

Symmetry-driven bulk-edge correspondence in electron magnetofluids at finite temperature

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We present a theoretical framework connecting the pseudo-Chern number in momentum space to the spectral flow index in phase space for continuous media, with specific applications to topological Langmuir-cyclotron waves (TLCWs) in magnetized plasmas at uniform finite temperatures. By deriving a rigorous correspondence between these two topological invariants, we provide a solid justification for previous studies that applied this relationship heuristically across various continuous media. For magnetized plasmas with finite-temperature effects, we confirm the existence of TLCWs through numerical computation of bulk Chern number differences and analytical calculation of the spectral flow index. These findings advance the understanding of topological phenomena in continuous media.

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I. INTRODUCTION

The principles of topological insulators [1–4], rooted in the quantum Hall effect, have garnered significant interest in condensed matter physics due to the promise of robust mode propagation immune to backscattering, even in the presence of defects. Recently, the study of topology has expanded into continuous media, enabling applications in fluid dynamics [5–7], acoustics [8,9], and plasma physics [10–15]. While the principles of topology have proven effective across diverse physical systems, applying topological invariants to continuous systems introduces unique theoretical challenges.

In periodic lattice systems, discrete translational symmetry gives rise to integer-valued topological invariants, such as Chern numbers, defined via Berry curvature integrals over the Brillouin zone in momentum space. The bulk-edge correspondence principle ensures these invariants predict localized edge modes at interfaces between distinct topological phases. However, in continuous media, where momentum space corresponds to noncompact and contractable manifolds like $\mathbb{R}^2$, Berry curvature integrals are often noninteger or divergent. Regardless of the numerous proposed regularizations to assign integer values to Chern-like quantities in continuous media, such bespoke integrals fail to qualify as topological invariants. The fundamental reason lies in the fact that momentum space in continuous media ($\mathbb{R}^n$) is topologically trivial. Consequently, these integrals cannot be regarded as true Chern numbers, as they lack the robust mathematical foundation required for a topological invariant. Consequently, phase-space

topology becomes central to describing topological phenomena in such systems [12,16,17].

In this context, the spectral flow index has been established by previous studies as the true topological invariant governing chiral edge modes in continuous media [12,16–18]. Faure's index theorem [16], an adaptation of the Atiyah-Patodi-Singer (APS) index theorem, relates the spectral flow index to the Berry monopole Chern number of eigenmode bundles over spherical surfaces that encompass Weyl points in phase space. This connection provides a rigorous framework for understanding edge modes in systems governed by phase-space topology, such as those featuring Weyl points.

Heuristic approaches employing pseudo-Chern numbers have achieved notable success in predicting topological modes. For example, in the shallow water equation model describing ocean waves, the Coriolis force induces distinct topological phases in the Northern and Southern Hemispheres, with band Chern numbers of ± 1 [5,16]. The difference in the gap Chern number between the two hemispheres is 2, which corresponds to two equatorial modes: the Kelvin waves and Yanai waves. These modes exhibit a spectral flow of 2. Similarly, in the electron magnetohydrodynamics (EMHD) model, the degeneracies between Langmuir and cyclotron waves generate Berry monopoles, resulting in a gap Chern number difference of -1 . This difference aligns precisely with the existence of one topological Langmuir-cyclotron wave (TLCW) in spectral flow across regions with varying densities [12].

In this study, we demonstrate that although pseudo-Chern numbers defined in momentum space are not true topological invariants, they are connected to the spectral flow index under specific conditions. By defining appropriate connections and curvatures, such as Berry connection and Berry curvature, we prove that the difference in pseudo-Chern numbers of bulk bands in momentum space equals the spectral flow index, the established true invariant. This involves regularizing eigenmodes for systems lacking symmetry in momentum space, enabling the pseudo-Chern numbers to be connected to the

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spectral flow index. In systems with symmetry in momentum space, this relationship holds even without regularization, despite the occurrence of noninteger bulk Chern numbers. These findings bridge the heuristic application of pseudo-Chern numbers with the rigorous framework of phase-space topology and spectral flow, offering a robust justification for previous results achieved via heuristic bulk-edge correspondence.

To investigate this correspondence, we integrate Faure's index theorem with the known correspondence between spectral flow and the Chern number defined on a spherical surface in phase space. Using the Levi-Civita connection, we explicitly derive the Berry curvature in phase space. Through Chern-Weil theory and Stokes' theorem, the spherical surface can be homotopically deformed into an arbitrary cylindrical surface enclosing the phase-space degeneracies. Under symmetric conditions, the spectral flow then equals the difference of momentum-space pseudo-Chern numbers.

The paper is structured as follows: Sec. II builds upon Faure's phase-space framework, implemented via the Hermitian Wigner symbol, to construct a Levi-Civita connection. We then homotopically deform the spectral flow–Chern number correspondence onto a cylindrical surface in phase space. This enables us to establish rigorous conditions under which the pseudo-Chern number difference corresponds to the spectral flow index. In Sec. III we apply this formalism to a $(2+1)$ -dimensional magnetized plasma model with finite temperature, demonstrating how symmetry circumvents regularization requirements and ensures coincidence between the spectral flow index and the gap Chern number difference. Section IV concludes with discussion of future research directions.

II. ESTABLISHING BULK-EDGE CORRESPONDENCE VIA FAURE'S INDEX THEOREM

We consider a linearized partial differential equation (PDE) governing the dynamics of a continuous medium and assume that it can be cast in a Schrödinger-like form:

$$i\partial_t \xi(x, t) = \hat{H}_\mu(x, \nabla) \xi(x, t), \quad (1)$$

where $\xi \in \mathbb{C}^m$ is the state vector, and $\hat{H}_\mu(x, \nabla) : \mathbb{C}^m \rightarrow \mathbb{C}^m$ is a Hermitian operator depending on a real control parameter μ . The Wigner symbol of $\hat{H}_\mu(x, \nabla)$, denoted by $H(x, k, \mu)$, is an $m \times m$ Hermitian matrix defined over the phase space $(x, k) \in \mathbb{R}^n \times \mathbb{R}^n$. We also assume that the symbol H is smooth with respect to its parameters (x, k, μ) . We order its eigenvalues and corresponding eigenvectors as $(\omega_1, \xi_1), (\omega_2, \xi_2), \dots, (\omega_m, \xi_m)$.

We assume that $H(x, k, \mu)$ satisfies the spectral gap condition [16] such that for $\|(x, k, \mu)\| \geq R_0 > 0$, the s th and $(s+1)$ -th eigenvalues can collapse only within the region $\|(x, k, \mu)\| \leq R_0$. By rescaling coordinates, R_0 is set to 1. Outside the sphere $B^{2n} = \{(x, k, \mu) \in \mathbb{R}^{1+2n}, \|(x, k, \mu)\| < 1\}$, a spectral gap exists in $T = \mathbb{R}^{1+2n} \setminus B^{2n}$, bounded by positive constants M_1 and M_2 , satisfying

$$\omega_s(x, k, \mu) < M_1 < M_2 < \omega_{s+1}(x, k, \mu). \quad (2)$$

A projector given by the eigenvectors of the first s eigenvalues is computed using the Cauchy integral:

$$\Pi = \frac{1}{2\pi i} \oint_L [z - H(x, k, \mu)]^{-1} dz, \quad (3)$$

where L is a contour in the complex plane enclosing $\omega_1, \omega_2, \dots, \omega_s$. Obviously, Π is an $m \times m$ matrix satisfies $\Pi^2 = \Pi$ and $\text{Tr}(\Pi) = s$. Π defines a vector bundle $F \rightarrow B^{2n} : \{(x, k, \mu) \in \mathbb{R}^{1+2n} | \|(x, k, \mu)\| \geq 1\}$, whose fiber $F_{(x,k,\mu)}$ is the vector space $\text{Ran} \Pi$. Let a clutching function $U(x, k)$ of $F|_{S^{2n}}$ on the equator $S^{2n-1} := \{(x, k, \mu) \in \mathbb{R}^{1+2n} | \|(x, k, 0)\| = 1\}$ corresponding to the vector bundle F . Extend it to the whole plane $P^0 : \{(x, k, \mu) \in \mathbb{R}^{1+2n} | \mu = 0\}$ by

$$\tilde{U}(x, k) = \begin{cases} U(x, k), & \text{if } (x, k) \in S^{2n} \cap P^0, \\ \|(x, k)\| U\left(\frac{(x, k)}{\|(x, k)\|}\right), & \text{if } (x, k) \in P^0 - S^{2n}. \end{cases} \quad (4)$$

The Weyl quantization of $\tilde{U}(x, k)$, denoted by $\hat{\tilde{U}} \equiv \text{Op}_\epsilon(\tilde{U})$, is an elliptic operator. While ellipticity ensures good symbolic properties, it is not sufficient alone to guarantee the Fredholm property. In our setting, the Fredholmness of $\text{Op}_\epsilon(\tilde{U})$ is ensured by considering suitable function spaces with decay conditions at infinity (e.g., Hermite function spaces), so that the operator acts on a domain where the analytic framework of [19, Theorem 3, p. 185] applies. In particular, we implicitly rely on the construction used in the proof in [16, Lemma 2.11]. Its algebraic index is expressed as

$$\text{Ind} \hat{\tilde{U}} = \dim \text{Ker} \hat{\tilde{U}} - \dim \text{Ker} \hat{\tilde{U}}^*. \quad (5)$$

According to Faure's theorem [16], the spectral flow index N , quantifying the net number of upward crossings of eigenmodes between bands ω_s and ω_{s+1} as the control parameter μ traverses the real line, is precisely equal to $\text{Ind} \hat{\tilde{U}}$.

By the index formula on Euclidean space of Fedosov-Hörmander [20,21], it holds

$$\text{Ind} \hat{\tilde{U}} = W_{2n-1} := c_n \int_{S^{2n} \cap P^0} \text{Tr}(\tilde{U}^{-1} d\tilde{U})^{2n-1}, \quad (6)$$

where $c_n = -2n!/((2n)!(2\pi i)^n)$.

If we choose a connection in Γ , then by [19, Sec. III.1.B, p. 271], this winding number has a Chern number representation:

$$W_{2n-1} = \int_{S^{2n}} \frac{1}{n!} \left(\frac{i}{2\pi} \right)^n \text{Tr} R^{\wedge n}, \quad (7)$$

where the connection here can be Levi-Civita connection $\nabla = \Pi d$ and the correspondent curvature is given by $R = \Pi d\Pi \wedge d\Pi$ [20], in which d is the differential in (x, k, μ) space. By applying Chern-Weil theory and utilizing the closedness of R , the integral region of Eq. (7) can be replaced by any surface Γ homotopic to S^{2n} . The homotopy requires the projector Π to remain well defined throughout the continuous deformation from S^{2n} to Γ , preserving both the trace property $\text{Tr}(\Pi) = s$ and the smoothness of Π .

This result enables us to choose appropriate Γ , to derive the desired bulk-edge correspondence for a continuous media system. Specifically, when Γ is a sphere centered at the origin enclosing all degeneracies, this result aligns with the boundary isomorphism theorem introduced in Ref. [12].

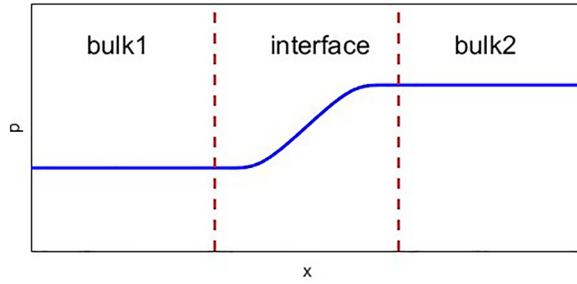


FIG. 1. Schematic representation of the bulk interface in a continuous medium. The x axis denotes spatial position, while the p axis represents the parameter values on which the operator $\hat{H}(x, \nabla)$ depends. In each bulk region, $\hat{H}$ remains uniform.

In the following, we use Eq. (7) to establish a necessary and sufficient condition for the validity of the bulk-edge correspondence. Consider the reduced $(2 + 1)$ -dimensional problem shown in Fig. 1. Here two uniform bulks with distinct uniform parameters are interfaced, and the parameter varies smoothly along the interface's spatial coordinate x .

Applying a Fourier transform along the uniform y direction introduces the wave number k_y , reducing the system operator to $\hat{H}(x, \partial_x, k_y)$.

In the bulk regions, where the parameters of $\hat{H}(x, \partial_x, k_y)$ are uniform and independent of x , the symbol $H(x, k_x, k_y)$ satisfies $\partial_x H(x, k_x, k_y) = 0$, as illustrated in Fig. 1. The parameter k_y acts as μ in the symbol $H(x, k, \mu)$. Suppose the interface between the two bulks is confined to the region $x \in [-d, d]$. We assume that all degeneracies are located within this interface and for sufficiently large $\|(x, k, \mu)\|$, $H(x, k_x, k_y)$ maintains a band gap larger than a constant.

We specify the deformed closed surface Γ as the cylinder Z_r , ensuring it contains all degeneracies inside. The cylindrical region Z_r is defined as

$$Z_r = \partial\{(x, k_x, k_y), \\ -d < x < d\} \cap \{(x, k_x, k_y), k_x^2 + k_y^2 \leq r^2\}.$$

The boundary sections of the cylinder can be written as

$$\begin{aligned} B^+(r) &= \{(x, k_x, k_y) \mid k_x^2 + k_y^2 \leq r^2 \text{ and } x = d\}, \\ B^-(r) &= \{(x, k_x, k_y) \mid k_x^2 + k_y^2 \leq r^2 \text{ and } x = -d\}, \\ B^0(r) &= \{(x, k_x, k_y) \mid k_x^2 + k_y^2 = r^2 \text{ and } -d < x < d\}. \end{aligned}$$

Thus, the region Z_r is expressed as

$$Z_r = B^+ \cup B^- \cup B^0,$$

as shown in Fig. 2.

Since we are considering a $n = 1$ model, the curvature is given by

$$F = \frac{i}{2\pi} \text{Tr} R = \frac{i}{2\pi} \text{Tr}(\Pi d\Pi \wedge d\Pi). \quad (8)$$

When H is Hermitian, the curvature corresponds to the total Berry curvature obtained from the eigenvectors of all bands

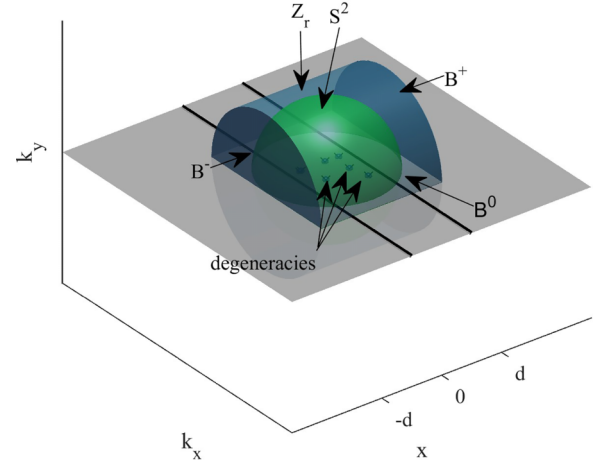


FIG. 2. Depiction of Z_r in phase space. The blue cylinder represents Z_r , whose circular face at $x = d$ is labeled B^+ and at $x = -d$ is labeled B^- , with the curved side boundary denoted by B^0 . Inside the cylinder lies the green sphere S^2 , encompassing all degeneracies of the symbol $H(x, k_x, k_y)$.

below the spectral gap (see Appendix G); i.e., we have

$$F = \sum_{i=1}^s d\xi_i^\dagger \wedge d\xi_i. \quad (9)$$

The integral in Eq. (7) over Z_r can be divided into three parts:

$$N = \frac{i}{2\pi} \int_{B^+} \text{Tr} R + \frac{i}{2\pi} \int_{B^-} \text{Tr} R + \frac{i}{2\pi} \int_{B^0} \text{Tr} R. \quad (10)$$

In the limit where the radius of the cylinder Z_r approaches infinity, $r \rightarrow \infty$, and if the integral over B^0 becomes negligible and those in $B^\pm$ converge, the index reduces to

$$N = \frac{i}{2\pi} \int_{B^+(\infty)} \text{Tr} R + \frac{i}{2\pi} \int_{B^-(\infty)} \text{Tr} R. \quad (11)$$

By consistently orienting B^+ and B^- in terms of (k_x, k_y) , the relation presented in (11) can be expressed in a more familiar form: the bulk-edge correspondence:

$$N = C^+ - C^-, \quad (12)$$

where C^+ and C^- represent integrals of the Berry curvature over momentum space (k_x, k_y) in the respective bulk regions $B^+(\infty)$ and $B^-(\infty)$. Although we refer to them as ‘‘bulk Chern numbers,’’ it should be emphasized that they are not true topological invariants in the rigorous sense, since Chern numbers are not well defined in the noncompact momentum space of continuous media. Rather, $C^\pm$ should be regarded as pseudo-Chern numbers, and their difference $\Delta C = C^+ - C^-$ defines a meaningful topological quantity, known as the bulk-difference invariant [22,23].

The result presented in Eq. (12) embodies the bulk-edge correspondence, linking the genuine topological invariant in phase space—namely, the spectral flow index—to the bulk Chern numbers (or pseudo-Chern numbers) evaluated in two separate bulk regions in momentum space. Two key conditions are required for this correspondence. First, the Berry curvature must be adopted to compute the bulk Chern numbers. Second, the contribution of the curvature integral

over B^0 must vanish as the radius of the cylinder goes to infinity.

In continuous media, the second condition is typically satisfied through various methods of regularization [6,11,14], which ensure that H asymptotically approaches a constant at infinity. Consequently, the corresponding momentum space can be compactified into a sphere, allowing the Berry Chern number over each bulk to become a well-defined integer. This is equivalent to the statement that, after regularization, the contribution of the integral over $B^0(\infty)$ vanishes. However, in systems featuring certain symmetries [11,14], the integral over B^0 vanishes on without explicitly regularizing the curvature. The plasma system discussed in the following section provides another example exhibiting this type of symmetry, that is,

$$PH_0(x, k_x, k_y) = H_0(x, -k_x, -k_y)P, \quad (13)$$

where P is a constant diagonal matrix, which ensures the curvature flux through B^0 is an odd function of (k_x, k_y) (see Appendix D). Here H_0 is the bulk symbol of H .

When a degenerate point exists at $k = \pm\infty$, the integral on the right-hand side of the equation may diverge from the spectral flow index derived from $\hat{H}$. This difference is attributed to an extra contribution to the spectral flow index at infinity, known as a ghost mode [24]. Additionally, violations of bulk-boundary correspondence can occur in systems with sharp boundaries [25], where the discontinuity in $\hat{H}$ disrupts the semiclassical framework. In this case, even with integrable C^+ and C^- , contributions from B^0 become significant and non-negligible.

As stated previously, the bulk-edge correspondence expressed in Eq. (12) holds, provided that the total integral over B^0 approaches zero as $|k| \rightarrow \infty$. In one case, the cancellation occurs when $H(x, k_x, k_y)$ shows symmetries that cause the Berry curvature contributions on B^0 to vanish. For instance, in the linearized EMHD framework, where a constant matrix commutes with its Wigner symbol, Berry curvatures at opposite k values cancel due to symmetry. This cancellation is demonstrated in Appendix D.

In the next section, we will examine the TLCW in magnetized plasma with finite temperature, or warm magnetized plasma, using the correspondence relation obtained in this section and a two-level approximation.

III. BERRY CURVATURE INTEGRAL OF ISOTHERMAL ELECTRON MAGNETOFLUIDS AND SPECTRAL FLOW CALCULATION

We extend the magnetized cold plasma framework through systematic incorporation of finite temperature effects to investigate topological Langmuir-cyclotron waves (TLCWs), following foundational studies of these waves in magnetized cold plasma systems [11–13]. The term “finite temperature” in this context specifically denotes the explicit inclusion of thermal pressure contributions within the momentum conservation equations. Crucially, all dissipative mechanisms arising from electron-ion momentum exchange are deliberately excluded from our formulation, justified by the substantial temporal scale separation between the wave dynamics under investigation and the characteristic timescale over which

dissipative processes become significant. This theoretical treatment preserves the Hermitian nature of the warm electron fluid dynamics, which are mathematically described by the following governing equations [26]:

$$\begin{aligned} m_e n \partial_t \mathbf{v} &= -ne(\mathbf{E} + \mathbf{v} \times \mathbf{B}) - \nabla P_e, \\ \partial_t n &= -n \nabla \cdot \mathbf{v} - \nabla n \cdot \mathbf{v}, \quad \partial_t \mathbf{B} = -\nabla \times \mathbf{E}, \\ \partial_t \mathbf{E} &= c^2 \nabla \times \mathbf{B} + c^2 \mu_0 n e \mathbf{v}. \end{aligned} \quad (14)$$

Here $\mathbf{v}$ represents the electron flow velocity, n the electron density, $\mathbf{E}$ the electric field, and $\mathbf{B}$ the magnetic field. The electron pressure P_e follows the isothermal relation $P_e = nk_B T_e$, where k_B is the Boltzmann constant and T_e is the constant electron temperature.

The equilibrium configuration satisfies $\pi en \mathbf{E}_0 + \nabla P_{e0} = 0$, with a uniform magnetic field $\mathbf{B}_0 = B_z \mathbf{e}_z$ and zero equilibrium flow ($\mathbf{v}_0 = 0$). We restrict the inhomogeneity of equilibrium quantities to the x direction, such that $n_0 = n_0(x)$ and $\mathbf{E}_0 = E_0(x) \mathbf{e}_x$.

Following standard linearization and normalization procedures (detailed in Appendix B), we obtain a Schrödinger-like equation:

$$i \partial_t \Psi = \hat{H} \Psi. \quad (15)$$

In our normalization scheme, time is normalized to the cyclotron frequency $\sigma = eB/m_e$, spatial variables to c/σ , electric field to a reference field E , magnetic field to E/c , velocity to $\sqrt{\epsilon_0 E^2/(m_e n_0)}$, and density to $\sqrt{n_0 \epsilon_0 E^2/(k_B T_e)}$. The state vector Ψ is defined as

$$\Psi = \begin{pmatrix} \tilde{v}_1 \\ \tilde{E}_1 \\ \tilde{B}_1 \\ \tilde{n}_1 \end{pmatrix}, \quad (16)$$

which is a 10×1 vector. The Hamiltonian operator takes the form

$$\hat{H} = \begin{pmatrix} i\sigma \mathbf{e}_x \times & -i\Omega_p \mathbf{I}_3 & 0 & \frac{1}{2} \frac{i u}{L_{n_0}} - ui \nabla \\ i\Omega_p \mathbf{I}_3 & 0 & i \nabla \times & 0 \\ 0 & -i \nabla \times & 0 & 0 \\ -\frac{1}{2} \frac{i u}{L_{n_0}} - ui \nabla & 0 & 0 & 0 \end{pmatrix}, \quad (17)$$

where $\Omega_p = \sqrt{[n_0 e^2/(m_e \epsilon_0)]}/(eB/m_e)$ is the normalized electron plasma frequency and $u = \sqrt{(k_B T_e/m_e)}/c$ is the normalized electron thermal velocity. The electron density scale length function $L_{n_0}(x)$ is determined by the equilibrium and isothermal conditions:

$$L_{n_0}^{-1}(x) = \frac{\nabla n_0}{n_0} = \frac{\nabla P_{e0}}{k_B T_e n_0} = -\frac{e E_0}{k_B T_e}. \quad (18)$$

To analyze the spectrum flow, we leverage the fact that L_n^{-1} varies only in the x direction due to the inhomogeneity of n_0 . It is convenient to apply a Fourier transform in the y and z spatial coordinates and time t :

$$\Psi = \psi(x, k_y, k_z) \exp[i(k_y y + k_z z - \omega t)]. \quad (19)$$

Equation (15) reduces to

$$\omega \psi = \hat{H}(x, \partial_x, k_y, k_z) \psi. \quad (20)$$

For our analysis, we treat k_z as a fixed constant and take $u < 1$, which is a reasonable assumption since the thermal velocity of the plasma typically remains well below the speed of light. Applying the Weyl-Wigner transform in the x direction yields the Wigner symbol $H(x, \mathbf{k})$ of $\hat{H}(x, \partial_x, k_y, k_z)$:

$$H(x, \mathbf{k}) = H_0(x, \mathbf{k}) + H_1(x, \mathbf{k}), \quad (21)$$

where $H_0(x, \mathbf{k})$ is defined as

$$H_0(x, \mathbf{k}) = \begin{pmatrix} ie_z \times & -i\Omega_p \mathbf{I}_3 & 0 & u\mathbf{k}^T \\ i\Omega_p \mathbf{I}_3 & 0 & -\mathbf{k} \times & 0 \\ 0 & \mathbf{k} \times & 0 & 0 \\ u\mathbf{k} & 0 & 0 & 0 \end{pmatrix}, \quad (22)$$

and $H_1(x, \mathbf{k})$ is given by

$$H_1(x, \mathbf{k}) = \begin{pmatrix} 0 & 0 & 0 & \frac{i u}{2L_{n_0}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{i u}{2L_{n_0}} & 0 & 0 & 0 \end{pmatrix}. \quad (23)$$

The Wigner symbol $H(x, \mathbf{k})$ preserves its Hermitian nature, consistent with the magnetized cold plasma model, where its upper-left 3×3 block defines the system's dynamics. In the limit $u \rightarrow 0$, the system simplifies to the magnetized cold plasma case discussed in [11–13].

We define a new interpolated symbol:

$$H_t(x, \mathbf{k}) = H_0(x, \mathbf{k}) + tH_1(x, \mathbf{k}), \quad t \in [0, 1]. \quad (24)$$

With finite-temperature effects included, $H_t(x, \mathbf{k})$ retains particle-hole symmetry $H^*(x, -\mathbf{k}) = -H(x, \mathbf{k})$ [11,13,27]. Its dispersion relation is invariant under $\mathbf{k} \rightarrow -\mathbf{k}$, ensuring symmetry in the energy bands with respect to $\omega = 0$. Thus, if ω is an eigenvalue, so is $-\omega$.

As detailed in Appendix D, the nonzero eigenvalues of $H_t(x, \mathbf{k})$ are denoted as $\pm\omega_1, \pm\omega_2, \pm\omega_3, \pm\omega_4$, ordered such that $0 < \omega_1 < \omega_2 < \omega_3 < \omega_4$. If $L^{-1}(x)$ is bounded, as k_x or $k_y \rightarrow \infty$, the eigenvalues asymptotically approach

$$(\omega_1, \omega_2, \omega_3, \omega_4) \rightarrow (k_\perp^s, uk_\perp, k_\perp, k_\perp), \quad (25)$$

where $k_\perp = \sqrt{k_x^2 + k_y^2}$ and $s < 0$. Thus, the gap condition between ω_1 and ω_2 persists throughout the interpolation range $t \in [0, 1]$, ensuring stability in the spectral gap.

Assuming that the density $n_0(x)$ follows the distribution outlined in Fig. 1, we find that within the bulk regions, where the density is approximately constant, the quantity $L_{n_0}^{-1}$ equals zero. Consequently, $H_1(x, \mathbf{k})$ vanishes in these regions, and thus H_t simplifies to $H_t = H_0$ for all $t \in [0, 1]$. The total Chern number associated with bands below the gap between ω_1 and ω_2 is expressed as

$$N = C^+ - C^- + \frac{i}{2\pi} \int_{B^0} \text{Tr} R. \quad (26)$$

Employing the Berry curvature to compute N in Eq. (26), note that the first two terms in Eq. (26), corresponding to the bulk Chern numbers C^+ and C^- , are determined entirely by H_0 . This is because they involve integrals evaluated over uniform bulk regions in phase space, where $H_1 = 0$.

For the last term in Eq. (26), since H_0 satisfies the symmetry $PH_0(x, -k_x, -k_y) = H_0(x, k_x, k_y)P$, where P is a constant

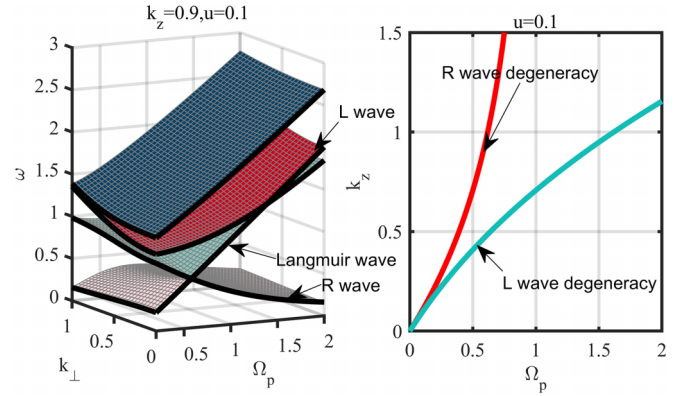


FIG. 3. Dispersion relations of H_0 . (a) Positive energy bands of H_0 in phase space for $k_z = 0.9$ and $u = 0.1$. (b) The degeneracies between the Langmuir wave and cyclotron waves in the k_z - Ω_p plane.

diagonal matrix written as

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \quad (27)$$

Given the symmetry, if $\xi(x, k_x, k_y)$ is a normalized eigenvector of $H_0(x, k_x, k_y)$, then $P\xi(x, k_x, k_y)$ is a normalized eigenvector of $H_0(x, -k_x, -k_y)$. This implies that

$$\xi(x, -k_x, -k_y) = e^{i\theta} P\xi(x, k_x, k_y),$$

where θ is a real constant. As a result, the Berry curvature flux density on $B^\pm$ is an even function of (k_x, k_y) , while on B^0 , the flux density is an odd function of (k_x, k_y) . The contribution of H_0 to the integral over B^0 vanishes (see Appendix D).

Now consider the full interpolated symbol H_t , where t varies from 0 to 1. This variation establishes a homotopy between the eigenbundles of H_0 and $H_0 + H_1$ formed by the eigenvectors below the gap between ω_1 and ω_2 . Importantly, no additional degeneracy arises between two non-zero bands of the symbol H_t for sufficiently large $\|(x, k_x, k_y)\|$. As a result, the spectral flow index N remains an invariant integer as t varies from 0 to 1. Combining this with the condition that $H_0 + H_1$ reduces to H_0 in the bulk regions, the Chern number of $H_0 + H_1$ remains $C^+ - C^-$. Therefore, the overall spectral flow index of the system is preserved.

To determine the band spectral flow index of H , we calculate $C^\pm$ using fixed parameters $k_z = 0.9$ and $u = 0.1$.

Figure 3(a) shows the energy band distribution of H_0 's symbol as a function of k and Ω_p . At the cross section where $k_\perp = 0$, we observe two degenerate points (Weyl points) in the energy bands. These points represent two distinct resonances: one between the R wave in cyclotron waves and the Langmuir wave (intersection of pink and light green bands), and another between the Langmuir wave and the L wave in cyclotron waves (intersection of red and light green bands).

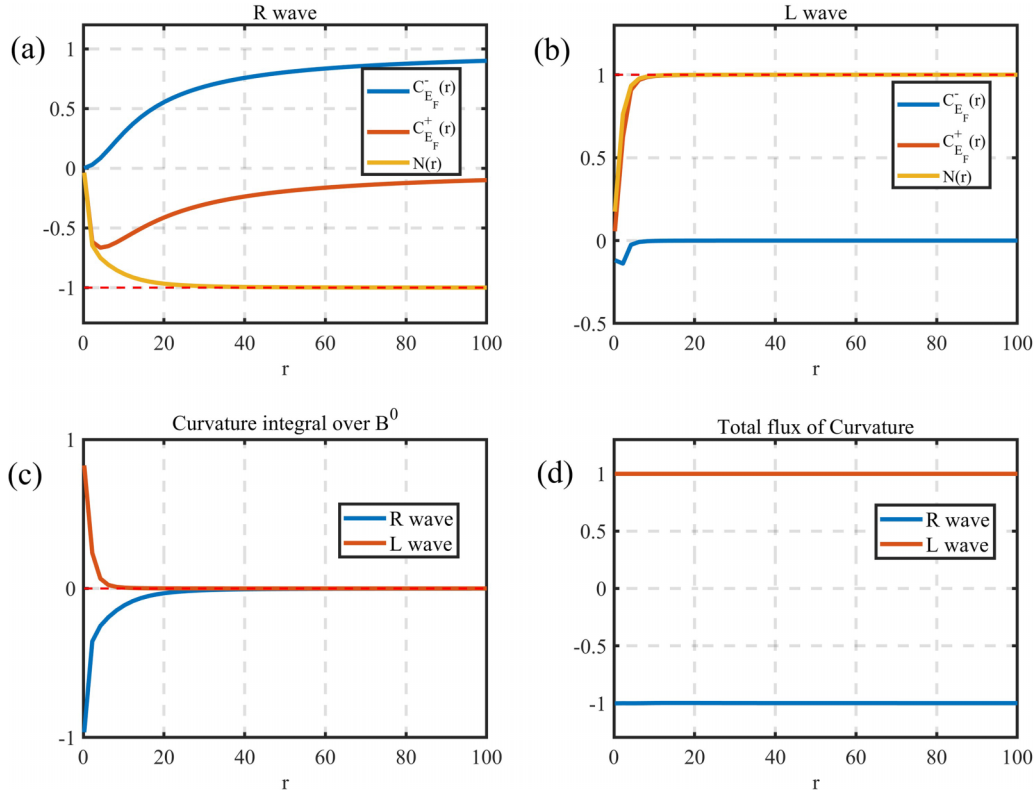


FIG. 4. Chern number as a Berry curvature integral. (a) The bulk Chern numbers of the R wave, and their difference, for $\Omega_p^- = 0.1$ in bulk 1 and $\Omega_p^+ = 0.9$ in bulk 2. (b) As in (a), but for the L wave with $\Omega_p^- = 1$ and $\Omega_p^+ = 2$. (c) Berry curvature flux through B^0 using the same parameters as in (a) and (b). (d) The total Berry curvature flux over $B^\pm$ and B^0 .

The calculation of $C^\pm$ requires integrating Berry curvature over the k_x - k_y plane in two bulk regions:

$$C^+ - C^- = \lim_{r \rightarrow +\infty} [C_{E_F}^+(r) - C_{E_F}^-(r)] := \lim_{r \rightarrow +\infty} N(r). \quad (28)$$

The gap Chern number in the bulk regions, representing the total Berry curvature integral of all eigenvectors of $H_0(x, k)$ below the gap (see Appendix D) is given by

$$C_{E_F}^\pm(r) = \frac{-1}{2\pi i} \sum_{i \leq s, j > s} \int_{k_x^2 + k_y^2 < r^2} \left(\frac{(\xi_i^\dagger \nabla_k H \xi_i) \times (\xi_j^\dagger \nabla_k H \xi_j)}{(\omega_i - \omega_j)^2} \right)_{\Omega_p = \Omega_p^\pm}. \quad (29)$$

In the eigenvalue ordering of the symbol H , the condition $i \leq s$ in the above expression denotes the summation over all bands below the gap, while $j > s$ refers to those above the gap, consistent with the notation used in the previous chapter. For the ten-band plasma model, s in the equation above is 7 (R wave) or 8 (L wave). For the analysis of R-wave resonance Weyl points shown in Fig. 3(a), we perform numerical integration of $C_{E_F}^\pm$ using $\Omega_p^- = 0.1$ in bulk 1 and $\Omega_p^+ = 0.9$ in bulk 2.

The plasma frequency profile across the interface is taken to be

$$\Omega_p(x) = \frac{\Omega_p^- + \Omega_p^+}{2} + \frac{\Omega_p^+ - \Omega_p^-}{2} \tanh \frac{x}{\delta}, \quad (30)$$

where the interface region spans $x \in [-10, 10]$, and $\delta = 1$ is used for calculating the Berry curvature integral over B^0 . The gap for this analysis is chosen to lie between the pink and light green bands.

Figure 4 illustrates the convergence of the total Berry curvature integral in the region $B^\pm$ of bands located below the gap to the gap Chern number as the integral region's radius r approaches infinity. The blue curve represents the Chern number for bulk 1 as r tends to infinity, while the orange curve corresponds to bulk 2. The yellow curve depicts the difference in Berry curvature between the two bulk bands. In the bulk, the difference in gap Chern number approaches an integer limit of -1 as the integral radius r increases, indicating that the spectral flow index is -1 and thus there is one net downward eigenmodes, TLCW, in spectral flow in the gap.

Using the numerically Hermitian-preserving discretization scheme introduced in [11] (see Appendix F), we obtain the results shown in Fig. 5. After incorporating finite temperature effects, we find that spectral flow remains consistent with that of the magnetized cold plasma case studied in [11]. Figures 5(a) and 5(b) demonstrate that at the interface of the two bulk structures, with $\Omega_p^- = 0.1$ and $\Omega_p^+ = 0.9$, there are chiral modes present at fixed k_y with a negative group velocity ($d\omega/dk_y < 0$), which directly corresponds to the spectral flow index of -1 .

It is also noticeable that although neither of the two gap Chern numbers $C^\pm$ is an integer, their difference is. Furthermore, if we consider $\Omega_p^- = 1$ for bulk 1 and $\Omega_p^+ = 2$ for bulk 2, we find that the spectral flow index of L-wave resonance

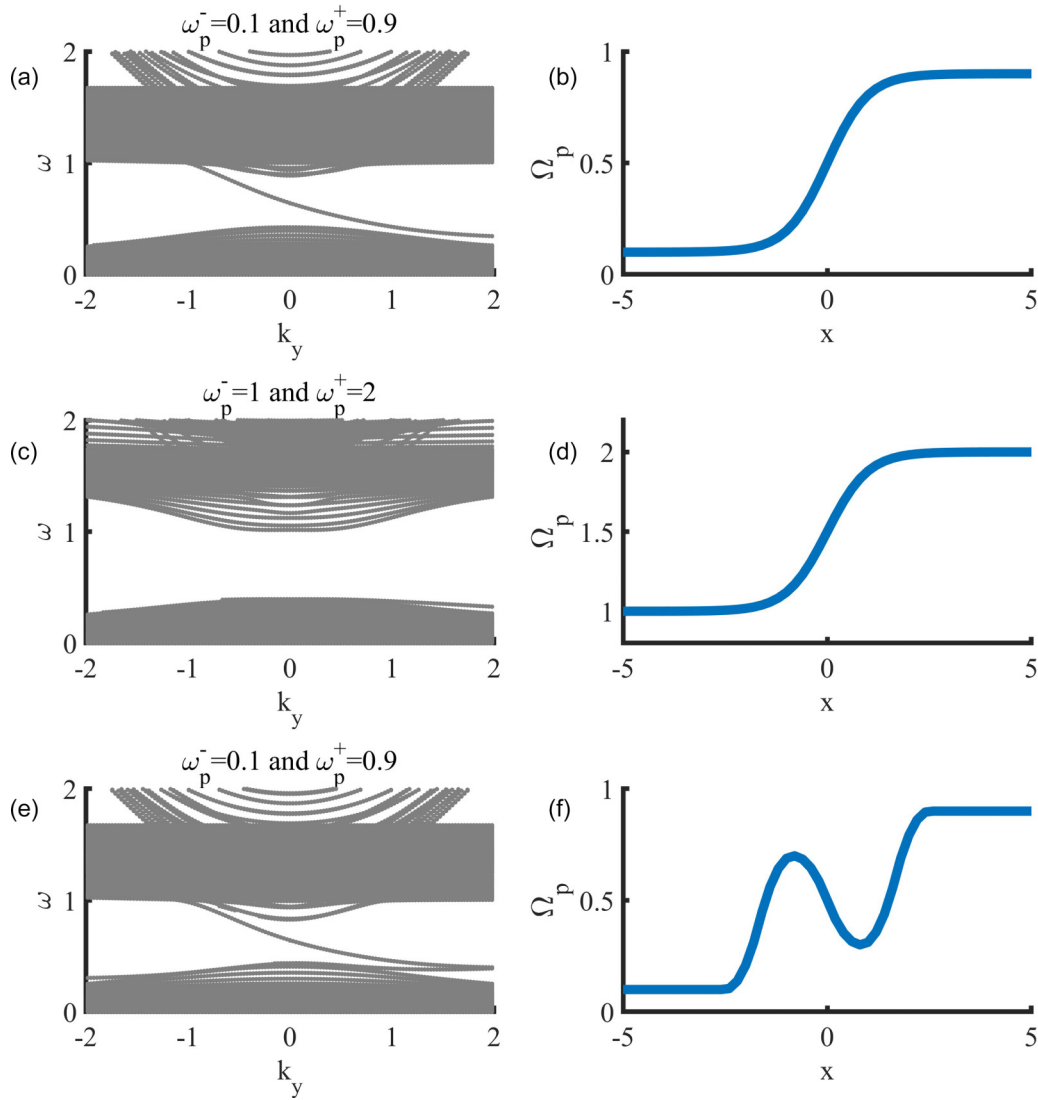


FIG. 5. Band structure for different plasma density profiles at $k_z = 0.9$ and $u = 0.1$, where $\Omega_p \sim \sqrt{n_0}$. (a) Band structure $\omega(k_y)$ corresponding to the density profile shown in (b), where the bulk density settings are $\Omega_p^- = 0.1$ and $\Omega_p^+ = 0.9$. (c) Band structure $\omega(k_y)$ associated with the density profile in (d), where the bulk density settings are $\Omega_p^- = 1$ and $\Omega_p^+ = 2$. (e) Band structure $\omega(k_y)$ linked to the density profile in (f), where the bulk density settings are again $\Omega_p^- = 0.1$ and $\Omega_p^+ = 0.9$.

Weyl points is $+1$, as shown in Fig. 4(b). These results are consistent with the analytic treatment via two-band approximation, which is employed to calculate the curvature flux through a vanishing sphere surrounding an isolated Weyl point (see Appendix E).

For magnetized plasma, when the density (and thus the plasma frequency) varies monotonically at the interface, the Langmuir wave of H and its resonant cyclotron wave will yield only one Weyl point. In this scenario, the two-bands approximation near the Weyl point and the gap Chern number difference between the two bulks are equivalent. Conversely, if the density varies nonmonotonically at the interface while the bulk density remains fixed, the gap Chern number will remain unchanged; however, the number of Weyl points will increase. As outlined in the previous section, the total number of chiral modes at the interface is given by the sum of the Chern numbers contributed by all Weyl points. Consequently,

the presence of a localized chiral mode does not necessarily occur near each Weyl point but rather depends on the overall topological properties of these points. Figures 5(e) and 5(f) illustrate that although the plasma frequency intersects the R-wave-resonance frequency multiple times, the number of chiral interface modes is always one, which corresponds to the total spectral flow index of all Weyl points. The continuity of Ω_p variation guarantees that adjacent nontrivial Weyl points exhibit opposite spectral flow indices.

In treating the magnetized plasma in this section, the Hamiltonian H_0 is not regularized. However, the Berry curvature on B_0 yields an overall zero result as $|k|$ tends toward $+\infty$. Notably, at B^0 , the curvature at the point (x, k_x, k_y) is canceled by the curvature at the point $(x, -k_x, -k_y)$. When incorporating H_1 , the Berry curvature integral over the entire B^0 still asymptotically approaches zero as r increases, as shown in Fig. 4(c).

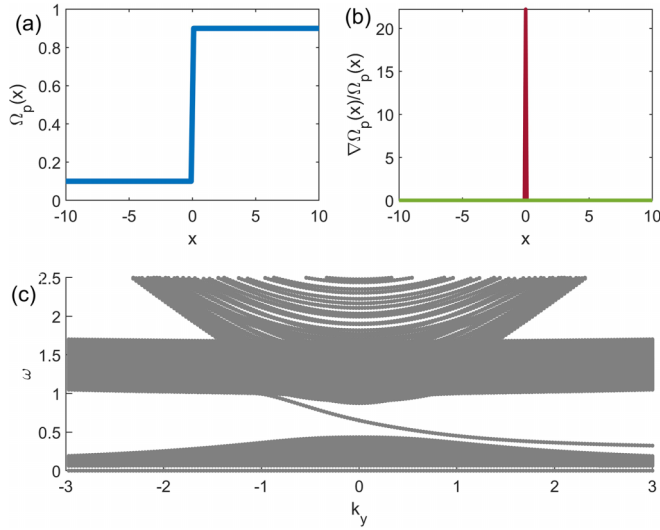


FIG. 6. Band structure for a sharp $\Omega_p(x)$ distribution at interface. (a) Distribution of $\Omega_p(x)$ with $\delta = 0.02$. (b) The distribution of $\nabla\Omega_p(x)/\Omega_p(x)$ corresponds to (a). (c) Band structure with the $\Omega_p(x)$ setting of (a), where $\Omega_p^- = 0.1$ and $\Omega_p^+ = 0.9$.

Since the model inherently incorporates a density gradient, we assume that the density profile is sufficiently smooth, ensuring the resulting gradient maintains smoothness. Once the density profile is specified, a finite interface width implies that the density gradient remains bounded. According to the previous discussion on the symbol $H_t = H_0 + tH_1$, a large but finite density gradient at the interface does not alter the asymptotic behavior of the eigenvalues at infinity [cf. Eq. (25)]. Therefore, the gap condition remains valid for sufficiently large $|k|$. Moreover, under the symmetry of H_0 [see Eq. (13)], the difference in the pseudo-Chern numbers of the bulk symbol [Eq. (28)] remains an integer. Given that N is also an integer, the Berry curvature flux through $B^0|_{|k|=\infty}$ can be regarded as a continuous integer-valued function of t , and thus must remain zero. This is consistent with the case $t = 0$, where $H_t = H_0$ fully respects the symmetry described in Eq. (13).

As shown in Fig. 6, we choose a sufficiently small $\delta = 0.02$ in the plasma frequency profile $\Omega_p(x)$ [see Eq. (30)] to verify that a smooth yet sufficiently large density gradient does not destroy the spectral flow index. According to the theory developed in Sec. II, as the interface width in Fig. 6 tends to zero, the spectral flow index remains preserved. In contrast, for models with an abrupt density discontinuity, the finite-temperature system becomes inappropriate, as they explicitly depend on the density gradient term, which would introduce singularities into the system equations.

For other values of the parameter k_z , the plasma frequency Ω_p corresponding to each of the two degenerate points shifts, yet these points continue to exist in the spectrum. Figure 3(b) illustrates how the two resonance points (the R wave in red and the L wave in blue) move as functions of k_z and Ω_p . For every fixed k_z , there are two intersections. However, when the overall density, i.e., Ω_p , sufficiently large, the R-wave resonance point disappears. In such a situation, no chiral interface modes emerge, as shown in Figs. 5(c) and 5(d).

Although the L-wave resonance point still gives rise to a locally nontrivial Berry monopole, producing an integer gap Chern number difference of $+1$ between the two bulks (which, as shown in the previous section, corresponds to the spectral flow index), no chiral modes can appear if the gap condition between the green and red bands in Fig. 3(a) is not satisfied.

The R-wave resonance frequency ω_{pc} in magnetized warm plasma system is given by

$$\begin{aligned}\omega_1^2 - \Omega_p^2 - k_z^2 u^2 &= 0, \\ \omega_2^3 - \omega_2(k_z^2 + \Omega_p^2) - (\omega_2^2 - k_z^2) &= 0, \\ \omega_1 = \omega_2 &:= \omega_{pc} > 0\end{aligned}\quad (31)$$

(see Appendix E). The condition for the existence of a chiral mode(TLCWs) in the gap is thus

$$(\Omega_p^- - \omega_{pc})(\Omega_p^+ - \omega_{pc}) < 0,$$

where the degenerate frequency ω_{pc} depends on both the plasma frequency Ω_p and the temperature u , implying that different temperature profiles can define distinct bulk regions. In other words, topologically Langmuir cyclotron waves (TLCWs) may also arise at locations corresponding to “resonant temperatures.” Hence, we consider a uniform background density n_0 and magnetic field B_0 , along with a one-dimensional nonuniform temperature profile $T_e(x)$. Assuming again the equilibrium condition $en\mathbf{E}_0 + \nabla P_{e0} = 0$, where the pressure gradient balances the electric field, and employing the ideal gas law $P_{e0} = n_0 k_B T_e(x)$, we note that the uniform background density eliminates density gradient terms from the governing wave equations. Furthermore, temperature gradient terms are removed during linearization.

Following the same normalization as shown in Appendix B, we find that under this setting, the symbol H takes the same form as H_0 [see Eq. (22)] throughout the entire phase space. Therefore, Eq. (30) can also be applied to analyze interface topological modes arising from nonuniform temperature distributions $u(x)$. However, since the expression of ω_2 in Eq. (30) does not explicitly depend on u , the common gap condition is not satisfied for all values of u when $k_\perp = 0$. In this case, the TLCW (R wave) disappears, and at the resonance point of the Langmuir and cyclotron modes, only a topological contact remains—no discrete edge mode appears within the gap (see Fig. 7).

For more comprehensive discussions on the topology and symmetry of TLCWs, the reader is referred to [12,15,18].

IV. CONCLUSIONS

This paper establishes a rigorous theoretical framework that connects the spectral flow index—a genuine topological invariant in phase space—to the pseudo-Chern numbers defined in momentum space. The key findings bridge the heuristic application of pseudo-Chern numbers with the rigorous framework of phase-space topology and spectral flow, providing a solid justification for previous results derived through the heuristic bulk-edge correspondence.

We demonstrate that although pseudo-Chern numbers are not true topological invariants, they can be related to the spectral flow index under specific conditions. By defining appropriate connections and curvatures, such as the Berry

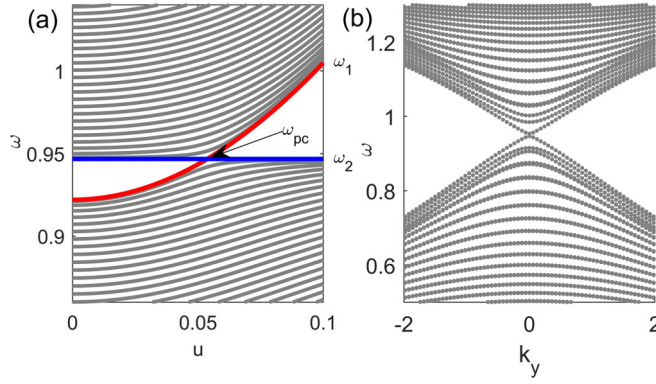


FIG. 7. Band structure relying on temperature profile. (a) Band of symbol H with fixed plasma frequency $\Omega_p = 0.922$ and $k_z = 4$. Lines in the upper and lower bands represent different $k_\perp$ values. (b) Band structure of plasma with a temperature profile as Fig. 1, where u is 0 and 0.1 in two bulks.

connection and Berry curvature, we prove that the difference in pseudo-Chern numbers of bulk bands in momentum space equals the spectral flow index. This relationship holds even without regularization in systems with symmetry in momentum space, despite the presence of noninteger bulk Chern numbers.

We also apply this framework to a $(2+1)$ -dimensional finite-temperature magnetized plasma model, emphasizing how symmetry negates regularization requirements while aligning the spectral flow index with differences in gap Chern numbers. It has been demonstrated that the bulk-edge correspondence expressed in our framework is valid, provided that the total integral over the phase-space surface at infinity approaches zero. Moreover, in warm magnetized plasma systems, topological Langmuir-cyclotron waves (TLCW) still exist, and the results are qualitatively similar to those found in studies of magnetized cold plasma. Finally, we present a generalized condition under which TLCW can exist. As a topological edge wave, TLCW can propagate unidirectionally and without reflection or scattering along complex boundaries. Due to its intrinsic topological stability, TLCW could be relatively easy to excite in experimental settings.

However, real-world plasmas—whether in laboratory or astrophysical environments—are influenced by numerous physical processes not accounted for in this model, such as collisions and nonuniform finite temperature effects. To enable practical applications, these factors must be thoroughly investigated through both experimental approaches and theoretical analyses.

The findings presented in this work provide a robust justification for previous results obtained through heuristic bulk-edge correspondence, bridging the gap between the heuristic use of pseudo-Chern numbers and the rigorous framework of phase-space topology and spectral flow. This research paves the way for a deeper understanding of topological phenomena in continuous media.

ACKNOWLEDGMENT

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: MATHEMATICAL TOOLS

1. Weyl quantization

For Weyl quantization [28], an operator $a(x, \xi) \in S(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{C})$, called a symbol on the phase space $T^*\mathbb{R}^n = \mathbb{R}^{2n}$, is associated with a pseudo-differential operator (PDO) denoted by $\hat{a} = Op_\epsilon(a)$. For a function $\psi \in S(\mathbb{R}^n)$, this operator is defined as

$$(Op_\epsilon(a)\psi)(x) = \frac{1}{(2\pi\epsilon)^n} \int a\left(\frac{x+y}{2}, \xi\right) \exp\left(\frac{i}{\epsilon}\xi(x-y)\right) \psi(y) dy d\xi. \quad (A1)$$

For a given operator $\hat{a}$, its symbol can be obtained through the Wigner transform:

$$a(x, \xi) = \int \left\langle x + \frac{y}{2} \left| \hat{a}(\hat{x}, \hat{\xi}) \right| x - \frac{y}{2} \right\rangle \exp\left(-\frac{i}{\epsilon}\xi y\right) dy, \quad (A2)$$

where $\hat{x} = x$ and $\hat{\xi} = -i\epsilon \nabla$.

2. Faure's index theorem

Faure's index theorem [16] relates the spectral flow index to the Chern number (Berry-Chern monopoles) arising from the degenerate points of two bands of a Weyl symbol.

(1) Setup:

Consider an operator $\hat{H}_\mu(x, -i\epsilon \nabla)$ with an Hermite symbol $H_\mu(x, \xi)$. The symbol is an $n \times n$ matrix with eigenvalues denoted by

$$\omega_1(\mu, x, \xi) \leq \omega_2(\mu, x, \xi) \leq \dots \leq \omega_n(\mu, x, \xi).$$

(2) Spectrum Gap Assumption:

Suppose there exists an index $r \in \{1, \dots, n-1\}$ and a constant $C > 0$ such that for all (μ, x, ξ) with $\|(\mu, x, \xi)\| \geq r > 0$, the following strict inequality holds:

$$\omega_s(\mu, x, \xi) < -C < C < \omega_{s+1}(\mu, x, \xi). \quad (A3)$$

In other words, there is a nonzero gap $[-C, C]$ separating ω_s from ω_{s+1} .

(3) Spectral Flow:

Vary the parameter μ continuously from $-\infty$ to $+\infty$. The *spectral flow* N is defined as the net number of eigenvalues of $\hat{H}_\mu(\hat{x}, \hat{\xi})$ that move from below the band gap $[g_1, g_2]$ to above it (counted with orientation).

(4) Chern Number:

Let $\omega_1, \omega_2, \dots, \omega_s$ be the eigenvalues of $H_\mu(x, \xi)$ lying below the band gap. The corresponding eigenvectors form a vector bundle F^- over a sphere in phase space that encloses the region where $\omega_s(\mu, x, \xi) = \omega_{s+1}(\mu, x, \xi)$ (i.e., the gapless points). The integral of the Berry curvature of these eigenvectors over that enclosing sphere yields an integer, called the *Chern number* of the eigenbundle F^- , denoted by C .

(5) Faure's Index Theorem:

Faure's result states that

$$N = C,$$

i.e., the spectral flow index N equals the Chern number C .

This theorem establishes a topological correspondence: the net count of how many eigenvalues cross the gap (spectral flow) matches the total Berry-Chern charge (Chern number) associated with the degenerate points in the Weyl symbol.

APPENDIX B: LINEARIZATION AND NORMALIZATION OF WEMHD

Substituting the isothermal relation $P_e = n k_B T_e$ into the WEMHD equations gives

$$m_e n \partial_t \mathbf{v} = -n e (\mathbf{E} + \mathbf{v} \times \mathbf{B}) - k_B T_e \nabla n, \quad (\text{B1a})$$

$$\partial_t n = -n \nabla \cdot \mathbf{v} - \nabla n \cdot \mathbf{v}, \quad (\text{B1b})$$

$$\partial_t \mathbf{B} = -\nabla \times \mathbf{E}, \quad (\text{B1c})$$

$$\partial_t \mathbf{E} = c^2 \nabla \times \mathbf{B} + c^2 \mu_0 n e \mathbf{v}. \quad (\text{B1d})$$

Next, consider the decomposition $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1$, $\mathbf{E} = \mathbf{E}_0(x) + \mathbf{E}_1$, $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1$, and $n = n_0(x) + n_1$, where $\mathbf{v}_0 = 0$, $n(x) e \mathbf{E}_0(x) + k_B T_e \nabla n(x) = 0$, and $\mathbf{B}_0 = B_z \mathbf{e}_z$. Linearizing the WEMHD equations around this equilibrium yields

$$\partial_t \tilde{\mathbf{v}}_1 = -[\omega_p(x) \tilde{\mathbf{E}}_1 + \tilde{\mathbf{v}}_1 \times \mathbf{e}_z] - \frac{v_s}{c} \nabla \tilde{n}_1 + \frac{1}{2} \frac{v_s}{c} \frac{\nabla n_0}{n_0} \tilde{n}_1, \quad (\text{B2a})$$

$$\partial_t \tilde{n}_1 = -\frac{v_s}{c} \nabla \cdot \tilde{\mathbf{v}}_1 - \frac{1}{2} \frac{v_s}{c} \frac{\nabla n_0}{n_0} \cdot \tilde{\mathbf{v}}_1, \quad (\text{B2b})$$

$$\partial_t \tilde{\mathbf{B}}_1 = -\nabla \times \tilde{\mathbf{E}}_1, \quad (\text{B2c})$$

$$\partial_t \tilde{\mathbf{E}}_1 = \nabla \times \tilde{\mathbf{B}}_1 + \omega_p(x) \tilde{\mathbf{v}}_1, \quad (\text{B2d})$$

where each variable with a tilde denotes a dimensionless, normalized perturbation. Time is normalized by $1/\Omega$, length by c/Ω , the electric field E_1 by a reference field E , where $\Omega = e B_z/m_e$ is the electron cyclotron frequency; the magnetic field B_1 by E/c , velocity v_1 by $\sqrt{\epsilon_0 E^2/m_e n_0}$, and density

n_1 by $\sqrt{n_0 \epsilon_0 E^2/k_B T_e}$; c is the speed of light; and the function $\Omega_p(x)$ is defined by $\Omega_p(x) = \sqrt{n_0 e^2/\epsilon_0 m_e}/(e B_z/m_e)$, and $v_s = \sqrt{k_B T_e/m_e}$.

APPENDIX C: CLOSEDNESS OF THE $(2n-1)$ FORM $\text{Tr}((U^{-1}dU)^{2n-1})$ FOR $U \in U(N)$

From the identity

$$0 = d(U^{-1}U) = d(U^{-1})U + U^{-1}(dU)$$

and the relation $U^{-1} = U^*$, we deduce

$$(U^{-1}dU)^* = (dU)^*(U^{-1})^* = (dU^{-1})U.$$

Hence $U^{-1}(dU)$ is a skew-symmetric one-form, denoted by A . We also have

$$d(U^{-1}) = -U^{-1}(dU)U^{-1}, \quad (\text{C1})$$

implying

$$\begin{aligned} dA &= d(U^{-1}dU) = d(U^{-1}) \wedge dU = U^{-1}(dU)U^{-1} \wedge dU \\ &= -(U^{-1}(dU))^{\wedge 2} = -A \wedge A. \end{aligned} \quad (\text{C2})$$

Applying the Leibniz rule to A^{2n-1} gives

$$\begin{aligned} d(A^{2n-1}) &= \sum_{k=1}^{2n-1} (-1)^{k-1} A^{k-1} \wedge (dA) \wedge A^{2n-k-1} \\ &= \sum_{k=1}^{2n-1} (-1)^k A^{2n} = -A^{2n}. \end{aligned} \quad (\text{C3})$$

Since A is a skew-symmetric form, we can write it locally as

$$A = \sum_{\mu=1}^{2n} A_{\mu} dx_{\mu}, \quad (\text{C4})$$

where each coefficient A_{μ} is a skew-symmetric matrix over the complex field. Consequently,

$$\begin{aligned} A^{2n} &= \sum_{\mu_1, \mu_2, \dots, \mu_{2n} \in \{1, \dots, 2n\}} A_{\mu_1} A_{\mu_2} \cdots A_{\mu_{2n}} dx_{\mu_1} \wedge dx_{\mu_2} \wedge \cdots \wedge dx_{\mu_{2n}} \\ &= \left(\sum_{\mu_1, \dots, \mu_{2n}} (-1)^{\tau(\mu_1, \mu_2, \dots, \mu_{2n})} A_{\mu_1} \cdots A_{\mu_{2n}} \right) dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{2n} \\ &\equiv G dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{2n}. \end{aligned} \quad (\text{C5})$$

Using the cyclic invariance of the trace, we obtain

$$\begin{aligned} \text{Tr} G &= \text{Tr} \left(\sum_{\mu_1, \dots, \mu_{2n}} (-1)^{\tau(\mu_1, \dots, \mu_{2n})} A_{\mu_1} \cdots A_{\mu_{2n}} \right) \\ &= \frac{1}{2} \sum_{\mu_1, \dots, \mu_{2n}} \text{Tr}((-1)^{\tau(\mu_1, \dots, \mu_{2n})} A_{\mu_1} \cdots A_{\mu_{2n}} + (-1)^{\tau(\mu_{2n}, \mu_1, \dots, \mu_{2n-1})} A_{\mu_{2n}} A_{\mu_1} \cdots A_{\mu_{2n-1}}) = 0. \end{aligned} \quad (\text{C6})$$

In the last step, we use $(-1)^{\tau(\mu_1, \dots, \mu_{2n})} = -(-1)^{\tau(\mu_{2n}, \mu_1, \dots, \mu_{2n-1})}$.

APPENDIX D: BERRY CURVATURE INTEGRAL IN B^0 AND GAP CONDITION OF H

Writing the symbol $H(x, k_x, k_y)$ in full matrix form yields

$$H = \begin{pmatrix} 0 & -i & 0 & -i\Omega_p(x) & 0 & 0 & 0 & 0 & 0 & \frac{i u}{2L_n} + u k_x \\ i & 0 & 0 & 0 & -i\Omega_p(x) & 0 & 0 & 0 & 0 & u k_y \\ 0 & 0 & 0 & 0 & 0 & -i\Omega_p(x) & 0 & 0 & 0 & u k_z \\ i\Omega_p(x) & 0 & 0 & 0 & 0 & 0 & 0 & k_z & -k_y & 0 \\ 0 & i\Omega_p(x) & 0 & 0 & 0 & 0 & -k_z & 0 & k_x & 0 \\ 0 & 0 & i\Omega_p(x) & 0 & 0 & 0 & k_y & -k_x & 0 & 0 \\ 0 & 0 & 0 & 0 & -k_z & k_y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_z & 0 & -k_x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -k_y & k_x & 0 & 0 & 0 & 0 & 0 \\ -\frac{i u}{2L_n} + u k_x & u k_y & u k_z & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (D1)$$

The Berry curvature $F_i(x, k_x, k_y)$ of a band ω_i , whose corresponding eigenvector is ξ_i , can be written as

$$F_i(x, k_x, k_y) = \mathbf{e}_n \cdot \nabla \times (\xi_i^* \nabla \xi_i), \quad (D2)$$

where ∇ is defined by $\mathbf{e}_n \partial_x + \mathbf{e}_{k_x} \partial_{k_x} + \mathbf{e}_{k_y} \partial_{k_y}$. The vector $\mathbf{e}_n$ takes the form $\frac{(0, k_x, k_y)}{\|(0, k_x, k_y)\|}$ at B^0 , $(1, 0, 0)$ at B^+ , and $(-1, 0, 0)$ at B^- .

In the bulk regions where $x > d$ or $x < -d$, H reduces to H_0 [see Eq. (21)]. The operator H_0 satisfies the symmetry

$$H_0(x, -k_x, -k_y)P = PH_0(x, k_x, k_y), \quad (D3)$$

where P is a real constant diagonal matrix with $P^2 = 1$, specifically

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \quad (D4)$$

If $\xi_i(x, k_x, k_y)$ is an eigenvector of H_0 with eigenvalue $\omega_i(x, k_x, k_y)$, then

$$\xi_i(x, -k_x, -k_y) = e^{i\phi} P \xi_i(x, k_x, k_y), \quad (D5)$$

$$\omega_i(x, -k_x, -k_y) = \omega_i(x, k_x, k_y), \quad (D6)$$

where ϕ is a real constant. Hence, on the surface B^0 , the Berry curvature satisfies

$$F_i(x, -k_x, -k_y) = -F_i(x, k_x, k_y). \quad (D7)$$

On $B^\pm$, we obtain

$$F_i(x, -k_x, -k_y) = F_i(x, k_x, k_y), \quad (D8)$$

by noting that $\mathbf{e}_n(x, k_x, k_y) = -\mathbf{e}_n(x, -k_x, -k_y)$ at B^0 , whereas $\mathbf{e}_n(x, k_x, k_y) = \mathbf{e}_n(x, -k_x, -k_y)$ at $B^\pm$. Consequently, the total Berry curvature contributed by H_0 on B^0 vanishes.

Next, let us use polar coordinates for (k_x, k_y) , so that

$$k_x = k \cos \theta, \quad k_y = k \sin \theta, \quad (D9)$$

where $k = \sqrt{k_x^2 + k_y^2}$. Set $\epsilon_k = \frac{t u}{2L_n k}$ and $\epsilon = \frac{t u}{2L_n}$. The eigenvalue polynomial of $H_t = H_0 + t H_1$, namely, $\det(H_t(x, k_x, k_y) - \omega I)$, is an even-degree polynomial in ω . It can be written as ω^2 multiplied by a fourth-degree polynomial in ω^2 . Because of the factor ω^2 , there are two zero eigenvalues, and the even-degree structure implies that $\pm\omega$ appear as eigenvalue pairs.

Consider the asymptotic behavior for large k . Suppose $\omega \sim k^s$. If $s > 1$, then $\det(H_t - \omega I) \sim \omega^{10}$ and there are no roots whose order exceeds 1. If $s = 1$, let $\omega = k\lambda$. Inserting $\omega = k\lambda$ into the characteristic polynomial and factoring out $k^{10}\lambda^4$ gives

$$k^{10}\lambda^4[\lambda^6 - (2 + u^2)\lambda^4 + (1 + 2u^2)\lambda^2 - u^2], \quad (D10)$$

which factors further as

$$k^{10}\lambda^4(\lambda^2 - 1)^2(\lambda^2 - u^2). \quad (D11)$$

Hence, for $\omega \sim k$, there are six nonzero asymptotic roots of $\det(H_t - \omega I) = 0$. For $0 \leq s < 1$, the polynomial is asymptotically dominated by $k^6 u^2 \omega^2$ and has no roots. Because $\det(H_t - \omega I) = 0$ is a tenth-degree polynomial containing the factor ω^2 , there are two roots if $s < 0$.

Focusing on the leading-order term of the discriminant of $\det(H_t - \omega I)/\omega^2$ (expanded as a fourth-degree polynomial in ω^2) reveals

$$k^{16}[u^4(-1 + u^2)^2[1 + 4k_z^2(-1 + u^2)^2]\Omega_p^4 + O(\epsilon_k^2)]. \quad (D12)$$

This expression does not vanish, as $\epsilon_k \ll 1$ for $k \rightarrow +\infty$. It follows that no additional degeneracies arise in H_t for large k , aside from the two zeros introduced by the factor ω^2 . Label the positive bands of H_t in ascending order by $\omega_1, \omega_2, \omega_3, \omega_4$. Only ω_1 and ω_2 maintain a nonzero gap, dictated by their asymptotic behavior. This conclusion holds for all $t \in [0, 1]$.

Therefore, for sufficiently large radii R of $B^\pm$, the eigenvector bundle of H_0 below the gap between ω_1 and ω_2 is homotopically equivalent to the corresponding eigenvector bundle of $H_0 + H_1$ on the surface Z_r .

APPENDIX E: ANALYTIC CALCULATION OF H^0 : CHERN NUMBER WITH TWO-BAND APPROXIMATION NEAR THE WEYL POINT

The Weyl point serves as a “Berry curvature monopole” in phase space thereby imparting a nontrivial topology to the eigenbundle. The Chern number of an eigenbundle hosting an isolated Weyl point can be computed analytically via a two-band approximation, as performed in the magnetized cold plasma case [12]. We employ this method for H_0 in the following.

In the phase space of $H_0(x, k_x, k_y)$, the Weyl points lie along the line $k_x = 0$ and $k_y = 0$, as seen in Fig. 3. Setting $k_x = k_y = 0$ reduces the eigenvalue formula (23) to

$$\begin{aligned} \det(H_0(x, k_x, k_y) - \omega I) \\ = \omega^2 (\omega^2 - \Omega_p^2 - k_z^2 u^2) (\omega^3 - \omega(k_z^2 + \Omega_p^2) \\ + (\omega^2 - k_z^2)) (\omega^3 - \omega(k_z^2 + \Omega_p^2) - (\omega^2 - k_z^2)), \end{aligned} \quad (E1)$$

which admits two classes of nonzero roots:

Class 1: $\omega_1^2 - \Omega_p^2 - k_z^2 u^2 = 0$.

The frequency ω_1 corresponds to the electron Langmuir wave at finite temperature.

Class 2: $\omega_2^3 - \omega_2(k_z^2 + \Omega_p^2) \pm (\omega_2^2 - k_z^2) = 0$.

The frequency ω_2 corresponds to the L wave (use +) or the R wave (use −), each exhibiting symmetry for positive and negative branches.

We focus on the L-wave resonance in the positive-frequency branch. Hence,

$$\begin{aligned} \omega_1^2 - \Omega_p^2 - k_z^2 u^2 &= 0, \\ \omega_2^3 - \omega_2(k_z^2 + \Omega_p^2) + (\omega_2^2 - k_z^2) &= 0, \\ \omega_1 = \omega_2 := \omega_{pc} > 0, \end{aligned} \quad (E2)$$

which is the resonance condition that must be met at the plasma interface. Consequently, for a given k_z , the resonance frequency lies between the plasma frequencies of the two bulk regions. Alternatively, one can rearrange the last condition to

$$\begin{aligned} (\sqrt{\Omega_p^2 + k_z^2 u^2})^3 - \sqrt{\Omega_p^2 + k_z^2 u^2} (k_z^2 + \Omega_p^2) \\ + (\Omega_p^2 + k_z^2 u^2 - k_z^2) = 0, \end{aligned} \quad (E3)$$

whose solution is $\Omega_p = \Omega_p(k_z)$.

The eigenvalue $\omega_1 = \sqrt{\Omega_p^2 + k_z^2 u^2}$ corresponds to the eigenvector

$$|v_1\rangle = \{0, 0, \omega_1, 0, 0, i\Omega_p, 0, 0, 0, k_z u\}, \quad (E4)$$

while the eigenvector for ω_2 is

$$|v_2\rangle = \{i(k_z^2 - \omega_{pc}^2), k_z^2 - \omega_{pc}^2, 0, \omega_{pc} \Omega_p, -i\omega_{pc} \Omega_p, 0, i k_z \Omega_p, k_z \Omega_p, 0, 0\}. \quad (E5)$$

We then construct a two-band system:

$$H_2 = \begin{pmatrix} \langle v_1 | H | v_1 \rangle & \langle v_1 | H | v_2 \rangle \\ \langle v_2 | H | v_1 \rangle & \langle v_2 | H | v_2 \rangle \end{pmatrix}, \quad (E6)$$

where

$$\begin{aligned} H &= (H_0 + (\partial_{k_x} H_0) \delta k_x \\ &+ (\partial_{k_y} H_0) \delta k_y + (\partial_{\omega} H_0) \delta \Omega_p) \big|_{k_x=0, k_y=0, \Omega_p=\omega_{pc}}, \end{aligned}$$

and $\delta > 0$ is small, with $\delta k_x^2 + \delta k_y^2 + \delta \Omega_p^2 = \delta^2$ defining the sphere S_δ where the topology of the eigenbundle is examined.

An explicit form of H_2 is

$$\begin{aligned} H_2 &= \begin{pmatrix} \omega_1 + \frac{\Omega_p}{\omega_1} \delta \Omega_p & \frac{(i \delta k_x + \delta k_y) k_z l(\omega_1^2 - u^2 \omega_2^2)}{2 \omega_1 \sqrt{(k_z^2 - \omega_2^2)^2 + \Omega_p^2 (k_z^2 + \omega_2^2)}} \\ \frac{(-i \delta k_x + \delta k_y) k_z (\omega_1^2 - u^2 \omega_2^2)}{2 \omega_1 \sqrt{(k_z^2 - \omega_2^2)^2 + \Omega_p^2 (k_z^2 + \omega_2^2)}} & \omega_2 + \frac{2 \omega_2 \Omega_p (\omega_2^2 - k_z^2)}{(k_z^2 - \omega_2^2)^2 + \Omega_p^2 (k_z^2 + \omega_2^2)} \delta \Omega_p \end{pmatrix} \\ &:= \begin{pmatrix} a & b^* \\ b & d \end{pmatrix}. \end{aligned} \quad (E7)$$

Near the Dirac point $\omega_1 = \omega_2 = \omega_{pc}$, the eigensystem of H_2 can be solved explicitly:

$$\begin{aligned} \lambda_1 &= \frac{1}{2}(a + d - \sqrt{(a - d)^2 + 4|b|^2}), \quad \lambda_2 = \frac{1}{2}(a + d + \sqrt{(a - d)^2 + 4|b|^2}), \quad \psi_1 = \left(\frac{a - d - \sqrt{(a - d)^2 + 4|b|^2}}{2b}, 1 \right) \\ &=: (\Psi_{11}, 1), \quad \psi_2 = \left(\frac{a - d + \sqrt{(a - d)^2 + 4|b|^2}}{2b}, 1 \right). \end{aligned} \quad (E8)$$

Imposing the resonance condition (31), one obtains

$$a - d = \frac{k_z^2 (1 - u^2) (k_z^2 + \omega_{pc}^2) \Omega_p}{\omega_{pc} (k_z^2 - \omega_{pc}^2)^2 + \omega_{pc} \Omega_p^2 (k_z^2 + \omega_{pc}^2)} \delta \omega := \alpha \delta \omega, \quad (E9)$$

where $\alpha > 0$. Hence, the only zero of Ψ_{11} on the sphere S_δ is $(\delta \omega, \delta k_x, \delta k_y) = (\delta, 0, 0)$. The Chern number associated with Ψ_1 equals the winding number of $\Psi_{11}(\delta, \delta k_x, \delta k_y)$ as $(\delta k_x, \delta k_y)$

encircle $(0, 0)$ in an anticlockwise manner. Noting that

$$\begin{aligned} \frac{\psi_{11}}{|\psi_{11}|} &= -\frac{1}{2b} \bigg/ \frac{1}{2b} = e^{-\frac{i\pi}{2}} e^{i\theta}, \\ \text{where } e^{i\theta} &= \frac{\delta k_x + i \delta k_y}{\sqrt{\delta k_x^2 + \delta k_y^2}}, \end{aligned} \quad (E10)$$

the phase of $\psi_{11}/|\psi_{11}|$ increases by 2π around that loop, giving a Chern number of $+1$ for Ψ_1 . Therefore, the L-wave Langmuir-cyclotron resonance has Chern number $+1$. By a similar analysis, the R-wave Langmuir-cyclotron resonance has Chern number -1 , although only the R wave satisfies the gap condition.

APPENDIX F: NUMERICAL METHOD OF BAND STRUCTURE IN FIGURE 4

Consider a density (or plasma frequency) profile along the x direction as an inhomogeneous parameter. We perform Fourier transforms in the y and z directions of Eq. (17), introducing wave numbers k_y and k_z . The computational domain extends from $-L$ to L , where the distribution in $[-L, 0]$ is shown on the right-hand side of Fig. 4. The parameters in $[0, L]$ exhibit a mirror-symmetric distribution with respect to $[-L, 0]$. Periodic boundary conditions are imposed on both ends of the interval $[-L, L]$. We discretize this interval as $x_i = -L + i\delta x$ for $i = 0, 1, \dots, 2N$, with $\delta x = L/N$.

Using this discretization, Eq. (17) takes the following discrete form:

$$\begin{aligned}
 \omega v_{x,i} &= -i v_{y,i} + i \omega_{p,i} E_{x,i} + \frac{i u}{2 L_i} \frac{n_{x,i+1/2} + n_{x,i-1/2}}{2} \\
 &\quad + i u \frac{n_{x,i+1/2} - n_{x,i-1/2}}{\Delta x}, \\
 \omega v_{y,i} &= i v_{x,i} + i \omega_{p,i} E_{y,i}, \\
 \omega v_{z,i} &= i \omega_{p,i} E_{z,i}, \\
 \omega E_{x,i} &= -i \omega_{p,i} v_{x,i} + k_z \frac{B_{y,i+1/2} + B_{y,i-1/2}}{2} \\
 &\quad - k_y \frac{B_{z,i+1/2} + B_{z,i-1/2}}{2}, \\
 \omega E_{y,i} &= -i \omega_{p,i} v_{y,i} - k_z \frac{B_{x,i+1/2} + B_{x,i-1/2}}{2} \\
 &\quad - i \frac{B_{z,i+1/2} - B_{z,i-1/2}}{\Delta x}, \\
 \omega E_{z,i} &= -i \omega_{p,i} v_{z,i} + k_y \frac{B_{x,i+1/2} + B_{x,i-1/2}}{2} \\
 &\quad + i \frac{B_{y,i+1/2} - B_{y,i-1/2}}{\Delta x}, \\
 \omega B_{x,i+1/2} &= -k_z \frac{E_{y,i+1} + E_{y,i}}{2} + k_y \frac{E_{z,i+1} + E_{z,i}}{2}, \\
 \omega B_{y,i+1/2} &= k_z \frac{E_{x,i+1} + E_{x,i}}{2} + i \frac{E_{z,i+1} - E_{z,i}}{\Delta x}, \\
 \omega B_{z,i+1/2} &= -k_y \frac{E_{x,i+1} + E_{x,i}}{2} - i \frac{E_{y,i+1} - E_{y,i}}{\Delta x}, \\
 \omega n_{x,i+1/2} &= -i v_{y,i} + i \omega_{p,i} E_{x,i} - \frac{i u}{2 L_i} \frac{E_{x,i+1} + E_{x,i-1}}{2} \\
 &\quad + i u \frac{E_{x,i+1} - E_{x,i-1}}{\Delta x}, \\
 \omega n_{y,i+1/2} &= i v_{x,i} + i \omega_{p,i} E_{y,i}, \\
 \omega n_{z,i+1/2} &= i \omega_{p,i} E_{z,i}.
 \end{aligned} \tag{F1}$$

We adopt the discrete numerical approach by Fu and Qin [11] for cold plasma, which preserves Hermiticity in the discrete system.

APPENDIX G: RELATION OF PROJECTOR Π AND BERRY CURVATURE

If H is Hermitian and $\xi_1, \xi_2, \dots, \xi_s$ denote the eigenvectors (as column vectors) corresponding to all bands below the gap of the m -dimensional symbol H , then defining the $m \times s$ matrix $\Psi = (\xi_1, \xi_2, \dots, \xi_s)$, the projector Π can be expressed, via the residue theorem, as

$$\Pi = \Psi \Psi^\dagger \quad \text{and} \quad \Psi^\dagger \Psi = \text{Id}_s. \tag{G1}$$

Hence

$$\begin{aligned}
 \text{Tr} \Pi d\Pi \wedge d\Pi &= \text{Tr}(\Psi \Psi^\dagger d(\Psi \Psi^\dagger) \wedge d(\Psi \Psi^\dagger)) \\
 &= \text{Tr}(d\Psi \Psi^\dagger)^2 + \text{Tr}(\Psi d\Psi^\dagger)^2 + \text{Tr}(\Psi \Psi^\dagger d\Psi \wedge d\Psi^\dagger) \\
 &\quad + \text{Tr}(\Psi d\Psi^\dagger \wedge d\Psi \Psi^\dagger) \\
 &= \text{Tr}(d\Psi \Psi^\dagger)^2 + \text{Tr}(\Psi d\Psi^\dagger)^2 - \text{Tr}(\Psi d\Psi^\dagger \Psi \wedge d\Psi^\dagger) \\
 &\quad + \text{Tr}(\Psi d\Psi^\dagger \wedge d\Psi \Psi^\dagger) \\
 &= 0 + \text{Tr}(\Psi d\Psi^\dagger \wedge d\Psi \Psi^\dagger) \\
 &= -\text{Tr}(d\Psi \wedge d\Psi^\dagger) \\
 &= -\sum_{i=1}^s \text{Tr}(d\xi_i \wedge d\xi_i^\dagger) \\
 &= \sum_{i=1}^s d\xi_i^\dagger \wedge d\xi_i := \sum_{i=1}^s \mathbf{F}_i \cdot d\mathbf{S},
 \end{aligned}$$

which is exactly the sum of all Berry curvature below the gap (where $d\mathbf{S}$ represent the area two-form.). The blue terms vanish because the trace of an even power of a matrix-valued one-form is zero. For detailed derivations, see Eqs. (C4)–(C6).

The following is the standard derivation of the Berry curvature expression in (29) for the Hermitian symbol $H(x, k_x, k_y)$, and we denote (x, k_x, k_y) by $\mathbf{R}$. Berry phase corresponding to n th eigenstate $|n(\mathbf{R})\rangle$ (which is ξ_n before, and we denote $\xi_n^\dagger$ by $\langle n|$) of symbol $H(\mathbf{R})$:

$$F_j^{(n)} = -\text{Im}[\varepsilon_{jkl} \partial_k \langle n | \partial_l n \rangle] = -\text{Im}[\varepsilon_{jkl} \langle \partial_k n | \partial_l n \rangle]$$

summation over repeated indices, and $\partial_l = \partial_{R_l}$. Inserting $\mathbb{I} = \sum_{n'} |n'\rangle \langle n'|$ (this holds for the sakes of all eigenvalues of H are nondegenerate, and H is Hermitian),

$$\mathbf{F}_n = -\text{Im} \left[\sum_{n' \neq n} \langle \nabla_{\mathbf{R}} n | n' \rangle \times \langle n' | \nabla_{\mathbf{R}} n \rangle \right].$$

Calculate $\langle n' | \nabla_{\mathbf{R}} n \rangle$ (both the symbol Hamiltonian H and the eigenstates $|n\rangle$ depend on $\mathbf{R}$):

$$\begin{aligned}
 H|n\rangle &= E_n|n\rangle, \\
 \nabla_{\mathbf{R}} H|n\rangle + H|\nabla_{\mathbf{R}} n\rangle &= (\nabla_{\mathbf{R}} E_n)|n\rangle + E_n|\nabla_{\mathbf{R}} n\rangle, \\
 \langle n' | \nabla_{\mathbf{R}} H|n\rangle + \langle n' | H|\nabla_{\mathbf{R}} n\rangle &= E_n \langle n' | \nabla_{\mathbf{R}} n \rangle.
 \end{aligned}$$

Since

$$\begin{aligned} \langle n'|H|n\rangle &= E_n \langle n'|n\rangle, \\ \langle n'|\nabla_{\mathbf{R}}H|n\rangle + E_n \langle n'|\nabla_{\mathbf{R}}|n\rangle &= E_n \langle n'|\nabla_{\mathbf{R}}|n\rangle, \\ \langle n'|\nabla_{\mathbf{R}}H|n\rangle &= (E_n - E_{n'}) \langle n'|\nabla_{\mathbf{R}}|n\rangle. \end{aligned}$$

Substituting this into

$$\mathbf{F}_n = -\text{Im} \left[\sum_{n' \neq n} \langle \nabla_{\mathbf{R}} n | n' \rangle \times \langle n' | \nabla_{\mathbf{R}} n \rangle \right],$$

one finds

$$\begin{aligned} \mathbf{F}_n &= -\text{Im} \left[\sum_{n' \neq n} \frac{\langle n | \nabla_{\mathbf{R}} H | n' \rangle \times \langle n' | \nabla_{\mathbf{R}} H | n \rangle}{(E_n - E_{n'})^2} \right] \\ &= -\text{Im} \left[\sum_{n' \neq n} \mathbf{f}_{nn'} \right]. \end{aligned}$$

If we sum the above $\mathbf{F}_n$ corresponding to bands that are below the gap and integrate over the region $k_x^2 + k_y^2 < r^2$, in the main text, this means summing over n from 1 to s (or equivalently, summing over all eigenvalues ω less than E_F), we obtain Eq. (29). During the summation, due to $\mathbf{f}_{nn'} = -\mathbf{f}_{n'n}$ and $\text{Re}(\mathbf{f}_{nn'}) = 0$ (because $\langle n | n \rangle = 1$, the Berry connection hence curvature is purely imaginary by taking gradient $\nabla_{\mathbf{R}}$ of

this identity), we get

$$\begin{aligned} C(r, x) &= \frac{1}{2\pi} \int_{k_x^2 + k_y^2 < r^2} \sum_{n \leq s} \mathbf{F}_n \cdot d\mathbf{k}_x d\mathbf{k}_y \\ &= \frac{-1}{2\pi i} \int_{k_x^2 + k_y^2 < r^2} \left[\sum_{n \leq s, n' \geq s+1} \mathbf{f}_{nn'} \right] \cdot d\mathbf{k}_x d\mathbf{k}_y. \end{aligned}$$

We only calculated the flux in (k_x, k_y) space, and x is the spatial parameter which determines the bulk $H(x, k_x, k_y)$ lies in. The “ $\sum_{n \leq s, n' \geq s+1}$ ” corresponds to “ $\sum_{\omega \leq E_F, \omega' \geq E_F}$ ” in the main text.

APPENDIX H: TOPOLOGICAL LANGMUIR-CYCLOTRON WAVES, CORRECTIONS DUE TO COLLISIONAL AND VISCOUS EFFECTS

This model in Sec. III does not include collisional damping or viscosity terms such as $\mathbf{v}/\tau$ or $\eta \nabla^2 \mathbf{v}$, this need that the plasma regime considered here satisfies the collisionless condition ($\omega \gg 1/\tau$) and is characterized by scale lengths much larger than the viscous diffusion length. In more general case, non-Hermitian terms like those associated with effective dissipation γ in [29] are involved.

If these two terms are introduced, the velocity equation in Eq. (14) is modified as

$$m_e n (\partial_t \mathbf{v} + \mathbf{v}/\tau - \eta \nabla^2 \mathbf{v}) = -ne(\mathbf{E} + \mathbf{v} \times \mathbf{B}) - \nabla P_e, \quad (\text{H1})$$

And the corresponding symbol H is modified to

$$H = \begin{pmatrix} i\eta & -i & 0 & -i\Omega_p(x) & 0 & 0 & 0 & 0 & 0 & \frac{i u}{2L_n} + u k_x \\ i & i\eta & 0 & 0 & -i\Omega_p(x) & 0 & 0 & 0 & 0 & u k_y \\ 0 & 0 & i\eta & 0 & 0 & -i\Omega_p(x) & 0 & 0 & 0 & u k_z \\ i\Omega_p(x) & 0 & 0 & 0 & 0 & 0 & 0 & k_z & -k_y & 0 \\ 0 & i\Omega_p(x) & 0 & 0 & 0 & 0 & -k_z & 0 & k_x & 0 \\ 0 & 0 & i\Omega_p(x) & 0 & 0 & 0 & k_y & -k_x & 0 & 0 \\ 0 & 0 & 0 & 0 & -k_z & k_y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_z & 0 & -k_x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -k_y & k_x & 0 & 0 & 0 & 0 & 0 \\ -\frac{i u}{2L_n} + u k_x & u k_y & u k_z & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (\text{H2})$$

where η is $\tau + \gamma(k_x^2 + k_y^2 + k_z^2)$. For simplicity, we omit the normalization procedure of the non-Hermitian term η , noting only that the temporal and spatial scales used for normalization are consistent with those in Appendix D.

Fixing $k_z = 0.9$ and $u = 0.1$, we consider two cases: $1/\tau = 0.5$, $\gamma = 0$, and $1/\tau = 0$, $\gamma = 0.5$, to evaluate the effects of the two non-Hermitian terms on the symbol bands shown in Fig. 3 and the spectral flow illustrated in Fig. 5. Due to the introduction of non-Hermitian terms, the bands of H are no longer purely real. As a result, the theoretical framework established in Sec. II for Hermitian symbols requires modification. This issue will be addressed in our future work. For now, it is sufficient to note that when the symbol satisfies the line gap condition, a correspondence between spectral flow and Chern number still holds [30]. However, in this case,

the Chern number must be computed using the Levi-Civita connection, and it can no longer be reduced to a simple sum over the Chern numbers of individual bands below each gap.

Under the above parameters, the results of our calculation are presented in Fig. 8, where Figs. 8(a)–8(f) illustrate the influence of non-Hermitian effects—specifically, collision time [Figs. 8(a)–8(c)] and viscosity [Figs. 8(d)–8(f)]—on the spectral flow in magnetized plasma systems.

Figures 8(a) and 8(d) show the line gap condition in phase space, where the symbol H possesses a line gap outside the spherical shell defined by $(\Omega_p - \omega_{pc})^2 + k_\perp^2 > \epsilon^2$, enclosing the R-wave resonance ring. The line gap corresponds to the orange-shaded region in Figs. 8(a) and 8(d), which aligns with the resonance gap of the R wave under Hermitian conditions.

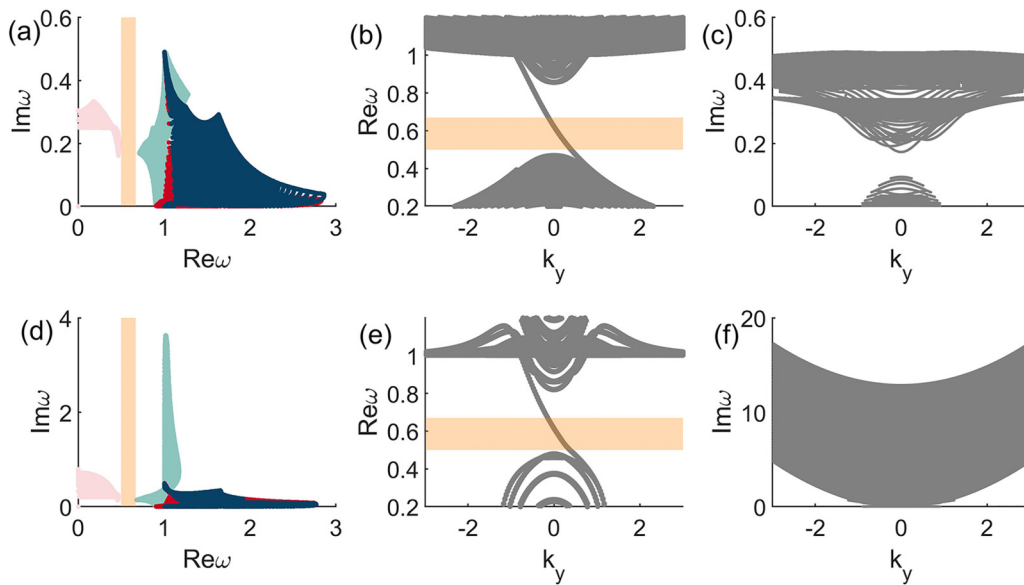


FIG. 8. Symbol bands and spectral flow under non-Hermitian effects induced by collisional and viscous terms. (a) Gapless band of symbol H with $\tau = 0.5$, $\gamma = 0$ in various (Ω_p, k_x, k_y) . (b) Real part of band structure of $\hat{H}:Op_1(H)$ with a plasma frequency distribution in equation (30), where $\delta = 1$. (c) Imaginary part of band structure corresponds to (b). (d)–(f) The case of $\tau = 0$, $\gamma = 0.5$.

Figures 8(b) and 8(e) display the corresponding one-dimensional spectral solutions, using the same parameters as in the Hermitian case (Fig. 5), with only the non-Hermitian terms introduced. These results demonstrate that spectral flow persists even in the presence of non-Hermitian perturbations.

However, since system symmetry is broken in this case, bulk-edge correspondence fails—meaning that the spectral flow can no longer be deduced solely from Berry curvature integrals in momentum space. Instead, it becomes necessary to evaluate the Chern number in phase space.

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