

Role of critical sets in the permeability of random fracture networks: A percolation scaling perspective

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Fracture networks in the subsurface often comprise multiple fracture sets, introducing anisotropy and heterogeneity due to wide variations in fracture orientations, sizes, and conductivities. In this study, we develop a finite-size scaling (FSS) hypothesis, rooted in percolation theory, to highlight the role of critical sets—those that conduct the majority of flow—in controlling the permeability of random fracture networks. Definitions of the FSS input quantities are provided, incorporating nondimensionalizations to account for the complexity of the networks, based on the statistical properties of the fracture parameters across all sets. For cases where critical sets are present, we emphasize their dominant role and propose modified definitions based solely on the parameters of these critical sets, in line with the idea that the density of critical sets determines the magnitude of permeability. FSS results demonstrate that the hypothesis serves as a universal model for characterizing anisotropic permeability in complex fracture networks, provided that the influence of critical sets is properly addressed. This finding can be extended to other complex systems where conductive objects can be categorized into distinct sets.

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I. INTRODUCTION

Fractures in the subsurface often intersect and form intricate, interconnected networks [1]. Characterizing the properties of these complex systems is challenging due to their inherent anisotropy and heterogeneity, which arise from the wide variability in fracture parameters such as orientation, length, aperture, and hydraulic conductivity. Accurately capturing these features is crucial, as they directly influence large-scale network properties like permeability—a key factor in various geophysical and environmental processes, including hydrocarbon extraction [2], geothermal energy production [3], and contaminant transport [4,5]. While analytical models [6,7] have been developed to describe such networks, they typically rely on simplifying assumptions, such as fracture network structures are known, sufficient network density or the existence of a representative elementary volume, which may not hold in significantly different cases.

Percolation theory [8–11] was developed to describe phase transitions in systems where the density of conductive elements increases, with applications in subsurface flow and transport [4,12], forest fires [13], epidemic spread [14], and

lake formation [15]. Scaling laws [16] in percolation theory establish relationships between the object density ρ'_o and any hydrodynamical or electrodynamical macroscopic behavior $\mathcal{A}'$. These relationships are typically expressed as a power-law proportionality: $\mathcal{A}' \propto (\rho'_o - \rho'_{o,c})^B$, where $\rho'_{o,c}$ is the critical value of ρ'_o (also called the percolation threshold) and B is the critical exponent associated with $\mathcal{A}'$. These approaches do not require detailed knowledge of network structures or rely on the aforementioned assumptions; instead, they only depend on the statistical distributions of conductive elements, which are typically accessible in practice. The finite-size scaling (FSS) hypothesis—derived from scaling laws [17,18]—has been successfully applied to various problems, including the runout distances of landslides [19] and the connectivity [20–23], permeability [24–27], and dispersion [28] of two- and three-dimensional (3D) random fracture networks. Compared to scaling laws, the FSS hypothesis directly provides an explicit functional relationship between $\mathcal{A}'$, ρ'_o , and the system size $\mathcal{L}'$, namely, $\mathcal{A}' = f(\rho'_o, \mathcal{L}')$. Thus, the FSS hypothesis offers a promising foundation for predictive modeling of complex system behaviors.

Previous studies applying percolation theory to fracture networks typically generate fracture-network models in which the orientation, size, and conductivity of fractures follow a single distribution type. However, in real fractured rock masses, multiple fracture sets can coexist in the subsurface, each characterized by distinct mean orientations, sizes, and conductivity [29–33]. Multiple fracture sets cause fracture networks to be strongly anisotropic and heterogeneous systems. This variability leads to significantly greater anisotropy

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and heterogeneity in the system compared to the fracture networks previously considered. Moreover, the presence of critical fracture sets significantly influences the macroscopic behavior of geological fracture media, affecting processes ranging from hydrogeology [34] to geomechanics [35]. In this study, we highlight the role of critical sets in random fracture networks based on the FSS hypothesis. Furthermore, we demonstrate that by properly defining the input parameters of the FSS, taking into account the dominant role of critical sets, the FSS hypothesis can serve as a universal model to describe anisotropic permeability in complex fracture networks.

This paper is organized as follows. In Sec. II, the generation of fracture networks and the flow simulation methodology are presented, along with the simulation scenarios. We assume that the permeability is primarily dominated by the spanning cluster formed by fractures in the critical sets, and that the density level—serving as the percolation parameter—together with other relevant dimensionless quantities, should be determined from the fracture parameters of these critical sets. In Sec. III, demonstration examples (see also Appendix) validate this assumption, and the results show that by accounting for the role of the critical set, the universality of the FSS hypothesis is confirmed. Finally, Sec. IV provides concluding remarks.

II. METHODS

A. Generation of fracture networks and flow simulation

3D random fracture networks are generated using cuDFNs [36], a GPU-parallelized code designed for generating fracture networks and simulating steady-state flow. Fractures are modeled as 3D squares. Previous work [37] has shown that, once appropriate percolation parameters are defined, the fracture shape has minimal influence. Fracture orientations follow a Fisher distribution for a single fracture set:

$$f_F(\theta, \phi) = \frac{1}{2\pi} \frac{\kappa \sin \theta}{\sinh \kappa} \cosh[\kappa \{\cos \theta_0 \cos \theta + \sin \theta_0 \sin \theta \cos(\phi - \phi_0)\}], \quad (1)$$

where $\phi \in (0, 2\pi]$ and $\theta \in [0, \pi/2]$ denote the azimuthal and polar angles, respectively, of the unit normal vector to a fracture. The pair (ϕ_0, θ_0) represents the mean orientation of the distribution, and κ , known as the Fisher constant, is the concentration parameter. The fracture sizes, i.e., radii (R) of the circumscribed circles, follow four distribution patterns for one single set: (1) power law:

$$f_P(R) = (1 - \alpha) \frac{R^{-\alpha}}{R_M^{1-\alpha} - R_m^{1-\alpha}}, \quad R_m < R < R_M, \quad (2)$$

where α is an exponent, and R_m and R_M are the minimum and maximum values, respectively; (2) lognormal:

$$f_L(R) = \frac{1}{R\sqrt{2\pi}v_L} e^{-(\ln R - m_L)^2/(2v_L^2)}, \quad R > 0, \quad (3)$$

where m_L is the mean of logarithmic values and v_L is the standard deviation of logarithmic values; (3) uniform:

$$f_U(R) = \frac{1}{R_M - R_m}, \quad R_m < R < R_M; \quad (4)$$

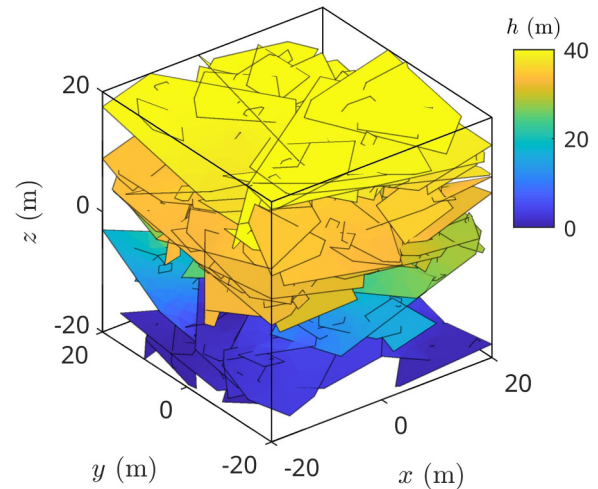


FIG. 1. A random fracture network consisting of two fracture sets. The mean azimuth and polar angles of the two orientation distributions are $(0, 0)$ and $(0, \pi/2)$, respectively, with concentration parameters κ (in Fisher distributions) of 15 and 22. The mean flow direction is along the z axis. The hydraulic head field h is visualized using a colormap.

and (4) monosize $f_M(R) = C$ (i.e., the sizes of all fractures are the same).

The cubic law [5,38,39] is used to characterize fracture conductivity k_f and aperture b_f , following a power-law relationship with fracture size:

$$k_f = \frac{b_f^3}{12} = \frac{(\gamma R^\beta)^3}{12}, \quad (5)$$

where γ and β are dimensionless constants and exponents, respectively. Steady-state flow through the fracture networks is solved three times independently, each time imposing a unit hydraulic head gradient ($\nabla h = 1$) along the x , y , or z direction, respectively, using the parallel-plate model [40], while impermeable boundary conditions are applied to the other domain surfaces. The governing equations are

$$\begin{cases} \mathbf{q} = -\frac{\rho_f g}{\mu} k_f \nabla h, \\ \nabla \cdot \mathbf{q} = 0, \end{cases} \quad (6)$$

where $\mathbf{q}$ is the local volumetric flow rate per unit width, h the hydraulic head, ρ_f the fluid density, g the gravitational acceleration, and μ the viscosity. Thus, the Darcian permeability along the three principal directions— $\mathcal{K}_1$, $\mathcal{K}_2$, and $\mathcal{K}_3$ —can be obtained for a network based on the volumetric flow rate through it. An example of a random fracture network is shown in Fig. 1. Besides, the governing equations are solved by a mixed hybrid finite element method [41] in cuDFNs, and the value of $\frac{\rho_f g}{\mu}$ is fixed at one throughout this study.

The designed simulation scenarios are summarized in Table I, including the domain sizes ($\mathcal{L}_m$) and the selected distribution types for fracture orientations, sizes, and conductivities. For each scenario, one to three fracture sets—differing or similar in density, conductivity, size, and/or orientation—are simultaneously generated within the networks, and the fracture density is progressively increased. For each chosen

TABLE I. Simulation scenarios. The parameter W denotes the density weight of each fracture set; $\mathcal{L}_m$ represents the domain size.

No.	W	Fracture orientation	Fracture size (m)	γ	β	$\mathcal{L}_m$ (m)	Marker
1	1	$(\phi_0, \theta_0) = (0, 0), \kappa = 15$	$f_M, R = 7.5$	2×10^{-7}	0.2	75	+
2	1	$(\phi_0, \theta_0) = (0, 0), \kappa = 30$	$f_P, \alpha = 1.8, (R_m, R_M) = (1, 15)$	8×10^{-5}	0.4	80	$\diamond$
3	1	$(\phi_0, \theta_0) = (0, 0), \kappa = 20$	$f_L, m_L = 8.5, v_L = 3.5, (R_m, R_M) = (1, 15)$	3×10^{-6}	0.3	90	$\square$
4	1	$(\phi_0, \theta_0) = (0, 0), \kappa = 22$	$f_U, (R_m, R_M) = (1, 15)$	1×10^{-8}	0.22	120	*
	0.5	$(\phi_0, \theta_0) = (0, 0), \kappa = 0$	$f_M, R = 1$	3×10^{-5}	0.22		
5	0.5	$(\phi_0, \theta_0) = (0, 0), \kappa = 0$	$f_M, R = 1$	1.5×10^{-9}	0.30	5, 7, 9, 11, 13	$\bigcirc$
	0.3	$(\phi_0, \theta_0) = (0, 90), \kappa = 22$	$f_P, \alpha = 2.4, (R_m, R_M) = (1, 25)$	1.2×10^{-8}	0.24		
6	0.7	$(\phi_0, \theta_0) = (0, 0), \kappa = 14$	$f_P, \alpha = 1.3, (R_m, R_M) = (1, 50)$	3.3×10^{-7}	0.31	250	$\triangle$
	0.3	$(\phi_0, \theta_0) = (0, 0), \kappa = 30$	$f_P, \alpha = 1.5, (R_m, R_M) = (1, 100)$	1.7×10^{-5}	0.23		
7	0.7	$(\phi_0, \theta_0) = (0, 90), \kappa = 15$	$f_L, m_L = 8.5, v_L = 5.5, (R_m, R_M) = (1, 50)$	2.2×10^{-6}	0.12	200	∇
	0.4	$(\phi_0, \theta_0) = (0, 0), \kappa = 25$	$f_P, \alpha = 1.5, (R_m, R_M) = (1, 100)$	1.7×10^{-5}	0.13		
8	0.6	$(\phi_0, \theta_0) = (0, 90), \kappa = 45$	$f_P, \alpha = 2.5, (R_m, R_M) = (1, 30)$	1.6×10^{-5}	0.12	300	$\times$
	0.5	$(\phi_0, \theta_0) = (0, 0), \kappa = 0$	$f_M, R = 1.5$	2.1×10^{-5}	0.12		
9	0.5	$(\phi_0, \theta_0) = (0, 0), \kappa = 0$	$f_M, R = 1.5$	2.1×10^{-5}	0.31	18	$\triangleleft$
	1/3	$(\phi_0, \theta_0) = (0, 0), \kappa = 0$	$f_M, R = 1.5$	2.1×10^{-5}	0.24		
	1/3	$(\phi_0, \theta_0) = (0, 0), \kappa = 0$	$f_M, R = 1.5$	2.1×10^{-5}	0.31		
10	1/3	$(\phi_0, \theta_0) = (0, 0), \kappa = 0$	$f_M, R = 1.5$	2.1×10^{-5}	0.43	17	$\triangleright$
	0.4	$(\phi_0, \theta_0) = (0, 0), \kappa = 17$	$f_P, \alpha = 1.5, (R_m, R_M) = (1, 100)$	1.7×10^{-7}	0.12		
11	0.6	$(\phi_0, \theta_0) = (0, 90), \kappa = 29$	$f_P, \alpha = 1.5, (R_m, R_M) = (1, 100)$	1.7×10^{-7}	0.14	350	$\star$

density level, 400 Monte Carlo (MC) iterations of network generation and flow simulation are conducted, and the MC-averaged permeability, e.g., $\langle \mathcal{K}_3 \rangle$, is calculated. Since (ϕ_0, θ_0) is always aligned with one of the principal axes, the MC-averaged permeability tensor reduces to a diagonal form, given by $\langle \mathcal{K} \rangle = \text{diag}(\langle \mathcal{K}_1 \rangle, \langle \mathcal{K}_2 \rangle, \langle \mathcal{K}_3 \rangle)$ [42]. This broad distribution of fracture parameters aims to verify the universality of the FSS hypotheses. Besides, Scenarios 1–4 involve a single fracture set, while Scenarios 5–8 contain multiple sets with significantly different fracture parameters. Scenarios 9–11 also involve multiple sets, but with similar fracture parameters. As will be highlighted later, the FSS quantities should be calculated differently for each situation.

B. Scaling parameters

To quantify the local connectivity of the networks for each scenario, we define the total dimensionless density ρ'_t [26,42] as

$$\rho'_t = \rho_t \cdot E[V_{\text{ex}}^t], \quad \text{with} \quad \rho_t = \sum_{i=1}^{N_s} \rho_i,$$

$$E[V_{\text{ex}}^t] = \frac{2}{\pi} \left(\sum_{i=1}^{N_s} \sum_{j=1}^{N_s} W_i W_j E[\sin \eta_{ij}] \right) \sum_{i=1}^{N_s} W_i \langle A_i \cdot P_i \rangle, \quad (7)$$

where N_s is the number of fracture sets and W_i denotes the density weight of the i th fracture set, while ρ_i represents its density. The operator $E[\cdot]$ denotes the expected value of a given property in the context of a network representation. The term $E[V_{\text{ex}}^t]$ refers to a characteristic volumetric measure of fracture networks, analogous to the excluded volume defined in Ref. [43]. The quantities A_i and P_i represent the fracture area and perimeter of the i th fracture set, respectively, and η_{ij} is the angle between the normals of two fractures belonging to the i th and j th fracture sets, respectively. For each scenario listed in Table I, approximately 30 values of ρ'_t are sampled,

ranging from $\rho'_t = 0$ up to about 3–4 times the percolation threshold $\rho'_c \approx 2.31$ of 3D fracture networks [37].

The FSS hypothesis for network permeability is written as

$$\langle \mathcal{K}' \rangle = \mathcal{L}'^{-\sigma/\nu} \mathcal{F}[(\rho' - \rho'_c) \mathcal{L}'^{1/\nu}], \quad (8)$$

where $\mathcal{F}[\cdot]$ is the FSS function, and $\mathcal{K}'$, $\mathcal{L}'$, and ρ' are dimensionless permeability, domain sizes, and densities, respectively. The exponents σ and ν are the critical exponents associated with the scaling of permeability and correlation length, respectively. Equation (8) presents the basic form of the FSS hypotheses [10].

In this study, the relevant quantities are defined according to different situations. The first case considers fracture sets with similar conductivity and size variations:

$$\langle \mathcal{K}' \rangle = \frac{\langle \mathcal{K} \rangle}{\mathcal{K}_0} \frac{\Phi}{\Psi}, \quad \mathcal{K}_0 = \frac{\sum_{i=1}^{N_s} W_i E[k_{f,i} \cdot l_i^2]}{(\sum_{i=1}^{N_s} W_i E[l_i^4])^{3/4}},$$

$$\Phi = \frac{4}{\pi} \sum_{i=1}^{N_s} \sum_{j=1}^{N_s} W_i W_j E[\sin \eta_{ij}], \quad (9)$$

$$\mathcal{L}' = \frac{\mathcal{L}_m}{\mathcal{L}_c}, \quad \mathcal{L}_c = \left(\frac{\sum_{i=1}^{N_s} W_i E[l_i^6]}{\sum_{i=1}^{N_s} W_i E[l_i^3]} \right)^{1/3},$$

where $\mathcal{K}$ is the Darcian permeability along one principal direction, which may refer to $\mathcal{K}_1$, $\mathcal{K}_2$, or $\mathcal{K}_3$, and l_i and $k_{f,i}$ are the fracture side length and conductivity belonging to i th set, respectively. The quantity $\mathcal{K}_0$ is a characteristic permeability scale that accounts for the greater contribution of larger fractures to the network permeability. The numerator captures the influence of both fracture conductivity and area. This scale has been shown to work well for networks containing only a single fracture set [26]. The parameter Φ is the ratio of excluded volumes of anisotropic and isotropic networks, and Ψ is the ratio of permeability between anisotropic and isotropic networks consisting of

infinite fractures [42]. To account for the tensorial nature of the Darcian permeability, Ψ varies with direction. If the mean orientation of the Fisher distribution aligns with the z axis, Ψ corresponds to a component of the tensor $\Psi_v = \text{diag}(\sum_{i=1}^{N_s} W_i \Psi_{1,i}, \sum_{i=1}^{N_s} W_i \Psi_{2,i}, \sum_{i=1}^{N_s} W_i \Psi_{3,i})$, where $\Psi_{1,i} = \Psi_{2,i} = \frac{3}{2}(1 + \frac{1 - \kappa_i \coth \kappa_i}{\kappa_i^2})$ and $\Psi_{3,i} = \frac{3}{\kappa_i^2}(\kappa_i \coth \kappa_i - 1)$ for i th set [42], depending on the mean flow direction. If the mean orientation aligns with another principal axis, $\Psi_{j,i} (j = 1, 2, 3)$ values can be adjusted accordingly by multiplying it with a rotation matrix. The characteristic length scale $\mathcal{L}_c$ is based on the statistical moment of l and accounts for the spread of fracture size distributions [23]. The dimensionless density ρ' in Eq. (8) can be replaced by ρ'_i in this case.

However, as will be shown, these quantities are effective for fracture networks in which the differences among fracture sets are relatively minor. In cases where certain fracture sets are significantly biased—exhibiting substantially larger sizes and/or conductivities than others—these dominant sets primarily govern the overall network behavior and must therefore be given greater emphasis. Accordingly, under the FSS hypothesis, the influence of less significant fracture sets becomes negligible, and their relative contributions can be reasonably disregarded. Thus, quantities in Eq. (8) should be computed based solely on this dominant set. We define

$$\begin{aligned} \langle \mathcal{K}'^d \rangle &= \frac{\langle \mathcal{K} \rangle}{K_0^d} \frac{\Phi^d}{\Psi^d}, \\ \mathcal{L}'^d &= \frac{\mathcal{L}_m}{\mathcal{L}_c^d}, \\ \rho'^d &= \rho^d \cdot E[V_{\text{ex}}^d], \quad \rho^d = \rho^t \cdot W_i^d, \end{aligned} \quad (10)$$

where quantities with the superscript d indicate that they are calculated based on the fracture parameters of the dominant set, rather than on weighted summations. For example, in Scenario 5, only the values of l and κ from Set 1 should be used to compute $\mathcal{L}_c$ and Ψ_v , without applying any weighting factors.

The formulation of Eq. (10) stems from the following consideration. When evaluating the permeability of fracture networks, it is crucial to identify critical fracture sets—subsets of fractures with substantially higher conductivities than others in the system. These dominant sets control fluid flow (if their density is large enough to form a spanning cluster in the network), serving as the primary pathways through the heterogeneous network. Consequently, the network's overall permeability is predominantly determined by the geometrical properties of these influential fractures, including their orientation, size, and conductivity. Thus, it is important to quantify the density level of the critical sets, which should represent the “true” percolation parameter governing flow conduction, as reflected in Eq. (10). This simplification requires a significant conductivity contrast between the critical set and other fractures. When the conductivity difference is less distinct, or when multiple sets exhibit similar conductivities and/or sizes, a more thorough analysis incorporating all fracture sets becomes necessary, as described by Eq. (9). It is also worth noting that the application of the FSS hypothesis to highlight the role of critical sets in the permeability of fracture networks

arises from the finite nature of the simulation domains, where the domain size $\mathcal{L}_m$ is typically 2–8 times larger than the maximum fracture radius. In such cases, the FSS hypothesis accounts for finite-size effects, whereas the power-law scaling proportionality discussed in Sec. I does not.

III. RESULTS AND DISCUSSION

Before presenting the results, we emphasize that the classical critical quantities, namely, $\rho'_c \approx 2.31$ [23], $\nu \approx 0.88$ [10], and $\sigma \approx 1.63$ [26], are adopted throughout this study. Note that the percolation threshold of fracture networks varies with the distribution of fracture parameters [37,44], with the variation remaining within a relatively narrow range. In our previous work, we assumed the critical quantities to be fixed (from universality) and found that these values performed well in the FSS analysis of fracture network connectivity and permeability [23,26], which was conducted for a single fracture set, regardless of variations in fracture size or orientation distributions. In contrast, the fracture networks considered here may contain multiple fracture sets, for which the classical values must be re-estimated using percolation probability tests over increasing domain sizes. Appendix demonstrates that the estimated percolation threshold differs substantially across scenarios. This arises because, in percolation probability, every individual fracture must be accounted for, irrespective of whether it belongs to the critical set. Further details are provided in Appendix. The situation differs for permeability, where the presence and properties of the critical set play the dominant role (if such a set exists). In summary, we contend that the geometric characteristics of the critical sets exert a dominant influence on the scaling behavior and should therefore serve as the primary parameters in percolation scaling analyses, which is why we propose Eq. (10) for Scenarios 5–8. Simulation data and scaling analyses further demonstrate the dominant role of critical sets in controlling the permeability of fracture networks.

We now validate the proposed definition [Eq. (10)]. As shown in Table I, Scenario 5 generates networks composed of two fracture sets that differ only in conductivity. The first fracture set has significantly higher conductivity, making it the dominant set. Figure 2 compares the FSS results for this scenario, with quantities calculated using Eqs. (9) and (10), respectively. It demonstrates that the scaling curves collapse well with Eq. (10), while the other does not. The role of the dominant set is emphasized: As the density increases, fractures from the first set form preferential paths in percolation clusters, which transport most of the flow. Even clusters containing more or fewer low-conductivity fractures have much less influence on the Darcian permeability. Thus, only the density and other parameters of fracture set 1 are considered, and ρ'^d along with other relevant quantities lead to a universal scaling curve.

Besides, in Scenario 5, the two fracture sets are identical in their fracture size distributions, orientations, and density weights, making the fracture network geometrically equivalent to one containing a single fracture set. The only difference lies in the conductivity distributions, which differ substantially between the two sets. Consequently, the percolation threshold should equal the classical value (if estimated by the

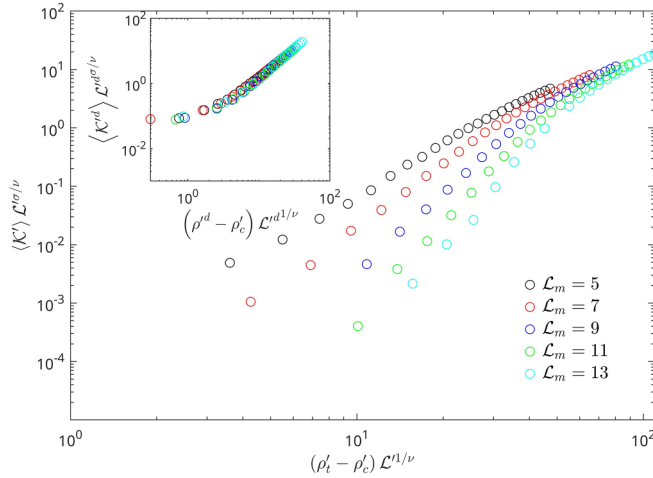


FIG. 2. FSS results for Scenario 5 using weighted summations and dominant set parameters (inset plot). The permeability values shown are directional, measured along the z axis, i.e., dimensionless $\mathcal{K}_3$.

standard percolation probability test as shown in Appendix). Using the classical value, the poor FSS results obtained with the weighted summation parameters [Eq. (9)], together with the successful collapse of the FSS curves based on Eq. (10), indicate that the critical set governs the permeability of fracture networks, even in the presence of another geometrically similar fracture set. This further highlights the necessity of considering only the parameters of the critical sets in FSS analyses. We next show that this applies in additional cases. For example, in Scenarios 6 and 7, the fracture sets with larger fracture sizes exhibit much higher conductivity. In Scenario 8, although the conductivity-related constants and exponents (i.e., γ and β) are similar, the larger fracture sizes still lead to higher conductivity because of their size dependence. Note that fractures with larger sizes generally exhibit higher conductivity [45]. In Scenarios 9–11, the fracture sets are similar in all respects, so these fracture networks can be regarded either as having no distinct critical sets or as having all fracture sets critical, with their scaling parameters calculated through a weighted summation as in Eq. (9).

The MC-averaged Darcian permeability $\mathcal{K}$ for all scenarios is shown in Fig. 3(a). Significant variations are observed, reflecting the large differences in fracture parameters among the scenarios. Even for the same fracture network scenario, the computed values of $\langle \mathcal{K} \rangle$ vary with direction due to the tensorial nature of permeability. After scaling by Ψ (note that for Scenarios 5–8 and Scenarios 9–11, Ψ^d and Ψ are used, respectively), the permeability-density curves for each scenario gradually converge, as shown in Fig. 3(b). This convergence suggests that Ψ is an appropriate factor to account for permeability anisotropy. The impact of strong heterogeneity in fracture conductivity and other parameters is incorporated through characteristic permeability scales. Figures 3(c) and 3(d) present the FSS results computed using only Eq. (9) and using both Eqs. (9) and (10), respectively. Figure 3(d) demonstrates a much better collapse of the FSS curves at large fracture densities compared to Fig. 3(c). This comparison clearly shows that, when critical fracture sets exist, permeability is primarily controlled by these sets—an important observation that should be taken into account in FSS analysis. In such cases, the “weaker” fracture sets can be neglected, and only the parameters of the critical sets should receive significant attention. For sparse networks, the FSS curves do not collapse in either Fig. 3(c) or Fig. 3(d), because when the dimensionless density approaches ρ'_c , the correlation length becomes very large, and the domain size becomes the dominant scale. Consequently, the permeability exhibits strong scale dependence, varying among different scenarios.

So far, we have used FSS curves as a reference to highlight the importance of critical sets in fracture networks. Specifically, defining scaling parameters solely based on the critical sets results in the corresponding FSS curves collapsing with others [e.g., Figs. 3(c) and 3(d)]. This indicates that only the density and other relevant geometrical and hydrodynamical parameters—when used to nondimensionalize the input quantities in the FSS hypothesis—can yield a universal FSS model. However, an open question remains: When should Eq. (9) be used versus Eq. (10), or more precisely, how different must the distribution parameters (particularly hydrodynamical characteristics) be for a fracture set to be considered distinct and require Eq. (10)? This question is not addressed in the present study. Instead, we generate highly contrasting cases—

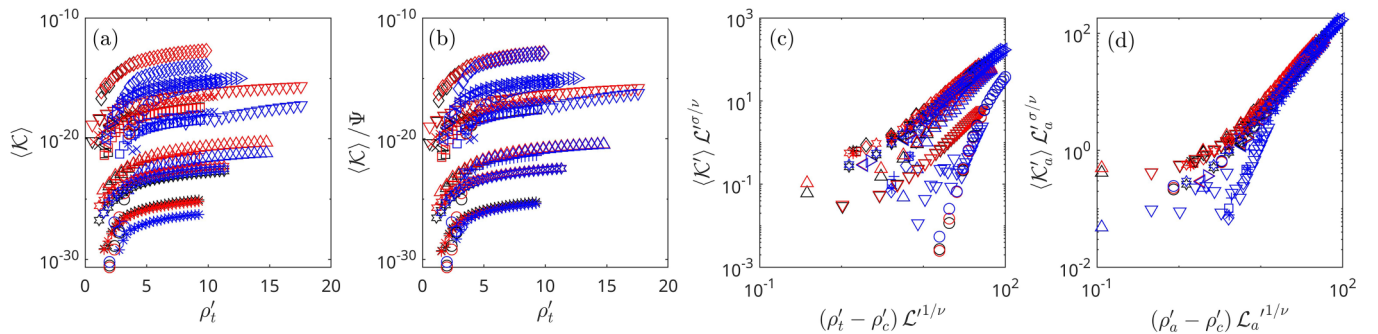


FIG. 3. Permeability and FSS results across all scenarios. Different markers represent the various scenarios listed in Table I. Colors indicate the direction of permeability: black for $\mathcal{K}_1$, red for $\mathcal{K}_2$, and blue for $\mathcal{K}_3$. (a) MC-averaged Darcy permeability vs dimensionless density. (b) MC-averaged Darcy permeability scaled by Ψ vs dimensionless density. (c) FSS results computed using Eq. (9) only. (d) FSS results computed using Eq. (9) for Scenarios 9–10, and Eq. (10) for Scenarios 5–8. The subscript a in panel (d) indicates that the quantities are derived using one of the two equations.

for Scenarios 5–8, either the conductivity-related constant γ or the fracture size distribution differs substantially between fracture sets, by at least one order of magnitude for γ —to demonstrate the dominant role of critical sets. A potential direction for future research is to establish clear criteria for selecting between these equations or even to develop a universal definition that encompasses both situations.

IV. CONCLUSION

The present study identifies the role of critical sets within a type of complex system—fracture networks. The FSS analysis is conducted. Definitions for the input quantities, incorporating proper nondimensionalization and scaling, are introduced to account for the influence of varying properties within these complex systems. When critical sets are present, only the parameters of these sets are considered, whereas for cases where all fracture sets are “competent,” summation-weighted calculations are used. These definitions successfully lead to the collapse of the FSS curves, demonstrating that the FSS hypothesis provides a universal function for describing the relationship between fracture parameters and anisotropic permeability, even in the presence of multiple fracture sets.

Our FSS results indicate that the permeability of a fracture network is predominantly controlled by its critical fracture sets. In networks with multiple sets, permeability is mainly governed by those with geometrical and hydrodynamical properties that carry the dominant flow. Accordingly, scaling analyses and permeability predictions should focus on the parameters of these critical sets, as they determine the network’s overall hydraulic behavior. That is, the dimensionless quantities in the FSS hypothesis should be computed solely based on the critical sets. For instance, only the density of the critical sets is meaningful for calculating the percolation parameter, as only the spanning cluster composed of fractures in these sets can yield a permeability comparable to the fracture conductivity. Also, the FSS hypothesis shows potential as a predictive model for characterizing permeability tensors in fracture networks and can be extended to more general complex systems containing multiple sets of conductive objects.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [46], embargo periods may apply.

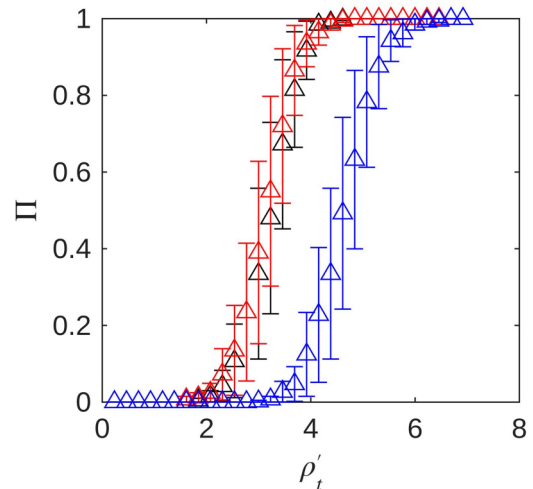


FIG. 4. Percolation probability Π as a function of ρ'_t for Scenario 6 with a domain size of 250 m, where colors indicate the direction of Π (black for x , red for y , and blue for z), and error bars represent the variance of the percolation state (percolated or not) of the 400 MC random realizations.

APPENDIX: PERCOLATION THRESHOLD ESTIMATION VIA PERCOLATION PROBABILITY TESTS

Scenarios 6, 7, 8, and 11 in Table I are selected for percolation probability tests, as their fracture sets differ in orientation or size distributions. For these tests, additional domain sizes are considered. Specifically, the five domain sizes used for the selected scenarios ranged from 100 to 300, 150 to 450, 200 to 450, and 200 to 450, respectively. As in the simulations described in Sec. II A, the fracture network density is gradually increased, and for each density level, 400 MC realizations are generated. It is worth noting that percolation probability tests are not computationally demanding, which makes it feasible to explore multiple domain sizes. In contrast, permeability simulations are extremely time consuming, which limits the number of cases that can be examined.

The percolation probability Π is defined as the fraction of fracture networks that contain spanning clusters among the 400 MC realizations at a given density level and fixed geometrical parameter distributions, with an example shown in Fig. 4. The percolation probability in Scenario 6 is direction dependent; as shown in Table I, the second fracture set has a higher-density weight and longer fracture sizes, which leads to a lower percolation probability in the z direction. Moreover, we use the definition of ρ'_t [Eq. (7)] as the percolation parameter because all fracture sets must be accounted for in percolation probability tests, even if the conductivity of one set may be much lower than that of another, which is irrelevant for these tests.

We follow the standard percolation probability test procedure to estimate the percolation threshold $\rho'_{c,t}$. In brief, the scattered points of Π versus ρ'_t are fitted with an error function, yielding two fitting parameters: the apparent percolation threshold and the width of the transition zone (from $\Pi = 0$ –1). These parameters are then used to estimate $\rho'_{c,t}$, which can be regarded as the apparent percolation threshold

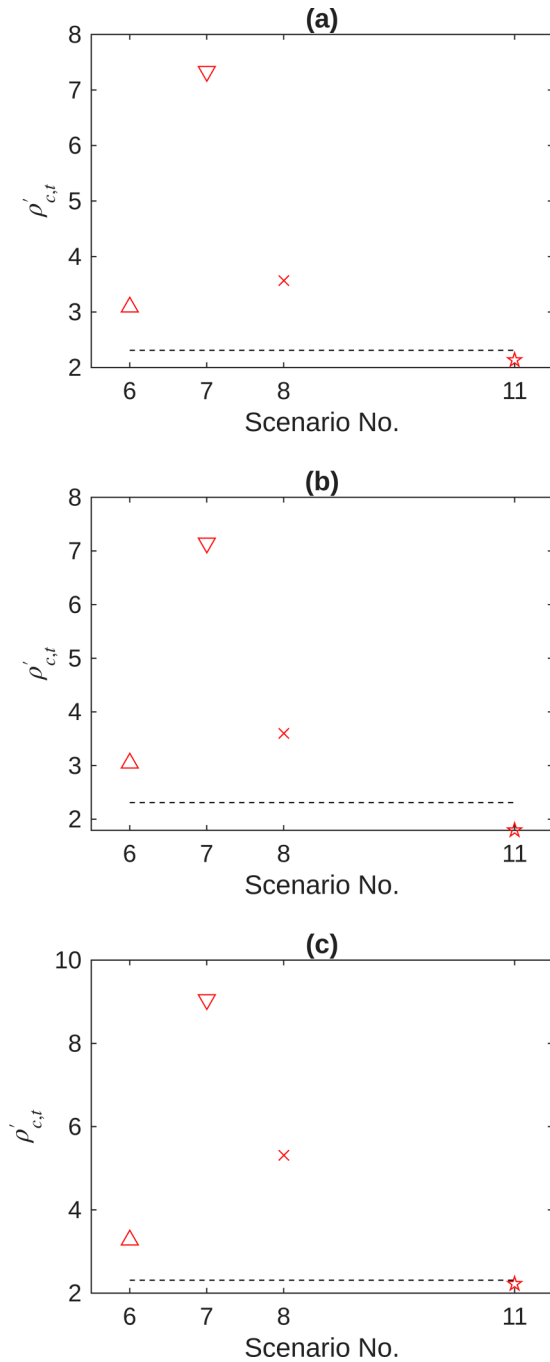


FIG. 5. Estimated $\rho'_{c,t}$ values for the four selected scenarios, with subplots (a), (b), and (c) showing the values along the x , y , and z directions, respectively; the black dashed lines indicate $\rho'_{c,t} = 2.31$.

in the limit of infinitely large domain sizes. For further details, we refer to Refs. [23,44,47].

The estimated $\rho'_{c,t}$ values for the four scenarios along the x , y , and z directions are shown in Fig. 5. Only Scenario 11 yields $\rho'_{c,t}$ values close to the classical value (2.31), whereas the estimated values for the other three scenarios deviate significantly. This is because the two fracture sets in Scenario 11 have very similar fracture geometry distributions, resulting in an estimated threshold close to that of a fracture network containing a single fracture set.

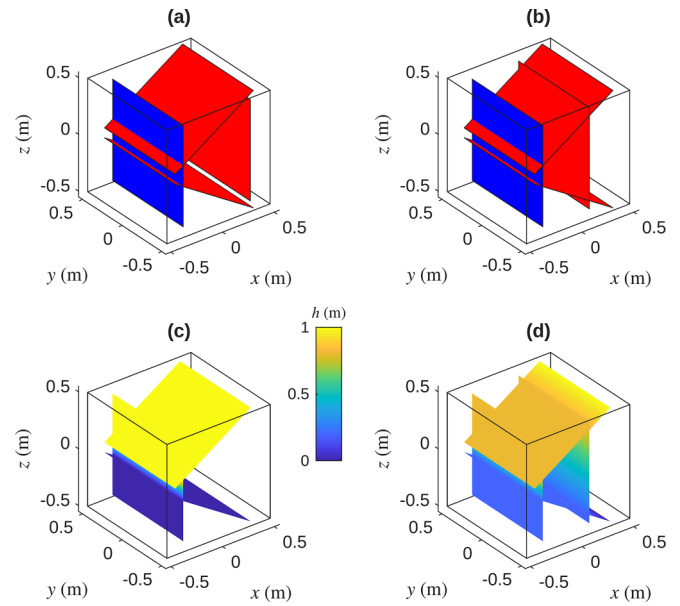


FIG. 6. Two fracture networks containing four fractures are shown in subplots (a) and (b). In both cases, the fractures colored in blue possess much lower conductivity. All four fractures have the same size. Subplots (c) and (d) show the steady-state flow simulation results corresponding to subplots (a) and (b), respectively, where the hydraulic head fields are represented by color. In subplot (c), as one of the fractures does not connect to the spanning cluster and therefore does not contribute to flow, it is removed from the domain.

For the other three scenarios, the distributions of the two fracture sets differ significantly, particularly in fracture sizes where the maximum size values vary greatly. There are two potential reasons for this: (1) the combined effect of multiple distinct fracture sets on percolation state cannot be accurately captured by a simple linear summation, as Eq. (7) treats their interactions as weighted linear combinations; and (2) as shown in Fig. 4, anisotropy causes differences in percolation probability along the three directions. Clearly, a prefactor accounting for this anisotropy should be introduced to reduce the percolation probability values, as done with the introduction of Φ and Ψ in Eq. (9).

In principle, if the interactions among multiple fracture sets and the resulting anisotropy could be fully accounted for in the definitions of percolation parameters and percolation probabilities, a universal percolation threshold would be expected. However, this issue is not addressed in the present study. Since permeability is assumed to be predominantly controlled by the critical sets (as confirmed by our results), only the distribution parameters of the critical sets are considered. Therefore, Eqs. (9) and (10) are applicable for permeability analyses. For Scenario 11 and similar cases, all fracture sets can be regarded as critical sets because their fracture geometries and hydrodynamic parameters are very similar.

Additionally, we provide a simple example to illustrate that percolation threshold values obtained from percolation probability tests have limited relevance for permeability scaling when critical sets exist. Two fracture networks, each containing four fractures, are generated as shown in Fig. 6. In both networks, fractures colored red have $\gamma = 1.5 \times 10^{-6}$

and $\beta = 0.2$, while the blue fracture has $\gamma = 1.5 \times 10^{-7}$ and $\beta = 0.15$. Using Eq. (5), one can calculate that the k_f values of the blue and red fractures are on the order of 10^{-22} and 10^{-19} , respectively, indicating that the blue fracture has much lower conductivity. For both networks, the percolation state is percolative. The difference is that in the first network, one higher-conductivity fracture does not connect to the spanning cluster, making the blue fracture a non-negligible part of the flow channel, whereas in the second network, all fractures participate in the spanning cluster. As the flow simulation shows, the permeability values of the two networks are 1.09×10^{-21} and 1.56×10^{-19} , respectively, differing by two orders of magnitude. In the second network, the three red fractures form a primary flow conduit that carries substantially more

flow than in the first network. Thus, even though the percolation state is identical, the permeabilities differ greatly. When permeability reaches a magnitude comparable to the fracture conductivity of the critical sets (as it will when fracture density increases), it is predominantly controlled by those critical sets. That is to say, it is important to know whether the density of critical sets is sufficient (or has a high enough probability) to form a spanning cluster within the domain. Otherwise, even if a spanning cluster forms with the participation of other fracture sets, as shown in Fig. 6(a), the resulting permeability would be much lower. Therefore, we conclude that in scaling analyses of permeability, only the parameters of the critical sets should be used to compute dimensionless quantities, as the contributions of other sets are negligible.

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