

Generalized synchronization in strong-mismatched, bidirectional-coupled systems: Empirical detection and experimental analysis

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Generalized synchronization, in which a nontrivial interdependence is established between the trajectories of two or more coupled units, has been observed in a wide range of simulated and physical systems. Although several numerical methods are available for its detection, visualization techniques have not been reported, and bidirectional coupling remains challenging, especially for experimental data. This paper introduces a simple empirical method for approximating the functional relation, based on expressing the current value of one system variable as a function of the state variable of the other system, reconstructed by time-delay embedding, along with suitable nonlinearities. Two strongly parametrically mismatched Rössler systems with weak bidirectional coupling have been considered. The proposed method allows the computation of a time series, namely, a “proxy signal,” which shows significantly stronger phase synchronization compared to the original system variables. The underlying approximated functional is visualized through regression coefficients, and its relationship with the Lyapunov exponents is demonstrated by simulated parameter sweeps in two dimensions. The results are validated via tunable analog electronic realization of the coupled systems. The effects of nonidealities in the physical apparatus are also investigated, demonstrating the robustness of the proposed method.

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I. INTRODUCTION

Synchronization, deduced from the Greek words $\sigma\upsilon\nu$ “together” and $\kappa\rho\omicron\nu\omicron\varsigma$ “time”, which traditionally refers to the simultaneity in periodic motion, has a much richer phenomenology in chaotic dynamics, particularly in systems driven by free oscillation amplitudes. A rise in the rate of energy transfer from two coupled systems creates different synchronization regimes [1,2]. These are usually to include phase synchronization, wherein the phase difference is constrained and the amplitudes have no correlation [3], advancing to lag synchronization [4], and eventually

leading to full synchronism, where complete overlap of the orbits of the coupled systems occurs [5]. The particular synchronization modes accessible to coupled nodes depend fundamentally upon their inherent discrepancies and attributes of the coupling mechanism. In the case of bidirectional, diffusive, and delayless coupling, continuous energy exchange is required to maintain a finite state difference in the presence of parametric discrepancies. While small discrepancies can affect the synchronization path in a way similar to initial conditions, large discrepancies often lead to large areas of incomplete synchrony or even prevent the achievement of complete synchronization [6,7]. In addition, structural nonequivalence generally renders complete synchronization of chaotic systems impossible [8]. This emphasizes the intricacies in achieving and handling robust synchronization in heterogeneous systems. In the given states, generalized synchronization (GS) is a phenomenon of great significance. Its emergence is distinguished by the absence of lower-order types of entrainment, thus emphasizing an intrinsic functional relationship between the state vectors of the systems, which can be achieved via delay-coordinate reconstruction

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[9,10]. In unidirectionally coupled systems, this relationship may be formulated in an explicit sense as $\mathbf{y}(t) = \Phi[\mathbf{x}(t)]$. In bidirectionally coupled equivalent oscillators, where there is reciprocal coupling between the systems, the functional relationship must be formulated in an implicit sense, namely, as $\Phi[\mathbf{x}(t), \mathbf{y}(t)] = 0$ [11]. Despite the development of a solid framework that accommodates a wide variety of synchronization regimes, including GS [12], empirical confirmation within real-world systems, specifically for bidirectionally coupled systems, still constitutes a major hurdle.

Several techniques are available for detecting generalized synchronization. The first method is the mutual false nearest-neighbor method, which quantifies the extent to which proximity in the response space implies proximity in the driving space [9]. This approach can be used for both unidirectional and bidirectional couplings. However, it does not provide a formal criterion for delineating the boundaries of generalized synchronization. When coupled systems are such that generalized synchronization is the strongest possible form of entrainment, the mean distance between the images of the nearest-neighbors is smaller than in this regime compared to phase synchronization, which can occur at lower coupling strengths. In contrast, in other situations, generalized synchronization is the weakest detectable form of entrainment before the energy exchange rate reaches a level sufficient to maintain a bounded phase difference, where the trend of the mean distance between neighbor images is the opposite [11,13,14]. These two possible relationships between the phase and generalized synchronizations are qualitatively different [15]. The mutual false nearest-neighbor method can fail in the presence of a complex attractor topology; however, this drawback was successfully addressed by selectively considering tubes of trajectories [16,17]. The second approach is the auxiliary system method, where a replica of a target system is set up as an indirect probe of synchronization. Generalized synchronization is inferred if the target and its replica exhibit phase or even complete synchronization, though neither is phase synchronized with the driver [18]. Although this approach is highly sensitive and endowed with some level of analytical tractability, it is inherently unsuitable for bidirectional coupling. Moreover, close matching between the target and replica systems is required [13,19]. The third approach is arguably the most fundamental and is based on the signs of the Lyapunov exponents. With appropriate adjustments for unidirectional and bidirectional couplings, this approach provides a formal criterion for inferring the boundaries of generalized synchronization [20]. However, estimating the Lyapunov spectrum from experimental data is problematic [21], which limits the practicability of this approach.

Although generalized synchronization underlies numerous phenomena in natural and engineered complex systems, including emergent computation paradigms, its detection in experimental recordings remains challenging [22,23]. In physical settings, this has been primarily reported for unidirectionally coupled systems, including chaotic electronic circuits [18,24,25], light modulators [26], lasers [27], and microwave oscillators [28]. For example, in neuroscience, brain signals from different cortical regions may only be synchronized in a general sense, with important implications for inferring functional connectivity using electroencephalography

and other recordings [29]. Other physiological processes, such as the heartbeat and respiratory cycle, interact without strict phase locking [30]. Such cross-disciplinary findings highlight the fact that generalized synchronization is a powerful concept for explaining emergent synchrony in complex, heterogeneous systems, from climate to living organisms [31].

Bidirectional coupled systems have been studied in the context of chaotic electronic circuits [32], optoelectronic oscillators [33], and thermoacoustic devices [34]. The experimental study of generalized synchronization, especially in the presence of bidirectional couplings, remains a topic of active interest and poses particular challenges given the inapplicability of the auxiliary system approach and the issues in estimating the Lyapunov spectrum from recorded time series. Because of these difficulties, detection techniques based on neural networks, such as echo-state networks, have recently been introduced. Although their accuracy is remarkable, these methods suffer from considerable complexity and limited explainability [35]. Bayesian inference techniques and other approaches based on information theory have also been developed. However, they are less commonly used [34,36].

To the authors' knowledge, to date, generalized synchronization has only been inferred indirectly, and suitable visualization techniques have not yet been developed. In this study, we propose a simple empirical method to detect generalized synchronization based on approximating the functional relation Φ using a general linear model that involves predictors containing both derivatives and a nonlinear transformation of the state variables of the system taken as driver, reconstructed by time-lag embedding. We focus on the paradigmatic case of two strongly mismatched and bidirectionally coupled Rössler systems. The implementation is straightforward, computationally inexpensive, and allows for easy delineation of the region of generalized synchronization. It also allows visualization of the approximated functional relation through the regression coefficients. Concordant results were obtained between numerical simulations and experiments based on the realization of coupled systems in the form of analog electronic circuits.

The remainder of the paper is organized as follows: Section II presents the proposed methodology that can be applied to detect generalized synchronization in bidirectionally coupled chaotic systems, focusing on phase-space reconstruction via augmented time-delay embedding and a regression-based approach for estimating the functional relationship between coupled systems. Section III applies this methodology to a pair of bidirectionally coupled Rössler systems via numerical simulations, including analyses of entropy measurements, influence of embedding parameters, phase synchronization value, root mean square error, generalized synchronization measures, visualization of the functional relationship Φ via regression coefficients, and calculation of the Lyapunov exponents. Section IV presents the experimental implementation to validate the numerical results, detailing the experimental setup and dataset, data analysis procedures, experimental visualization of the functional relationship, and effects of nonideality in electronic components. Section V presents an in-depth discussion of the results, compares the numerical simulations with experimental observations, and highlights the implications of nonidealities in practical applications.

II. PROPOSED METHOD

In this section, we propose an empirical method for detecting generalized synchronization. Let us consider two coupled systems described by the following equations:

$$\begin{aligned}\dot{\mathbf{y}}_1 &= \mathbf{F}(\mathbf{y}_1) + \mathbf{K}_{\mathbf{y}_1\mathbf{y}_2}(\mathbf{y}_2 - \mathbf{y}_1), \\ \dot{\mathbf{y}}_2 &= \mathbf{G}(\mathbf{y}_2) + \mathbf{K}_{\mathbf{y}_2\mathbf{y}_1}(\mathbf{y}_1 - \mathbf{y}_2).\end{aligned}\quad (1)$$

where $\mathbf{y}_1(t) \in \mathbb{R}^n$ and $\mathbf{y}_2(t) \in \mathbb{R}^n$ are the state vectors of the two systems, where n denotes the number of dimensions, $\mathbf{F}$ and $\mathbf{G}$ represent the intrinsic dynamics of each system, and $\mathbf{K}_{\mathbf{y}_1\mathbf{y}_2} \in \mathbb{R}^{n \times n}$ and $\mathbf{K}_{\mathbf{y}_2\mathbf{y}_1} \in \mathbb{R}^{n \times n}$ represent the couplings, which are bidirectional, diffusive, and lag-free.

Briefly recall that $\mathbf{y}_1(t)$ and $\mathbf{y}_2(t)$ are in generalized synchronization if any deterministic function Φ exists such that $\Phi[\mathbf{y}_1(t), \mathbf{y}_2(t)] = 0$. Although in this case the coupling is bidirectional, we make the simplifying assumption of considering each direction separately. In other words, we consider both directions $\mathbf{y}_1 \rightarrow \mathbf{y}_2$ and $\mathbf{y}_2 \rightarrow \mathbf{y}_1$, knowing that $\mathbf{y}_i \rightarrow \mathbf{y}_j$ means a unidirectional coupling between $\mathbf{y}_i$ and $\mathbf{y}_j$, where $\mathbf{y}_i$ is the node taken as driver and $\mathbf{y}_j$ the node taken as target. As mentioned above, in a unidirectional configuration, $\mathbf{y}_i(t)$ and $\mathbf{y}_j(t)$ are in generalized synchronization if any functional relation exists such that $\mathbf{y}_j(t) = \Phi[\mathbf{y}_i(t)]$. Therefore, the idea is to estimate Φ and assess the extent to which $\mathbf{y}_j(t)$ predicts $\mathbf{y}_i(t)$, and vice versa; in other words, the implicit functional relation is approximated with two explicit functions. To achieve this, we followed the simple empirical method described below.

A. Time series and time-lag embedding

First, we generate the time series of one of the dynamic variables of each system $[x_1(t)$ and $x_2(t)]$ by numerically solving system (1) using a suitable integration algorithm. Next, we reconstruct the phase spaces underlying the dynamics of each time series to identify the potential functional relationships between the two systems. Adopting Takens's approach [37], that is, by time-delay embedding, we obtain the reconstructed vectors $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ defined by

$$\begin{aligned}\mathbf{x}_1(t) &= [x_1(t), x_1(t + \tau), \dots, x_1(t + (m - 1)\tau)], \\ \mathbf{x}_2(t) &= [x_2(t), x_2(t + \tau), \dots, x_2(t + (m - 1)\tau)],\end{aligned}\quad (2)$$

where τ is the time delay, selected as the first minimum of the average mutual information function calculated between the signal and its time-shifted version, m is the embedding dimension, determined using the false nearest-neighbor algorithm, and $\mathbf{x}_1(t) \in \mathbb{R}^m$ and $\mathbf{x}_2(t) \in \mathbb{R}^m$ are the reconstructed vectors of $x_1(t)$ and $x_2(t)$ at time t , respectively. In this case, the notation $\mathbf{x}$ is used instead of $\mathbf{y}$ to distinguish the time-delay reconstruction from the original state vector in Eq. (1). As the primary objective of the present study is to realize generalized synchronization detection, augmenting the time-delay reconstruction with derivatives and suitable nonlinear transformations is necessary. The usage of first derivatives in time-delay embedding has been previously found to be advantageous toward attractor reconstruction [38]. This can be achieved by adding the first derivatives (approximated by the differences) and the absolute values of these

derivatives,

$$\begin{aligned}\mathbf{X}_1(t) &= [\mathbf{x}_1(t), \Delta\mathbf{x}_1(t), |\Delta\mathbf{x}_1(t)|], \\ \mathbf{X}_2(t) &= [\mathbf{x}_2(t), \Delta\mathbf{x}_2(t), |\Delta\mathbf{x}_2(t)|],\end{aligned}\quad (3)$$

where $\Delta\mathbf{x}_i(t) = [\mathbf{x}_i(t + \delta) - \mathbf{x}_i(t)]/\delta$ stands for the approximate derivatives, $|\Delta\mathbf{x}_i(t)|$ are the corresponding absolute values, and δ is the integration step size. The absolute value here is merely a choice of convenience, as the range of values is not changed, only the sign. Other even functions will also work in the present case (omitted for brevity). $\mathbf{X}_i(t)$ with $i \in \{1, 2\}$ are vectors embedding the enriched reconstructed vectors at time t .

The obtained vector embeddings are finally used to assess the possible functional correspondence between the two systems using a linear regression approach applied in both directions, as detailed below.

B. Estimation by linear regression

Here, we assume the existence of a functional relation that can be estimated by performing a multiple linear regression from the vector embedding of $x_i(t)$ on the instantaneous state of $x_j(t)$:

$$\hat{x}_j(t) \simeq \hat{\beta}\mathbf{X}_i(t), \quad (4)$$

where $\hat{x}_j(t)$ is an estimation, which we choose to name the “proxy signal” of variable $x_j(t)$, $\mathbf{X}_i(t)$ is the vector embedding of variable $x_i(t)$, and $\hat{\beta}$ is a vector containing the estimated regression coefficients. In matrix notation, if $\mathbf{X}_i \in \mathbb{R}^{N \times 3m}$ is the matrix containing the N vectors of $3m$ dimensions reconstructed from the node taken as the driver, and $\mathbf{x}_j \in \mathbb{R}^N$ is the target variable from the node taken as the target, then the coefficients are estimated by the least squares technique. One has

$$\hat{\beta} = \arg \min_{\hat{\beta}} \|\mathbf{x}_j - \mathbf{X}_i\hat{\beta}\|^2 \Rightarrow \hat{\beta} = (\mathbf{X}_i^T \mathbf{X}_i)^{-1} \mathbf{X}_i^T \mathbf{x}_j.$$

The next step is to assess the quality of the proxy signal reconstruction by comparing it with the original target signal.

We used two measures to assess the quality of the proxy signal reconstruction: the differences in the phase synchronization value r_{ij} and synchronization error e_{ij} between each proxy signal and the corresponding target. Recall that for two signals $s_1(t)$ and $s_2(t)$, the phase synchronization value r and synchronization error e , respectively, are defined as

$$r(s_1, s_2) = \left| \frac{1}{N} \sum_{t=1}^N e^{i(\phi_{s_1(t)} - \phi_{s_2(t)})} \right|, \quad (5)$$

$$e(s_1, s_2) = \frac{1}{N} \sqrt{\sum_{t=1}^N (s_1(t) - s_2(t))^2}, \quad (6)$$

where $\phi_{s_i}(t) = \arg(\mathcal{H}[s_i(t)])$ denotes the instantaneous phase of signal $s_i(t)$, $\mathcal{H}[\cdot]$ denotes the Hilbert transform, and N denotes the total number of temporal samples.

These two measurements enable us to determine the level of correspondence between the proxy and original signals in terms of both amplitude and phase. Low reconstruction error and strong phase coherence suggest the existence of a robust functional relationship, which is indicative of generalized

synchronization. Using these two quantities, we can assess the generalized synchronization between the driver and target signals.

C. Assessment of the generalized synchronization

To assess the presence and intensity of generalized synchronization, we calculated the generalized synchronization index $\Delta r_{ij} = r_{ij} - r$ and generalized synchronization error $\Delta e_{ij} = e - e_{ij}$. Here, r_{ij} is the phase synchronization value between the proxy and target signals, and r corresponds to the phase synchronization value between the target and original driver signals. e_{ij} is the synchronization error between the proxy and target signals, and e denotes the synchronization error between the target and original driver signals. The generalized synchronization is significant when Δr_{ij} (resp. Δr_{ji}) and Δe_{ij} (resp. Δe_{ji}) are appreciably greater than zero. The overall methodology is described in Figs. 19 and 20 (Appendix A).

To confirm that the generalized synchronization relationship detected between the two coupled systems is not the result of chance or artifact related to the structure of the signals (e.g., their autocorrelation or power spectrum), the same analysis (i.e., phase-space reconstruction, linear regression, and generalized synchronization measures) was performed on surrogate signals [39–41]. Surrogate generation retains the autocorrelation and value distribution of the time series, while destroying all the nonlinear structure [42,43]. This step tested the null hypothesis that the detected relationships could emerge even in the absence of real coupling between the systems.

III. APPLICATION TO TWO COUPLED RÖSSLER OSCILLATORS

The feasibility of detecting generalized synchronization is investigated in two coupled Rössler oscillators—which are henceforth referred to as System 1, with variables (x_1, y_1, z_1) , and System 2 with variables (x_2, y_2, z_2) —adopting the approach described in Sec. II. In the following, we use the index $s = \{1, 2\}$ to refer to the two systems alternatively.

The dynamics are described as

$$\begin{aligned}\dot{x}_1 &= -y_1 - z_1 + k_1(x_2 - x_1), \\ \dot{y}_1 &= x_1 + a_1 y_1, \\ \dot{z}_1 &= b_1 + z_1(x_1 - c_1), \\ \dot{x}_2 &= -y_2 - z_2 + k_2(x_1 - x_2), \\ \dot{y}_2 &= x_2 + a_2 y_2, \\ \dot{z}_2 &= b_2 + z_2(x_2 - c_2),\end{aligned}\tag{7}$$

where a_s , b_s , and c_s are the parameters of the two coupled Rössler systems [44], and k_1 and k_2 are the coupling coefficients. Throughout this study, we considered only symmetrical bidirectional coupling, such that $k = k_1 = k_2$. As explained below, the choice of coupling on the x variable, which realizes the class 2 master stability function (bounded stable region), was motivated by issues related to the realization of a tunable electronic implementation [45].

The trajectories were obtained by integrating the differential equations in Eq. (7) via Runge-Kutta Dormand-Prince

pair [46] until $t_{\max} = 2000$ using an adaptive step size, but recording the solution with $t_{\text{step}} = 0.04$. The trajectories had a length N of approximately 26 000 points, matching the size of the experimental recordings (Sec. IV A). According to the block diagram in Fig. 20, the two halves of the datasets are obtained by repeating the simulations twice using two different sets of initial conditions, namely: $x_1(0) = -x_2(0) = 10$, $y_1(0) = y_2(0) = z_1(0) = z_2(0) = 0$ and $y_1(0) = -y_2(0) = 10$, $x_1(0) = x_2(0) = z_1(0) = z_2(0) = 0$. In the experiments, the two halves were obtained by splitting each recording into two halves over time.

The dataset consists of two sets of 48×48 trajectories. The b and c parameters are equal for both oscillators and are fixed: $b_1 = b_2 = 0.2$ and $c_1 = c_2 = 5.7$. This mismatch is due to the different values of the a_s parameters. Two datasets were obtained, corresponding to $a_1 = 0.1$ and $a_1 = 0.2$. These two values correspond to the different behaviors of System 1 when no coupling exists; in the former case, the evolution is period-1 and thus nonchaotic, whereas in the latter, the dynamics have a positive maximum Lyapunov exponent (MLE) and are therefore chaotic. Each trajectory has different values of a_2 and k , which take 48 uniformly distributed values in intervals $a_2 \in [0.1, 0.35]$ and $k \in [0.001, 0.2]$.

Next, a measure of entropy H_s was calculated for each system separately, as well as the phase synchronization value, synchronization error, and generalized synchronization measures between the two systems. All of these analyses are based on the trajectories of the x_s variable of each system (i.e., x_1 for System 1 and x_2 for System 2). Furthermore, as detailed below, the Lyapunov exponent spectrum was analyzed, focusing on the first and second MLEs.

A. Entropy measure

The entropy measure $H_s(x_s, y_s)$ essentially consists of the Shannon entropy over the sample joint probability distribution of x_s and y_s . Let us consider one generic Rössler system and omit the index s . To evaluate the sample joint probability distribution via a two-dimensional histogram, the number of bins is first set as $N_x = N_y = 10$ for, respectively, the $x[n]$ and $y[n]$ sequences, where $n = 1, \dots, N$ and N is the number of points in the trajectory. For each point $(x[n], y[n])$, the counter $q_{k,l}$ of the corresponding two-dimensional bin is incremented by $k = 1, \dots, N_x$ and $l = 1, \dots, N_y$ is the bin index. $H(x, y)$ is therefore defined as

$$H(x, y) = - \sum_{k=1}^{N_x} \sum_{l=1}^{N_y} \frac{p_{k,l} \log(p_{k,l})}{\log(N_x N_y)},\tag{8}$$

where $p_{k,l} = q_{k,l}/N_x N_y$ is the population rate of bin (k, l) . This measure was introduced as a bridge between numerical simulations and experiments, considering the well-known difficulty in reliably estimating the Lyapunov exponents from instrumental recordings affected by discretization and noise.

Figures 1(a) and 1(c) presents the results of the aforementioned measures when $a_1 = 0.1$. In general, the values of entropy H_2 grow once the parameter a_2 exceeds a critical value, which corresponds to System 2 transitioning from the limit cycle evolution at $a_2 = 0.1$ to chaotic behavior; accordingly, H_1 increases when both a_2 and coupling k are

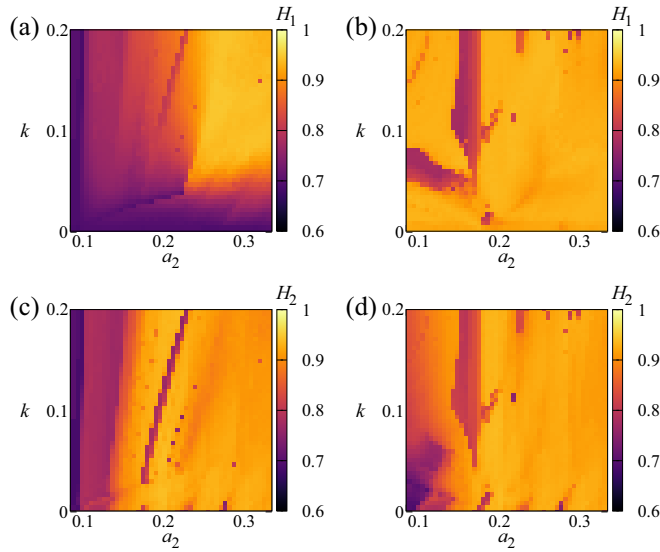


FIG. 1. Heatmaps of entropies of the two numerically simulated attractors computed for (a)–(c) $a_1 = 0.1$ and (b)–(d) $a_1 = 0.2$. The rest of parameter values are $b_1 = b_2 = 0.2$, and $c_1 = c_2 = 5.7$.

sufficiently large ($a_2 > 0.2$ and $k > 0.05$). These results point to chaos-transfer dynamics between the two systems, and assuming that the increase in H_1 is an indication of chaotic behavior (as discussed in the following sections), we can state that System 1 transitions to chaotic behavior owing to mutual coupling with System 2.

In Figs. 1(b) and 1(d), we present the results for the same measure when $a_1 = 0.2$. In this case, System 1 exhibits chaotic behavior when no coupling is present, and the measure of H_1 has an overall higher value than in the previous case with $a_1 = 0.1$. The entropy H_2 of System 2 is lower at $a_2 \simeq 0.1$ and low coupling, and then increases when the coupling is elevated and when $a_2 \simeq a_1 = 0.2$.

B. Influence of the embedding parameters

To evaluate the influence of the embedding parameters on the detection of generalized synchronization, we computed the phase synchronization value r_{ij} and the synchronization error e_{ij} in both directions between the two bidirectionally coupled Rössler systems (i.e., $x_i \rightarrow x_j$ with $i, j \in \{1, 2\}$ and $i \neq j$) by simultaneously varying the embedding dimension m and the time delay τ over the range $[1, 40]$. The results are presented in Fig. 2. The coupling strength was fixed at $k = 0.1$ and two values of the bifurcation parameter $a_1 \in \{0.1, 0.2\}$ were considered. The synchronization error and phase synchronization values were obtained by applying the formulas given in Eqs. (5) and (6), respectively. Panels (a) and (b) show the results for $a_1 = 0.1$, whereas panels (c) and (d) show the results for $a_1 = 0.2$. The others parameter values are $b_1 = b_2 = 0.2$, $c_1 = c_2 = 5.7$, and $a_2 = 0.3$.

The heatmaps reveal how both synchronization measures vary across the (m, τ) plane. Higher values of r_{ij} and lower values of e_{ij} indicated a stronger ability to predict system j from the reconstructed attractors of system i , suggesting a directional influence.

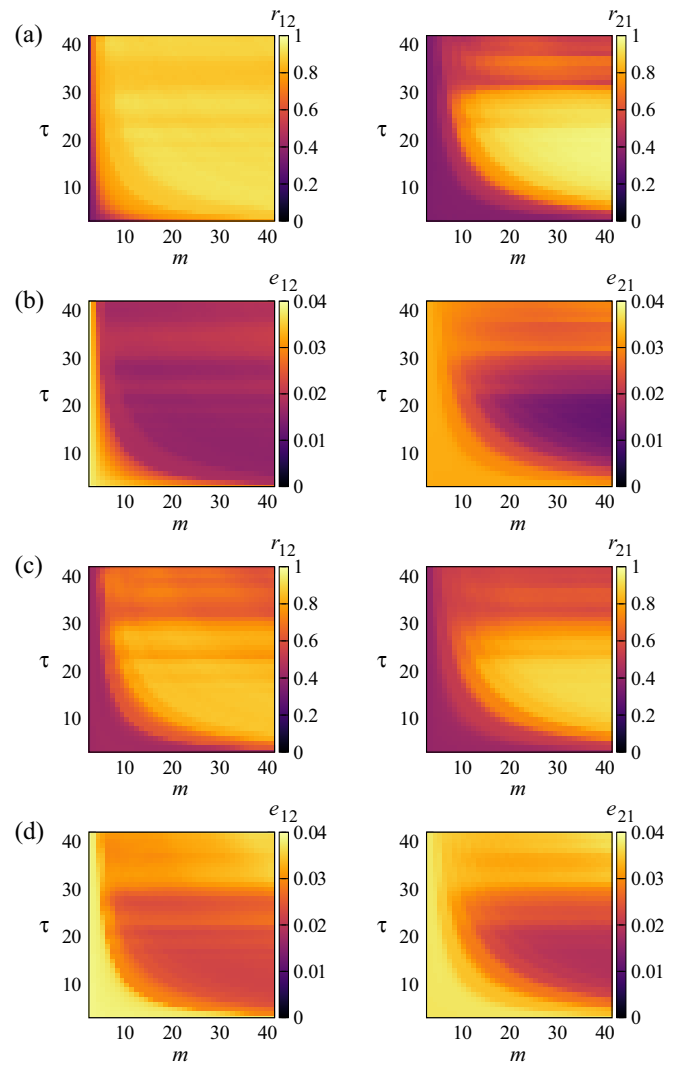


FIG. 2. Heatmaps of phase synchronization values r_{12}, r_{21} and synchronization errors e_{12}, e_{21} computed across a grid of embedding dimensions $m \in [1, 40]$ and time delays $\tau \in [1, 40]$, for a fixed coupling strength $k = 0.1$. Panels (a) and (b) correspond to $a_1 = 0.1$, while panels (c) and (d) correspond to $a_1 = 0.2$. High values of r_{ij} and low values of e_{ij} indicate a stronger synchronization from system j to system i . These maps guide the selection of optimal embedding parameters for generalized synchronization analysis. The rest of parameter values are $b_1 = b_2 = 0.2$, $c_1 = c_2 = 5.7$, and $a_2 = 0.3$.

We observe that when $a_1 = 0.1$, phase coherence is high in the $x_1 \rightarrow x_2$ direction ($r_{12} > 0.8$) and squared error is very low ($e_{12} < 0.01$) for a large area of the (m, τ) plane: $m \geq 10$ and $\tau \geq 10$. The opposite direction shows a lower phase synchronization and higher errors. In this direction, the areas of high synchronization and low error are concentrated in smaller regions of (m, τ) space, $m \in [15, 40]$ and $\tau \in [10, 30]$.

When $a_1 = 0.2$, r_{12} values become similar to r_{21} , indicating stronger reciprocity in predictive ability. Unlike when $a_1 = 0.1$, phase coherence is elevated in a smaller region (m, τ) : $m \in [20, 40]$ and $L \in [10, 28]$. A relative increase in e_{12} and closeness to e_{21} are observed. For $a_1 = 0.2$, the cou-

pling leads to more symmetrical synchronization but narrower parameter windows (m , τ).

In summary, these results indicate that when $a_1 = 0.1$, the synchronization is more pronounced in the direction $x_1 \rightarrow x_2$, implying a relatively better ability to predict System 2 from System 1. However, when $a_1 = 0.2$, the measures became more symmetric, suggesting a transition toward bidirectional synchronization. It should be emphasized that this observation merely represents an empirical fact and does not necessarily correspond to the direction of preferential causal influence.

These results also provide practical insights for selecting embedding parameters. Regions with simultaneously high r_{ij} and low e_{ij} help identify the optimal combinations of m and τ to detect generalized synchronization and estimate the functional relation between the coupled systems. For example, as illustrated in Fig. 2, one could choose the values for (m, τ) in the range $\tau \in [18, 22]$ and $m \in [20, 40]$.

C. Generalized synchronization measures

The measures of generalized synchronization (Δr and Δe) in the two coupled Rössler systems were evaluated using the methodology described in Sec. II C. Figures 3 and 4 present the results obtained by varying a_2 and k within the range defined in Sec. III. Phase-space reconstruction was performed using the embedding parameters $\tau = 20$ and $m = 20$. As in the previous analysis, two cases are considered: $a_1 = 0.1$ (left panels) and $a_1 = 0.2$ (right panels). The first heatmaps of each panel in Figs. 3(a) and 3(d) (resp. Fig. 4) show the phase synchronization value r (resp. synchronization error e) between the two systems, calculated based on target and original driver signals, whereas the others, which are based on target and proxy signals, demonstrate the evolution of Δr (resp. Δe) in both directions (Δr_{12} and Δr_{21} in Fig. 3) and (Δe_{12} and Δe_{21} in Fig. 4).

The synchronization measures r and e provide deeper insights into the interactions between the two systems. We observe that when $a_1 = 0.1$, the region in the (a_2, k) space such that $a_2 > 0.2$ and $k > 0.05$ does not seem to correspond to the two systems being phase synchronized, as the phase coherence measure shows. Furthermore, the synchronization error is sufficiently high to exclude detectable synchronization from the x_1 and x_2 components alone. Therefore, if a synchronization mechanism is present, it should fall under a generalized synchronization regime.

Panels (d) of Figs. 3 and 4 show the results of the same measures when $a_1 = 0.2$. From the values of r and e , we also observe that when the parameter mismatch is close to zero, i.e., $a_2 \simeq a_1 = 0.2$, the two systems are phase synchronized at low coupling, and as the coupling increases, they reach complete synchronization as the synchronization error approaches zero.

At values $a_2 > 0.2$, the phase coherence r shows that the two systems are not phase synchronized at low coupling, but increasing the coupling leads to an increase in the phase synchronization. Furthermore, the synchronization error is sufficiently high to exclude detectable synchronization from the x_1 and x_2 components alone. If a synchronization

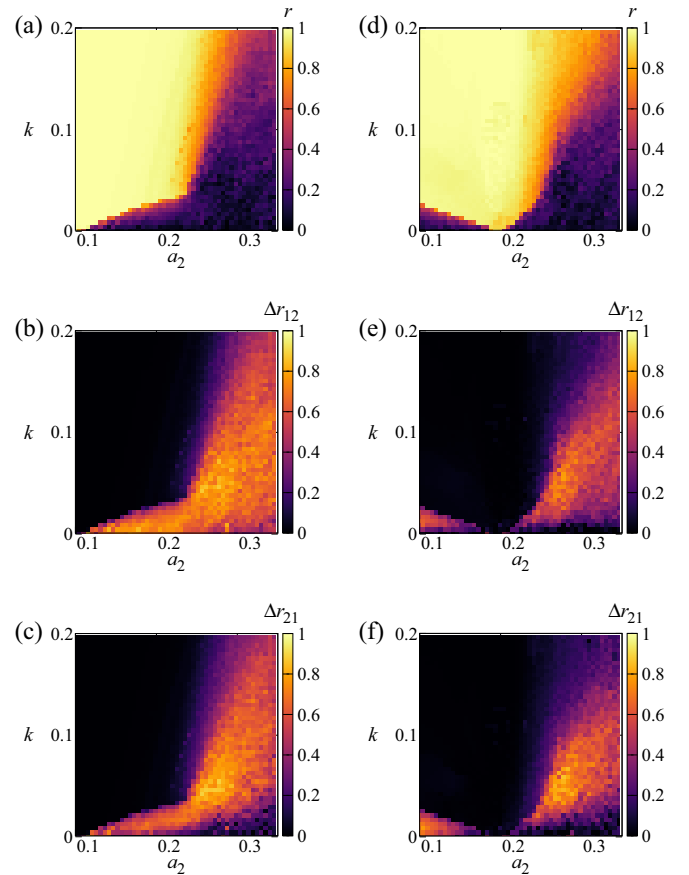


FIG. 3. Heatmaps of phase synchronization value r and generalized synchronization measures Δr_{12} and Δr_{21} computed as a function of the intrinsic parameter a_2 and the coupling strength k . Panels (a)–(c) correspond to $a_1 = 0.1$, while panels (d)–(f) correspond to $a_1 = 0.2$. A clear transition toward synchronization is observed beyond a critical coupling threshold, with a marked directional asymmetry between the systems (in some small regions). The variation of a_1 influences both the extent and the intensity of the synchronized regions. The rest of parameter values are $b_1 = b_2 = 0.2$, and $c_1 = c_2 = 5.7$.

mechanism is present, it should fall under the regime of generalized synchronization.

For $a_1 = 0.1$ [Fig. 3(b)], we observe that in direction $x_1 \rightarrow x_2$, in regions with high phase synchronization value r , Δr_{12} values are close to zero, whereas in regions of low phase coherence, Δr_{12} is greater than zero, suggesting generalized synchronization between the systems. Δr_{12} is also close to zero for weak couplings ($k \approx 0.001$) in the absence of interaction between systems, and similar results are obtained in other directions ($x_2 \rightarrow x_1$) [Fig. 3(c)]. The same observations are evident for $a_1 = 0.2$ [Figs. 3(b) and 3(c)]. These interpretations are supported by generalized synchronization error measurement results shown in Figs. 4(b) and 4(c) for $a_1 = 0.1$ and Figs. 4(e) and 4(f) for $a_1 = 0.2$. For both values of a_1 , the systems are in generalized synchronization in both directions when coupling is nonzero and in the absence of phase synchronization over an extended parameter space.

To gain deeper insight into the generalized synchronization detection between the bidirectionally coupled Rössler systems, phase portraits are plotted in Fig. 5. The coupling

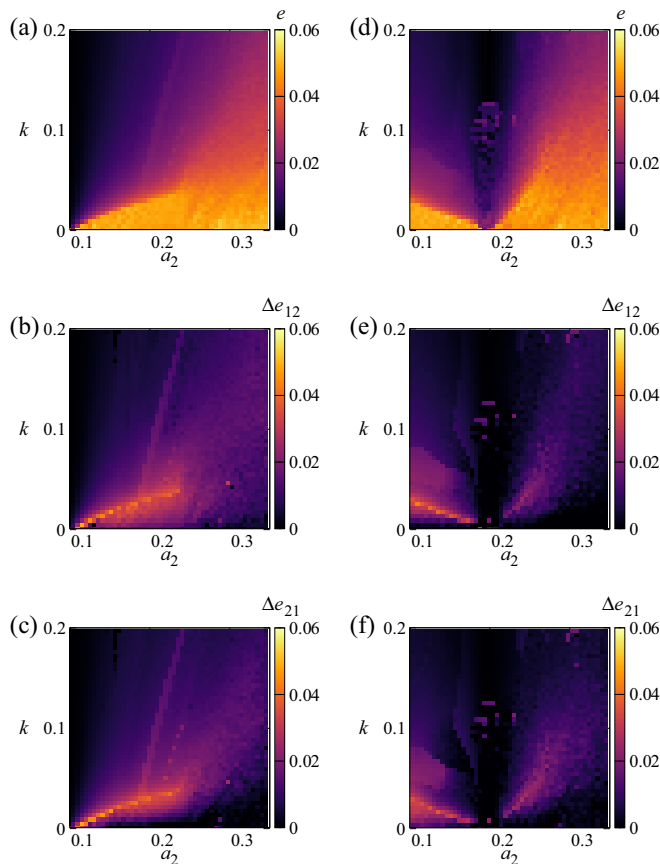


FIG. 4. Heatmaps of synchronization error e and generalized synchronization error measures Δe_{12} and Δe_{21} computed as a function of the intrinsic parameter a_2 and the coupling strength k . Panels (a)–(c) correspond to $a_1 = 0.1$, while panels (d)–(f) correspond to $a_1 = 0.2$. The rest of parameter values are $b_1 = b_2 = 0.2$, and $c_1 = c_2 = 5.7$.

strength is fixed at $k = 0.1$, whereas the parameter a_1 of the two systems varies; panel (a) corresponds to $a_1 = 0.1$, whereas panel (b) corresponds to $a_1 = 0.2$. In all panels, x_1 and x_2 denote the original time series from System 1 and System 2, respectively, and $\hat{x}_1$ and $\hat{x}_2$ represent the proxy signals. For $a_1 = 0.1$, the phase synchronization values between the two original time series, target and proxy signals in the direction $x_2 \rightarrow x_1$, and target and proxy signals in the direction $x_1 \rightarrow x_2$ are $r = 0.2499$, $r_{21} = 0.9145$, and $r_{12} = 0.8966$, respectively. For $a_1 = 0.2$, the values of the measures are $r = 0.5640$, $r_{21} = 0.8784$, and $r_{12} = 0.8140$. A considerable increase in the phase synchronization values between the target and proxy signals was observed in both directions. These results further confirmed the existence of a functional relationship between the two systems, which could be to some extent successfully approximated through the proposed linear regression scheme.

To assess the robustness of generalized synchronization detection between coupled Rössler systems, different methods are compared for reconstructing attractors from simulated signals, and a validation test is also carried out using surrogated data. Figure 6 shows heatmaps of the generalized synchronization index Δr and the mean value of the two directions

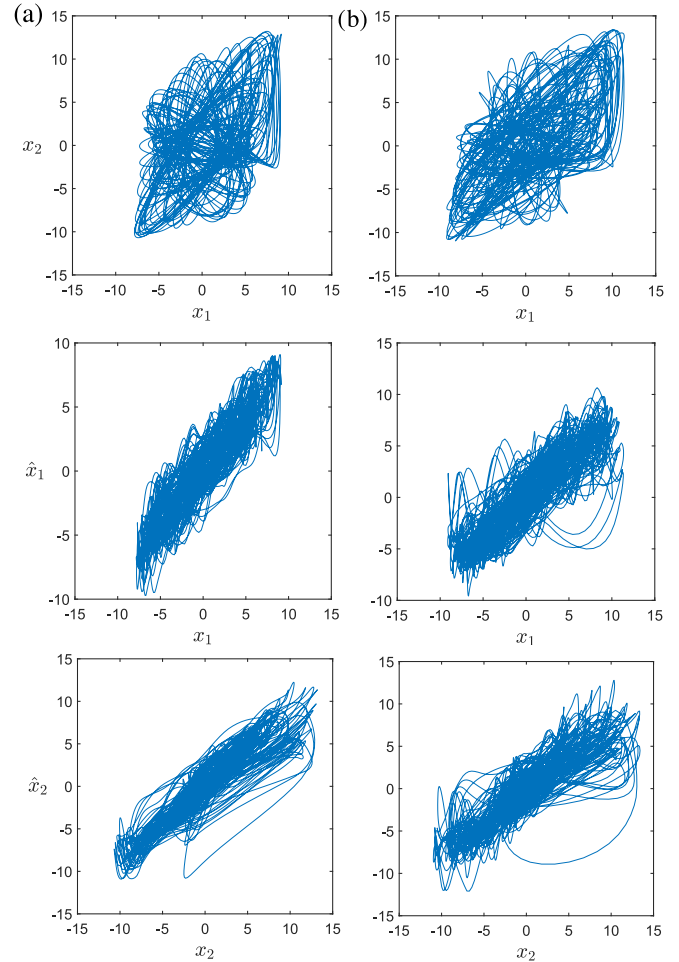


FIG. 5. Phase portraits plotted in the planes (x_1, x_2) , $(x_1, \hat{x}_1)$, and $(x_2, \hat{x}_2)$. The coupling strength is set to $k = 0.1$. Panel (a) corresponds to the case $a_1 = 0.1$, while panel (b) corresponds to $a_1 = 0.2$. In all panels, x_1 and x_2 denote the original time series, while $\hat{x}_1$ and $\hat{x}_2$ represent the proxy signals. The rest of parameter values are $b_1 = b_2 = 0.2$, $c_1 = c_2 = 5.7$, and $a_2 = 0.3$.

Δr_{12} and Δr_{21} as functions of the coupling parameters k and a_2 in four distinct configurations. In panel (a), the attractors were reconstructed using the derivatives and absolute values of the signals. Δr increases in the region where a_2 exceeds approximately 0.2, for moderate to high values of k , suggesting significant generalized synchronization in this region of the parameter plane. In panel (b), only derivatives are used. Although similar structures appeared, the area of strong synchronization was considerably weaker. Conversely, panel (c) shows the results obtained using absolute values only (without derivatives), where the effect vanishes entirely.

Finally, panel (d) presents the results for the surrogate data, which are univariate linear without any preservation of statistical properties, but with zero dynamic dependencies. As expected, the Δr values observed in this context were low and devoid of significant structures. This observation validates the hypothesis that the synchronization detected in the other cases is not an artifact but the result of genuine dynamic interactions between the two systems. These complementary analyses enabled us to select the most effective reconstruction method for

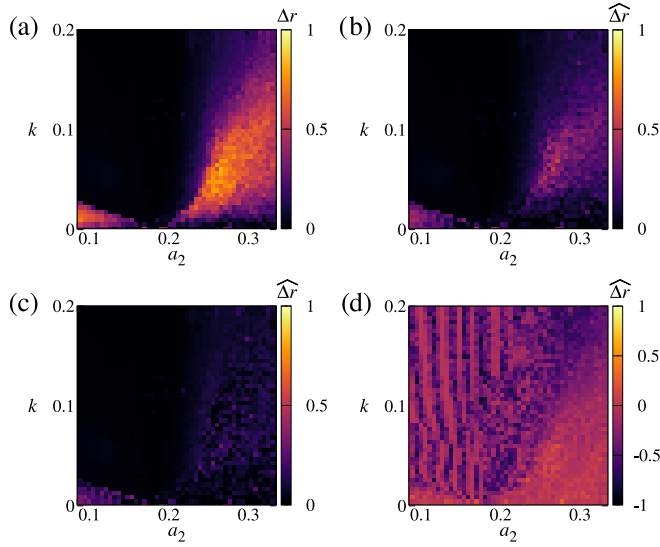


FIG. 6. Comparative analysis of the generalized synchronization index $\widehat{\Delta r}$ for different attractor reconstruction methods in the (a_2, k) parameter space. (a) Reconstruction based on derivatives and absolute values of the signals. (b) Reconstruction based on derivatives only. (c) Reconstruction based on absolute values only. (d) Results obtained from surrogate data. The lighter regions indicate stronger generalized synchronization. These results help evaluate the effectiveness of reconstruction techniques and confirm that the detected synchronization is not a statistical artifact. The rest of parameter values are $b_1 = b_2 = 0.2$, $c_1 = c_2 = 5.7$, and $a_1 = 0.2$.

detecting generalized synchronization (here, derivatives and absolute values are used in combination).

D. Visualization of the functional relation Φ through regression coefficients

In order to visualize the approximated functional relation between the two coupled Rössler systems, the regression coefficients $\hat{\beta}$, average values evaluated over 100 iterations, as a function of the index l of the regressor components obtained from the reconstruction of one system from another in both directions are plotted in Fig. 7. In all panels, the coupling strength was fixed at $k = 0.1$. The left panels (a) and (c) correspond to $a_1 = 0.1$, and the right panels (b) and (d) correspond to $a_1 = 0.2$. Panels (a) and (b) show the regression coefficients for predicting System 2 from System 1 (direction: $x_1 \rightarrow x_2$). Panels (c) and (d) show the reverse direction ($x_2 \rightarrow x_1$).

The first case shows that generalized interdependence has a short memory effect, where coefficients show small fluctuations and decay to near zero beyond $l \approx 10$. In the second case, persistent oscillatory behavior indicates a lag term effect. These traces resembled a three-phase-shifted wave packet, gradually decaying with increasing delay. Each curve corresponds to a different regressor construction: the time-delay embedding of variable x , its derivative x' , and absolute value $|x'|$, offering complementary perspectives on the underlying dynamics. If we consider the position, then they appear to shift with respect to their derivatives (velocity). The contributions of derivatives and absolute values are in phase, demonstrating

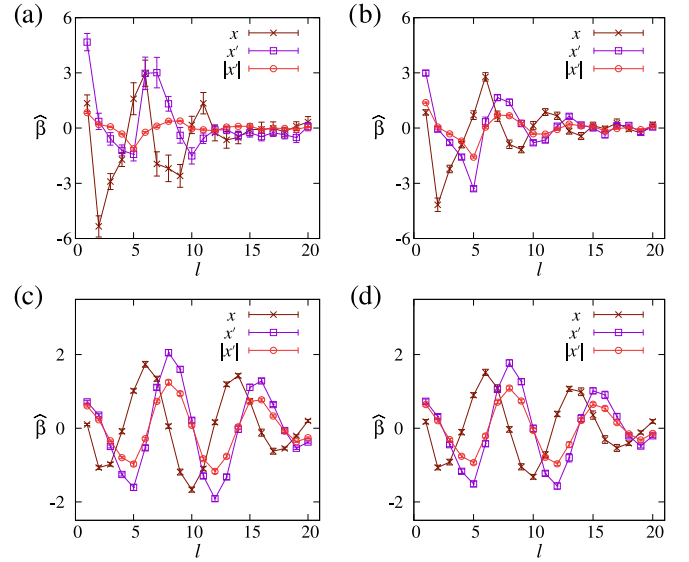


FIG. 7. Plots of the regression coefficients $\hat{\beta}$ as a function of the index l of the regressor components, providing a visual representation of the functional relationship between the two systems. Panels (a) and (c) correspond to $a_1 = 0.1$, and panels (b) and (d) to $a_1 = 0.2$, with fixed coupling $k = 0.1$. The top row (a, b) shows the direction from System 1 to System 2, while the bottom row (c, d) shows the reverse direction. Each curve represents the average coefficient values obtained using original signals x , their derivatives x' , and their absolute values $|x'|$, with error bars indicating the standard deviation. The rest of parameter values are $b_1 = b_2 = 0.2$, $c_1 = c_2 = 5.7$, and $a_2 = 0.3$.

the establishment of asymmetric functional interdependence. This represents the generalized nonlinear interdependence.

E. Lyapunov spectrum and MLE

To study the Lyapunov spectrum, two different approaches were used. First, the Lyapunov spectrum via the so-called standard method [47,48] was computed, considering Eq. (7) as a six-dimensional system, thereby obtaining six Lyapunov exponents. Second, the MLE was estimated via the divergence exponent algorithm [49–52] using time-delay embeddings [53] of the x_1 and x_2 time series, thus obtaining an estimate of the MLE for the two systems separately. To study the Lyapunov exponents, trajectories of size $N = 10^6$ points were used to improve the estimates. Furthermore, the evaluation was performed by taking the average value of ten trajectories, each starting from randomly chosen boundary conditions.

To investigate generalized synchronization, the two largest Lyapunov exponents obtained via the standard method were computed and displayed in Fig. 8; according to Moskalenko *et al.* [11], the onset of generalized synchronization between the two coupled chaotic systems can be detected when the MLE is positive and the second-largest one (LE2) is zero. This case typically occurs when the two uncoupled systems are both independently chaotic, that is, each has a positive MLE. Panels (a) and (b) show MLE and LE2, respectively, of the coupled system in the case $a_1 = 0.1$, for which System 1 is, as discussed above, natively (i.e., without coupling) in a limit cycle evolution with a vanishing MLE, and thus nonchaotic.

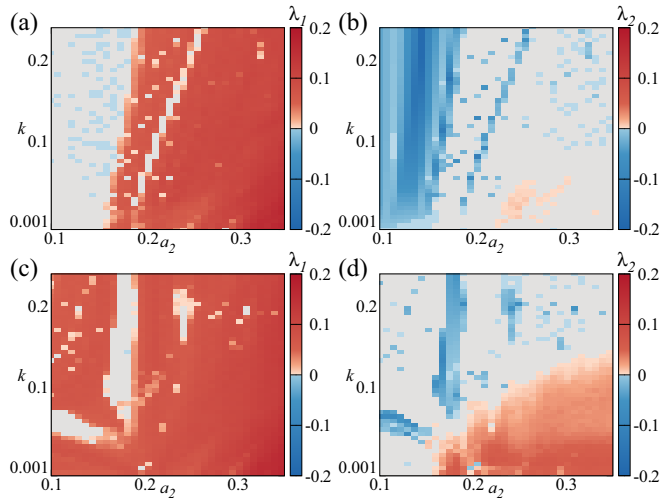


FIG. 8. Heatmaps of the first two Lyapunov exponents, evaluated via standard method on the numerically simulated data. Panels (a) and (b) show, respectively, the first and second Lyapunov exponents in the case of $a_1 = 0.1$. Panels (c) and (d) show, respectively, the first and second Lyapunov exponents in the case of $a_1 = 0.2$. The rest of parameter values are $b_1 = b_2 = 0.2$, and $c_1 = c_2 = 5.7$.

If $a_2 > 0.2$ and at any coupling value k , the coupled-system MLE becomes positive, whereas LE2 remains zero. This situation can be interpreted as System 2 becoming chaotic and transferring the chaos to the coupled system. By definition, this process does not correspond to generalized synchronization. However, the entropy analysis shown in Fig. 1, and in particular, the region defined by $a_2 > 0.2$, $k > 0.05$, two facts become apparent: First, the entropy H_1 shown in panel (a) of System 1 increases and becomes comparable to the entropy of System 2; second, at the same time, the phase coherence r shown in panel (a) of Fig. 3 does not evidence any phase synchronization. In summary, while no standard generalized synchronization can be called in a strict sense due to the native nonchaoticity of System 1, a “transfer” of chaos, in terms of entropy, from System 2 to System 1 clearly takes place.

To support this interpretation, we use a time-series analytical approach and compute the MLE using a numerical method [54,55]. We set $a_2 = 0.3$ and select a set of values for coupling k ranging from 0 to 0.25. For each value of k , we simulated the trajectories of the two systems and extracted x_s sequences. We then performed time-delay embedding on each of the x_s sequences, with embedding dimension m and time lag τ chosen on a “lattice” [54,55] $\{m, \tau\}$ defined by $2 \leq m \leq 20$, $1 \leq \tau \leq 20$. Given an embedding pair (m, τ) , let $\mathbf{Y}_s[i]$ be the embedded sequence of vectors, such that $\mathbf{Y}_s[i] = x_s[i + l\tau]$, where $0 \leq l \leq m - 1$ and $0 \leq i \leq N - (m - 1)\tau$. For a given delay k , the divergence exponent $\hat{\Lambda}(n)$ is estimated as follows:

$$\hat{\Lambda}(k) \equiv \left\langle \log \frac{\|\mathbf{Y}_s[i + k] - \mathbf{Y}_s[j + k]\|}{\|\mathbf{Y}_s[i] - \mathbf{Y}_s[j]\|} \right\rangle,$$

where the average is computed on a set of 1000 randomly selected “neighboring” embedding vectors $\mathbf{Y}_s[i]$, $\mathbf{Y}_s[j]$ (further details of the procedure can be found in Ref. [55]).

For each embedding pair, the MLE of the source system was estimated by determining the slope of the plot $\hat{\Lambda}(k)$ vs

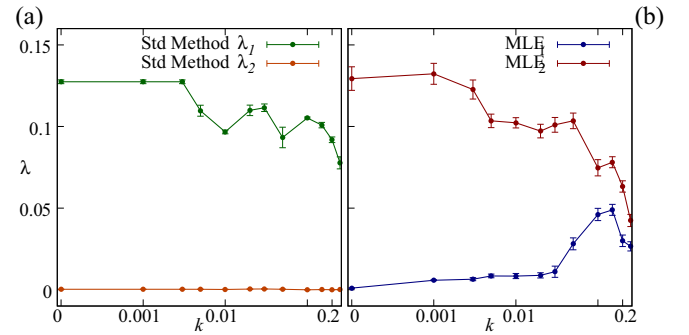


FIG. 9. Panel (a) shows the first and second Lyapunov exponents, computed via the standard method, considering the whole system made up of the two attractors, with parameters $a_1 = 0.1$ and $a_2 = 0.3$. The dotted lines correspond to the averages on ten trajectories with random initial conditions; the shaded area corresponds to one standard deviation. Panel (b) shows, for the same trajectories and parameters, the MLE evaluated via divergence exponent on the embedding of x coordinates, separately for System 1 (blue) and System 2 (red).

k , where the plot increases linearly. A histogram is made of the MLE estimates on each embedding pair (m, l) (setting the bin size via Freedman-Diaconis rule [56]) and the value of the most populated bin is taken as final estimate of the MLE of the starting sequence x_s , with the half-width of the bin as the MLE uncertainty.

Figure 9 shows the results of this procedure and the corresponding values of the MLE computed using the standard method. Evidently, at low coupling, only System 2 is in a chaotic state, and the value of its MLE, computed using the divergence exponent, agrees well with the largest exponent obtained using the standard method. Once the coupling increases, the MLE of System 1 starts to increase, and at high coupling values, the values of the MLE of the two separate systems tend to converge. Notably, with increasing coupling, the overall value of the largest exponent computed via the standard method decreases, and this decrease is well matched by the values of the MLEs via the divergence method. Studying the MLE of the two systems separately allows us to uncover a chaos-transfer dynamic that cannot be seen by the Lyapunov spectrum computed via the standard method. Although the MLE decreases as the coupling becomes stronger, the chaotic behavior is transferred from one system to the other owing to the coupling.

In Fig. 8, panels (c) and (d) refer to a situation in which $a_1 = 0.2$, such that System 1 is natively chaotic. In this case, when $a_2 > 0.2$ such that System 2 is also natively chaotic, the increase in coupling k leads to LE2 going abruptly to zero, thus providing clear evidence of generalized synchronization in a canonical sense.

IV. EXPERIMENTAL IMPLEMENTATION

A. Experimental setup and dataset

As shown in Fig. 21(b), the circuit was realized on a breadboard, while ensuring adequate separation and shielding to avoid interference between the signals. The signals

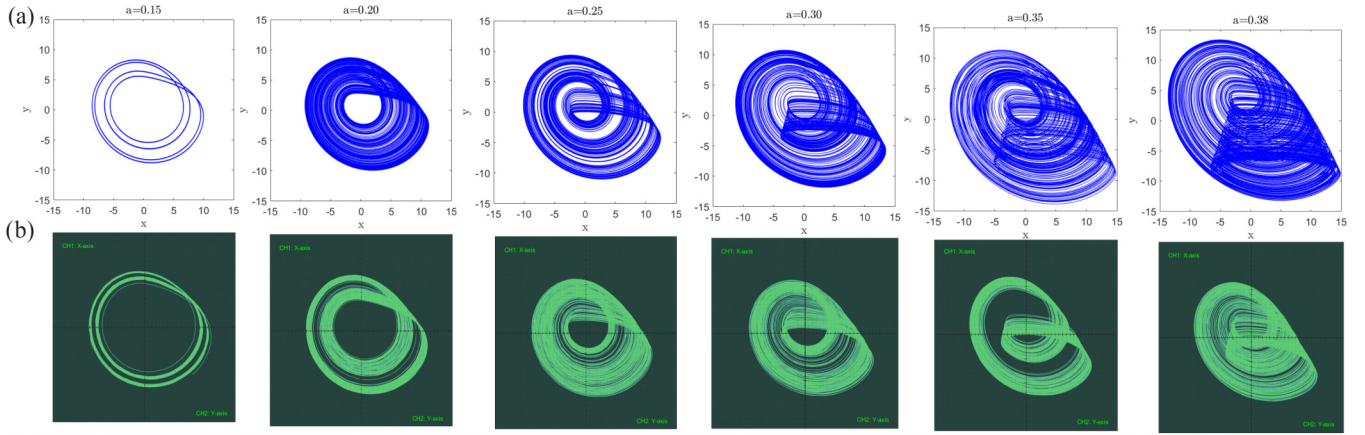


FIG. 10. Phase portraits (x, y coordinates) of the Rössler system. (a) Numerical simulations. (b) Electronically implemented. Plots are generated for different values of parameter a (with $b = 0.2$ and $c = 5.7$ fixed). Experimental figures in panel (b) are oscilloscope-like computer renderings.

corresponding to the x_1, y_1, x_2 , and y_2 variables were recorded at 625 kHz, 16 bits, using a high-performance data acquisition card (type PCIe-6259, National Instruments Inc., Austin, TX, USA), and filtered with a moving-average filter to reduce the sampling noise. A total of 25 000–40 000 points after an undersampling between a factor of 1 and 25 were recorded per run, depending on the choice of the integration capacitor. All the data are freely available in Ref. [57]; while only $a_1 = \{0.1, 0.2\}$ were considered in this study, additional recordings for $a_1 = \{0.15, 0.25, 0.3, 0.35, 0.4\}$ were performed and are available upon request to the authors. The digital potentiometers were controlled via an serial bus interface, and prior to each run, the voltage across all integration capacitors was reset to zero by pulsing the corresponding relays, thereby avoiding spurious memory effects. The resistance and capacitor values have tolerances, respectively, of $\pm 1\%$ and $\pm 10\%$. To exclude the possible bias effects related to parametric tolerances rendering the two electronic circuits heterogeneous, all acquisitions were repeated twice, assigning System 1 and System 2 to each of them and then swapping, and the final results were averaged.

For each capacitor value, the dataset comprises two sets of 48×48 trajectories, with each set corresponding to $a_1 = 0.1$ and $a_1 = 0.2$. The trajectories have values of a_2 and k that are uniformly distributed over the intervals $a_2 \in [0.1, 0.4]$ and $k \in [0.001, 0.25]$. Each trajectory comprises the x and y components of the attractors and has a size of approximately 26 000 points. Each signal was normalized to have a zero mean value and a unitary standard deviation.

Figure 10 shows phase portraits in the (x, y) plane of the Rössler system, comparing results obtained by numerical simulations (upper panel, a) with those obtained by experimental setup based on an analog Rössler electronic circuit (lower panel, b). Resulting phase portraits are plotted for increasing values of the control parameter a , keeping the values of $b = 0.2$ and $c = 5.7$ as constant parameters. The experimental data were visualized by simulating the display of an analog oscilloscope [58], through generating a two-dimensional representation that faithfully reproduces the visual properties of an analog oscilloscope screen. In the attempt to mimic the dynamic brightness of a tube screen, a local estimate of the

signal's “speed” was performed by differentiating the waveform, smoothing it, and then multiplying it by the normalized gradient level. The resulting brightness envelope was then transformed through a logistic curve, modulated by tunable parameters for brightness and contrast, thus closely mimicking the nonlinear relationship characteristic of phosphor excitation.

A visual inspection shows that the numerical simulations (panel a) demonstrate a gradual and clear change in the Rössler system's chaotic dynamics. The process begins with a periodic attractor at $a = 0.15$ and proceeds through increasingly chaoticity as the system parameter a increases, in particular starting at $a = 0.30$, where the chaotic band grows. In panel (b) of the experiment are the phase portraits that mostly represent the numerical attractors' structure, thus offering proof of the successful electronic realization of the system. However, noticeable distortions and irregularities are present, mostly at lower values of a . At $a = 0.15$ (as shown in the leftmost panel of b), the experimental attractor seems to exhibit reduced stable periodicity, thus exhibiting a less stable and less “clean” (when compared with their numerical counterparts) trajectory. The appearance of such an attractor can be explained by the inherent noise within the circuit.

With increasing values of the parameter a , the general structure of the attractor is better maintained in the electronic realization, even though with scale discrepancies and loss of finer features compared to the simulation. For instance, at values of $a = 0.20, a = 0.25, a = 0.30$, and $a = 0.35$, typical loops of characteristic appearance typical of the Rössler attractor are clearly visible in the corresponding pairs of figures, although the experimental attractors are denser in structure and less “stretched” in the y direction compared to the corresponding numerical ones. Scale discrepancies and the slight geometric distortions of the attractor thus obtained in the electronic realization may be traced mainly to the nonideal nature of the electronic circuitry of the operational amplifiers, namely, offsets and parasitic capacitances. These results thus testify to the need to take such nonideal components into consideration during the design phase as well as when analyzing analog chaotic circuits.

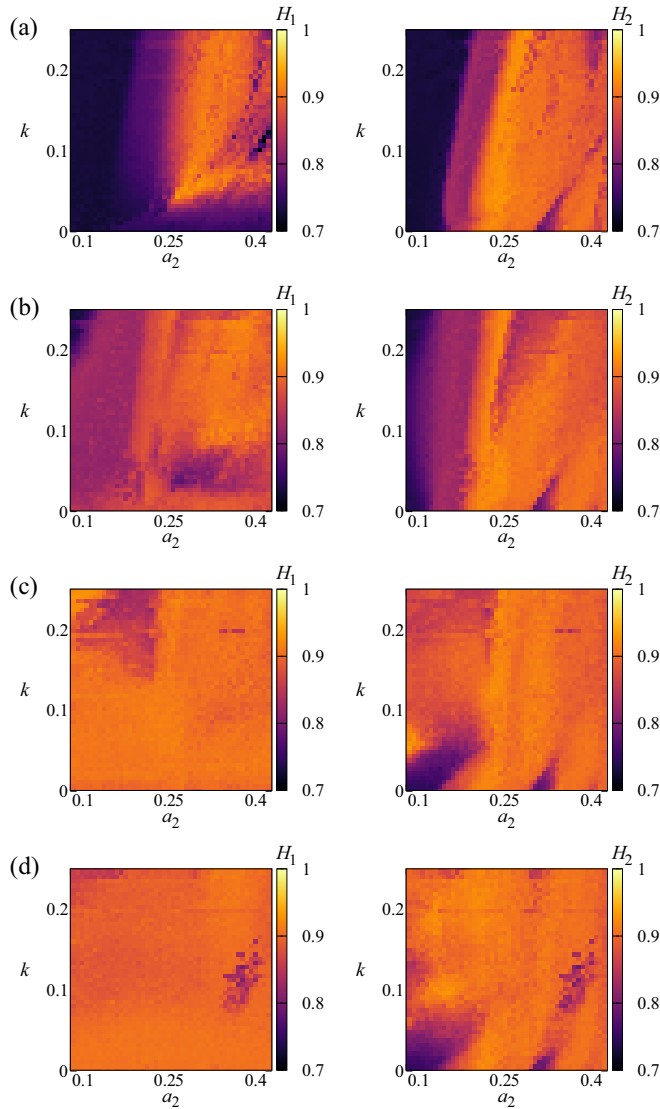


FIG. 11. Heatmaps of entropies of the two electronically implemented attractors in panel (a) when $a_1 = 0.1$, panel (b) when $a_1 = 0.2$, panel (c) when $a_1 = 0.3$, and panel (d) when $a_1 = 0.4$.

B. Data analysis

For the experimental data, we measured the phase synchronization r , synchronization error e , and entropy H_s using Eqs. (5), (6), and (8). The case of 100 nF was considered for the majority of the analyses, and recordings with other capacitors were used for comparison. A discussion of the differences between circuits with other capacitor values is presented in Sec. IV D.

Figure 11 shows the results of the entropy H_s , when $a_1 = 0.1$, $a_1 = 0.2$, $a_1 = 0.3$, and $a_1 = 0.4$. For $a_1 = 0.1$, the plots show a good overall agreement between the experimental data and numerical simulations; nonetheless, a few differences emerge. The entropy heatmaps show that the entropy of both systems is generally lower in the experimental data than in the numerical simulations, particularly on the left side of the plots, corresponding to lower values of a_2 . Furthermore, the values of r were generally higher in the experimental data. These

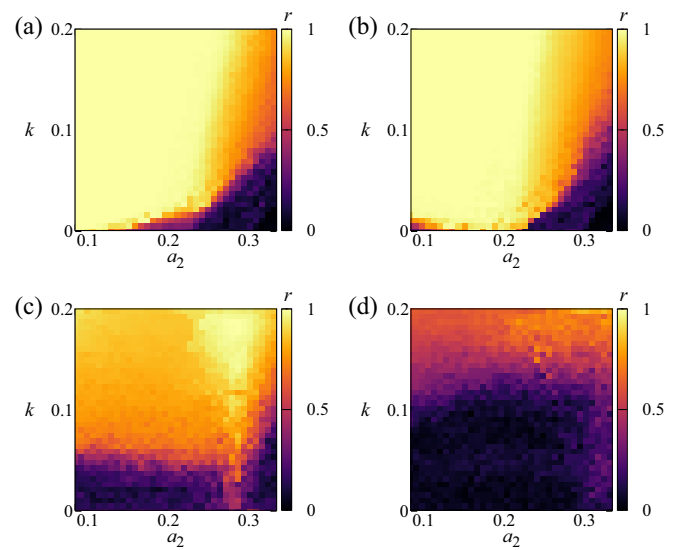


FIG. 12. Experimental heatmaps of phase synchronization value r computed as a function of the couple of parameter (a_2, k) for some distinct values of a_1 : panel (a) $a_1 = 0.1$, panel (b) $a_1 = 0.2$, panel (c) $a_1 = 0.3$, and panel (d) $a_1 = 0.4$.

two pieces of evidence suggest that the two experimental systems achieved phase synchronization more easily than the corresponding simulated systems. One possible explanation is that we do not exactly realize the theoretical parameters owing to the component tolerances. In System 2, the emergence of chaotic behavior is suppressed when $a_2 \leq 0.15$. Similarly, panel (b) presents the results of the same measure when $a_1 = 0.2$. The same observations were made in the previous case (i.e., when $a_1 = 0.1$). For $a_1 = 0.30$ and $a_1 = 0.4$, System 1 is chaotic whatever the value of k considered, while System 2 exhibits a low entropy for low values of a_2 ($a_2 \in [0.1, 0.15]$) and coupling strength k ($k \in [0, 0.05]$).

Figure 12 shows the experimental heatmaps of the phase synchronization value between the two coupled Rössler oscillators plotted in the parameter space (a_2, k) for four distinct values of a_1 : $a_1 = 0.1$ in panel (a), $a_1 = 0.2$ in panel (b), $a_1 = 0.3$ in panel (c), and $a_1 = 0.4$ in panel (d). For $a_1 = 0.1$, the heatmap shows that the two oscillators are highly synchronized ($r \simeq 1$) for low values of a_2 ($a_2 < 0.2$). When $a_2 > 0.2$ and the coupling $k < 0.1$, the two systems are not synchronized because the phase coherence is close to zero; however, increasing k synchronizes the two oscillator phases. The same observations were made for $a_1 = 0.2$. When $a_1 = 0.3$, the two oscillators are phase synchronized only for values of $k > 0.1$, and when a_2 is increased to 0.4, the two systems are not synchronized for a large band of the parameter space (a_2, k) and are phase synchronized only in a small region.

The formulas given in Sec. II C were applied to obtain the generalized synchronization index Δr between the two oscillators. The corresponding results are shown in Fig. 13, where we have displayed the heatmaps of Δr computed for four values of a_1 as in Fig. 12. It is observed that Δr is close to zero in regions where the two oscillators are phase synchronized and greater than zero when they are not synchronized, which

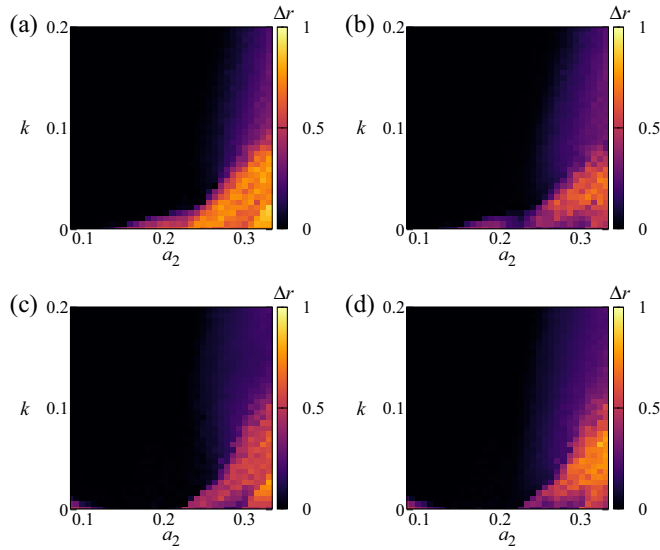


FIG. 13. Experimental heatmaps of generalized synchronization index Δr computed as a function of the couple of parameter (a_2, k) for some distinct values of a_1 : panel (a) $a_1 = 0.1$, panel (b) $a_1 = 0.2$, panel (c) $a_1 = 0.3$, and panel (d) $a_1 = 0.4$.

confirms the existence of a generalized functional relation between the two oscillators.

Furthermore, to gain deeper insights into the generalized synchronization detection between the bidirectionally coupled Rössler oscillators, experimental phase portraits in the planes (x_1, x_2) , $(x_1, \hat{x}_2)$, and $(x_2, \hat{x}_1)$ are plotted in Fig. 14. Here, the coupling strength is set to $k = 0.1$. Panel (a) corresponds to case $a_1 = 0.1$, and panel (b) corresponds to $a_1 = 0.2$. In all the panels, x_1 and x_2 denote the original time series and $\hat{x}_1$ and $\hat{x}_2$ represent the proxy signals.

For $a_1 = 0.1$, the phase synchronization values between the two original time series, target and proxy signals in the direction $x_2 \rightarrow x_1$, and target and proxy signals in the direction $x_1 \rightarrow x_2$ are $r = 0.1068$, $r_{21} = 0.7867$, and $r_{12} = 0.8901$, respectively. For $a_1 = 0.2$, the values of the measures are $r = 0.1261$, $r_{21} = 0.9241$, and $r_{12} = 0.8639$. Similar to the numerical data, a considerable increase in the phase synchronization values between the target and proxy signals is observed in both directions for the simulated data. These results further confirm the existence of a functional relation between the two Rössler oscillators.

C. Experimental visualization of the functional relation Φ

We aimed to visualize the functional relation Φ between two coupled Rössler oscillators using the experimental recording data and following the same empirical approach described in Sec. III D. Figure 15 shows the distribution of the regression coefficients β as a function of the index l of the regressor components.

Each subfigure represents the regression coefficients obtained for the four distinct values of a_1 : $a_1 = 0.1$ (a), $a_1 = 0.2$ (b), $a_1 = 0.3$ (c), and $a_1 = 0.4$ (d), where the error bars correspond to the standard deviation of the regression coefficients. From this figure, a good concordance between the results of the experimental data and numerical simulations is observed.

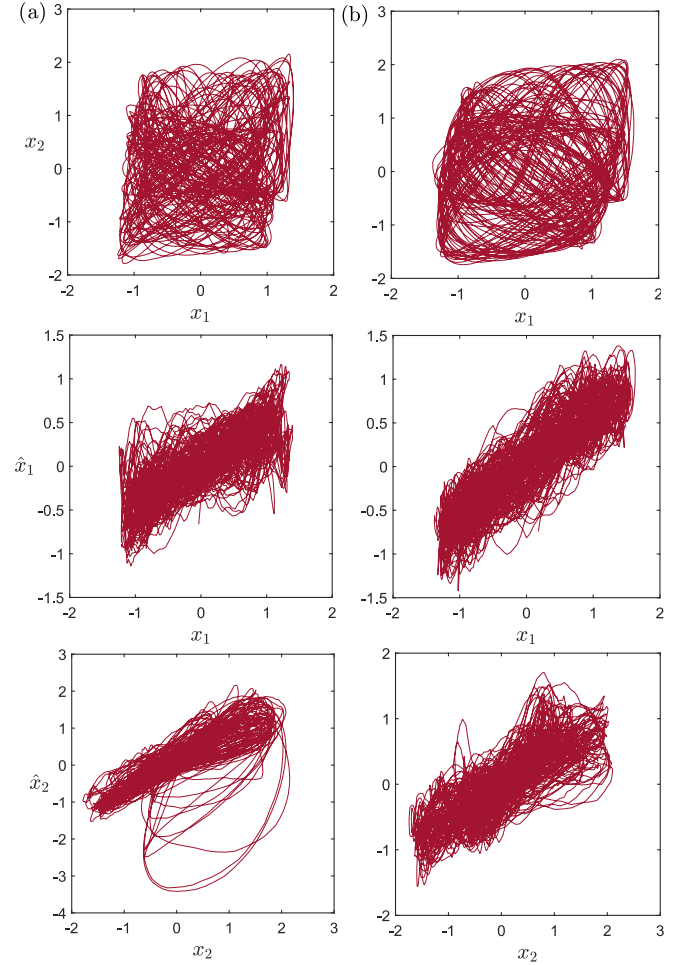


FIG. 14. Experimental phase portraits plotted in the planes (x_1, x_2) , $(x_1, \hat{x}_1)$, and $(x_2, \hat{x}_2)$. The coupling strength is set to $k = 0.1$. Panel (a) corresponds to the case $a_1 = 0.1$, while panel (b) corresponds to $a_1 = 0.2$. In all panels, x_1 and x_2 denote the original time series, while $\hat{x}_1$ and $\hat{x}_2$ represent the proxy signals. The rest of parameter values are $b_1 = b_2 = 0.2$, $c_1 = c_2 = 5.7$, and $a_2 = 0.3$.

Therefore, the practical feasibility of our method for detecting generalized synchronization in two coupled Rössler systems is validated. In general, they appeared to decay faster than in the simulated case.

D. Effects of nonideality of electronic components

The effects of component nonidealities are often neglected, yet are intriguing aspects of experimental analog implementations of dynamic systems. Such problems are due to the physical limitations intrinsic to the components (i.e., operational amplifiers, resistors, capacitors, and analog multipliers), which together influence the overall behavior of the circuit. In this work, these effects are manifested in the chaotic dynamics of two analog coupled Rössler circuits.

A particularly salient effect of these nonidealities is that the operational amplifiers employed for integrators and summers possess a constrained bandwidth, thereby implying that the operational amplifier response is not instantaneous. This

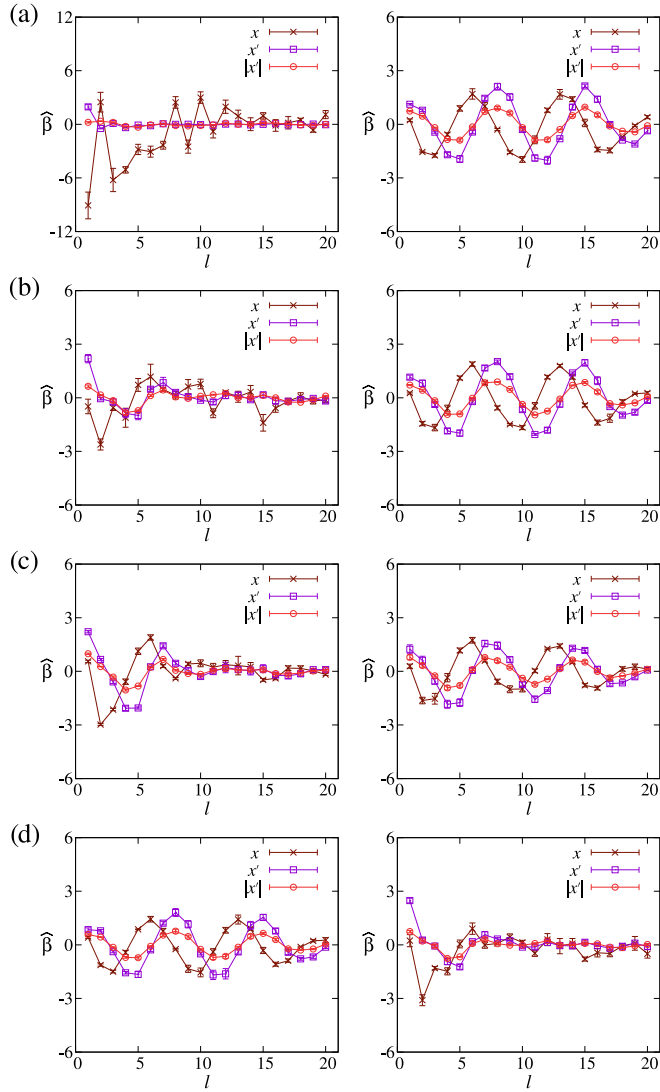


FIG. 15. Experimental visualization of the regression coefficients $\hat{\beta}$ as a function of the index l of the regressor components when $k = 0.1$ and $a_2 = 0.3$. Each subfigure represents the average regression coefficients obtained for four distinct values of a_1 : $a_1 = 0.1$ (a), $a_1 = 0.2$ (b), $a_1 = 0.3$ (c), and $a_1 = 0.4$ (d), with the error bars corresponding to the standard deviation of the regression coefficients.

response latency has the potential to induce spurious oscillations or unstable behaviors in specific coupling ranges.

This behavior is observed in the synchronization heatmaps shown in Fig. 16(c), where some fluctuations appear in $k \in [0.15, 0.16]$ and $k \in [0.18, 0.2]$. These fluctuations are attributed to undesirable oscillations within the coupling loop, a phenomenon induced by the noninstantaneous response of the operational amplifiers. To minimize these undesirable effects, a simple and efficient solution is implemented. This consists of adding a small filter capacitor C_4 between the inverter and noninverter pins of the operational amplifier of the x variable.

The insertion of a capacitor C_4 directly between the positive and negative terminals of an operational amplifier creates a link between two high-impedance nodes with an impedance that decreases with increasing frequency (i.e., $1/s$, where s is

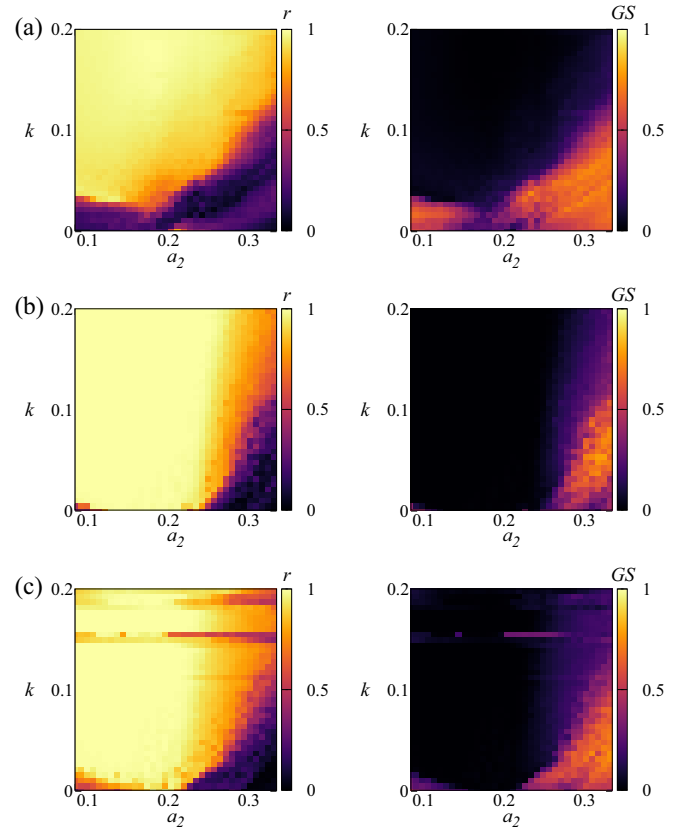


FIG. 16. Heatmaps showing the phase synchronization value r and the generalized synchronization index Δr (denoted here as GS) computed for $C = 33$ nF (a), $C = 220$ nF (b), and $C = 220$ nF (c) with no filter capacitor. Parameter a_1 is set to 0.1 here.

the complex frequency variable in the Laplace analysis). At the direct current (DC) level, the capacitor acts as an open circuit, leaving the original resistor-defined gains G_P and G_N , representing the open-loop gains from the signal sources, to the respective positive and negative inputs of the operational amplifier unchanged.

In the case of loop transfer, the addition of C_4 makes the gain paths frequency-dependent, with $G_P/[1 + (s/\omega_c)]$ and $G_N/[(1 + (s/\omega_c))]$, where $\omega_c = 1/R_x C_4$ and R_x is the resistance from an input pin to the corresponding source. This setup creates a pole in ω_c , thereby reducing the loop gain beyond a specific frequency. In the absence of a capacitor, the dynamics include a series of fast activities that appear as isolated but large-amplitude transients, reminiscent of interictal activity in an electroencephalogram. This instability completely vanishes after the installation of the additional filter capacitor. Analysis of the coupling circuit shown in Fig. 21(a) (Appendix B) shows that the instability can be attributed to a mismatch between the cross-coupling gain and internal feedback gain, which may originate from component tolerances.

The resultant system dynamics are more stable, in accordance with theoretical expectations. The experimental results obtained following this modification demonstrated the almost complete disappearance of the observed irregularities [Fig. 16(b)], thereby confirming the effectiveness of this

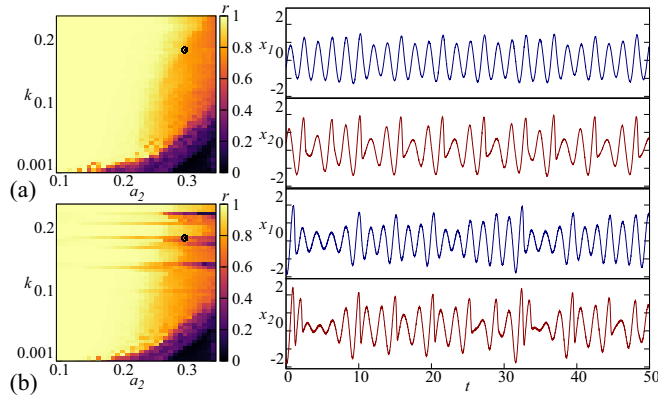


FIG. 17. Heatmaps of phase synchronization value in two experimental settings and time series of the x_1 and x_2 components. Panel (a) shows the setting with the coupling capacitor, and panel (b) shows the setting without the coupling capacitor. Both settings correspond to the circuit capacitance of 100 nF and $a_1 = 0.1$. The time series shown in the right-side plots correspond to points of the heatmaps marked with the black circle, namely, $k = 0.197$ and $a_2 = 0.3$.

approach. Figure 17 shows the heatmaps of phase coherence in two experimental settings and time series of the x_1 and x_2 components for another circuit capacitance, namely, $C = 100$ nF. Panel (a) shows the settings with the coupling capacitor, and panel (b) the setting without the coupling capacitor. The time series shown in the right-side plots correspond to the points of the heatmaps marked with black circles, namely, $k = 0.197$ and $a_2 = 0.3$. The same effects were observed in the absence of the filter capacitance. In addition, by using the corresponding parameter settings for the horizontal streaks shown in Fig. 17 (b), self-sustained oscillation of the x variables at a 150 kHz could ensue.

Another manifestation of nonidealities in analog implementation is the sensitivity of the system dynamics to time scaling, which is directly affected by the values of the integration capacitors. By adjusting the capacitor value (e.g., from 33 nF to 2.2 μ F), the chaotic dynamics of the system can be slowed or accelerated with a direct impact on the synchronization [Figs. 16(a) and 16(b)]. Although predictable, this phenomenon has rarely been highlighted in experimental studies.

Theoretically, in an ideal mathematical model, a chaotic system should exhibit invariance under temporal rescaling; slowing down or speeding up the dynamics should not alter the qualitative behavior of the system. That is, the attractors, synchronization regimes, and entropy characteristics should remain unchanged because the system explores the same phase-space trajectory at a different rate. However, this temporal invariance does not hold perfectly in real-world electronic circuits, where components have limited bandwidths and nonideal frequency responses. This process is illustrated in Fig. 18, which shows the entropy heatmaps (H_1) of the experimental implementations of System 1 plotted over the coupling strength k and parameter a_2 for different values of the integration capacitors, ranging from 33 nF to 2.2 μ F. These capacitors determine the system's time constants and thus affect how fast or slow the dynamics evolve.

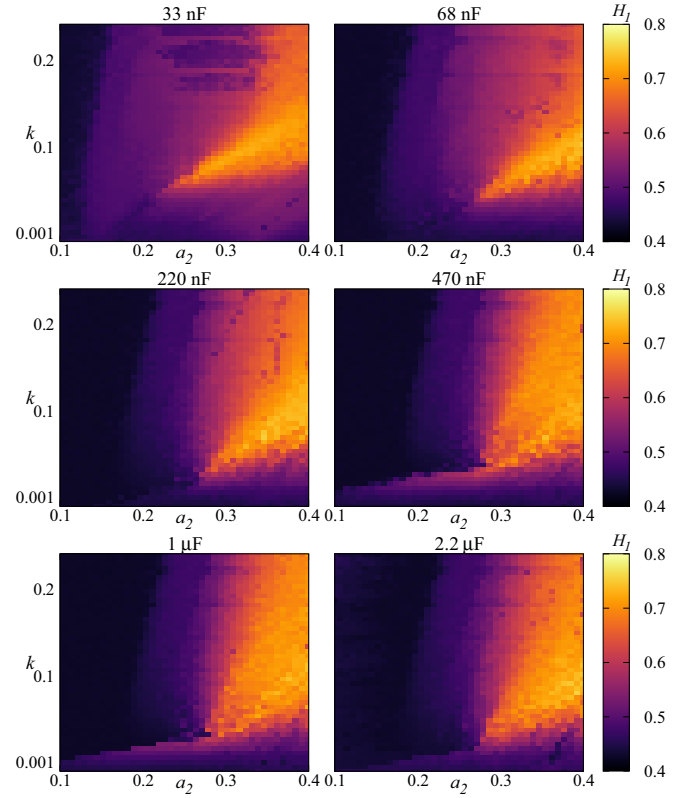


FIG. 18. Heatmaps of entropies of experimental implementations of System 1 with different values of capacitor. Parameter a_1 is here set to 0.1.

Although the general structure of the entropy regions remains broadly similar across the different panels, notable differences emerge, particularly for the smallest and largest capacitor values. In particular, the high-entropy region appears to cover a relatively narrow range of coupling parameters k when operating at the highest frequency. This region extends considerably when the timescale of the oscillation is low. By contrast, the corresponding range of parameters a_2 remains more stable. For a low capacitance (e.g., 33 nF), the system operates at high frequencies, pushing the analog multipliers and op-amps closer to their bandwidth limits. This results in signal distortion and degradation of the subtler features of chaotic dynamics. Conversely, at high capacitance values (e.g., 2.2 μ F), the system slows down considerably, and while the components track the signals better, low-frequency effects, such as $1/f$ noise and DC offset integration, may interfere with the dynamics.

V. DISCUSSION

A. Methodological uniqueness: Empirical functional relationship reconstruction

One main finding of this study is the establishment of an empirical approach aimed at rebuilding the functional interaction between two bidirectionally coupled chaotic systems based on time-series data directly. Traditional generalized synchronization detection methods, such as the auxiliary system test [9,31,59] and Lyapunov exponent

analysis [10,11,17], often experience significant limitations, particularly when dealing with experimental data series and bidirectional coupling cases. The auxiliary system test is inherently insufficient in the case of mutual coupling, while extraction of the Lyapunov spectrum based on experimental data faces significant difficulty due to a requirement for extensive, low-noise time series and proper models [21]. In addition, imprecision in the estimation of the Lyapunov exponent can often make the detection of generalized synchronization difficult.

The approach taken resolves the above issues with a simpler method: empirically finding the functional relation Φ through regression analysis. This approach includes a mixture of delay coordinates, derivatives thereof, nonlinear transforms, and a variety of other features in an attempt to predict the current condition of the alternative system. By developing a universal nonlinear system with linearity in the estimated parameters, the problems inherent in function approximation are considerably reduced. In addition, the data-driven approach improves computational efficiency and interpretability.

The reconstruction performance is evaluated through the proxy time series, which exhibit a marked increment in phase locking over the raw unprocessed signals. This finding indicates that our model effectively maintains the intrinsic coupling, revealed through a clear expression of coherence. For the resulting mappings derived, the transformation outputs clearly depict the phenomenon of synchronization. This provides our first empirical evidence for generalized synchronization, going beyond detection, since it defines and applies concepts of functional coupling for designing measurable estimates in synchronization.

Another benefit in our methodology is through interpretability provided in the regression coefficients. As opposed to cryptic machine learning methods [35], our linear in-parameter model provides coefficients, easy for humans to interpret (see Figs. 7 and 15). From the identified trends within these coefficients, we find indications that the present system is mostly determined through a well-defined set of Φ -state variables, providing a realization of Φ , a limiting manifold in a domain comprehensible to humans. In addition, relative stability exhibited through the coefficients for different parameter discrepancies reinforces the finding that our approach uncovers a stable, deterministic correspondence. Such transparency is difficult to realize through other methods, making our approach a union between complexity and interpretability, aiding in validation as well as illuminating the synchronization domain.

Methodologically, our empirical reconstruction is based on nonlinear time-series analysis in the classical tradition, specifically through the claim for attractor reconstructability provided by Takens's theorem [37,38]. We generalize this approach toward cross-system reconstruction, where embedded coordinates in System 1 are transferred for modeling toward predictions in System 2, and vice versa. This approach, conceptually, aligns with other tactics like convergent cross mapping [60]; yet, it provides a more explicit, constructive representation of mutual coupling, where the optimal mapping is determined directly. One main innovative feature is in adding time derivatives, in addition to absolute values, as elements for modeling toward predictions. Such

physically relevant metrics (e.g., derivatives for momentum and absolute values for amplitude envelopes) enrich the embedding space, where a linear model, in a surprising manner, can effectively summarize nonlinear, complex relationships. The “handcrafted feature mapping,” based on the physics of the problem, leads to detection of GS in challenging conditions, including situations with highly mismatched oscillators with weak coupling. Robustness of the approach has been established with successful application on data from analog circuitry, successfully revealing function relationships in the presence of tolerances in components and noise. In summary, our empirical approach for functional relationships offers a highly effective, model-agnostic tool for revealing hidden synchrony in coupled systems, simplifying analysis without a deep understanding of the basic equations for motion, giving broad experimental contexts.

B. Implications for stability and geometry of attractors

The emergence of generalized synchronization plays a crucial role in determining the stability and attractor configuration in coupled systems, consequently exerting stringent constraints on their dynamical action toward a stable synchronization manifold [9,61,62]. Even though existing postulations are toward unidirectional coupling, giving rise to generalized synchronization with asymptotic stability for the above manifold with negative transverse Lyapunov exponents [10,17,20], the exact occurrence and identification in circumstances with highly mismatched, bidirectional coupling in chaotic systems, especially through empirical measurement, pose a significant challenge.

The quantitative results provide robust support for the theoretical conjecture for the onset of generalized synchronization in these complex systems: with increasing coupling strength between the noncompatible Rössler oscillators, an independent positive Lyapunov exponent exhibits a significant decrease. This finding implies a decrease in dimensionality in the approach toward a lower-dimensional chaotic attractor, in accordance with the expected onset of generalized synchronization (Fig. 9) and consistent with earlier research suggesting convergence toward, or a negative value for, the second-largest Lyapunov exponent during the onset of generalized synchronization [11,17,20]. From a geometric perspective, an interconnectivity between the two attractors follows, since a continuous map Φ would also imply a diffeomorphic correspondence. However, evidence for a significant disparity often suggests the existence of an added, potentially fractal, mapping [9].

Notably, the current research provides empirical evidence demonstrating a persistent and continuous function mapping onto a complex space with significant mismatches, a feature hitherto never explored with similar precision through direct function approximations. By our approach, we systematically determined a specific functional correspondence under different parameter conditions. Such a finding could be verified through the relatively stable regression coefficients (see Figs. 7 and 15) as well as through systematically improved phase locking for the phase-transformation measures. Such evidence strongly implies that, in spite of significant intrinsic variation, the generalized synchronization manifold features a

very high intrinsic stability and structural organization, revealing a definite intrinsic coupling between the two oscillators in a regime with chaos.

In addition, our approach clearly revealed and highlighted the intrinsic asymmetry typical of this functional interaction: The association of System 1 with System 2 significantly differed from that of System 2 with System 1, due to the asymmetric patterns witnessed in the regressions. Such a result stands particularly relevant for nonisomorphic systems connected in a bidirectional fashion, where in general, such asymmetries are rarely put under experimental examination and quantified in a fashion favorable for immediate interpretation. Such a finding tends toward the conclusion that even for examples with a result of bidirectional coupling toward generalized synchronization, an intrinsic measurable directionality for the influence might still hold, evoking a situation in which one system plays a leading “driving” role toward the state variables for the other. By providing a visual representation, complemented, in a following analysis, based on functional representation, our research provides a unique experimental insight into the complicated geometric interdependency characteristic for two highly mismatched chaotic systems being driven toward a common attractor. Such an extensive characterization within the functional interaction considerably advances the empirical understanding for complex synchronization phenomena.

C. Influence of experimental nonideal behaviors on chaotic dynamics

In this work, the effects of nonidealities on the physical apparatus were also investigated. Two major aspects of experimental nonidealities are highlighted: (1) instabilities induced by the limited response frequency of analog amplifiers, corrected by the addition of a filter capacitor, and (2) a temporal scaling effect linked to the integration capacitance that directly influences the system dynamics. These observations emphasize the necessity of considering experimental nonidealities in the physical implementation of coupled chaotic systems. These imperfections are not merely technical in nature; rather, they offer valuable insights into the disparities between the model and the real world.

D. Conclusions

This paper presented a framework for detecting generalized synchronization in bidirectionally coupled systems, based on estimating the functional relationship between the systems using a linear regression model that includes the reconstructed phase space, time derivatives, and appropriate nonlinear transformations. This setup allowed unambiguous estimation and visualization of a correlation between the trajectories of two coupled Rössler systems (taken as a canonical example of low-dimensional chaos), as observed in models of complex systems in various fields. Regression coefficients facilitate direct visualization of the complex interdependence between systems.

The next step is to determine the generality of the findings concerning other low-dimensional chaotic systems with different geometries, such as Lorenz and double-scroll systems,

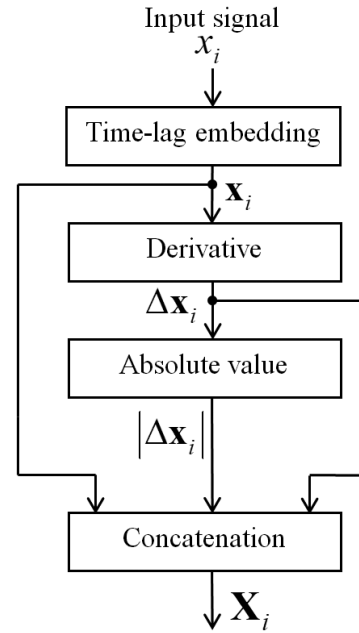


FIG. 19. Block diagram showing the computational steps to obtain the vector embedding X_i with $i \in \{1, 2\}$.

and higher-dimensional dynamics, such as those observed in physiological systems. To generalize the results, a more systematic approach for the selection of nonlinearity and other components to be included in the construction of the prediction must be applied. Furthermore, the application of the local synchronization approach proposed in Ref. [12] to the present empirical data could reveal partial or intermittent synchronization regions, providing further insight into the causative mechanisms underlying the establishment of generalized synchronization.

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The authors declare that there are no conflicts of interest.

DATA AVAILABILITY

The data that support the findings of this article are openly available [57].

APPENDIX A: FLOW CHARTS

The block diagram in Fig. 19 summarizes the computational steps to obtain the vector embedding X_i .

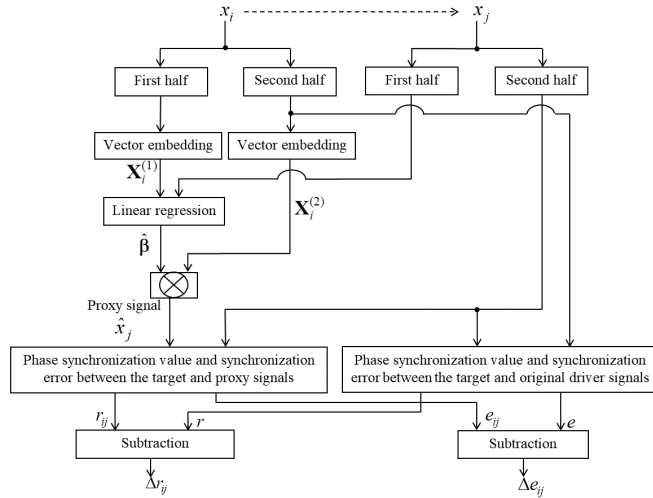


FIG. 20. Block diagram showing the computational steps to obtain the generalized synchronization measures $\Delta r_{ij} = r_{ij} - r$ and $\Delta e_{ij} = e - e_{ij}$. $\mathbf{X}_i^{(1)}$ and $\mathbf{X}_i^{(2)}$ represent the embedding vectors coming from the first and second halves of the signal taken as driver (x_i). The process for obtaining the vector embedding is detailed in Fig. 19.

The overall procedure for assessing the generalized synchronization is detailed in the block diagram shown in Fig. 20. As depicted, half of the data of x_i and x_j are used to perform a linear regression and find the regression coefficients $\hat{\beta}$. The remaining half of the data are to calculate the proxy signals and different synchronization measures. This approach was used to avoid circularity, which can lead to statistical bias.

APPENDIX B: EXPERIMENTAL IMPLEMENTATION DETAILS

In this study, an analog electronic emulator [Figs. 21(a) and 21(b)] of the system of Eq. (7) is designed and implemented to demonstrate the feasibility of the proposed method. It consists of several operational amplifiers (U1–U3, U5, and U6) of type TL082 (ST Microelectronics SPA, Agrate Brianza MI, Italy), several resistors and capacitors, and a four-quadrant analog multiplier (U4) of type AD633 (Analog Devices Inc., Wilmington, MA, USA). The power-supply voltage was set to ± 15 V. Digitally controlled relays were used to reset the voltage of the integrated capacitors before each acquisition to ensure a known initial condition. (x , y , and z) represent the variables of one system, and ($\bar{x}$, $\bar{y}$, and $\bar{z}$) represent those of the other. The two systems were structurally symmetrical.

Each uncoupled analog circuit emulates the dynamics of the classical Rössler system and is coupled through the coupling strength k , which is adjustable via the variable resistance $RV1$. As previously mentioned, the variable x was chosen for coupling because selecting y would further increase the number of digital potentiometers necessary owing to interactions with the adjustment of parameters (a , k). AD7376-type potentiometers were used to control the resistors, shown as variable resistors, that set both the variable a and the coupling coefficient k in both directions. In addition, coupling may be enabled/disabled via another relay. Applying Kirchhoff's laws to the entire designed coupled analog Rössler circuit, we

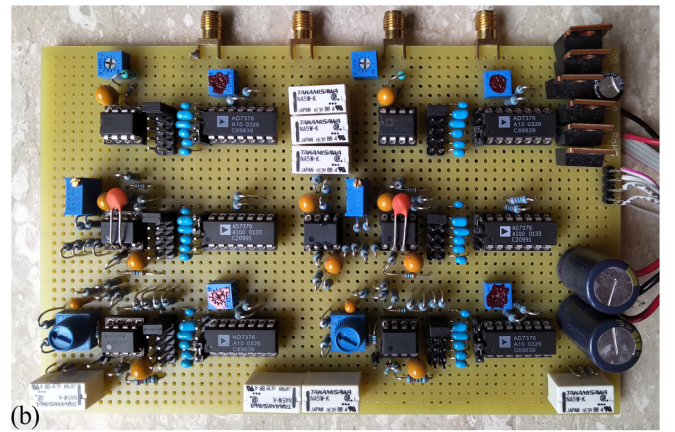
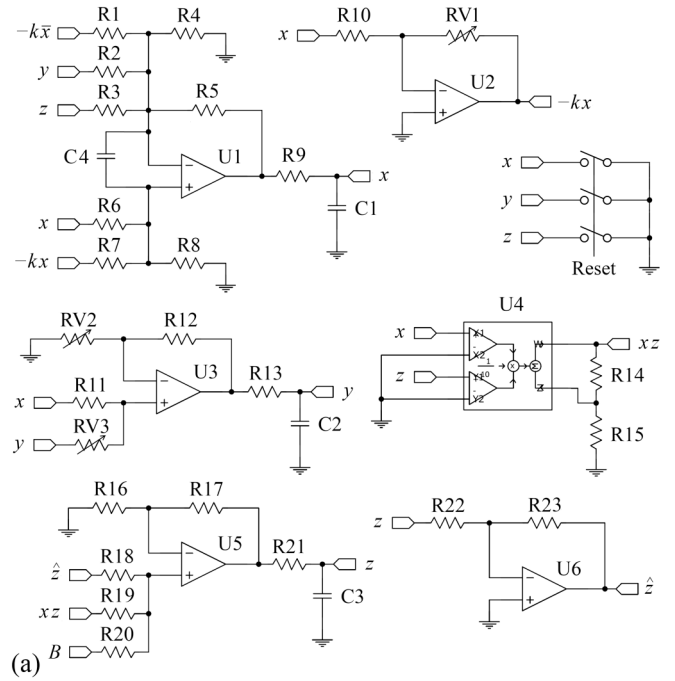


FIG. 21. (a) Analog electronic implementation of two bidirectionally coupled Rössler systems designed to experimentally investigate generalized synchronization and (b) picture of the physical realization. The circuit includes integrators, analog multipliers, and operational amplifiers to simulate the dynamics of each oscillator and their cross coupling. The adjustable resistor $RV1$ enables tuning of the coupling strength k , and $RV2$ and $RV3$ tuned a . Analog multipliers (U4) are used to generate the nonlinear terms xz . Capacitors $C1$ – $C3$ define the integration time constants. A reset switch is also included to initialize the state variables. The component values used for experiments are provided in the text.

obtain the following state equations:

$$\begin{aligned} \frac{dv_{x_1}}{dt} &= \frac{-\frac{R_5}{R_2} v_{y_1} - \frac{R_5}{R_3} v_{z_1} + \left(\frac{R_5}{R_6} - 1\right) v_{x_1} + k \left(\frac{R_5}{R_1} v_{x_2} - \frac{R_5}{R_7} v_{x_1}\right)}{C_1 R_9}, \\ \frac{dv_{y_1}}{dt} &= \frac{\frac{R_{12}}{R_{11}} v_{x_1} + \left(\frac{R_{12}}{RV_3} - 1\right) v_{y_1}}{C_2 R_{13}}, \\ \frac{dv_{z_1}}{dt} &= \frac{\frac{R_{17}}{R_{18}} v_{z_1} + \frac{R_{17}}{R_{19}} v_{x_1 z_1} + \frac{R_{17}}{R_{20}} v_B - v_{z_1}}{C_3 R_{21}}, \end{aligned}$$

$$\begin{aligned}
\frac{dv_{x_2}}{dt} &= \frac{-\frac{R_5}{R_2}v_{y_2} - \frac{R_5}{R_3}v_{z_2} + (\frac{R_5}{R_6} - 1)v_{x_2} + k(\frac{R_5}{R_1}v_{x_1} - \frac{R_5}{R_7}v_{x_2})}{C_1 R_9}, \\
\frac{dv_{y_2}}{dt} &= \frac{\frac{R_{12}}{R_{11}}v_{x_2} + (\frac{R_{12}}{RV_3^{(2)}} - 1)v_{y_2}}{C_2 R_{13}}, \\
\frac{dv_{z_2}}{dt} &= \frac{\frac{R_{17}}{R_{18}}v_{z_2} + \frac{R_{17}}{R_{19}}v_{x_2 z_2} + \frac{R_{17}}{R_{20}}v_B - v_{z_2}}{C_3 R_{21}}, \quad (B1)
\end{aligned}$$

with $k = RV_1/R_{10}$, $v_{z_2} = -(R_{23}/R_{22})v_{z_1}$, $v_{z_1} = -(R_{23}/R_{22})v_{z_1}$, $v_{x_1 z_1} = [(R_{14} + R_{15})/10R_{14}]v_{x_1}v_{z_1}$, and $v_{x_2 z_2} = [(R_{14} + R_{15})/10R_{14}]v_{x_2}v_{z_2}$. Here, v_{x_1} , v_{y_1} , and v_{z_1} (resp. v_{x_2} , v_{y_2} , and v_{z_2}) are the voltages across the capacitors C_1 , C_2 , and C_3 , respectively. As detailed below, C_4 is a filter capacitor used to correct the instabilities induced by the limited response frequency of analog amplifiers. System (B1) is equivalent to the dimensionless system (7) when the following parameter and variable changes are performed: $t = RC\tau$, $R = R_9 = R_{13} = R_{21}$, $C = C_1 = C_2 = C_3$, $v_{x_1} = x_1 V_0$, $v_{x_2} = x_2 V_0$, $v_{y_1} = y_1 V_0$, $v_{y_2} = y_2 V_0$, $v_{z_1} = z_1 V_0$, $v_{z_2} = z_2 V_0$, $a_1 = (R_{12}/RV_3^{(1)}) - 1$, $a_2 = (R_{12}/RV_3^{(2)}) - 1$, $c = 1 + (R_{23}/R_{22})$, $b = v_B/V_0$, $R_1 = R_2 = R_3 = R_5 = R_6 = R_7$, $R_{11} = R_{12}$, and $R_{17} = R_{18} = R_{20}$.

where τ is the dimensionless time and $RV_3^{(1)}$ and $RV_3^{(2)}$ correspond to RV_3 in System 1 and System 2, which serves to create a mismatch between the two coupled oscillators as we did in Sec. III with a_1 and a_2 . Owing to the limited voltage swing available to the operational amplifiers, to limit saturation, the system variables are rescaled by considering $V_0 = 1V/5$ [63]. We set $R_{23} = 47\text{ k}\Omega$ and $R_{22} = 10\text{ k}\Omega$ to obtain $c = 5.7$ and $v_B = 40\text{ mV}$ to get $b = 0.2$. The rest of component values used for experiments are $R_9 = 1\text{ k}\Omega$, $R_{13} = 1\text{ k}\Omega$, $R_{21} = 1\text{ k}\Omega$, $R_1 = 100\text{ k}\Omega$, $R_2 = 100\text{ k}\Omega$, $R_3 = 100\text{ k}\Omega$, $R_4 = 100\text{ k}\Omega$, $R_5 = 100\text{ k}\Omega$, $R_6 = 100\text{ k}\Omega$, $R_7 = 100\text{ k}\Omega$, $R_8 = 100\text{ k}\Omega$, $R_{17} = 100\text{ k}\Omega$, $R_{16} = 16.7\text{ k}\Omega$, $R_{18} = 100\text{ k}\Omega$, $R_{10} = 300\text{ k}\Omega$, $R_{11} = 50\text{ k}\Omega$, $R_{12} = 50\text{ k}\Omega$, $R_{19} = 20\text{ k}\Omega$, and $R_{20} = 100\text{ k}\Omega$. Different integration capacitors C were tried, 33 nF, 68 nF, 100 nF, 220 nF, 470 nF, 1 μF , and 2.2 μF , to show the effects of time scaling on the dynamics of the coupled oscillators (see Sec. IV D for more detailed explanations). Unless otherwise indicated, the analyses were performed using an integration capacitor $C = 220\text{ nF}$ and the filter capacitor was set to $C_4 = 1\text{ nF}$.

APPENDIX C: FREQUENCY SPECTRA ANALYSIS

Another manifestation of nonideality (imperfect time scaling) is presented in Fig. 22. This figure shows graphs, on a log-log scale, of the magnitudes of the frequency spectra

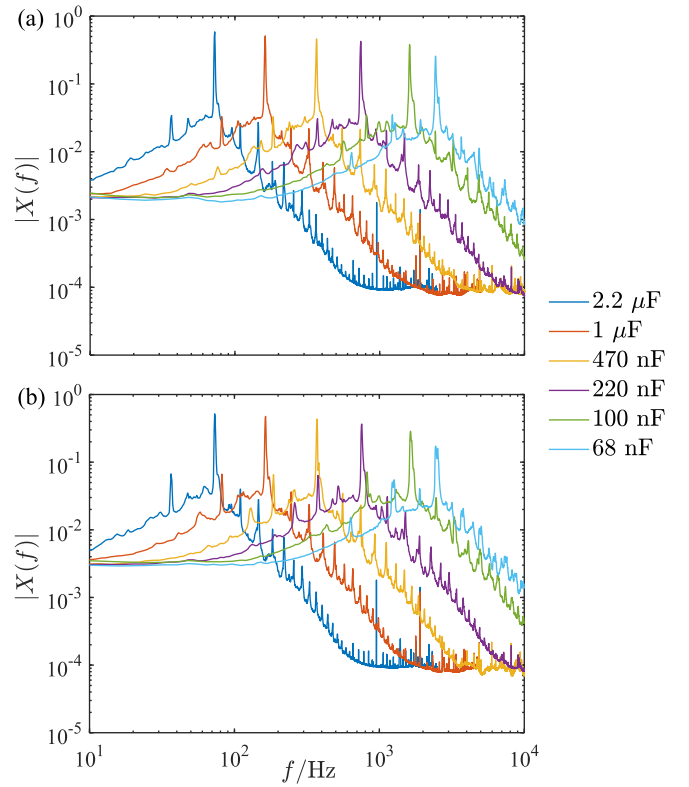


FIG. 22. Plots, in a log-log scale, of the frequency spectra, averaged over the entire 48×48 trajectories, both nodes, calculated for several capacitances ranging from 68 nF to 2.2 μF under two different conditions: panel (a) $a_1 = 0.1$ and panel (b) $a_1 = 0.2$.

$X(f)$, averaged over the entire 48×48 trajectories, at both nodes (x_1 and x_2), and calculated for several capacitances ranging from 68 nF to 2.2 μF under two different conditions: panel (a) $a_1 = 0.1$ and panel (b) $a_1 = 0.2$. From this figure, for the two considered values of $a_1 \in \{0.1, 0.2\}$, it can be seen that the general shape of the frequency spectrum is similar across all capacitances considered; however, the rescaling does not yield a perfect overlap. Another observation is that, as capacitance decreases, spectral peaks expectably shift to higher frequencies. As a result, the frequencies of the predominant peaks increase as the capacitance decreases. When a_1 is set to 0.1, the frequencies of the predominant peaks are 72.5, 162, 366, 740, 1608, and 2431 Hz for the capacitances 2.2 μF , 1 μF , 470 nF, 220 nF, 100 nF, and 68 nF, respectively. Similarly, when $a_1 = 0.2$, the values of the predominant amplitudes are 73, 164, 372, 756, 1639, and 2439 Hz, for the capacitances 2.2 μF , 1 μF , 470 nF, 220 nF, 100 nF, and 68 nF, respectively.

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