

Twist-driven interlayer hybridization and multiple mobility edges in bilayer quasiperiodic lattices

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We investigate two coupled one-dimensional Aubry-André-Harper rings with N sites, where a site-dependent angular twist in the second ring generates a spatially varying mosaic of distance-modulated couplings. By varying the quasiperiodic potential strength V and twist periodicity k , we find multiple closely spaced mobility edges separating localized, extended, and critical eigenstates within narrow energy intervals. Fractal dimension analysis shows that k strongly controls the spectral distribution of these states, with larger k enhancing the coexistence of different localization characters. In addition, we find that for $k = 2$, most eigenstates remain nearly equally distributed between the two rings, whereas larger k drives pronounced layer polarization. We further examine real-time wave-packet dynamics and show that increasing k accelerates the crossover from nearly ballistic spreading to subdiffusive motion and lowers the long-time saturation of the normalized mean-square displacement. At the same time, an initially layer-balanced wave packet remains strongly hybridized for $k = 2$, whereas the interlayer overlap is reduced for $k = 3$ and $k = 4$. Thus, the site-dependent twist provides a tunable mechanism to control spectral localization, wave-packet transport, and interlayer mixing in quasiperiodic bilayer lattices.

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I. INTRODUCTION

Anderson localization is one of the most intriguing phenomena in condensed matter physics, describing how disorder-induced interference suppresses wave propagation [1]. First proposed by Philip W. Anderson in 1958, this phenomenon plays a significant role in electronic conduction, optical media, and ultracold atomic systems. The central idea behind Anderson's theory is that sufficiently strong disorder in a system can lead to quantum interference that halts the spread of wavelike excitations, resulting in eigenstates that are exponentially localized. Since its inception, localization theory has been extensively explored across diverse settings, with active progress continuing in recent years [2–7]. In one-dimensional systems with uncorrelated disorder, all states are localized for any nonzero disorder strength. However, in higher dimensions, a metal-insulator transition can occur at a critical disorder strength, where extended and localized states coexist, separated by an energy-dependent boundary known as the mobility edge [8,9].

A particularly intriguing manifestation of localization arises in quasiperiodic systems. The Aubry-André-Harper (AAH) model provides a quintessential example of such

behavior, describing a one-dimensional tight-binding lattice with an incommensurate potential. The model is given by the Hamiltonian

$$H = -t \sum_n (c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n) + \sum_n V_n c_n^\dagger c_n, \quad (1)$$

where t is the hopping amplitude and V_n is a quasiperiodic on-site potential of the form

$$V_n = V \cos(2\pi\beta n + \phi), \quad (2)$$

with V being the potential strength, β an irrational modulation parameter, and ϕ a phase factor. Unlike purely random disorder, quasiperiodicity induces a sharp localization transition at a critical potential strength $V_c = 2t$. For $V < 2t$, all eigenstates remain extended, whereas for $V > 2t$, all states become localized. This self-duality is a unique feature of the AAH model, distinguishing it from conventional Anderson localization.

In its standard form, the AAH model does not exhibit a mobility edge, as all states transition simultaneously at V_c . However, generalized versions introduce energy-dependent transitions, allowing for the coexistence of localized and extended states at different energy levels [10,11]. Localization phenomena in other quasiperiodic systems have been extensively studied, with the AAH model serving as a fundamental backbone. The AAH model has been extended by incorporating interactions [12–14], long-range correlations [15], and mosaic lattice structures [16–19], leading to richer localization and delocalization transitions. Furthermore, studies have shown that mobility edges can also arise through potential modulations [20–22] and have been experimentally verified using optical [23,24] and photonic lattices [25]. These

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advances collectively deepen our understanding of localization physics in quasiperiodic systems and provide promising directions for future experimental and theoretical research.

In recent years, the discovery of moiré patterns in van der Waals heterostructures, particularly in twisted bilayer graphene, has reignited interest in systems where a relative twist between layers produces large-scale interference patterns [26–28]. These moiré superlattices can lead to dramatic changes in the electronic structure, including the emergence of flat bands and correlated insulating states. Twist angle engineering has transformed two-dimensional materials by allowing dramatic changes in band structures as seen in magic-angle twisted bilayer graphene. However, these systems use a uniform twist that creates periodic patterns across the entire material, limiting control over specific energy states. We extend this concept to quasiperiodic systems using a site-dependent twist in coupled Aubry-André-Harper rings. This creates a synthetic moiré pattern with quasiperiodic character, where the spatially varying twist produces a mosaic of interlayer couplings that can be precisely adjusted. Unlike uniform twist systems, our geometry provides eigenstate localization and enables the creation of multiple mobility edges. Furthermore, it offers a unique tuning parameter to continuously modulate states from strongly hybridized regimes to strong layer polarization phenomena that remain mostly unexplored in quasiperiodic lattices.

II. MODEL

We consider two coupled AAH ring lattices, each consisting of N sites arranged on a one-dimensional ring of radius R . The composite system lives in a Hilbert space of dimension $2N$, with sites indexed globally as $i = 1, \dots, 2N$. We partition this into two layers: layer 1 ($l = 1$) contains sites $i = 1, \dots, N$, and layer 2 ($l = 2$) contains sites $i = N + 1, \dots, 2N$. We enforce a one-to-one interlayer correspondence so that site i in layer 1 couples to site $i + N$ in layer 2.

The initial angular positions are equally spaced and aligned between layers. For any site i in layer l , we define its local index within the layer as $j = i - (l - 1)N$, where $j = 1, \dots, N$. The initial angular position on each ring is

$$\theta_j^{(l)} = \frac{2\pi(j-1)}{N}, \quad j = 1, \dots, N, \quad l = 1, 2.$$

We then introduce a site-dependent twist by modifying only layer 2 and, from this point on, use θ_j to denote the *twist offset* at site j :

$$\theta_j = (\text{mod}(j-1, k) + 1)\theta, \quad j = 1, \dots, N, \quad (3)$$

so that the twist follows a k -periodic mosaic along the ring.

Both intralayer and interlayer hoppings decay with chord distance on a circle of radius R . The intralayer hopping from site j to $j + 1$ in layer ℓ is

$$t_{j,j+1}^{(\ell)} \propto [d_{j,j+1}^{(\ell)}]^{-\alpha},$$

where for layer 1

$$d_{j,j+1}^{(1)} = \sqrt{2R^2 \left(1 - \cos \frac{2\pi}{N}\right)},$$

and for layer 2 (using the twist-offset convention for θ_j)

$$d_{j,j+1}^{(2)} = \sqrt{2R^2 \left(1 - \cos \left(\theta_j - \theta_{j+1} - \frac{2\pi}{N}\right)\right)}.$$

The interlayer hopping between site j in layer 1 and site j in layer 2 is

$$t_j^{\text{inter}} \propto [d_j^{\text{inter}}]^{-\alpha}, \quad d_j^{\text{inter}} = 2R \left| \sin \left(\frac{\theta_j}{2} \right) \right|.$$

Here, $\alpha > 0$ controls the distance decay of all hoppings. This geometry-induced modulation of both intralayer and interlayer couplings captures the essence of moiré superlattice physics in a quasiperiodic setting. From this setup, we obtain the following mosaic pattern for coupling strength:

$$t_{j,j+1}^{(\ell)} = \frac{1}{(d_{j,j+1}^{(\ell)})^\alpha} = \begin{cases} \frac{1}{d^\alpha}, & \ell = 1, \\ \frac{1}{d^\alpha}, & \ell = 2, \text{ for } \text{mod}(j, k) \neq 0, \\ \frac{1}{d_k^\alpha}, & \ell = 2, \text{ for } \text{mod}(j, k) = 0, \\ \frac{1}{d_{\text{mod}(j-1, k)+1}^\alpha}, & \ell = \text{inter}. \end{cases} \quad (4)$$

This structure indicates that the distance between nodes in layer 1 remains constant. In layer 2, the distance is also constant for the first $k - 1$ pairs and differs at every k th pair. For the interlayer coupling, the distances follow a periodic pattern of length k , which repeats along the ring. To facilitate analysis, we normalize each distance, including $d_{ij}^{(1)}$, $d_{ij}^{(2)}$, and $d_{ij}^{(\text{inter})}$, by the average value within its respective coupling group, allowing us to work on a near-unit coupling scale. In principle, the site-dependent twist could be approximated in optical waveguide arrays, where spatial placement controls both intralayer and interlayer couplings [29,30]. Likewise, ultracold atoms in bichromatic lattices or synthetic dimensions may emulate tunable tunneling and bilayer coupling [31,32]. These platforms thus offer plausible, if idealized, routes to explore the interplay of quasiperiodicity and mosaic inhomogeneity in coupled-layer systems. In Fig. 1, we schematically illustrate the bilayer model incorporating a site-dependent twist with periodicity k . Each panel depicts two concentric N -site rings: the first layer L_1 (blue), whose sites are positioned at angular coordinates $\theta_i^{(1)} = \frac{2\pi}{N}(i-1)$, and the second, twisted layer L_2 (red), whose site positions are displaced according to the mosaic modulation defined in Eq. (3). For $k = 1$, the displacement corresponds to a uniform angular shift $\Delta\theta$ across all sites in L_2 , resulting in a simple concentric offset where both the interlayer distances between corresponding nodes and the intralayer spacing within L_2 remain constant. In contrast, for $k > 1$, the twist introduces spatial inhomogeneity: both the interlayer distances between L_1 and L_2 and the intralayer distances within L_2 vary according to the mosaic structure described by Eq. (4). Collecting all contributions, the full tight-binding Hamiltonian on the combined $2N$ -site system is

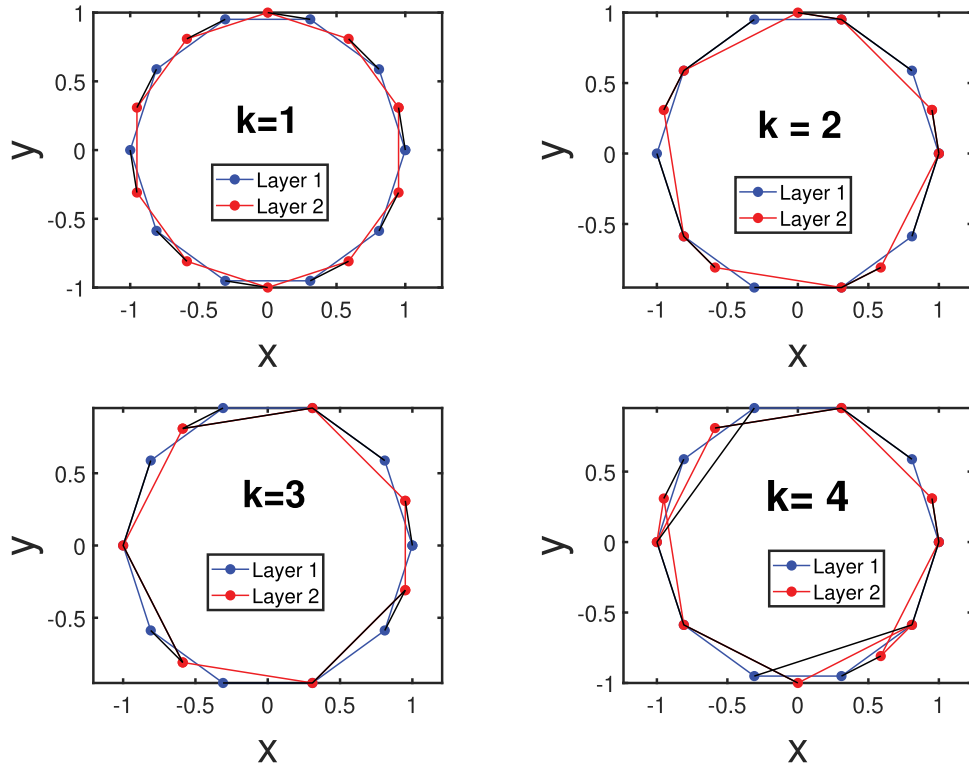


FIG. 1. Schematic of two concentric N -site ring lattices with a site-dependent twist of period k . In each panel, the blue dots and polygon denote layer 1 (no twist), while the red dots and polygon denote layer 2 after applying the mosaic twist. As k increases, the inhomogeneity of interlayer registry grows, leading to increasingly complex moiré patterns. The figure is shown for $N = 10$ for different k .

given by

$$\begin{aligned}
 H = & - \sum_{j=1}^N \frac{1}{[d_{j,j+1}^{(1)}]^\alpha} (c_{j,1}^\dagger c_{j+1,1} + \text{H.c.}) \\
 & - \sum_{j=1}^N \frac{1}{[d_{j,j+1}^{(2)}]^\alpha} (c_{j,2}^\dagger c_{j+1,2} + \text{H.c.}) \\
 & + \sum_{j=1}^N \sum_{\ell=1}^2 V \cos(2\pi\beta j + \phi) c_{j,\ell}^\dagger c_{j,\ell} \\
 & - \sum_{j=1}^N \frac{1}{[d_j^{\text{inter}}]^\alpha} (c_{j,1}^\dagger c_{j,2} + \text{H.c.}). \quad (5)
 \end{aligned}$$

III. RESULTS

A. Spectral analysis

We first present the results for $k = 2$, and later report and compare the corresponding results for other values of k . For convenience, we set $\alpha = 1$, $R = 1$, fix the phase $\phi = 0$, and choose the modulation parameter $\beta = \lim_{n \rightarrow \infty} \frac{F_{n-1}}{F_n} = \frac{\sqrt{5}-1}{2}$, where F_n denotes the n th Fibonacci number. For finite-size systems, it is customary to take $N = F_n$ and approximate β by $\beta = \frac{F_{n-1}}{F_n}$, which provides an optimal rational approximation to the irrational golden mean. This choice ensures compatibility with periodic boundary conditions and facilitates efficient numerical diagonalization of the Hamiltonian. To introduce

structural asymmetry, we apply a small angular twist in the second layer by setting $\theta = \pi/10$.

A widely used metric to measure the extent of localization is the Inverse Participation Ratio (IPR), defined for the j th eigenstate as $\text{IPR}_j = \sum_{i=1}^{2N} |\phi_i^{(j)}|^4$, where $\phi_i^{(j)}$ denotes the i th component of the normalized eigenstate ϕ_j . However, computing the IPR for an eigenstate alone is not enough to ascertain whether an eigenstate is localized or extended. Instead, to substantiate true characteristics, one must evaluate the fractal dimension (FD), defined as

$$\text{FD} = - \lim_{N \rightarrow \infty} \frac{\ln(\text{IPR})}{\ln(2N)},$$

which, in the thermodynamic limit, approaches 1 for extended states and 0 for localized states, while $0 < \text{FD} < 1$ for critical states.

In Fig. 2, we plot the FD as a function of eigenenergy λ of the Hamiltonian in Eq. (5) for various values of the potential strength V . Note that the plot is shown only for $\lambda < 0$, as the spectrum is symmetric, and identical distribution is obtained for $\lambda > 0$. For $V = 1$, the FD approaches 1 for all eigenenergies as N increases, indicating that all eigenstates are delocalized. When $1 < V \leq 1.3$, multiple mobility edges appear, separating localized and delocalized regimes (see $V = 1.2$ in Fig. 2). For $V > 1.3$, the FD curve shows additional structure: in addition to fully localized and fully extended regions, there is a narrow interval around $\lambda = \pm 1.33$ (e.g., for $V = 1.5$), where $0 < \text{FD} < 1$, signaling a band of multifractal states. Finally, for $V \gtrsim 2.2$, all eigenstates become localized,

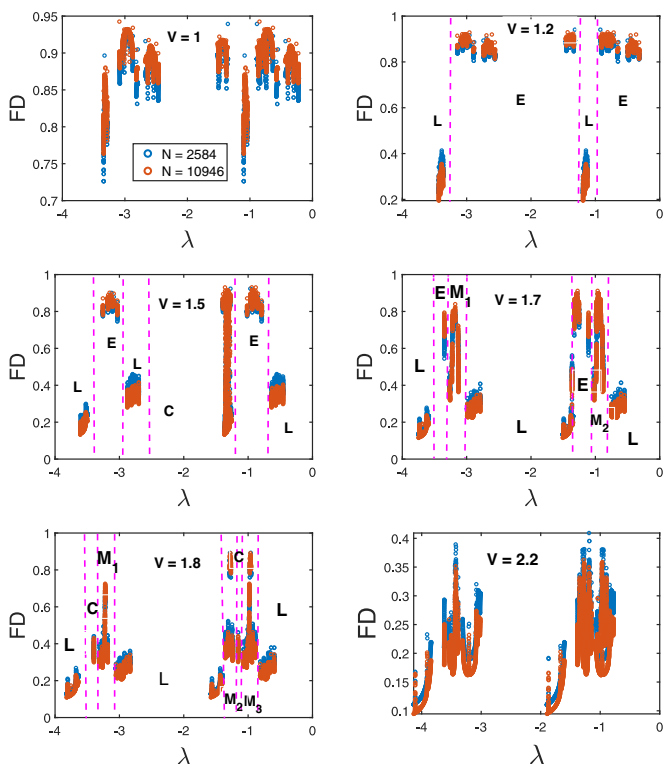


FIG. 2. Fractal dimension as a function of eigenenergy for various potential strengths V at twist periodicity $k = 2$. The magenta dashed line indicates the mobility edge, separating regions of localized (L), extended (E), and critical (C) states. Each layer consists of $N = 2584$ lattice sites.

as evidenced by the FD tending toward 0 with increasing system size (Fig. 2).

Furthermore, the situation is more nuanced in the transitional region; see, for example, the cases $V = 1.7$ and $V = 1.8$ in Fig. 2. In addition to well-defined localized and extended states, we observe regions—labeled as M_1 , M_2 , and M_3 in the figure, where the fractal dimension spans a broad range of values in the immediate vicinity of certain eigenenergies. We refer to these as *mixed regions*, characterized by the coexistence of localized, extended, and critical states. These should not be confused with the multifractal band observed for $V = 1.5$, where although the FD also varies significantly, the transitions lack clearly defined boundaries separating localized and extended behavior. Therefore, to investigate the true nature of these regions, in Fig. 3, we present a magnified view of the corresponding segments of the FD spectrum. As evident from the figure, unlike the critical regime observed at $V = 1.5$, the mixed regions exhibit a clear separation between eigenstates of different character—localized, extended, and critical. To further substantiate these observations, we extrapolate the fractal dimension to the thermodynamic limit ($N \rightarrow \infty$) by computing the average FD in each region as a function of $1/n$, where n denotes the Fibonacci index. As shown in the figure, even within a narrow interval of the eigenenergy spectrum, the FD approaches 0 or 1, or remains strictly between 0 and 1, signaling the coexistence of localized, extended, and critical states with well-defined boundaries. This behavior provides compelling evidence for the existence of closely spaced

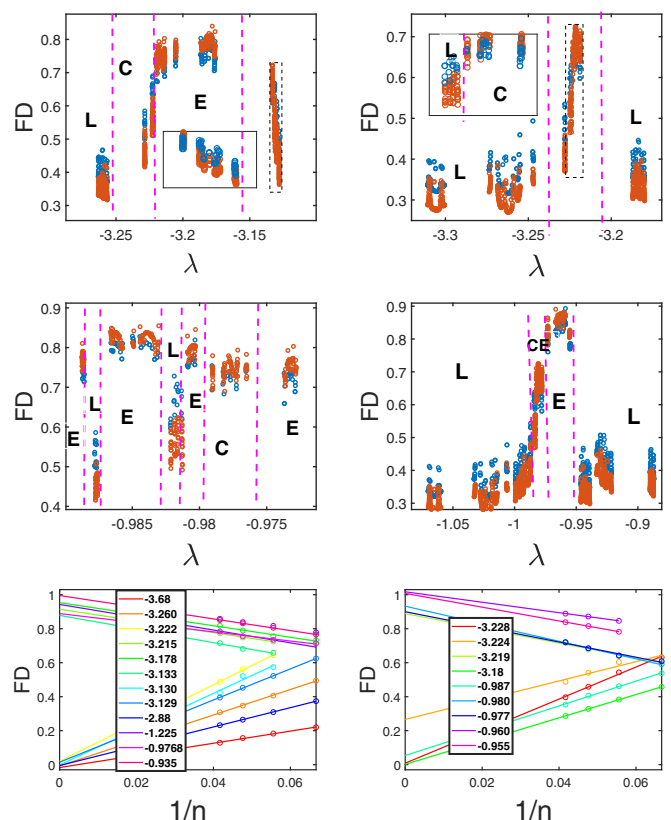


FIG. 3. Magnified view of the fractal dimension FD vs eigenenergy λ for $V = 1.7$ (left column) and $V = 1.8$ (right column). The top panels reveal localized (L), extended (E), and critical (C) states within narrow eigenenergy intervals separated by magenta dashed lines. The bottom panels show finite-size scaling of the region-averaged FD, obtained by averaging over selected eigenenergies within the corresponding regions identified in the top panels. The scaling confirms the coexistence of distinct types of eigenstates and closely spaced mobility edges. Each layer consists of $N = 2584$ lattice sites.

mobility edges residing in close proximity within the eigenenergy spectrum. These results place our model within the broader class of quasiperiodic systems that support simultaneous localized, critical, and extended eigenstates in different parts of the spectrum [33–36].

Next, in Fig. 4, we plot the fractal dimension FD_j of single-particle eigenstates with $\lambda_j < 0$ as a function of the potential strength V and eigenenergy λ_j for different twist periodicities k . For each k , the spectrum can be broadly separated into different localization regimes, with approximate crossover boundaries marked by V_{c1} , V_{c2} , and V_{c3} . At weak potential, $V \lesssim V_{c1}$, most eigenstates have $FD_j \approx 1$, indicating extended behavior. At strong potential, $V \gtrsim V_{c3}$, the eigenstates become predominantly localized, with $FD_j \approx 0$. In the intermediate region, the spectrum shows a richer structure. For $V_{c1} < V < V_{c2}$, extended, critical, and localized eigenstates can appear within nearby energy windows, indicating the coexistence of different localization characters in the same spectrum. For $V_{c2} < V < V_{c3}$, the extended contribution is strongly reduced, and the spectrum is mainly composed of localized and critical eigenstates. In this regime, the boundaries separating localized

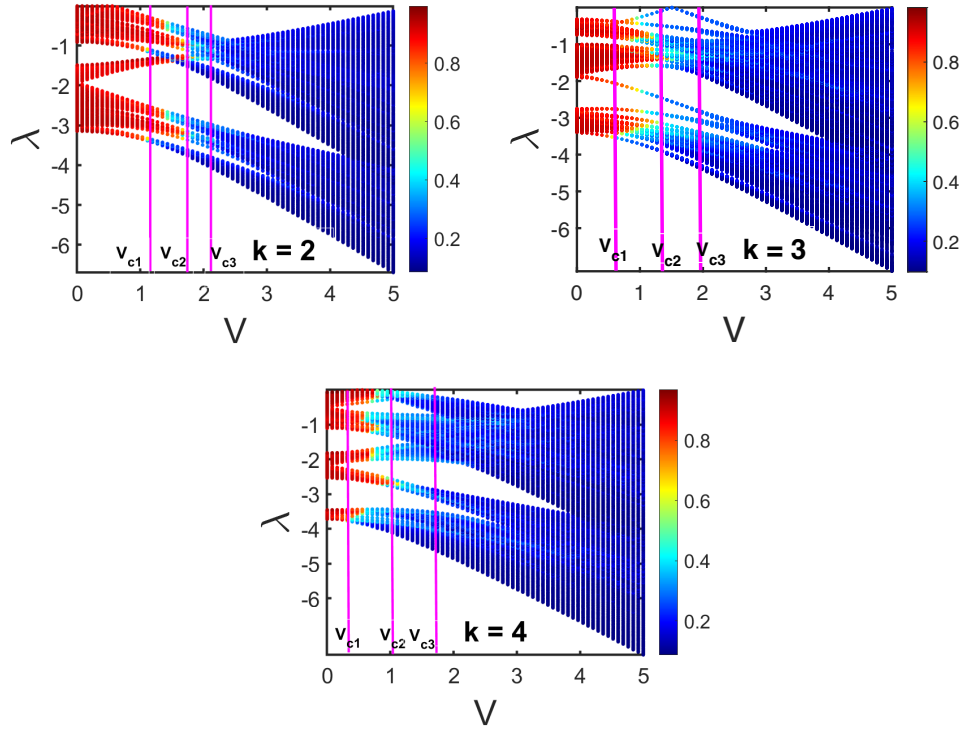


FIG. 4. Fractal dimension FD as a function of potential strength V and eigenenergy λ , shown for twist periodicities $k = 2, 3$, and 4 . Each panel shows the distribution of FD across the spectrum and distinguishes extended, critical, and localized eigenstates. For $k = 2$ and $k = 4$, each layer contains $N = 2584$ sites, whereas for $k = 3$, the system size is $N = 987$.

and critical states can be viewed as anomalous mobility edges, since they involve critical states and are distinct from the conventional mobility edge between localized and extended states. Increasing k enhances the intermediate regimes and makes the localized and critical regions more prominent. This shows that the site-dependent twist periodicity controls the structure of the localization crossover. In particular, for larger k , the extended regions shrink more rapidly with increasing V , while critical and localized eigenstates occupy a larger portion of the spectrum. Thus, Fig. 4 demonstrates that the twist periodicity provides an additional control parameter for tuning the coexistence and separation of extended, critical, and localized eigenstates.

We characterize the degree of interlayer hybridization of the j th eigenstate $\phi^{(j)}$ of the $2N$ -site bilayer Hamiltonian using two complementary metrics. The first is the layer-resolved IPR ratio, defined as

$$f_j^{(\ell)} = \frac{\text{IPR}_j^{(\ell)}}{\text{IPR}_j} = \frac{\sum_{i=(\ell-1)N+1}^{\ell N} |\phi_i^{(j)}|^4}{\sum_{i=1}^{2N} |\phi_i^{(j)}|^4} \quad (\ell = 1, 2),$$

where $\text{IPR}_j = \sum_{i=1}^{2N} |\phi_i^{(j)}|^4$ is the total inverse participation ratio and $\text{IPR}_j^{(\ell)} = \sum_{i=1}^N |\phi_i^{(\ell)}(j)|^4$ is the IPR restricted to layer ℓ . The ratio $f_j^{(\ell)}$ thus quantifies the fraction of the localization weight confined to a specific layer, offering insight into whether a state is layer polarized or evenly spread across the bilayer. Subsequently, the quantity $H_j = 4 f_j^{(1)} f_j^{(2)}$ serves as a normalized mixing metric that varies between $0 \leq H_j \leq 1$. By construction, $H_j = 0$ if an eigenstate j is confined entirely

in one layer ($f_j^{(1)} = 0$ or $f_j^{(2)} = 0$), while $H_j = 1$ if it is equally shared between the two layers ($f_j^{(1)} = f_j^{(2)} = \frac{1}{2}$). In Fig. 5, we plot the binned average of H_j (color scale) as a

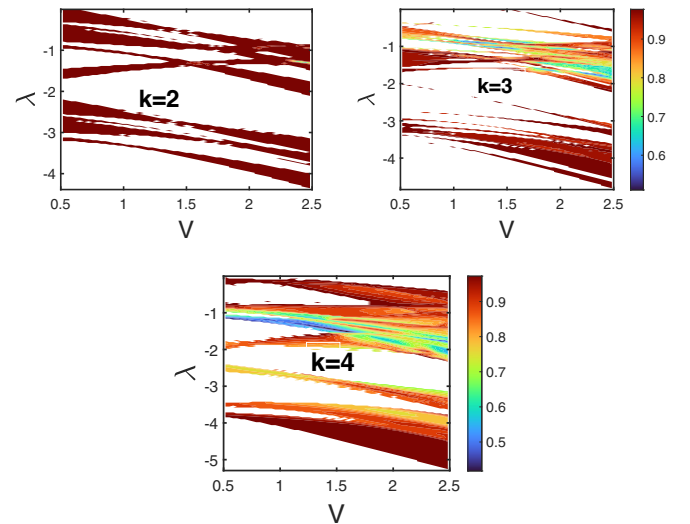


FIG. 5. Contour plots of the normalized mixing metric H_j vs potential V and eigenenergy λ for twist periodicities $k = 2, 3, 4$, showing the crossover from bilayer hybridization ($H_j \approx 1$) to reduced interlayer mixing and increased layer polarization with increasing k . For $k = 2$ and $k = 4$, each layer contains $N = 2584$ sites, whereas for $k = 3$, the system size is $N = 987$. To improve visualization, we averaged data over small bins in the (V, λ) plane before constructing the color map.

function of eigenenergy λ and potential strength V for periodicities $k = 2, 3, 4$. For $k = 2$, most states lie near $H_j \approx 1$, indicating almost complete bilayer hybridization. Increasing k progressively breaks this symmetry: at $k = 3$, one observes bands of reduced $H_j \lesssim 0.8$ (partial layer polarization), and by $k = 4$ large regions of the spectrum exhibit H_j moderately below 1, signaling the emergence of twist periodicity-induced layer localization. Thus, H_j efficiently captures how the twist periodicity k controls the crossover from fully hybridized bilayer states to more layer-polarized, less mixed eigenstates.

Furthermore, to quantify the correlations and leakage of each single-particle eigenstate $|\phi_j\rangle$ between the two layers, we compute its bipartite (von Neumann) entanglement entropy under a layer bipartition between layer 1 and layer 2. This mode (partition) entanglement construction is standard for free-fermion lattice models and directly tied to single-particle occupation probabilities [37–39]. Defining the reduced density matrix of eigenstate j on layer 1 as

$$\rho_j^{(1)} = \text{Tr}_2\{|\phi_j\rangle\langle\phi_j|\},$$

the entanglement entropy is given by the von Neumann formula

$$S_j = -\text{Tr}[\rho_j^{(1)} \ln \rho_j^{(1)}].$$

In our two-ring (bilayer) model, $\rho_j^{(1)}$ has only two diagonal blocks corresponding to the probability $p_j = \sum_{i=1}^N |\phi_j(i)|^2$ that the particle resides in layer 1, and $1 - p_j$ that it resides in layer 2. Hence, $\rho_j^{(1)} = \text{diag}(p_j, 1 - p_j)$ and the entropy reduces to the binary form

$$S_j = -[p_j \ln p_j + (1 - p_j) \ln(1 - p_j)].$$

Physically, $S_j = 0$ indicates a completely layer-polarized (unhybridized) state, while $S_j = \ln 2$ (the maximum for a two-level mixture) signals perfect layer delocalization. Intermediate values $0 < S_j < \ln 2$ quantify the degree of hybridization or leakage between the rings. In the contour plots shown in Fig. 6, we display the bin-averaged entropy S_j (color scale) as a function of eigenenergy λ and AAH potential strength V for twist periodicity $k = 2, 3, 4$. For $k = 2$ (top left panel), most states lie close to $S_j \approx 0.69$, indicating almost complete bilayer mixing. At $k = 3$ (top right), one sees lowered entropy bands $S_j \approx 0.65 - 0.68$, signaling partial layer polarization over significant spectral regions. By $k = 4$ (bottom panel), extended regions of the spectrum show $S_j \lesssim 0.62$, indicating stronger layer imbalance and reduced interlayer mixing, although the states are not fully confined to a single layer. Thus, the entanglement-entropy map is consistent with a crossover from nearly fully hybridized bilayer states to increasingly layer-polarized states as the twist periodicity k increases, in agreement with the results obtained from H_j .

Previous studies of moiré heterobilayers and quasicrystals have examined hybridization in various contexts [40–42]. Here we show that a site-dependent k -periodic twist geometrically tunes interlayer mixing. For $k > 1$, structural mismatch generally prevents complete layer decoupling, with strong broadband hybridization for $k = 2$ and suppression within

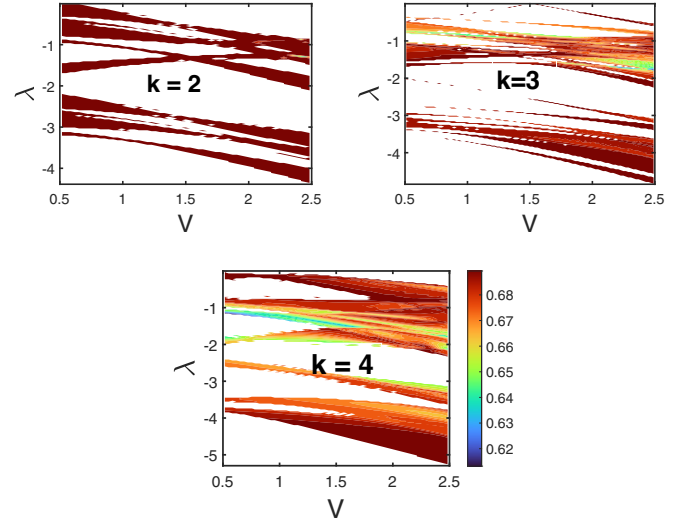


FIG. 6. Contour plots of bipartite entanglement entropy S_j vs potential V and eigenenergy λ for twist periodicities $k = 2, 3, 4$. The moderate decrease of S_j from $\ln 2$ with increasing k signals reduced layer leakage and an increasing tendency toward layer polarization. For $k = 2$ and $k = 4$, each layer contains $N = 2584$ sites, whereas for $k = 3$, the system size is $N = 987$. To improve visualization, we averaged data over small bins in the (V, λ) plane before constructing the color map.

selected energy windows for $k = 3, 4$. Thus, the mosaic twist acts as an energy-selective filter and discrete geometric control parameter for spectral engineering. Additional robustness checks for the hybridization under variations of the twist angle and distance-decay exponent are provided in the Supplemental Material [43].

B. Dynamical Analysis

To complement the spectral analysis, we next investigate the real-time evolution of a single-particle wave packet in the full bilayer Hilbert space governed by the Hamiltonian in Eq. (5). Since the on-site potential is quasiperiodically modulated, different lattice sites generally correspond to different local potential energies. Thus, choosing a different initial position effectively probes a different local-energy environment. At the same time, a state localized on a single site is not an eigenstate of the full Hamiltonian; rather, it can be expanded as a superposition of the Hamiltonian eigenstates. The subsequent dynamics therefore contains information about the spectral properties of the system as well as the spatial spreading of the wave packet.

We first consider a local excitation placed at the central site of layer 1,

$$|\psi(0)\rangle = |j_0, 1\rangle, \quad (6)$$

where $j_0 = \lfloor N/2 \rfloor$ denotes the central site of the ring. This initial condition represents a particle initially localized on a single site and allows us to study how the excitation spreads through the coupled bilayer system. Expanding the initial state

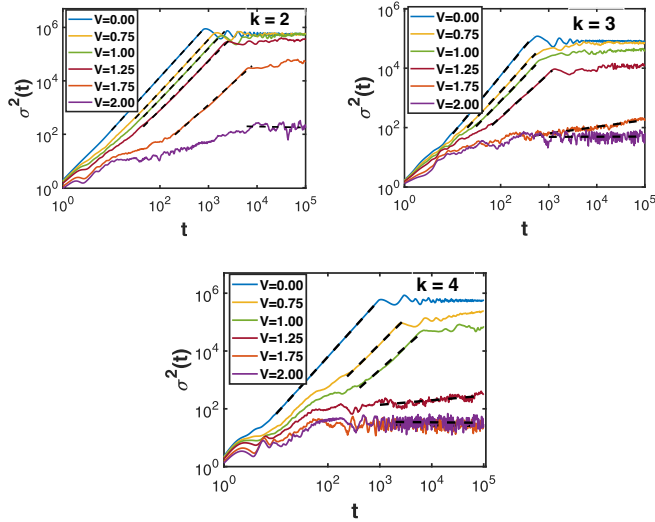


FIG. 7. Log-log plot of the mean-square displacement $\sigma^2(t)$ for the bilayer AAH model with the wave packet initially localized on the layer 1 at the central site $j_0 = \lfloor N/2 \rfloor$, shown for twist periodicities $k = 2, 3, 4$ and several values of the quasiperiodic potential V . The system size is $N = 2584$ for $k = 2$ and $k = 4$, whereas $N = 987$ for $k = 3$. The black dashed lines correspond to power-law fits in the expansion regime.

in the eigenbasis $\{|\phi_n\rangle\}$ of the full Hamiltonian H ,

$$|\psi(0)\rangle = \sum_{n=1}^{2N} c_n |\phi_n\rangle, \quad c_n = \langle \phi_n | \psi(0) \rangle, \quad (7)$$

the time-evolved state is

$$|\psi(t)\rangle = e^{-iHt} |\psi(0)\rangle = \sum_{n=1}^{2N} c_n e^{-i\lambda_n t} |\phi_n\rangle, \quad (8)$$

where λ_n are the eigenenergies of H . One important quantity for characterizing the spatial spreading of the wave packet is the mean-square displacement (MSD). Since the system has periodic boundary conditions, the distance from site j to the initially excited site j_0 is measured along the ring as the shortest path,

$$d(j, j_0) = \min(|j - j_0|, N - |j - j_0|). \quad (9)$$

The MSD is then defined as

$$\sigma^2(t) = \sum_{j=1}^N d^2(j, j_0) \rho_j(t), \quad (10)$$

where

$$\rho_j(t) = |\psi_{j,1}(t)|^2 + |\psi_{j,2}(t)|^2 \quad (11)$$

is the total probability density on rung j , summed over the two layers. Here, $\psi_{j,\ell}(t)$ denotes the time-dependent wavefunction amplitude at site j in layer $\ell = 1, 2$. Thus, $\sigma^2(t)$ measures how far the probability distribution has spread away from the initial site, irrespective of which layer the particle occupies.

In Fig. 7, we present log-log plots of the mean-square displacement $\sigma^2(t)$ as a function of time t for different values of the quasiperiodic potential strength V and twist periodicity

TABLE I. Transport exponent μ obtained from the power-law fit $\sigma^2(t) \sim t^\mu$ in the expansion regime with the wave packet initially localized on the first layer at the central site $j_0 = \lfloor N/2 \rfloor$. Ballistic, superdiffusive, and subdiffusive transport correspond to $\mu \approx 2$, $1 < \mu < 2$, and $0 < \mu < 1$, respectively.

V	$\mu(k=2)$	$\mu(k=3)$	$\mu(k=4)$
0.00	1.99	1.98	1.99
0.75	1.95	1.76	1.70
1.00	1.90	1.65	1.60
1.25	1.82	1.47	0.15
1.50	1.74	0.43	0.02
1.75	1.62	0.22	0.02
2.00	0.16	0.01	0.01

k . The MSD initially grows approximately as a power law. At later times, it saturates and oscillates around a finite value due to the finite system size and the restricted spatial spreading of the wave packet. To quantify the transport behavior, we fit the MSD in the expansion regime using

$$\sigma^2(t) \sim t^\mu, \quad (12)$$

where the exponent μ characterizes the nature of the dynamics. In particular, $\mu \approx 2$ corresponds to ballistic transport, $1 < \mu < 2$ to superdiffusive transport, $\mu = 1$ to diffusive spreading, $0 < \mu < 1$ to subdiffusive transport, and $\mu = 0$ to a localized wave packet with negligible spatial spreading. The fitted values of μ are summarized in Table I.

For all values of k , the transport exponent decreases as the quasiperiodic potential strength V is increased, indicating a gradual suppression of wave-packet spreading. For $k = 2$, the dynamics remains nearly ballistic up to $V \approx 1$ and stays superdiffusive over a broad intermediate range of V , before becoming strongly subdiffusive at the largest potential strength considered. For $k = 3$, the reduction of μ occurs more rapidly, with the dynamics evolving from near-ballistic motion at weak potential to subdiffusive transport at larger V . This suppression is even more pronounced for $k = 4$, where the wave packet becomes strongly subdiffusive already around $V = 1.25$. These trends are consistent with the spectral analysis and highlight the role of k in the localization characteristics of the eigenstates. Since the initial localized wave packet is a superposition of many eigenstates, the observed MSD dynamics reflects the combined contribution of extended, critical, and localized states. Comparing the dynamical results with the spectral analysis, a value of $\mu > 1$ suggests that extended states continue to make a dominant contribution to wave-packet spreading, while the deviation from the ballistic limit $\mu \simeq 2$ reflects the growing influence of critical states. In the subdiffusive regime $0 < \mu < 1$, the extended contribution is reduced and the dynamics is increasingly governed by critical and localized components. Thus, increasing either V or k tends to suppress the ballistic part of the motion, leading to slower spreading and earlier saturation of the MSD.

In Fig. 8, we also present the normalized mean-square displacement,

$$\frac{\sigma^2(t)}{\sigma_{\max}^2}, \quad \sigma_{\max}^2 = \max_j d^2(j, j_0). \quad (13)$$

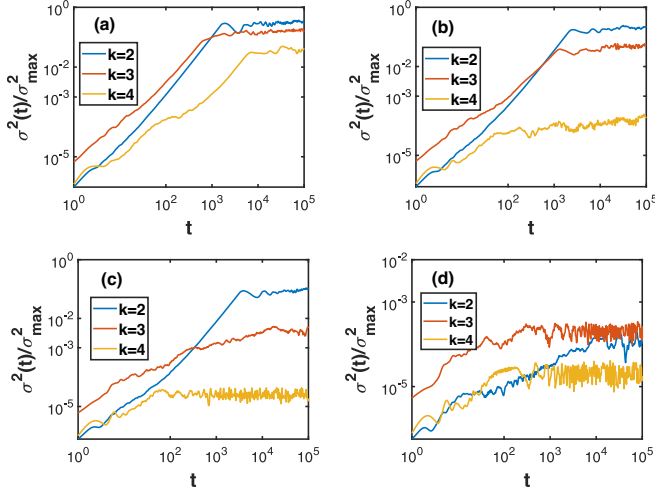


FIG. 8. Normalized mean-square displacement $\sigma^2(t)/\sigma_{\max}^2$ for the bilayer AAH model with the wave packet initially localized on the first layer at the central site $j_0 = \lfloor N/2 \rfloor$, shown for $k = 2, 3, 4$ and representative values of the quasiperiodic potential V . The system size is $N = 2584$ for $k = 2$ and $k = 4$, whereas $N = 987$ for $k = 3$. (a) $V = 1$, (b) $V = 1.25$, (c) $V = 1.5$, and (d) $V = 2$.

This normalization removes the trivial dependence on the maximum possible ring distance and allows a direct comparison of transport for different values of k . From Fig. 8, we observe that, for a fixed value of V , the normalized MSD saturates at a lower value as k increases. This indicates that the wave packet explores a smaller fraction of the ring for larger twist periodicity. The result further supports the spectral picture that increasing k enhances the contribution of critical and localized eigenstates at a given potential strength, thereby suppressing long-range transport.

Next, we probe the interlayer mixing of an initially localized wave packet. For this purpose, we choose the antisymmetric layer-balanced initial state

$$|\psi(0)\rangle = \frac{|j_0, 1\rangle - |j_0, 2\rangle}{\sqrt{2}}. \quad (14)$$

An analogous conclusion holds for the symmetric layer-balanced state as well, so the observed behavior is robust within the natural layer-balanced basis. This layer-balanced excitation does not favor either layer and therefore probes the intrinsic ability of the coupled bilayer Hamiltonian to preserve or suppress interlayer hybridization, rather than measuring the injection of amplitude from one layer into the other. To quantify interlayer mixing dynamically, we monitor the hybridization measure

$$C(t) = 2 \sum_{j=1}^N |\psi_{j,1}(t)\psi_{j,2}(t)|, \quad 0 \leq C(t) \leq 1. \quad (15)$$

By construction, $C(t) = 0$ corresponds to a fully layer-polarized state, while $C(t) = 1$ indicates maximal local sharing of the wave function between the two layers. The hybridization dynamics shown in Fig. 9 provides a complementary dynamical view of the interlayer mixing. Here, the initial wave packet is chosen to be equally shared between

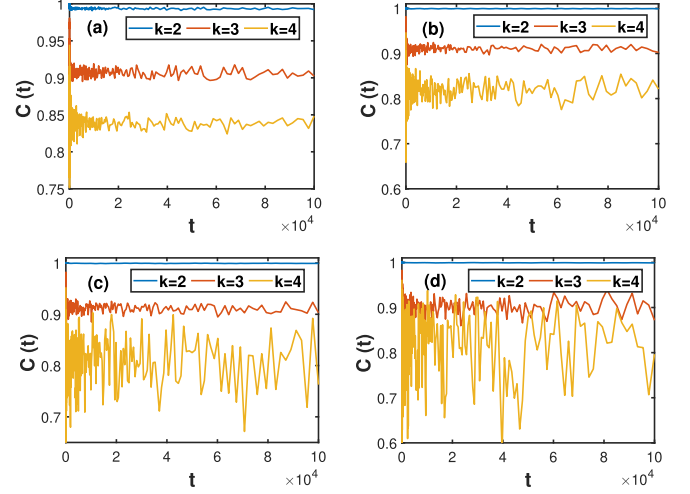


FIG. 9. Time evolution of the hybridization measure $C(t) = 2 \sum_j |\psi_{j,1}(t)\psi_{j,2}(t)|$ for an initial state equally shared between the two layers at the central rung $j_0 = \lfloor N/2 \rfloor$, shown for $k = 2, 3, 4$ and representative values of the quasiperiodic potential V . The system size is $N = 2584$ for $k = 2$ and $k = 4$, whereas $N = 987$ for $k = 3$. (a) $V = 0.5$, (b) $V = 0.75$, (c) $V = 1$, and (d) $V = 1.25$.

the two layers at the central rung. For $k = 2$, $C(t)$ remains close to unity over the entire time window, indicating that the initially layer-balanced wave packet preserves strong interlayer hybridization during the evolution. In contrast, for $k = 3$ and $k = 4$, $C(t)$ oscillates and remains below 1 over the entire time window, showing that the site-dependent twist weakens the interlayer mixing between the two layers and promotes partial layer separation. Therefore, these observations are consistent with the spectral results where twist periodicity controls the localization character of the eigenstate.

IV. CONCLUSION

To conclude, we have studied a bilayer extension of the Aubry-André-Harper model in which the intralayer and interlayer hoppings are controlled by chord distances on a twisted ring geometry. A site-dependent twist in the second layer, characterized by the periodicity k , generates a mosaic pattern of distance-modulated couplings and provides a tunable route to control localization and interlayer mixing. Using fractal dimension analysis, we show that varying the quasiperiodic potential strength V produces multiple closely spaced mobility edges and narrow spectral windows where localized, extended, and critical eigenstates coexist. Comparing different k reveals that larger twist periodicity suppresses the extended portions of the spectrum more rapidly and expands the spectral range occupied by critical and localized eigenstates. The same twist periodicity also controls interlayer hybridization. For $k = 2$, most eigenstates remain nearly equally distributed between the two rings, whereas larger k leads to pronounced layer polarization. The real-time dynamics supports this spectral picture. The mean-square displacement shows that $k = 2$ sustains nearly ballistic or superdiffusive spreading over a broad range of V , while increasing k produces an earlier

crossover to subdiffusive motion and a lower long-time saturation of the normalized MSD. The dynamical hybridization measure further shows that an initially layer-balanced wave packet remains strongly mixed for $k = 2$, whereas the interlayer overlap is progressively reduced for $k = 3$ and $k = 4$. Overall, our results establish the site-dependent k -periodic twist as a practical tuning mechanism for controlling spectral localization, wave-packet transport, and interlayer hybridization in quasiperiodic bilayer lattices.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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