, this discrepancy all but disappears through regularization procedures that we introduce in Sec. SIII [30]. Strikingly, the particle density inside the cloak $R_1 < r' \leq R_2$ is markedly different from that seen for thermal cloaking, where the density typically decays smoothly from the outer region $r' > R_2$ to the inner cloak boundary $r' = R_1$ [7]. Here, there is a distinct “halo” of high particle density in the cloak to accommodate those particles that would otherwise be inside the core $r' < R_1$. This also suggests why the smooth decay in density for thermal cloaking comes at the cost of penetration of the core [7].

Discussion and conclusion. Inspired by optical and thermal cloaking [6,7], we demonstrated “stochastic cloaking” of a

region from diffusive particles. This is in spite of the lack of form invariance of the Fokker-Planck equation under a general coordinate transformation, as well as the inherent difficulties introduced by stochasticity. Our setup involved a circular core that we attempted to conceal by surrounding it with an annular metamaterial of spatially varying diffusivity. To quantify the cloaking performance, we measured the arrival distribution, Eq. (14), of particles along a tangential line downstream of the cloak. The hallmark of cloaking in this case is an arrival distribution that matches that of a simulation involving no core for all times. Our key result is a demonstration of near-perfect cloaking for the diffusion tensor D' in Eq. (13) if the Langevin equation is interpreted under the Itô convention [17], see Fig. 2. The cloaking performance can be further improved through regularization procedures introduced in Sec. SIII [30]. These procedures also soften the requirements for the cloak to be implemented in an experiment by removing any singular behavior. While we demonstrated near-perfect cloaking, the specific coordinate transformation we considered resulted in invariance of the Fokker-Planck equation only on a piecewise basis. Therefore, future work should focus on establishing a firmer theoretical basis for the cloaking that results from the Itô convention. Another interesting avenue would be to test stochastic cloaking on other cloak geometries, such as diamonds [37].

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Data availability. The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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