

Synchronization stability in simplicial complexes of near-identical systems

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Assessing the stability of synchronization is a fundamental task when studying networks of dynamical systems. However, this becomes challenging when the coupled systems are not exactly identical, as is always the case in practical settings. Here we introduce an extension of the Master Stability Function to determine near-synchronization stability within simplicial complexes of nearly identical systems coupled by synchronization-noninvasive functions. We validate our method on a simplicial complex of Lorenz oscillators, finding a good correspondence between the predicted regions of stability and those observed via direct simulation. This confirms the correctness of our approach, making it a valuable tool for the evaluation of real-world systems, in which differences between the constitutive elements are unavoidable.

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I. INTRODUCTION

The understanding of the evolution of coupled dynamical systems has been greatly advanced by the use of the network representation, where discrete nodes interact across edges [1–5]. Within the vast phenomenology of dynamical networks, particular attention has been given to the study of synchronization, which occurs when a non-negligible fraction of the elements of the system, or even their totality, follow the same trajectory [6–8]. Synchronized states have been shown to be of fundamental importance for a variety of different systems, across the natural world and the engineered one [9–14]. Thus, a large amount of research has arisen to determine how specific structural and dynamical properties affect the general features of synchronization [15–25].

Because of the broad relevance of synchronization, much attention has been specifically devoted to the methods by means of which one can assess its stability. In the case of complete synchronization among identical systems, the Master Stability Function (MSF) has proved to be a very effective approach for this goal [26]. The main idea behind this method is to estimate the largest Lyapunov exponent of a given system for perturbations in the subspace transverse to the synchronization manifold. Then a necessary condition for the synchronized state to be stable is that the sign of this exponent be negative. In fact, the procedure is equivalent to deriving an expression for the evolution of a state vector that quantifies how far the system elements are from being synchronized. The components of this vector are decoupled from each other and correspond to eigenmodes of the dynamics. Thus, for synchronization to be stable, they must all asymptotically vanish, which is possible only if all the Lyapunov exponents are negative. The power and elegance of this method have quickly caused it to be one of the main approaches used to study the stability of synchronized states, and thus it has been applied to a broad range of dynamical systems and network structures [27–36].

However, the interactions that occur in numerous real-world complex systems are often not limited to pairwise ones,

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and the importance of higher-order interactions has been recognized in a large variety of scenarios [37,38]. In turn, this means that the structure of such higher-order networks cannot be represented by a graph. Instead, hypergraphs must be used, in which edges can link more than just two nodes at a time. Among the different possible types of hypergraphs, extensive research efforts have been dedicated to ones in which the presence of an edge of size k implies the existence of all possible edges of smaller sizes on the same nodes. These hypergraphs take the name of *simplicial complexes*. Note that, in the commonly used terminology, edges of size $k + 1$ are called k -simplices. Thus, an edge between two nodes is a 1-simplex, whereas an edge between three nodes is a 2-simplex, and so on.

Recently, the MSF formalism has been generalized to both static and time-varying simplicial complexes [39,40]. However, the dynamical systems considered in previous works have always been assumed to be entirely identical, even though this condition is seldom encountered in real-world situations, where even small variations in the dynamical parameters of the system elements are unavoidable. Consequently, existing techniques may not be directly applicable to them.

In this article, we extend the use of the MSF to simplicial complexes consisting of nearly identical dynamical systems. Our approach is based on introducing the idea of *near synchronization* and deriving a MSF formalism based on it. We then validate our theory studying a network of nearly identical Lorenz oscillators and verifying our findings with numerical simulations. These results constitute a valuable tool for the study of synchronization in real-world systems, where noise and other stochastic effects make it impossible to have a perfect identity of the elements.

II. NEAR SYNCHRONIZATION IN NEARLY IDENTICAL SYSTEMS

Consider a simplicial complex of maximum size D on N nearly identical m -dimensional systems. The evolution of the i th element can be written as

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i, \mu_i) + \sum_{k=1}^D \sigma_k \sum_{j_1=1}^N \cdots \sum_{j_k=1}^N a_{i,j_1,\dots,j_k}^{(k)} \times \mathbf{g}^{(k)}(\mathbf{x}_i, \mathbf{x}_{j_1}, \dots, \mathbf{x}_{j_k}). \quad (1)$$

Here the local dynamics of the systems is described by $\mathbf{f}$; also, for k -simplices, σ_k is their coupling strength, $\mathbf{g}^{(k)}$ is their coupling function, and $\mathbf{a}^{(k)}$ is their adjacency tensor, whose elements are 1 if the $k + 1$ corresponding nodes form an edge, and 0 otherwise. Note that we are explicitly writing the dependence of $\mathbf{f}$ on a generic parameter μ , whose value changes between the nodes. In the following, we illustrate our results for $D = 2$, and then generalize them to any value of D .

If the dynamical parameters of the system differ, the network cannot reach an identical synchronous trajectory. Thus, one cannot directly apply the MSF formalism as developed so far to obtain the stability conditions of the synchronized state. To attack this problem, we define a *nearly synchronous state* as one in which, given a small positive constant c bounded from below, there exists a time t^* such that $\|\mathbf{x}_j(t) - \mathbf{x}_i(t)\| \leq c$ for all $t \geq t^*$. Note that c necessarily has a lower bound, because if it were arbitrarily small, then the system would be in identical synchronization. But then, in the nearly synchronous state, the global dynamics of the system can be approximated by the average trajectory $\bar{\mathbf{x}} = \frac{1}{N} \sum_{i=1}^N \mathbf{x}_i$, whose evolution, for $D = 2$, is given by

$$\dot{\bar{\mathbf{x}}} = \frac{1}{N} \sum_{i=1}^N \mathbf{f}(\mathbf{x}_i, \mu_i) + \frac{\sigma_1}{N} \sum_{i=1}^N \sum_{j_1=1}^N a_{i,j_1}^{(1)} \mathbf{g}^{(1)}(\mathbf{x}_i, \mathbf{x}_{j_1}) + \frac{\sigma_2}{N} \sum_{i=1}^N \sum_{j_1=1}^N \sum_{j_2=1}^N a_{i,j_1,j_2}^{(2)} \mathbf{g}^{(2)}(\mathbf{x}_i, \mathbf{x}_{j_1}, \mathbf{x}_{j_2}). \quad (2)$$

We can now introduce a measure of the deviation of the state of the i th node from the average trajectory, $\delta \mathbf{x}_i = \mathbf{x}_i - \bar{\mathbf{x}}$. Similarly, we indicate with $\delta \mu_i = \mu_i - \bar{\mu}$ the difference between the value of the changing parameter μ for the i th node and its average. Then the dynamics of the trajectory deviations is

$$\begin{aligned} \delta \dot{\mathbf{x}}_i &= \mathbf{f}(\bar{\mathbf{x}} + \delta \mathbf{x}_i, \bar{\mu} + \delta \mu_i) + \sigma_1 \sum_{j_1=1}^N a_{i,j_1}^{(1)} \mathbf{g}^{(1)}(\bar{\mathbf{x}} + \delta \mathbf{x}_i, \bar{\mathbf{x}} + \delta \mathbf{x}_{j_1}) + \sigma_2 \sum_{j_1=1}^N \sum_{j_2=1}^N a_{i,j_1,j_2}^{(2)} \mathbf{g}^{(2)}(\bar{\mathbf{x}} + \delta \mathbf{x}_i, \bar{\mathbf{x}} + \delta \mathbf{x}_{j_1}, \bar{\mathbf{x}} + \delta \mathbf{x}_{j_2}) \\ &- \frac{1}{N} \sum_{i=1}^N \mathbf{f}(\bar{\mathbf{x}} + \delta \mathbf{x}_i, \bar{\mu} + \delta \mu_i) - \frac{\sigma_1}{N} \sum_{i=1}^N \sum_{j_1=1}^N a_{i,j_1}^{(1)} \mathbf{g}^{(1)}(\bar{\mathbf{x}} + \delta \mathbf{x}_i, \bar{\mathbf{x}} + \delta \mathbf{x}_{j_1}) - \frac{\sigma_2}{N} \sum_{i=1}^N \sum_{j_1=1}^N \sum_{j_2=1}^N a_{i,j_1,j_2}^{(2)} \\ &\times \mathbf{g}^{(2)}(\bar{\mathbf{x}} + \delta \mathbf{x}_i, \bar{\mathbf{x}} + \delta \mathbf{x}_{j_1}, \bar{\mathbf{x}} + \delta \mathbf{x}_{j_2}). \end{aligned} \quad (3)$$

Since we are interested in the stability of the near-synchronized state, we can assume that the deviations from the average trajectory are small. The same assumption is reasonable for the deviations of the μ_i from their average value, since the elements of the system are nearly identical. Then, expanding the previous equation around $\bar{\mathbf{x}}$ and $\bar{\mu}$, and stopping at first order, we get

$$\begin{aligned} \delta \dot{\mathbf{x}}_i &= J_{\mathbf{x}} \mathbf{f}(\bar{\mathbf{x}}, \bar{\mu}) \delta \mathbf{x}_i + J_{\mu} \mathbf{f}(\bar{\mathbf{x}}, \bar{\mu}) \delta \mu_i + \sigma_1 \sum_{j_1=1}^N a_{i,j_1}^{(1)} \left(\left. \frac{\partial}{\partial \mathbf{x}_i} \mathbf{g}^{(1)} \right|_{(\bar{\mathbf{x}}, \bar{\mathbf{x}})} \delta \mathbf{x}_i + \left. \frac{\partial}{\partial \mathbf{x}_{j_1}} \mathbf{g}^{(1)} \right|_{(\bar{\mathbf{x}}, \bar{\mathbf{x}})} \delta \mathbf{x}_{j_1} \right) \\ &+ \sigma_2 \sum_{j_1=1}^N \sum_{j_2=1}^N a_{i,j_1,j_2}^{(2)} \left(\left. \frac{\partial}{\partial \mathbf{x}_i} \mathbf{g}^{(2)} \right|_{(\bar{\mathbf{x}}, \bar{\mathbf{x}}, \bar{\mathbf{x}})} \delta \mathbf{x}_i + \left. \frac{\partial}{\partial \mathbf{x}_{j_1}} \mathbf{g}^{(2)} \right|_{(\bar{\mathbf{x}}, \bar{\mathbf{x}}, \bar{\mathbf{x}})} \delta \mathbf{x}_{j_1} + \left. \frac{\partial}{\partial \mathbf{x}_{j_2}} \mathbf{g}^{(2)} \right|_{(\bar{\mathbf{x}}, \bar{\mathbf{x}}, \bar{\mathbf{x}})} \delta \mathbf{x}_{j_2} \right), \end{aligned} \quad (4)$$

where $J_{\mathbf{x}}$ and J_{μ} are the Jacobian operators with respect to the state vectors and to the dynamical parameter μ , respectively, and we have used $\sum_{i=1}^N \delta \mathbf{x}_i = \sum_{i=1}^N \delta \mu_i = 0$.

Now, assume that the coupling functions $\mathbf{g}$ are synchronization-noninvasive, so that there is no self-coupling regardless of the order of the interactions. This implies that the total derivatives of the coupling functions vanish when they are computed on identical states. This is true in particular when the identical state is the one corresponding to the average trajectory of the system. Then, up to the a difference in the Jacobians, the equation above has the same functional form and properties as the one for the evolution of the synchronization perturbations described in Ref. [39]. So one can follow the steps outlined there and simplify it, rewriting it as

$$\begin{aligned} \dot{\delta \mathbf{x}}_i &= J_{\mathbf{x}} \mathbf{f}(\bar{\mathbf{x}}, \bar{\mu}) \delta \mathbf{x}_i + J_{\mu} \mathbf{f}(\bar{\mathbf{x}}, \bar{\mu}) \delta \mu_i \\ &- \sigma_1 \sum_{j=1}^N L_{i,j}^{(1)} J_1 \mathbf{g}^{(1)} \delta \mathbf{x}_{j1} - \sigma_2 \sum_{j=1}^N L_{i,j}^{(2)} J_2 \mathbf{g}^{(2)} \delta \mathbf{x}_{j1}. \end{aligned} \quad (5)$$

In this equation, the operators J_1 and J_2 are defined as $J_1 = \frac{\partial}{\partial \mathbf{x}_{j1}} |_{(\bar{\mathbf{x}}, \bar{\mu})}$ and $J_2 = \frac{\partial}{\partial \mathbf{x}_{j1}} |_{(\bar{\mathbf{x}}, \bar{\mu})} + \frac{\partial}{\partial \mathbf{x}_{j2}} |_{(\bar{\mathbf{x}}, \bar{\mu})}$, $\mathbf{L}^{(1)}$ is the graph Laplacian, and $\mathbf{L}^{(2)}$ is the second-order generalized Laplacian, whose elements are given by

$$L_{i,j}^{(2)} = \begin{cases} \sum_{k=1}^N \sum_{l=1}^N a_{i,j,k}^{(2)} & \text{if } i = j, \\ 0 & \text{if } i \neq j \text{ and } a_{i,j}^{(1)} = 0, \\ -T_{(i,j)} & \text{otherwise,} \end{cases} \quad (6)$$

where $T_{(i,j)}$ is the number of triangles that include the link (i, j) .

We can then introduce the global trajectory deviation vector $\delta \mathbf{X} = (\delta \mathbf{x}_1^T, \dots, \delta \mathbf{x}_N^T)^T$ and the global parameter deviation vector $\delta \mu = (\mu_1, \dots, \mu_N)^T$, which allow us to express the evolution of all the $\delta \mathbf{x}_i$ vectors in a single equation as

$$\begin{aligned} \dot{\delta \mathbf{X}} &= (\mathbb{1}_N \otimes J_{\mathbf{x}} \mathbf{f}(\bar{\mathbf{x}}, \bar{\mu}) - \sigma_1 \mathbf{L}^{(1)} \otimes J_1 \mathbf{g}^{(1)} \\ &- \sigma_2 \mathbf{L}^{(2)} \otimes J_2 \mathbf{g}^{(2)}) \delta \mathbf{X} + (\mathbb{1}_N \otimes J_{\mu} \mathbf{f}(\bar{\mathbf{x}}, \bar{\mu})) \delta \mu. \end{aligned} \quad (7)$$

In the expressions above, $\mathbb{T}$ indicates transposition, $\mathbb{1}_N$ is the N -dimensional identity matrix, and $\otimes$ is the Kronecker product. This equation can be then rewritten in the reference frame given by the eigenvectors of $\mathbf{L}^{(1)}$, which is a real symmetric matrix, and, therefore, it can be diagonalized by an orthonormal basis of $\mathbb{R}^N$ as $\mathbf{L}^{(1)} = \mathbf{V} \mathbf{\Lambda} \mathbf{V}^T$. Introducing the transformed variable $\delta \boldsymbol{\eta} = (\mathbf{V}^T \otimes \mathbb{1}_m) \delta \mathbf{X}$, this results in

$$\begin{aligned} \dot{\delta \boldsymbol{\eta}} &= (\mathbf{V}^T \otimes \mathbb{1}_m) (\mathbb{1}_N \otimes J_{\mathbf{x}} \mathbf{f}(\bar{\mathbf{x}}, \bar{\mu}) - \sigma_1 \mathbf{L}^{(1)} \otimes J_1 \mathbf{g}^{(1)} \\ &- \sigma_2 \mathbf{L}^{(2)} \otimes J_2 \mathbf{g}^{(2)}) (\mathbf{V} \otimes \mathbb{1}_m) \delta \boldsymbol{\eta} \\ &+ (\mathbf{V}^T \otimes \mathbb{1}_m) (\mathbb{1}_N \otimes J_{\mu} \mathbf{f}(\bar{\mathbf{x}}, \bar{\mu})) \delta \mu \\ &= (\mathbb{1}_N \otimes J_{\mathbf{x}} \mathbf{f}(\bar{\mathbf{x}}, \bar{\mu}) - \sigma_1 \mathbf{\Lambda} \otimes J_1 \mathbf{g}^{(1)} \\ &- \sigma_2 \tilde{\mathbf{L}}^{(2)} \otimes J_2 \mathbf{g}^{(2)}) \delta \boldsymbol{\eta} + (\mathbf{V}^T \otimes J_{\mu} \mathbf{f}(\bar{\mathbf{x}}, \bar{\mu})) \delta \mu, \end{aligned} \quad (8)$$

where we have put $\tilde{\mathbf{L}}^{(2)} = \mathbf{V}^T \mathbf{L}^{(2)} \mathbf{V}$ and used the identity $(\mathbf{M}_1 \otimes \mathbf{M}_2)(\mathbf{M}_3 \otimes \mathbf{M}_4) = (\mathbf{M}_1 \mathbf{M}_3 \otimes \mathbf{M}_2 \mathbf{M}_4)$. It is useful to

write the equation above by components:

$$\begin{aligned} \dot{\delta \eta}_i &= (J_{\mathbf{x}} \mathbf{f}(\bar{\mathbf{x}}, \bar{\mu}) - \sigma_1 \lambda_i J_1 \mathbf{g}^{(1)}) \delta \eta_i \\ &- \sigma_2 \sum_{j=1}^N \tilde{L}_{i,j}^{(2)} J_2 \mathbf{g}^{(2)} \delta \eta_j + J_{\mu} \mathbf{f}(\bar{\mathbf{x}}, \bar{\mu}) \sum_{j=1}^N V_{j,i} \delta \mu_j, \end{aligned} \quad (9)$$

where we indicated the eigenvalues of $\mathbf{L}^{(1)}$ as λ_i . These are all non-negative, and the first one of them is 0, since the sum of the elements of each row of the matrix vanishes. The corresponding mode of the dynamics is then along the average trajectory, and the other ones are transverse to it. Thus, the stability of the near-synchronized state can be assessed by verifying whether the norm of the vector consisting of all the components of $\delta \boldsymbol{\eta}$ except the first one remains asymptotically bounded. In turn, this can be estimated via the time average

$$\Omega = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \sum_{i=2}^N \|\delta \boldsymbol{\eta}_i\|^2 dt. \quad (10)$$

Note that the derivation above, which we have written for $D = 2$ for the sake of simplicity, can be straightforwardly generalized to any value of D , yielding

$$\begin{aligned} \dot{\delta \boldsymbol{\eta}}_i &= [J_{\mathbf{x}} \mathbf{f}(\bar{\mathbf{x}}, \bar{\mu}) - \sigma_1 \lambda_i J_1 \mathbf{g}^{(1)}] \delta \boldsymbol{\eta}_i \\ &- \sum_{k=2}^D \sigma_k \sum_{j=1}^N \tilde{L}_{i,j}^{(k)} J_k \mathbf{g}^{(k)} \delta \boldsymbol{\eta}_j + J_{\mu} \mathbf{f}(\bar{\mathbf{x}}, \bar{\mu}) \sum_{j=1}^N V_{j,i} \delta \mu_j, \end{aligned} \quad (11)$$

where $\tilde{\mathbf{L}}^{(k)} = \mathbf{V}^T \mathbf{L}^{(k)} \mathbf{V}$ and we define the differential operator $J_k = \sum_{r=1}^k \frac{\partial}{\partial \mathbf{x}_{jr}} |_{(\bar{\mathbf{x}}, \dots, \bar{\mathbf{x}})}^{\text{k times}}$.

III. NUMERICAL VALIDATION

To verify the applicability of our method, we studied a network of $N = 5$ nearly identical Lorenz oscillators, whose evolution is described by

$$\begin{aligned} \dot{x} &= s(y - x), \\ \dot{y} &= x(\rho - z) - y, \\ \dot{z} &= xy - \beta z. \end{aligned} \quad (12)$$

Here we put $s = 10$ and $\beta = \frac{8}{3}$, and introduce mismatches in the values of ρ , which we draw from a Gaussian distribution with mean 28 and varying standard deviation ζ_{ρ} . Moreover, we consider natural diffusive cubic coupling between the oscillators, so that, when they are coupled in the $x \rightarrow x$ coupling, the system becomes

$$\begin{aligned} \dot{x}_i &= s(y_i - x_i) + \sigma_1 \sum_{j=1}^N a_{i,j}^{(1)} (x_j^3 - x_i^3) \\ &+ \sigma_2 \sum_{j=1}^N \sum_{k=1}^N a_{i,j,k}^{(2)} (x_j^2 x_k - x_i^3), \\ \dot{y}_i &= x_i(\rho_i - z_i) - y_i, \\ \dot{z}_i &= x_i y_i - \beta z_i, \end{aligned} \quad (13)$$

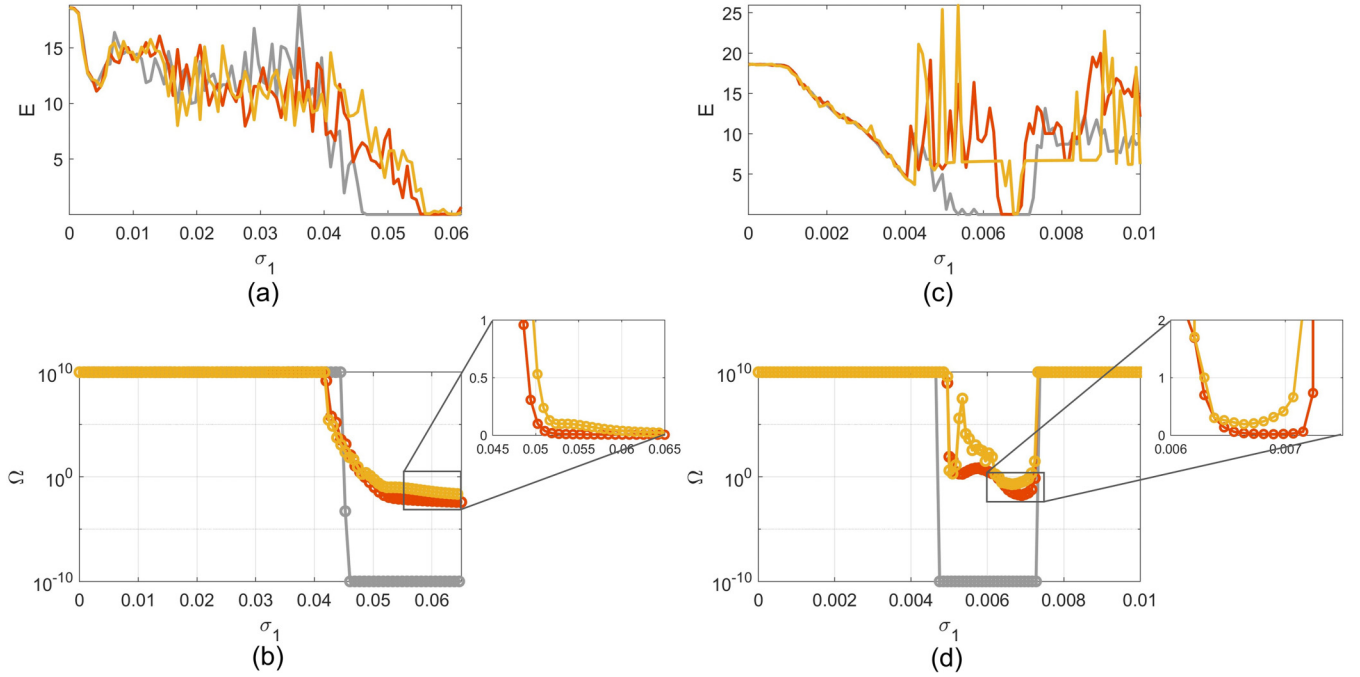


FIG. 1. The MSF correctly predicts the stable near-synchronization region at fixed coupling-strength ratios. In a simplicial complex of coupled near-identical Lorenz oscillators with fixed ratio of coupling strengths $r = 10^{-3}$, the synchronization error E , measured by direct simulation, vanishes or reaches values close to 0 within a range of first-order coupling strengths σ_1 , as shown in panel (a) for $x \rightarrow x$ coupling and in panel (c) for $y \rightarrow x$ coupling. This confirms the MSF prediction, shown in panel (b) for $x \rightarrow x$ coupling and in panel (d) for $y \rightarrow x$ coupling. Note that the value of Ω reaches 0 only when $\zeta_\rho = 0$ (gray points), which corresponds to identical oscillators. When, instead, $\zeta_\rho = 10^{-4}$ (red points) or 5×10^{-4} (yellow points), Ω converges only to a small but strictly positive value, since synchronization is not complete, but only nearly so.

and when they are coupled in the $y \rightarrow x$ coupling, it is

$$\begin{aligned} \dot{x}_i &= s(y_i - x_i) + \sigma_1 \sum_{j=1}^N a_{i,j}^{(1)}(y_j^3 - y_i^3) \\ &\quad + \sigma_2 \sum_{j=1}^N \sum_{k=1}^N a_{i,j,k}^{(2)}(y_j^2 y_k - y_i^3), \\ \dot{y}_i &= x_i(\rho_i - z_i) - y_i \\ \dot{z}_i &= x_i y_i - \beta z_i \end{aligned} \quad (14)$$

The evolution of the perturbations for the coupled Lorenz oscillators is

$$\delta \dot{\eta}_i = \begin{pmatrix} -s - 3\alpha_i \bar{x}^2 & s & 0 \\ \bar{\rho} - \bar{z} & -1 & -\bar{x} \\ \bar{y} & \bar{x} & -\beta \end{pmatrix} \delta \eta_i + \begin{pmatrix} 0 \\ \bar{x} \\ 0 \end{pmatrix} \sum_{j=1}^N V_{j,i} \delta \mu_j \quad (15)$$

for the $x \rightarrow x$ coupling and

$$\delta \dot{\eta}_i = \begin{pmatrix} -s - 3\alpha_i \bar{y}^2 & s & 0 \\ \bar{\rho} - \bar{z} & -1 & -\bar{x} \\ \bar{y} & \bar{x} & -\beta \end{pmatrix} \delta \eta_i + \begin{pmatrix} 0 \\ \bar{x} \\ 0 \end{pmatrix} \sum_{j=1}^N V_{j,i} \delta \mu_j \quad (16)$$

for the $y \rightarrow x$ coupling, where α_i denotes the i th eigenvalue of the matrix $\sigma_1 L^{(1)} + \sigma_2 L^{(2)}$. Then we use these

expressions to compute Ω , and we calculate the synchronization error

$$E = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \sqrt{\frac{1}{N(N-1)} \sum_{i,j=1}^N \|\mathbf{x}_j - \mathbf{x}_i\|^2} dt, \quad (17)$$

comparing the behavior of the two quantities. All simulations were performed using the fourth-order Runge-Kutta method. The total integration time for each simulation was 10000 time units, with a fixed time step of 0.01. Initial conditions were selected randomly for each run, and the observables were computed across multiple sets of initial conditions. The results shown are averages over these different realizations.

When the ratio between the coupling strengths $r = \frac{\sigma_2}{\sigma_1}$ is kept fixed at a value of 10^{-3} , the system achieves near synchronization if σ_1 lies within a finite interval, behaving similarly to classic networks in class III of synchronizability, for which stable synchronization can be obtained only within a bounded region of coupling strength [3]. As shown in Fig. 1, when ζ_ρ is 0, making the oscillators identical, the synchronization is complete and Ω vanishes, as expected. When ζ_ρ is positive, instead, Ω converges to a small nonzero value within the stability region, which corresponds to values of E that are themselves close to 0. The same phenomenology is observed regardless of the variable of coupling, even though coupling in y reduces the size of the stability region. Notably, the predictions of the ranges of σ_1 yielding stable

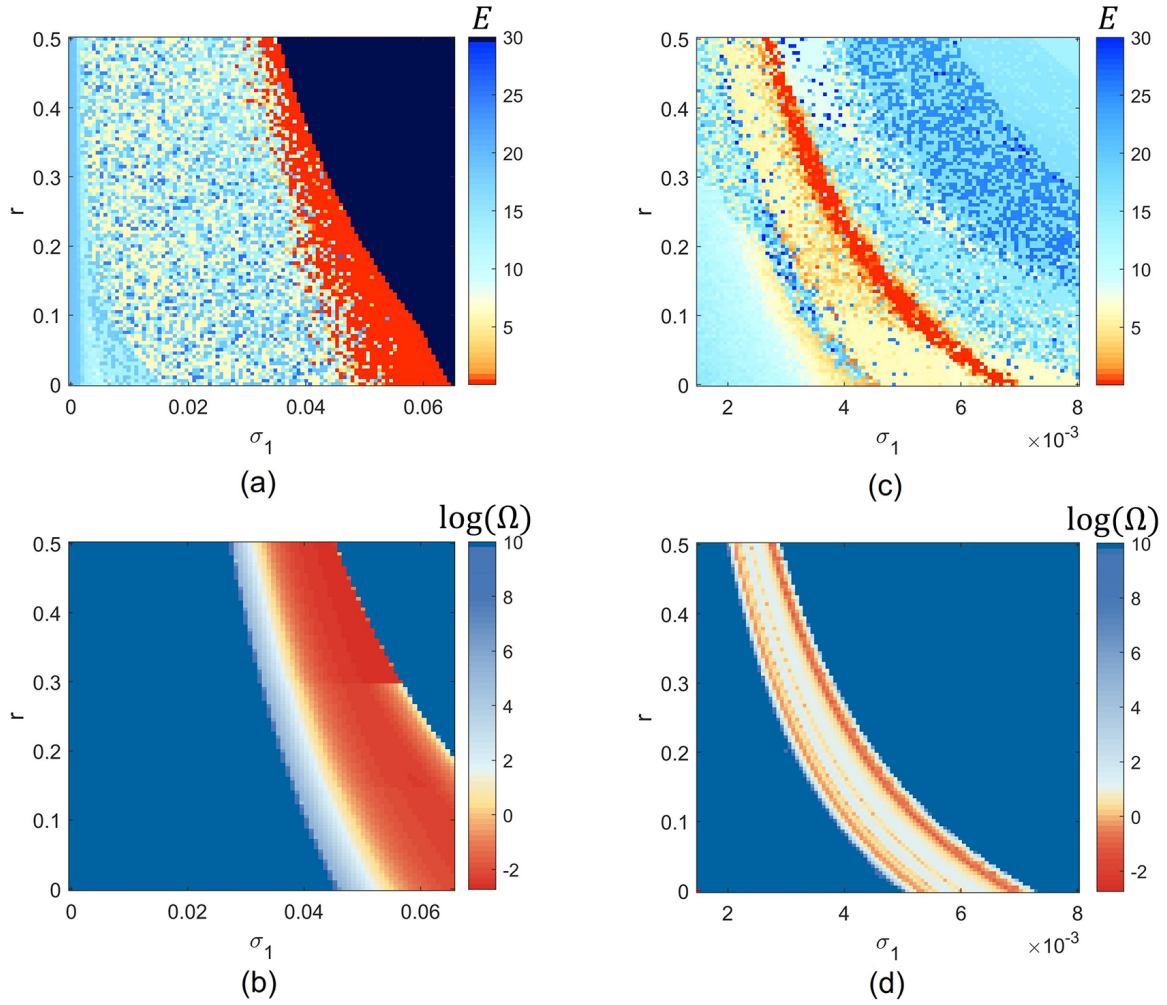


FIG. 2. The MSF correctly predicts the stable near-synchronization region for varying coupling strengths. In a simplicial complex of x -coupled near-identical Lorenz oscillators with parameter standard deviation $\zeta_\rho = 10^{-4}$, higher coupling-strength ratios r cause the region of stable near synchronization to decrease in size and its onset to shift towards lower values of σ_1 , as evidenced by the synchronization error E , shown in panel (a). The dark blue region in panel (a) shows the instability region in numerical simulations. The shift to lower first-order coupling strengths is also found when the coupling is in $y \rightarrow x$, but the size of the stable region remains almost constant regardless of the value of r , as shown in panel (c). The corresponding measurements of Ω , illustrated in panels (b) and (d), respectively, show that our method correctly predicts the location of the stable region. Note the logarithmic (base-10) scale for Ω .

near synchronization are confirmed by the calculation of the synchronization error, confirming the predictive power of our method.

To estimate the relative importance of the second-order interactions, we computed Ω and E for different values of σ_1 and r , fixing ζ_ρ at 10^{-4} . Also in this case, the stable region we predicted corresponds well to the one detected via direct simulation of the system. The results, illustrated in Fig. 2, show that, for the x coupling, higher values of σ_2 shift the onset of the stable near-synchronization region towards smaller first-order coupling strengths, while reducing its size at the same time. When the coupling is $y \rightarrow x$, higher ratios of coupling strengths simply shift the regions towards lower values of σ_1 , but they leave its width mostly unaffected. This suggests that, while multibody interactions facilitate synchronization, they also have the potential to limit the range of parameters for which it is stable, depending on the specific coupling between the network elements.

It is important to note that the MSF framework provides a necessary but not sufficient condition for synchronization. That is, while low values of Ω suggest that synchronization may be possible, they do not guarantee its actual occurrence when the full dynamics of a nonlinear system is accounted for. In particular, when the coupling strength becomes sufficiently large, the Lorenz oscillators enter regimes of instability, preventing full synchronization and leading to large values of E . The instability region is shown in dark blue in Fig. 2(a) and explains the apparent discrepancy between Figs. 2(a) and 2(b).

While the previous numerical results, shown in Fig. 1 and Fig. 2, were obtained by varying the coupling strengths at fixed parameter mismatch levels, Fig. 3 illustrates how increasing the parameter heterogeneity ζ_ρ impacts the ability of our method to correctly predict the region of synchronization stability. To explore this, we fixed the ratio of coupling strengths to 0.1 and varied ζ_ρ . The results, reported in Fig. 3,

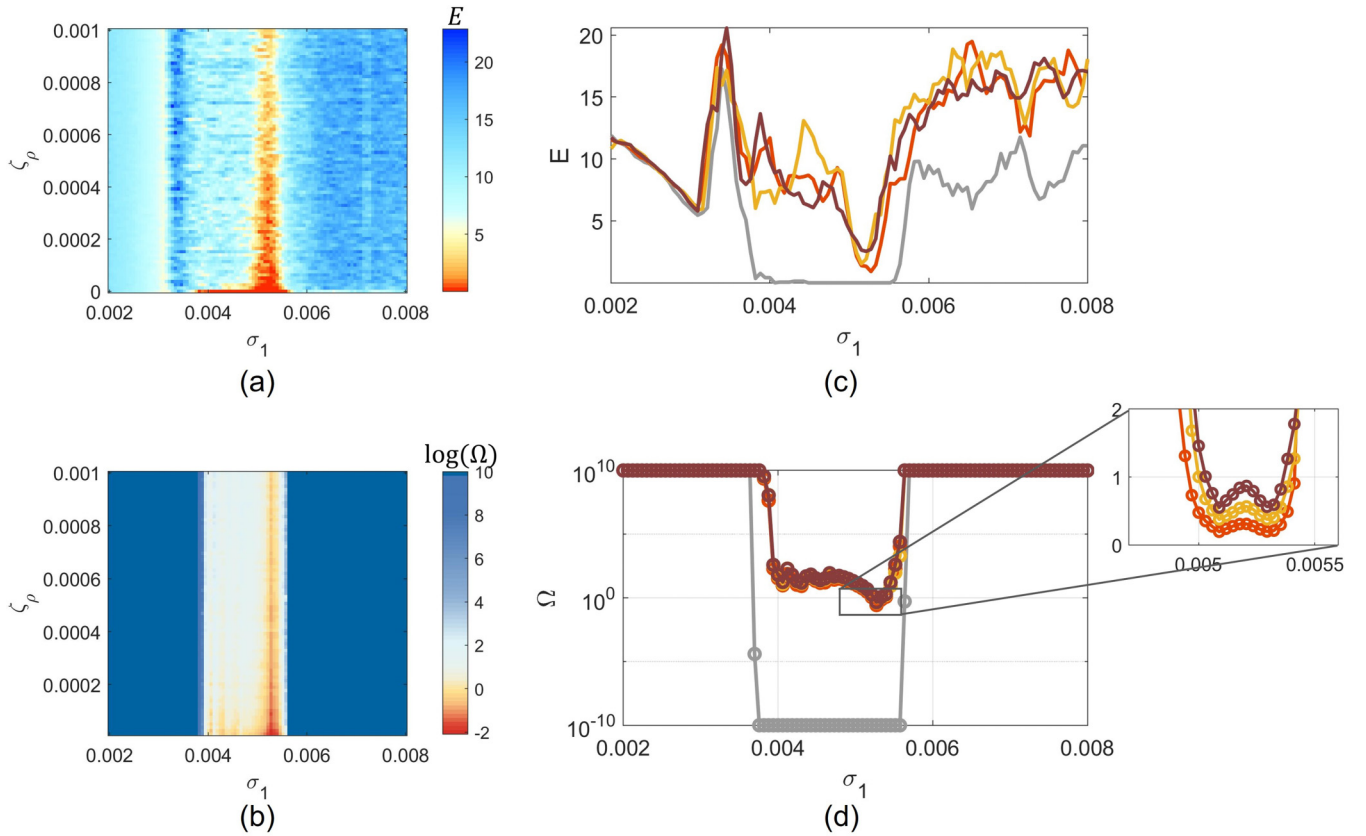


FIG. 3. The MSF correctly predicts the stable near-synchronization region for varying parameter mismatches. In a simplicial complex of $y \rightarrow x$ -coupled near-identical Lorenz oscillators with fixed ratio of coupling strengths $r = 0.1$, the size of the region of stable synchronization decreases sharply as soon the oscillators are no longer identical ($\zeta_\rho > 0$), as shown in panel (a). An increase of the lowest value of the synchronization error E can also be noticed as ζ_ρ increases. As in the previous cases, the regions of stability for near synchronization are correctly predicted by our MSF method, as shown in panel (b). To better demonstrate this behavior, panels (c) and (d) show the synchronization error and Ω for four specific values of ζ_ρ , namely, 0 (gray), 4×10^{-4} (red), 8×10^{-4} (yellow), and 10^{-3} (brown), clearly illustrating the reduction in size of the stable region with increasing parameter mismatch. Note the logarithmic (base-10) scale for Ω .

show that even very slight deviations from a complete identity of the oscillators cause a considerable shrinking of the stability region and a corresponding increase of the minimum value of the synchronization error E . Also, the results of the simulations correspond once more to the predictions obtained using our method.

Further numerical validations, using a Lorenz system with $z \rightarrow z$ coupling and a network of Rössler oscillators, can be found in the Supplemental Material [41].

IV. CONCLUSIONS

In summary, we have extended the use of the Master Stability Function to the case of simplicial complexes of nonidentical oscillators. Because of the differences in the dynamical parameters of the system elements, it is not possible, in principle, for them to reach a state of identical synchronization. However, provided their parameters are close enough to each other, they can still be in a state of *near synchronization*, in which each oscillator does not deviate too much from the average trajectory of all the nodes. Using our extension of the MSF results in a system of coupled linear differential equations that involve generalized Jacobian operators and

higher-order graph Laplacians. Solving the system, one can estimate the stability of the near-synchronization state in a simpler way than solving the full original nonlinear system.

We verified the predictive power of our method by studying a simplicial complex of nearly identical Lorenz oscillators, and analyzing its behavior under different conditions. In all cases, the results of our method corresponded very well to the regions of near-synchronization stability revealed by direct simulation of the network. Also, they revealed that, while facilitating the onset of near synchronization, higher-order interactions can also have the effect of decreasing the range of parameters for which it is stable. In future work, it will be valuable to explore additional metrics to quantitatively assess the agreement between theoretical predictions and direct simulation of the systems.

As our method is generally applicable, regardless of network structure or nature of the individual dynamical systems, we believe it will find application in numerous real-world scenarios, where stochastic discrepancies between the elements of a system are unavoidable. In all such cases, our method offers the advantage of having to solve only a system of linear differential equations in order to estimate near-synchronization stability. This will make it a valuable

tool, particularly since, as mentioned above, the presence of higher-order interactions can shrink the region in which near synchronization is stable, having a significant impact on the operation of natural and engineered complex systems.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [42].

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