

Theory and simulations of few-photon Fock state pulses strongly interacting with a single qubit in a waveguide: Exact population dynamics and time-dependent spectra

Sofia Arranz Regidor^{1,*}, Andreas Knorr², and Stephen Hughes^{1,†}

¹*Department of Physics, Engineering Physics and Astronomy, Queen's University, Kingston, Ontario, Canada, K7L 3N6*

²*Institut für Theoretische Physik, Nichtlineare Optik und Quantenelektronik, Technische Universität Berlin, Berlin 10623, Germany*



(Received 8 October 2024; accepted 1 June 2025; published 23 June 2025)

We present a detailed quantum theory and simulations of few-photon Fock state pulses interacting with a two-level system (TLS) in an open waveguide-QED system. For a rectangular pulse shape, we present an exact temporal scattering theory to derive analytical expressions for the TLS population, using one-photon and two-photon pulses, for both chiral and symmetric emitters. We also derive the stationary (long-time) and time-dependent spectra for one-photon excitation, and show how these differ at a fundamental level when considering TLS population effects. Numerically, we also present matrix product state (MPS) simulations, which allow us to compute more general photon correlation functions for arbitrary quantum pulses, and we use this approach to also show results for Gaussian quantum pulses and to confirm the accuracy of our analytical solutions. In the case of a chiral TLS, we show how a one-photon pulse, of any temporal shape, yields a transmitted long-time spectrum that is identical to the input pulse, despite significant TLS population effects. However, the dynamical spectra and spectral intensity show rich population effects, allowing for their detection in spectroscopic experiments. We also show the striking differences between one-photon and two-photon excitation, where the latter shows clear signatures of nonlinear saturation effects. In addition, we demonstrate how significant population TLS dynamics also occur for pulses that are relatively long compared to the radiative decay time (showing that a weak excitation approximation, which neglects finite TLS population effects, cannot be made) and investigate the population signatures, nonlinear features and dynamical behavior as a function of quantum pulse length.

DOI: [10.1103/lp1b-yswm](https://doi.org/10.1103/lp1b-yswm)

I. INTRODUCTION

Confined waveguide modes can couple to localized quantum emitters in ways that are not possible with discrete cavity modes, opening up rich light-matter coupling regimes [1–10]. In a quantum optics picture, such systems facilitate *waveguide-QED*, which can be experimentally realized in various material systems, including waveguide systems with atoms, molecules, quantum dots, and superconducting qubits [11–22].

With superconducting waveguides (transmission lines, which use microwave photons), qubits can be arranged as *giant atoms*, where multiple qubits can be collectively coupled with great precision at various spatial points, typically separated in such a way as to cause either destructive or constructive interference [23–26]. In atomic systems, laser-cooled atoms can also be spatially trapped in nanofibers, evanescently coupled to the propagating fiber modes. From a practical perspective, semiconductor quantum dots (grown

within semiconductor waveguides) behave as ideal quantum emitters, emitting optical photons on-chip with very large waveguide beta factors [27–29] (i.e., almost all of the emitted photons are into the waveguide mode of interest). Such waveguides also offer rich design flexibility, for example, one can easily tune the chirality of the emitter coupling [30,31], opening up other areas of study such as chiral networks [32–37].

From a theoretical perspective, various approaches have been used to describe waveguide-QED systems, often with different levels of approximation. For example, with a weak excitation approximation (WEA), one neglects finite populations. In this case, the problem can be solved within the domain of linear response, e.g., using Green function techniques, allowing easy access to the polariton states, even for collective multiemitter systems [38]. However, such approximations cannot be done in the ultrastrong coupling regime [39–41], where unique quantum waveguide-QED effects appear from not making a rotating wave approximation [42–45]. For the regimes of interest in this paper, we will safely assume the rotating wave approximation, since we are interested in a single emitter whose waveguide coupling rate is much smaller than the emitter transition frequency.

At the quantum field level, other simplified approaches also exist, for example, instantaneous master equations, and second-order Born-Markov solutions where the waveguide modes can be traced out [14,46]. Unfortunately, such approaches are not well suited to look at some of the unique

*Contact author: 18sar4@queensu.ca

†Contact author: shughes@queensu.ca

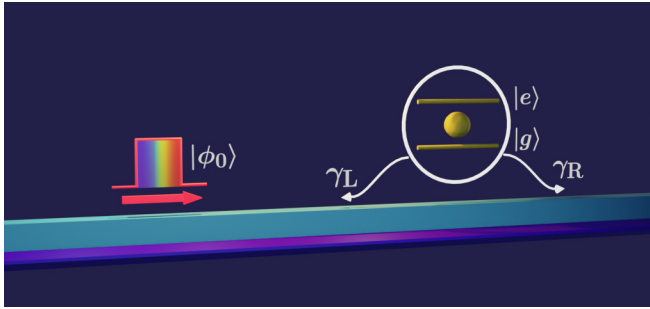


FIG. 1. Quantum field wave packet interacting with a waveguide-QED system. Schematic visualization of a single quantum emitter coupled to an open waveguide, excited with an incident rectangular pulse, for example, which could be a one-photon or two-photon Fock state pulse.

quantum light-matter entanglement that can occur when one treats the waveguide photons exactly, in a regime where the photons and emitters interact strongly.

One of the simplest waveguide-QED systems is a single qubit coupled to an open waveguide, as shown schematically in Fig. 1, which already admits various nontrivial solutions. Fan and coworkers have developed powerful scattering theories to model such systems, usually specialized to model the long-time output states with respect to the input states, allowing for a powerful description of the transmitted and reflected quantum fields, while fully accounting for the two level system (TLS) interaction [47–50]. In a waveguide-QED single TLS system, when the waveguide mode interacts strongly with the TLS, the often used WEA can drastically fail, even when one uses temporally short quantum fields at the level of one photon (a single Fock state). Indeed, it has been shown that one can fully invert a TLS when the appropriate one-photon quantum pulse excitation is employed [51]. Despite this, the WEA is still commonly employed in many scattering theories, and it is generally not well known when such an approximation works in terms of the excitation fields. It is also not well appreciated if the single excitation, which can still yield significant population transfer and photon-matter entanglement, shows any spectral signatures of population transfer, as one still expects a spectral response that resembles the linear response from a single photon excitation, which is usually well described at the level of a classical simple harmonic oscillator.

More direct time-dependent modeling of few photon scattering can also be solved from the time-dependent Schrödinger equation [52–54] or the relevant Heisenberg equations of motion [55,56]. For easier access to the multiphotons, time retardation effects and various causality-modified correlation effects, matrix product states (MPS) offers a powerful tensor network approach to solve the entire waveguide-QED system in a numerically exact way [37,57–62], in principle only limited by numerical step sizes, etc, which can easily be checked for accurate numerical convergence.

In this work, we present a detailed theoretical study of how a few-photon pulse interacts with a single TLS in a waveguide. We first present a real-space time-dependent solution to obtain

analytical results for both one-photon and two-photon (Fock state) pulses, for the TLS population. We also describe our MPS approach, and use both these methods to connect to the stationary as well as the time-dependent spectra, presenting a selection of results for both symmetric and chiral TLSs.

One of our interesting findings is that the population dynamics of a one-photon-pulse excitation completely cancels out in the transmitted stationary spectra, *but* it remains an observable for the time-dependent spectra, which directly relates to similar observables that were measured recently with classical pulses in the regime of semiconductor quantum dot cavity systems [63,64], a regime termed *dynamical resonance fluorescence*. We also show how a two-photon pulse can yield significantly different stationary spectra, as well as dynamic (time-dependent) spectra. In addition, we study how the peak TLS populations depend on the pulse duration, and present spectral solutions for various top-hat pulse durations as well as Gaussian pulses.

The rest of our article is arranged as follows. First, in Sec. II, we derive an analytical solution for the TLS population using a rectangular (top-hat) temporal quantum pulse containing one photon, for both chiral and symmetrical quantum emitters. We then extend this approach in Sec. III to a two-photon quantum pulse with the same temporal shape.

In Sec. IV, we derive an analytical solution for the stationary spectrum (i.e., the long-time limit) with a one-photon excitation pulse, again for both the symmetric and chiral emitter cases. Here, we see how all the TLS population effects that we observed in the previous sections cancel out, in the stationary spectra, independent of the ratio of TLS decay and pulse duration. In the chiral TLS case, we simply recover a transmitted spectrum that is identical to the input pulse. In order to further investigate this effect and go beyond the calculation of stationary spectra, in Sec. V, we derive an analytical solution for the chiral *time-dependent spectrum*. We start by deriving the transmitted mean photon population in Sec. VA. Then, in Sec. VB, we derive the transmitted first-order photon correlation functions, knowing that in the limit when $\tau = 0$, for the correlation time, these simply recover the mean photon population previously calculated. Here, we also verify our analytical results by numerically calculating the same observable using MPS, and show excellent agreement. Next, in Sec. VC, we show the subtle connection between the first-order photon correlations, the time-dependent spectrum, and the frequency-dependent spectral intensity.

In Sec. VI, we present the MPS theoretical approach we use that can be used to calculate one-photon and two-photon quantum pulses with any given temporal shape.

In Sec. VII, we first study a chiral TLS system with short pulse excitation (where the pulse duration is comparable to the inverse decay time), and show the population dynamics as well as the stationary spectra. We benchmark our analytical results versus MPS and show perfect agreement. In addition, we show stationary spectra and demonstrate that the one-photon results are identical to the input spectra, while the two-photon results differ significantly.

Further population results (for the TLS and transmitted fields) are shown in Sec. VIII. We begin with a comparison between chiral and symmetric TLSs (Sec. VIII A), and then study the influence of pulse duration on the TLS populations

in Sec. VIII B, where we observe that populations are considerable, even for relatively long pulses, still showing results that clearly deviate from any WEA results. Indeed, we observe that one does not recover the WEA results until the pulse durations are beyond the order of $500\gamma^{-1}$, where γ is the total TLS decay rate into the waveguide.

In Sec. IX, we show example analytical results for the time-dependent spectra and spectral intensity for a one-photon rectangular quantum pulse with a chiral emitter, and confirm that the results agree with the calculations performed using MPS. In addition, we use MPS to show the symmetrical TLS case with rectangular pulses containing one and two photons. For chiral TLSs, we also show the time-dependent spectra for one-photon and two-photon pulses, for several longer pulse durations ($t_p = 10\gamma^{-1}$ and $t_p = 50\gamma^{-1}$), demonstrating pulse breakup effects for intermediate pulse lengths ($t_p = 10\gamma^{-1}$).

Finally, in Sec. X, we further exploit MPS to study Gaussian temporal pulses, by calculating similar observables, populations, and time-dependent spectral observables, and showing results for both one and two photons. Our conclusions are presented in Sec. XI.

II. ANALYTICAL SOLUTIONS FOR THE TWO-LEVEL-SYSTEM POPULATION WITH A ONE-PHOTON (FOCK STATE) PULSE

We consider an infinite waveguide with a single quantum emitter, as shown in Fig. 1. In the frequency regime of interest, the waveguide supports a target forward and backward waveguide mode below the light line, and is thus lossless. This could be in the form of a semiconductor nanowire, a circuit-QED waveguide, or a photonic crystal waveguide—which allows lots of flexibility in how one couples to artificial atoms such as quantum dots [11,16,32,65–69]. Since we are dealing with forward (e.g., to the right, the $+z$ direction) and backward (to the left) propagating modes for waveguides, we can derive space-time solutions for the (quasi) one-dimensional (1D) envelopes of the waveguide mode operators, which satisfy the following propagation equation [70–72]:

$$(\partial_t \pm v_g \partial_z) a_{R/L}^\dagger(z, t) = i\omega_0 a_{R/L}^\dagger(z, t), \quad (1)$$

where R and L label “right” and “left” propagating fields, and we have expanded the waveguide dispersion solution as $\omega_k = \omega_0 + v_g(k - k_0)$ ($k > 0$) or $\omega_k = \omega_0 - v_g(k + k_0)$ ($k < 0$), i.e., we assume linear dispersion at some frequency of interest (near ω_0).

Defining $A(z, t) = a(z, t)e^{-i\omega_0 t}$, we can also write in the rotating frame

$$(\partial_t \pm v_g \partial_z) a_{R/L}^\dagger(z, t) = 0, \quad (2)$$

which are the 1D envelopes for the *slowly-varying* waveguide mode operators. The boson field operators satisfy the fundamental commutation relations,

$$[a_{R/L}(z, t), a_{R/L}^\dagger(z', t)] = \delta(z - z'), \quad (3)$$

where $a^\dagger(z, t)$ and $a(z, t)$ have units of $1/\sqrt{\text{m}}$. These operators describe the creation of a right- and left-moving state at position z , respectively, and allow us to introduce a real space picture for the Hamiltonian description below.

For the TLS-waveguide system, we start with the following total Hamiltonian:

$$H = H_{\text{wg}} + H_{\text{TLS}} + H_{\text{int}}, \quad (4)$$

where the free (uncoupled) waveguide term is

$$H_{\text{wg}} = \hbar \int dz [(a_R^\dagger(z, t)(\omega_0 - iv_g \partial_z) a_R(z, t)) + (a_L^\dagger(z, t)(\omega_0 + iv_g \partial_z) a_L(z, t))], \quad (5)$$

and the free TLS term, in the Heisenberg picture (where operators are time-dependent in general), is

$$H_{\text{TLS}} = \hbar \delta \sigma^+(t) \sigma^-(t), \quad (6)$$

where $\delta = \omega_a - \omega_0 = \omega_a - \omega_p$ is the detuning between the TLS (or atom), with a resonant frequency ω_a , and the waveguide mode *pulse*, with the center frequency ω_p ; thus, for on resonance conditions, $\delta = 0$.

For the TLS-field interaction term, after using a dipole and rotating wave approximation, we have

$$H_{\text{int}} = i\hbar \int dz [(g_R^* \sigma^+(t) a_R(z, t) \delta(z - z_0)) + (g_L^* \sigma^+(t) a_L(z, t) \delta(z - z_0)) - \text{H.c.}], \quad (7)$$

which, for a single TLS at z_0 , yields

$$H_{\text{int}} = i\hbar [(g_R^* \sigma^+(t) a_R(z_0, t) + g_L^* \sigma^+(t) a_L(z_0, t)) - \text{H.c.}], \quad (8)$$

where we introduced the photon-TLS coupling term $g_{R/L}$ (with right and left coupling indices), for an atom at position $\mathbf{r}_0(x_0, y_0, z_0)$. This waveguide photon-TLS coupling term is defined from

$$g_R = g_R(k_0) = \sqrt{c_0} \sqrt{\frac{\omega_0}{2\hbar\epsilon_0}} \mathbf{d} \cdot \mathbf{f}_{k_0}(\mathbf{r}_0) = d \sqrt{\frac{c_0 \omega_0}{2\epsilon_0 \hbar V_{\text{eff}}^R}}, \quad (9)$$

$$g_L = g_L(k_0) = \sqrt{c_0} \sqrt{\frac{\omega_0}{2\epsilon_0 \hbar}} \mathbf{d} \cdot \mathbf{f}_{k_0}^*(\mathbf{r}_0) = d \sqrt{\frac{c_0 \omega_0}{2\epsilon_0 \hbar V_{\text{eff}}^L}}, \quad (10)$$

where $V_{\text{eff}}^{L/R} = V_{\text{eff}}^{L/R}(\mathbf{r}_0)$ is an effective mode volume (per unit cell), $\mathbf{d}$ is the dipole moment, and $|\mathbf{f}_k|^2$ has units of inverse volume, with the normalization $\int d^3 r \epsilon(\mathbf{r}) |\mathbf{f}_k(\mathbf{r})|^2 = 1$ [27]; in addition, c_0 is the pitch (unit cell length) in the case of a photonic crystal waveguide, or a length associated with periodic boundary conditions.

Note that we write $g_{R/L} = g_{R/L}(k_0)$ for brevity, and these coupling strengths have units of $[\sqrt{\text{m}}/\text{s}]$. We can also relate these to the TLS population decay rate, $\gamma_{R/L} = |g_{R/L}|^2 / v_g$, so that

$$\gamma_{R/L} \equiv \gamma_{R/L}(\mathbf{r}_0) = \frac{d^2 c_0 \omega_0}{2v_g V_{\text{eff}}^{R/L}(\mathbf{r}_0) \hbar \epsilon_0}, \quad (11)$$

in agreement with known results for photonic crystal waveguides [27,31].

The transmitted and reflected fields can be calculated from the corresponding equations of motion for the photon operators,

$$\begin{aligned} \frac{da_{R/L}(z, t)}{dt} &= \frac{i}{\hbar} [H, a_{R/L}(z, t)] \\ &= \frac{i}{\hbar} [H_{\text{wg}} + H_{\text{int}}, a_{R/L}(z, t)]. \end{aligned} \quad (12)$$

Thus, using the result of the unperturbed field from Eq. (2), and together with Eqs. (8) and (3), we obtain

$$\begin{aligned}(\partial_t - v_g \partial_z) a_R(z, t) &= -\frac{g_R}{v_g} \sigma^-(t) \delta(z - z_0), \\ (\partial_t + v_g \partial_z) a_L(z, t) &= -\frac{g_L}{v_g} \sigma^-(t) \delta(z_0 - z),\end{aligned}\quad (13)$$

and after integrating,

$$\begin{aligned}a_R(z, t) &= a_{\text{in},R} \left(t - \frac{z}{v_g} \right) - \frac{g_R}{v_g} \sigma^- \left(t - \frac{z - z_0}{v_g} \right) \theta(z - z_0), \\ a_L(z, t) &= a_{\text{in},L} \left(t + \frac{z}{v_g} \right) - \frac{g_L}{v_g} \sigma^- \left(t - \frac{z + z_0}{v_g} \right) \theta(z_0 - z).\end{aligned}\quad (14)$$

Note that the input fields contain noise fields (free fields) as well as any considered quantum source.

Next, considering a right input field operator, and neglecting vacuum noise terms as they will not contribute in normal ordering, $a_{\text{in},R}(t - z/v_g) = a_{\text{in}}(t - z/v_g)$ and $a_{\text{in},L}(t + z/v_g) = 0$, then the transmitted field is

$$a_R(z, t) = a_{\text{in}} \left(t - \frac{z}{v_g} \right) - \frac{g_R}{v_g} \sigma^- \left(t - \frac{z - z_0}{v_g} \right) \theta(z - z_0),\quad (15)$$

while the reflected field (to the left of the TLS) is

$$a_L(z, t) = -\frac{g_L}{v_g} \sigma^- \left(t - \frac{z_0 - z}{v_g} \right) \theta(z_0 - z).\quad (16)$$

It is important to note again that $a(z, t)$ has units of $1/\sqrt{\text{m}}$ and, consequently, a_{in} has the same units.

Let us now assume $z_0 = 0$, with a right (forward) propagating input field, and using $\theta(0) = 1/2$, we derive

$$\begin{aligned}a_R(0, t) &= a_{\text{in}}(t) - \frac{g_R}{2v_g} \sigma^-(t), \quad z = 0, \\ a_R(z, t) &= a_{\text{in}} \left(t - \frac{z}{v_g} \right) - \frac{g_R}{v_g} \sigma^- \left(t - \frac{z}{v_g} \right), \quad z > 0,\end{aligned}\quad (17)$$

$$\begin{aligned}a_L(0, t) &= -\frac{g_L}{2v_g} \sigma^-(t), \quad z = 0, \\ a_L(z, t) &= -\frac{g_L}{v_g} \sigma^- \left(t + \frac{z}{v_g} \right), \quad z < 0.\end{aligned}\quad (18)$$

For any temporal input pulse shape $f(t)$ (which modulates the input quantum state), normalized with

$$\int dt |f(t)|^2 = 1,\quad (19)$$

we can define the input operator as

$$a_{\text{in}}(t) = a_0 \beta_1 f(t),\quad (20)$$

with $a_0^\dagger |0\rangle = 1$, and β_1 is a constant which has to be determined in a way that the initial photon number is fixed. The temporal pulse has a center frequency $\omega_p = \omega_0$, and propagates freely in the waveguide in the absence of a TLS. The pulse bandwidth is then determined by the quantum pulse time duration and temporal profile.

For a one-photon pulse,

$$\begin{aligned}\int_{-\infty}^{\infty} dz \langle a_{\text{in}}^\dagger(t_z) a_{\text{in}}(t_z) \rangle &= \int_{-\infty}^{\infty} dz \langle a_{\text{in}}^\dagger(t - z/v_g) a_{\text{in}}(t - z/v_g) \rangle \\ &= 1,\end{aligned}\quad (21)$$

which enables us to obtain β_1 for a one-photon input state.

Note also that

$$\int_{-\infty}^{\infty} \langle a_0^\dagger a_0 \rangle |f(t)|^2 dt = \int_{-\infty}^{\infty} |f(t)|^2 dt = v_g,\quad (22)$$

and thus to obtain a normalized pulse containing one photon, $\beta_1 = -1/\sqrt{v_g}$, and the input boson operator is

$$a_{\text{in}}(t) = -a_0 \frac{f(t)}{\sqrt{v_g}},\quad (23)$$

which naturally depends also the normalized pulse shape $f(t)$, and the annihilation operator a_0 , where $a_0 |0\rangle = 0$.

A. Symmetrical emitter (two-level system) solution

To obtain the expectation value of the TLS population, we seek to solve the appropriate differential equations of motion in the Heisenberg representation. We first consider the case of symmetrical TLS coupling. The equation of motion for $\sigma^-(t)$ is obtained from:

$$\begin{aligned}\frac{d\sigma^-(t)}{dt} &= \frac{i}{\hbar} [H(t), \sigma^-(t)] \\ &= i\delta\sigma^-(t) - g_R^* \sigma^+(t) a_R(0, t) \sigma^-(t) - g_L^* \sigma^+(t) a_L(0, t) \sigma^-(t) + g_R^* \sigma^-(t) \sigma^+(t) a_R(0, t) + g_L^* \sigma^-(t) \sigma^+(t) a_L(0, t) \\ &= i\delta\sigma^-(t) - \left(\frac{\gamma_R}{2} + \frac{\gamma_L}{2} \right) \sigma^-(t) - g_R^* a_{\text{in}}(t) \sigma^z(t),\end{aligned}\quad (24)$$

with $\gamma_R = \gamma_L = \gamma/2$ and $g_R = g_L = g_0$. Assuming g_0 is real, then we obtain

$$\frac{d\sigma^-(t)}{dt} = i\delta\sigma^-(t) - \frac{\gamma}{2} \sigma^-(t) - g_0 a_{\text{in}}(t) \sigma^z(t).\quad (25)$$

We can also derive the differential equation for the TLS population,

$$\begin{aligned}\frac{d\sigma^+(t)\sigma^-(t)}{dt} &= \frac{i}{\hbar}[H, \sigma^+(t)\sigma^-(t)] = g_R a_R^\dagger(0, t)\sigma^-(t) + g_L a_L^\dagger(0, t)\sigma^-(t) + g_R^* \sigma^+(t)a_R(0, t) + g_L^* \sigma^+(t)a_L(0, t) \\ &= g_R a_{in}^\dagger(t)\sigma^-(t) + g_R^* \sigma^+(t)a_{in}(t) - (\gamma_R + \gamma_L)\sigma^+(t)\sigma^-(t) \\ &= -\gamma\sigma^+(t)\sigma^-(t) + (g_0 a_{in}^\dagger(t)\sigma^-(t) + \text{H.c.}),\end{aligned}\quad (26)$$

which shows explicitly the role of the radiative emission rate, γ , on the population, even in the absence of any quantum input pulse.

Now, if we apply the differential equation to an input pulse containing one photon, we obtain

$$\begin{aligned}\frac{d}{dt} \langle 1, g | \sigma^+(t)\sigma^-(t) | 1, g \rangle &= -\gamma \langle 1, g | \sigma^+(t)\sigma^-(t) | 1, g \rangle + [\langle 1, g | g_0 a_{in}^\dagger(t)\sigma^-(t) + \text{H.c.} | 1, g \rangle] \\ &= -\gamma \langle 1, g | \sigma^+(t)\sigma^-(t) | 1, g \rangle - \sqrt{\frac{\gamma}{2}} [f^*(t) \langle 0, g | \sigma^-(t) | 1, g \rangle + f(t) \langle 1, g | \sigma^+(t) | 0, g \rangle]\end{aligned}\quad (27)$$

and

$$\begin{aligned}\frac{d}{dt} \langle 0, g | \sigma^-(t) | 1, g \rangle &= \left(i\delta - \frac{\gamma}{2}\right) \langle 0, g | \sigma^-(t) | 1, g \rangle - g_0 \langle 0, g | a_{in}(t)(2\sigma^+(t)\sigma^-(t) - \mathbb{1}) | 1, g \rangle \\ &= \left(i\delta - \frac{\gamma}{2}\right) \langle 0, g | \sigma^-(t) | 1, g \rangle - \sqrt{\gamma/2} f(t) [1 - 2 \langle 0, g | \sigma^+(t)\sigma^-(t) | 0, g \rangle].\end{aligned}\quad (28)$$

Considering the resonant case, i.e., $\delta = 0$, then we have

$$\frac{d}{dt} \langle 0, g | \sigma^-(t) | 1, g \rangle = -\frac{\gamma}{2} \langle 0, g | \sigma^-(t) | 1, g \rangle - \sqrt{\gamma/2} f(t) [1 - 2 \langle 0, g | \sigma^+(t)\sigma^-(t) | 0, g \rangle].\quad (29)$$

This general approach can be extended to pulses with a higher number of photons (n) [56]:

$$\frac{d}{dt} \langle n, g | \sigma^+(t)\sigma^-(t) | n, g \rangle = -\gamma \langle n, g | \sigma^+(t)\sigma^-(t) | n, g \rangle - \sqrt{\frac{n\gamma}{2}} [f^*(t) \langle n-1, g | \sigma^-(t) | n, g \rangle + \text{H.c.}],\quad (30)$$

where n represents the photon number and $f(t)$ is the pulse shape. To solve Eq. (30), we also need the equation of motion for the coherence

$$\frac{d}{dt} \langle n-1, g | \sigma^-(t) | n, g \rangle = -\frac{\gamma}{2} \langle n-1, g | \sigma^-(t) | n, g \rangle - \sqrt{\frac{n\gamma}{2}} f(t) [1 - 2 \langle n-1, g | \sigma^+(t)\sigma^-(t) | n-1, g \rangle].\quad (31)$$

For the case of a rectangular pulse, of length t_p , containing one photon, the pulse *envelope* is defined from

$$\begin{aligned}f_{\text{rect}}(t) &= \text{rect}\left(\frac{t}{t_p}\right) = 0, \quad t > t_p \\ \text{rect}\left(\frac{t}{t_p}\right) &= \frac{1}{\sqrt{t_p}} \quad 0 < t \leq t_p,\end{aligned}\quad (32)$$

and Eq. (30) transforms to

$$\begin{aligned}\frac{d}{dt} \langle 1, g | \sigma^+(t)\sigma^-(t) | 1, g \rangle &= -\gamma \langle 1, g | \sigma^+(t)\sigma^-(t) | 1, g \rangle - 2\sqrt{\frac{\gamma}{2t_p}} \langle 0, g | \sigma^-(t) | 1, g \rangle, \quad t \leq t_p \\ \frac{d}{dt} \langle 1, g | \sigma^+(t)\sigma^-(t) | 1, g \rangle &= -\gamma \langle 1, g | \sigma^+(t)\sigma^-(t) | 1, g \rangle, \quad t > t_p.\end{aligned}\quad (33)$$

Solving for Eq. (29), yields

$$\frac{d}{dt} \langle 0, g | \sigma^-(t) | 1, g \rangle = -\frac{\gamma}{2} \langle 0, g | \sigma^-(t) | 1, g \rangle - \sqrt{\frac{\gamma}{2t_p}},\quad (34)$$

which, after integrating, we obtain

$$\langle 0, g | \sigma^-(t) | 1, g \rangle = \sqrt{\frac{2}{\gamma t_p}} [e^{-\gamma t/2} - 1].\quad (35)$$

Using this solution in Eq. (33) gives

$$\begin{aligned}\frac{d}{dt} \langle 1, g | \sigma^+(t)\sigma^-(t) | 1, g \rangle &= -\gamma \langle 1, g | \sigma^+(t)\sigma^-(t) | 1, g \rangle \\ &\quad - \frac{2}{t_p} [e^{-\gamma t/2} - 1], \quad t \leq t_p,\end{aligned}\quad (36)$$

$$\frac{d}{dt} \langle 1, g | \sigma^+(t)\sigma^-(t) | 1, g \rangle = -\gamma \langle 1, g | \sigma^+(t)\sigma^-(t) | 1, g \rangle, \quad t > t_p.\quad (37)$$

Finally, integrating Eqs. (36) and (37), gives the analytical TLS population solution, for a symmetrical emitter, as a function of time

$$\begin{aligned}\langle 1, g | \sigma^+(t) \sigma^-(t) | 1, g \rangle &= \frac{2}{\gamma t_p} [e^{-\gamma t/2} - 1]^2, \quad t \leq t_p, \\ \langle 1, g | \sigma^+(t) \sigma^-(t) | 1, g \rangle &= \frac{2}{\gamma t_p} [e^{\gamma t_p/2} - 1]^2 e^{-\gamma t}, \quad t > t_p.\end{aligned}\quad (38)$$

B. Chiral emitter solution

In the case of having a chiral TLS system, where $\gamma_R = \gamma$ and $\gamma_L = 0$, the relevant interaction equations of motion are

$$\frac{d\sigma^+(t)\sigma^-(t)}{dt} = -\gamma\sigma^+(t)\sigma^-(t) + (g_R a_{\text{in}}^\dagger(t)\sigma^-(t) + \text{H.c.}) \quad (39)$$

and

$$\frac{d\sigma^-(t)}{dt} = \left(i\delta - \frac{\gamma}{2}\right)\sigma^-(t) - g_R a_{\text{in}} \sigma^z(t). \quad (40)$$

Considering again the system on resonance, then

$$\frac{d\sigma^-(t)}{dt} = -\frac{\gamma}{2}\sigma^-(t) - g_R a_{\text{in}} \sigma^z(t). \quad (41)$$

Now using this with the one-photon rectangular pulse, and considering that in the chiral case, $|g_R|^2/v_g = \gamma_R = \gamma$, (for $t < t_p$, i.e., while the pulse is being applied), we obtain

$$\langle 0, g | \frac{d\sigma^-(t)}{dt} | 1, g \rangle = -\frac{\gamma}{2} \langle 0, g | \sigma^-(t) | 1, g \rangle - \sqrt{\frac{\gamma}{t_p}}. \quad (42)$$

Integrating this equation gives

$$\langle 0, g | \sigma^-(t) | 1, g \rangle = \frac{2}{\sqrt{\gamma t_p}} [e^{-\gamma t/2} - 1], \quad (43)$$

and

$$\frac{d}{dt} \langle 1, g | \sigma^-(t) | 2, g \rangle = -\frac{\gamma}{2} \langle 1, g | \sigma^-(t) | 2, g \rangle - \sqrt{\frac{\gamma}{t_p}} [1 - 2 \langle 1, g | \sigma^+(t) \sigma^-(t) | 1, g \rangle], \quad (50)$$

where the last term contains the one-photon solution, derived explicitly in the previous subsection. Using Eq. (48), then

$$\frac{d}{dt} \langle 1, g | \sigma^-(t) | 2, g \rangle = -\frac{\gamma}{2} \langle 1, g | \sigma^-(t) | 2, g \rangle - \sqrt{\frac{\gamma}{t_p}} \left[1 - \frac{4}{\gamma t_p} (e^{-\gamma t/2} - 1)^2 \right], \quad (51)$$

and after integrating Eq. (51), we obtain

$$\langle 1, g | \sigma^-(t) | 2, g \rangle = \frac{1}{\sqrt{\gamma t_p}} \left[-\frac{4}{\gamma t_p} e^{-\gamma t} + \left(\frac{4t}{t_p} + 1 \right) e^{-\gamma t/2} + \left(\frac{4}{\gamma t_p} - 1 \right) \right]. \quad (52)$$

and the time-derivative for the population is

$$\frac{d\sigma^+(t)\sigma^-(t)}{dt} = -\gamma\sigma^+(t)\sigma^-(t) - (g_R a_{\text{in}}^\dagger(t)\sigma^-(t) + \text{H.c.}). \quad (44)$$

Thus we obtain

$$\begin{aligned}\langle 1, g | \frac{d\sigma^+(t)\sigma^-(t)}{dt} | 1, g \rangle &= -\gamma \langle 1, g | \sigma^+(t)\sigma^-(t) | 1, g \rangle \\ &\quad - \frac{4}{t_p} [e^{-\gamma t/2} - 1].\end{aligned}\quad (45)$$

Subsequently, the relevant differential equations of motion in this chiral TLS case are

$$\begin{aligned}\frac{d}{dt} \langle 1, g | \sigma^+(t)\sigma^-(t) | 1, g \rangle &= -\gamma \langle 1, g | \sigma^+(t)\sigma^-(t) | 1, g \rangle - \frac{4}{t_p} [e^{-\gamma t/2} - 1], \quad t \leq t_p, \\ \frac{d}{dt} \langle 1, g | \sigma^+(t)\sigma^-(t) | 1, g \rangle &= -\gamma \langle 1, g | \sigma^+(t)\sigma^-(t) | 1, g \rangle, \quad t > t_p.\end{aligned}\quad (46)$$

Integrating Eqs. (46) and (47) then gives the desired time-dependent population for the chiral TLS,

$$\begin{aligned}\langle 1, g | \sigma^+(t)\sigma^-(t) | 1, g \rangle &= \frac{4}{\gamma t_p} [e^{-\gamma t/2} - 1]^2, \quad t \leq t_p, \\ \langle 1, g | \sigma^+(t)\sigma^-(t) | 1, g \rangle &= \frac{4}{\gamma t_p} [e^{\gamma t_p/2} - 1]^2 e^{-\gamma t}, \quad t > t_p,\end{aligned}\quad (48)$$

which yields precisely double the population of the symmetrical case [Eq. (38)], as expected (since the coupling strength of the chiral-field interaction is exactly double the symmetric case).

III. ANALYTICAL SOLUTION FOR THE EMITTER POPULATION USING A TWO-PHOTON PULSE

A. Symmetrical emitter population dynamics with two-photon excitation

Using the same top-hat pulse profile as before, we now consider a two-photon excitation pulse. While $t \leq t_p$, Eqs. (30) and (31) transform to

$$\frac{d}{dt} \langle 2, g | \sigma^+(t)\sigma^-(t) | 2, g \rangle = -\gamma \langle 2, g | \sigma^+(t)\sigma^-(t) | 2, g \rangle - \sqrt{\frac{\gamma}{t_p}} [\langle 1, g | \sigma^-(t) | 2, g \rangle + \text{H.c.}], \quad (49)$$

and

$$\frac{d}{dt} \langle 1, g | \sigma^-(t) | 2, g \rangle = -\frac{\gamma}{2} \langle 1, g | \sigma^-(t) | 2, g \rangle - \sqrt{\frac{\gamma}{t_p}} [1 - 2 \langle 1, g | \sigma^+(t)\sigma^-(t) | 1, g \rangle], \quad (50)$$

where the last term contains the one-photon solution, derived explicitly in the previous subsection. Using Eq. (48), then

$$\frac{d}{dt} \langle 1, g | \sigma^-(t) | 2, g \rangle = -\frac{\gamma}{2} \langle 1, g | \sigma^-(t) | 2, g \rangle - \sqrt{\frac{\gamma}{t_p}} \left[1 - \frac{4}{\gamma t_p} (e^{-\gamma t/2} - 1)^2 \right], \quad (51)$$

and after integrating Eq. (51), we obtain

$$\langle 1, g | \sigma^-(t) | 2, g \rangle = \frac{1}{\sqrt{\gamma t_p}} \left[-\frac{4}{\gamma t_p} e^{-\gamma t} + \left(\frac{4t}{t_p} + 1 \right) e^{-\gamma t/2} + \left(\frac{4}{\gamma t_p} - 1 \right) \right]. \quad (52)$$

We can use this result to solve Eq. (49), so that:

$$\frac{d}{dt} \langle 2, g | \sigma^+(t) \sigma^-(t) | 2, g \rangle = -\gamma \langle 2, g | \sigma^+(t) \sigma^-(t) | 2, g \rangle - \frac{4}{t_p} \left[-\frac{4}{\gamma t_p} e^{-\gamma t} + \left(\frac{4t}{t_p} + 1 \right) e^{-\gamma t/2} + \left(\frac{4}{\gamma t_p} - 1 \right) \right], \quad (53)$$

and then the TLS population, while $t \leq t_p$, is

$$\langle 2, g | \sigma^+(t) \sigma^-(t) | 2, g \rangle = \frac{4}{t_p} \left[\left(\frac{4t}{t_p} + \frac{20}{\gamma t_p} + 1 \right) e^{-\gamma t} + \left(\frac{8t}{t_p} - \frac{16}{\gamma t_p} - 2 \right) e^{-\gamma t/2} + \left(-\frac{4}{\gamma t_p} + 1 \right) \right]. \quad (54)$$

For $t > t_p$, Eq. (30) yields

$$\frac{d}{dt} \langle 2, g | \sigma^+(t) \sigma^-(t) | 2, g \rangle = -\gamma \langle 2, g | \sigma^+(t) \sigma^-(t) | 2, g \rangle. \quad (55)$$

Integrating and ensuring continuity at $t = t_p$, we obtain

$$\langle 2, g | \sigma^+(t) \sigma^-(t) | 2, g \rangle = \frac{4}{t_p} \left[\left(-\frac{4t}{t_p} + 1 \right) e^{\gamma t_p} + \left(-\frac{16}{\gamma t_p} + 6 \right) e^{\gamma t_p/2} + \left(\frac{20}{\gamma t_p} + 5 \right) \right] e^{-\gamma t}. \quad (56)$$

Finally, combining Eqs. (54) and (56), we get the symmetric TLS population solution for the two-photon top-hat pulse:

$$\begin{aligned} \langle 2, g | \sigma^+(t) \sigma^-(t) | 2, g \rangle &= \frac{4}{t_p} \left[\left(\frac{4t}{t_p} + \frac{20}{\gamma t_p} + 1 \right) e^{-\gamma t} + \left(\frac{8t}{t_p} - \frac{16}{\gamma t_p} - 2 \right) e^{-\gamma t/2} + \left(-\frac{4}{\gamma t_p} + 1 \right) \right], \quad t \leq t_p, \\ \langle 2, g | \sigma^+(t) \sigma^-(t) | 2, g \rangle &= \frac{4}{t_p} \left[\left(-\frac{4t}{t_p} + 1 \right) e^{\gamma t_p} + \left(-\frac{16}{\gamma t_p} + 6 \right) e^{\gamma t_p/2} + \left(\frac{20}{\gamma t_p} + 5 \right) \right] e^{-\gamma t}, \quad t > t_p. \end{aligned} \quad (57)$$

B. Chiral emitter population dynamics with two-photon excitation

The population equation of motion in a chiral emitter system transforms to

$$\frac{d}{dt} \langle 2, g | \sigma^+(t) \sigma^-(t) | 2, g \rangle = -\gamma \langle 2, g | \sigma^+(t) \sigma^-(t) | 2, g \rangle - 2\sqrt{\frac{\gamma}{t_p}} [\langle 1, g | \sigma^-(t) | 2, g \rangle + \text{H.c.}]. \quad (58)$$

Using the chiral solution for the one-photon pulse, Eq. (50) gives

$$\frac{d}{dt} \langle 1, g | \sigma^-(t) | 2, g \rangle = -\frac{\gamma}{2} \langle 1, g | \sigma^-(t) | 2, g \rangle - \sqrt{\frac{\gamma}{t_p}} \left[1 - \frac{8}{\gamma t_p} (e^{-\gamma t/2} - 1)^2 \right]. \quad (59)$$

Integrating this, using Eq. (58) and proceeding in the same manner as in the symmetrical case, we obtain the two-photon TLS population solution for the chiral system:

$$\begin{aligned} \langle 2, g | \sigma^+(t) \sigma^-(t) | 2, g \rangle &= \frac{8}{t_p} \left[\left(\frac{8t}{t_p} + \frac{40}{\gamma t_p} + 1 \right) e^{-\gamma t} + \left(\frac{16t}{t_p} - \frac{32}{\gamma t_p} - 2 \right) e^{-\gamma t/2} + \left(-\frac{8}{\gamma t_p} + 1 \right) \right], \quad t \leq t_p, \\ \langle 2, g | \sigma^+(t) \sigma^-(t) | 2, g \rangle &= \frac{8}{t_p} \left[\left(-\frac{8t}{t_p} + 1 \right) e^{\gamma t_p} + \left(-\frac{32}{\gamma t_p} + 14 \right) e^{\gamma t_p/2} + \left(\frac{40}{\gamma t_p} + 9 \right) \right] e^{-\gamma t}, \quad t > t_p. \end{aligned} \quad (60)$$

IV. ANALYTICAL SOLUTION FOR THE ONE-PHOTON STATIONARY SPECTRA

A. One-photon transmitted stationary spectra for the symmetrical emitter

We next calculate the transmitted stationary (long time) spectrum, $S_{\text{sym}}(\omega) = S_{\text{sym}}(\omega, t \rightarrow \infty)$, from

$$\begin{aligned} S_{\text{sym}}(\omega) &= v_g \langle a_R^\dagger(\omega) a_R(\omega) \rangle = v_g \left\langle \left(a_{\text{in}}^\dagger(\omega) - \frac{g_R}{v_g} \sigma^+(\omega) \right) \left(a_{\text{in}}(\omega) - \frac{g_R}{v_g} \sigma^-(\omega) \right) \right\rangle \\ &= v_g \langle a_{\text{in}}^\dagger(\omega) a_{\text{in}}(\omega) \rangle - g_R \langle a_{\text{in}}^\dagger(\omega) \sigma^-(\omega) \rangle - g_R \langle \sigma^+(\omega) a_{\text{in}}(\omega) \rangle + \frac{|g_R|^2}{v_g} \langle \sigma^+(\omega) \sigma^-(\omega) \rangle. \end{aligned} \quad (61)$$

The first term of Eq. (61) is

$$v_g \langle a_{\text{in}}^\dagger(\omega) a_{\text{in}}(\omega) \rangle = |f(\omega)|^2, \quad (62)$$

which is just the spectrum of the incident pulse. For the symmetrical coupling case, where $g_R = g_0$, the

second term is

$$g_0 \langle a_{\text{in}}^\dagger(\omega) \sigma^-(\omega) \rangle = \frac{-g_0}{\sqrt{v_g}} f^*(\omega) \langle 0, g | \sigma^-(\omega) | 1, g \rangle. \quad (63)$$

In order to solve $\langle 0, g | \sigma^-(\omega) | 1, g \rangle$, we need to use the equation of motion for σ^- [Eq. (25)] and perform a Fourier transform to obtain

$$i\omega \sigma^-(\omega) = i\delta \sigma^-(\omega) - \frac{\gamma}{2} \sigma^-(\omega) - g_0 \mathcal{F}[a_{\text{in}}(t) \sigma^z(t)], \quad (64)$$

and thus

$$\sigma^-(\omega) = \frac{-g_0}{i(\omega - \delta) + \frac{\gamma}{2}} \mathcal{F}[a_{\text{in}}(t) \sigma^z(t)]. \quad (65)$$

Applying this to Eq. (63) gives

$$g_0 \langle a_{\text{in}}^\dagger(\omega) \sigma^-(\omega) \rangle = \frac{-\gamma/2}{i(\omega - \delta) + \frac{\gamma}{2}} |f(\omega)|^2 \mathcal{F} \langle 0, g | \sigma^z(t) | 0, g \rangle. \quad (66)$$

To calculate the right-hand side, we use

$$\langle 0, g | \sigma^z(t) | 0, g \rangle = \langle 0, g | 2\sigma^+(t)\sigma^-(t) - \mathbb{1} | 0, g \rangle = -1. \quad (67)$$

Inserting this result in Eq. (66) gives

$$g_0 \langle a_{\text{in}}^\dagger(\omega) \sigma^-(\omega) \rangle = \frac{\gamma/2}{i(\omega - \delta) + \frac{\gamma}{2}} |f(\omega)|^2. \quad (68)$$

The next term of Eq. (61) is simply the conjugate of Eq. (68),

$$g_0 \langle \sigma^+(\omega) a_{\text{in}}(\omega) \rangle = \frac{\gamma/2}{-i(\omega - \delta) + \frac{\gamma}{2}} |f(\omega)|^2. \quad (69)$$

Finally, the last term of Eq. (61) can be solved using the previous results, and we obtain

$$\frac{|g_0|^2}{v_g} \langle \sigma^+(\omega) \sigma^-(\omega) \rangle = \frac{\gamma^2/4 |f(\omega)|^2}{(\omega - \delta)^2 + \gamma^2/4}. \quad (70)$$

Adding up all the previous terms, we obtain the transmitted stationary spectrum for the symmetrical emitter:

$$S_{\text{sym}}(\omega) = |f(\omega)|^2 \left(1 - \frac{\gamma^2/4}{(\omega - \delta)^2 + \gamma^2/4} \right), \quad (71)$$

where the second term represents a drop on the transmitted spectrum due to reflection from the TLS.

This is exactly the same result as a bosonic emitter, namely, a harmonic oscillator model for the atom or qubit, which also coincides with the result from a WEA, for this spectral function. However, clearly the terminology of a “weak excitation” is not appropriate here, as the atom can indeed be efficiently excited [for example, see Eq. (38)]; it just happens to yield the same stationary spectrum result as a bosonic atom in the strict one quantum subspace. Indeed, we could also simply have replaced $\sigma_z = -\mathbb{1}$ much earlier in the derivation, and identical final results will be obtained for the 1-photon case, but such an approximation would get the wrong dynamical result, discussed in more detail later.

Note to obtain Eq. (71), we have exploited the convolution theorem: $\mathcal{F}(h * g) = \mathcal{F}(h)\mathcal{F}(g)$, and $h * g = \int_{-\infty}^{\infty} h(t')g(t - t')dt'$. In this case, $h = \sigma^+(t)\sigma^-(t)f(t)$ and $g = \sigma^+(t - t')\sigma^-(t - t')f^*(t - t')$, which leads to

$$\begin{aligned} h * g &= \int_{-\infty}^{\infty} \sigma^+(t')\sigma^-(t')f(t')\sigma^+(t - t')\sigma^-(t - t')f^*(t - t')dt' = \int_{-\infty}^{\infty} \sigma^+(t')\sigma^-(t')f(t')[\sigma^+(\omega)\sigma^-(\omega)f^*(\omega)e^{i\omega(t-t')}d\omega]dt' \\ &= \int_{-\infty}^{\infty} \sigma^+(\omega)\sigma^-(\omega)f(\omega)e^{i\omega t'}\sigma^+(\omega)\sigma^-(\omega)f^*(\omega)e^{i\omega(t-t')}d\omega dt' \\ &= \int_{-\infty}^{\infty} (\sigma^+(\omega)\sigma^-(\omega))^2 |f(\omega)|^2 e^{i\omega t} d\omega = (\sigma^+(t)\sigma^-(t))^2 |f(t)|^2 = \sigma^+(t)\sigma^-(t)|f(t)|^2. \end{aligned} \quad (72)$$

B. One-photon transmitted stationary spectra for the chiral emitter

As before, we can calculate the stationary spectrum, $S_{\text{chir}}(\omega)$, from Eq. (61), where the first term is again

$$v_g \langle a_{\text{in}}^\dagger(\omega) a_{\text{in}}(\omega) \rangle = |f(\omega)|^2, \quad (73)$$

and the equation of motion for σ^- is

$$\frac{d\sigma^-(t)}{dt} = i\delta \sigma^-(t) - \frac{\gamma}{2} \sigma^-(t) - g_R a_{\text{in}}(t) \sigma^z(t). \quad (74)$$

If we perform a Fourier transform to Eq. (74),

$$i\omega \sigma^-(\omega) = i\delta \sigma^-(\omega) - \frac{\gamma}{2} \sigma^-(\omega) - g_R \mathcal{F}(a_{\text{in}}(t) \sigma^z(t)), \quad (75)$$

then

$$\sigma^-(\omega) = \frac{-g_R}{i(\omega - \delta) + \frac{\gamma}{2}} \mathcal{F}(a_{\text{in}}(t) \sigma^z(t)) \quad (76)$$

and

$$g_R \langle a_{\text{in}}^\dagger(\omega) \sigma^-(\omega) \rangle = \frac{\gamma}{i(\omega - \delta) + \frac{\gamma}{2}} |f(\omega)|^2. \quad (77)$$

The third term takes the form

$$g_R \langle \sigma^+(\omega) a_{\text{in}}(\omega) \rangle = \frac{\gamma}{-i(\omega - \delta) + \frac{\gamma}{2}} |f(\omega)|^2, \quad (78)$$

and the last term is

$$\frac{|g_R|^2}{v_g} \langle \sigma^+(\omega) \sigma^-(\omega) \rangle = \frac{\gamma^2 |f(\omega)|^2}{(\omega - \delta)^2 + \gamma^2/4}. \quad (79)$$

Adding up all the previous terms, we simply get the trivial result for the transmitted stationary spectrum,

$$S_{\text{chir}}(\omega) = |f(\omega)|^2, \quad (80)$$

and thus *the transmitted spectrum is identical to the input spectrum in this chiral case* where the emitter radiates only to the right (at least for the stationary spectrum).

It is important to stress that the cancellation of the final three terms *only occurs for the long-time spectra* as we show below, since the population dynamics of the chiral TLS indeed affect the time-dependent spectra, even for a one-photon input pulse.

V. SOLUTION FOR THE TRANSMITTED TIME-DEPENDENT SPECTRA FROM THE CHIRAL EMITTER

Here we will first derive the population terms in Sec. V A, the temporal field correlations in Sec. V B, and then the transmitted *time-dependent* spectra (Sec. V C) for the field after interacting with the quantum chiral emitter. Results for a nonchiral emitter can be derived in a similar way.

A. Derivation of the transmitted photon mean population (chiral TLS)

In general, the transmitted and reflected photon flux can be calculated from

$$n_{R/L}^{\text{out}}(t) = v_g \langle a_{R/L}^\dagger(t) a_{R/L}(t) \rangle, \quad (81)$$

where R and L , as before, refer to the right (transmitted) and left (reflected) photons, respectively. For the chiral emitter, we only need to deal with the right propagating field after the TLS interaction,

$$\begin{aligned} v_g \langle a_R^\dagger(t) a_R(t) \rangle &= v_g \left\langle \left(a_{\text{in}}^\dagger(t) - \frac{g_R}{v_g} \sigma^+(t) \right) \left(a_{\text{in}}(t) - \frac{g_R}{v_g} \sigma^-(t) \right) \right\rangle \\ &= v_g \langle a_{\text{in}}^\dagger(t) a_{\text{in}}(t) \rangle - g_R \langle a_{\text{in}}^\dagger(t) \sigma^-(t) \rangle - g_R \langle \sigma^+(t) a_{\text{in}}(t) \rangle + \frac{|g_R|^2}{v_g} \langle \sigma^+(t) \sigma^-(t) \rangle. \end{aligned} \quad (82)$$

The first term of Eq. (82) is

$$v_g \langle a_{\text{in}}^\dagger(t) a_{\text{in}}(t) \rangle = |f(t)|^2, \quad (83)$$

while the second term is

$$g_R \langle a_{\text{in}}^\dagger(t) \sigma^-(t) \rangle = -\frac{g_R}{\sqrt{v_g}} f^*(t) \langle 0, g | \sigma^-(t) | 1, g \rangle. \quad (84)$$

The coherence term evolves as

$$\begin{aligned} \langle 0, g | \frac{d\sigma^-(t)}{dt} | 1, g \rangle &= (i\delta - \gamma/2) \langle 0, g | \sigma^-(t) | 1, g \rangle \\ &\quad - \frac{g_R}{\sqrt{v_g}} f(t), \end{aligned} \quad (85)$$

and after integrating,

$$\begin{aligned} \langle 0, g | \sigma^-(t) | 1, g \rangle &= -\frac{g_R}{\sqrt{v_g}} e^{(i\delta - \gamma/2)t} \int_{-\infty}^t f(t') e^{(-i\delta + \gamma/2)t'} dt'. \end{aligned} \quad (86)$$

Introducing this in Eq. (84), gives

$$\begin{aligned} g_R \langle a_{\text{in}}^\dagger(t) \sigma^-(t) \rangle &= \frac{|g_R|^2}{v_g} f^*(t) e^{(i\delta - \gamma/2)t} \int_{-\infty}^t f(t') e^{(-i\delta + \gamma/2)t'} dt'. \end{aligned} \quad (87)$$

Similarly, for the third term, we obtain

$$\begin{aligned} g_R \langle \sigma^+(t) a_{\text{in}}(t) \rangle &= -\frac{|g_R|^2}{v_g} f(t) e^{(-i\delta - \gamma/2)t} \int_{-\infty}^t f(t') e^{(i\delta + \gamma/2)t'} dt'. \end{aligned} \quad (88)$$

Finally, to calculate the last term in Eq. (82), we can use Eq. (27), and for the chiral case, we have

$$\begin{aligned} \langle 1, g | \frac{d\sigma^+(t)\sigma^-(t)}{dt} | 1, g \rangle &= -\gamma \langle 1, g | \sigma^+(t) \sigma^-(t) | 1, g \rangle + \gamma e^{-\gamma t/2} \left[f^*(t) e^{i\delta t} \int_{-\infty}^t f(t') e^{(-i\delta + \gamma/2)t'} dt' + f(t) e^{-i\delta t} \int_{-\infty}^t f(t') e^{(i\delta + \gamma/2)t'} dt' \right]. \end{aligned} \quad (89)$$

Considering the resonant case ($\delta = 0$), then

$$\langle 1, g | \frac{d\sigma^+(t)\sigma^-(t)}{dt} | 1, g \rangle = -\gamma \langle 1, g | \sigma^+(t) \sigma^-(t) | 1, g \rangle + \gamma e^{-\gamma t/2} (f^*(t) + f(t)) \int_{-\infty}^t f(t') e^{(\gamma/2)t'} dt'. \quad (90)$$

To further check that this general solution is correct, we can calculate the chiral TLS population for a rectangular pulse (since we worked this out earlier),

$$\langle 1, g | \frac{d\sigma^+(t)\sigma^-(t)}{dt} | 1, g \rangle = -\gamma \langle 1, g | \sigma^+(t)\sigma^-(t) | 1, g \rangle - \frac{4}{t_p} [e^{-\gamma t/2} - 1], \quad (91)$$

and after integrating,

$$\langle 1, g | \sigma^+(t)\sigma^-(t) | 1, g \rangle = \frac{4}{\gamma t_p} [e^{-\gamma t/2} - 1]^2, \quad (92)$$

which agrees perfectly with the result calculated in Eq. (47).

The more general solution (in the resonant chiral case), for arbitrary pulse shapes, is

$$\langle 1, g | \sigma^+(t)\sigma^-(t) | 1, g \rangle = \gamma e^{-\gamma t} \int_{-\infty}^t dt' e^{-\gamma t'/2} (f^*(t') + f(t')) \int_{-\infty}^{t'} f(t'') e^{\gamma t''/2} dt''. \quad (93)$$

Finally, putting everything together, we obtain the transmitted photon population:

$$\begin{aligned} v_g \langle a_R^\dagger(t) a_R(t) \rangle &= |f(t)|^2 + \gamma f^*(t) e^{-\gamma t/2} \int_{-\infty}^t f(t') e^{\gamma t'/2} dt' - \gamma f(t) e^{-\gamma t/2} \int_{-\infty}^t f^*(t') e^{\gamma t'/2} dt' \\ &\quad + \gamma^2 e^{-\gamma t} \int_{-\infty}^t dt' e^{-\gamma t'/2} (f^*(t') + f(t')) \int_{-\infty}^{t'} f(t'') e^{\gamma t''/2} dt''. \end{aligned} \quad (94)$$

We observe that, in general, the chiral TLS interaction terms do not cancel out, even for the nonresonant case ($\delta \neq 0$). Thus (apart from the case of relatively very long pulses, quantified later) a WEA does not make sense here, and would produce the wrong result, even if it happens to yield the same stationary spectrum.

The transmitted photon flux for the rectangular pulse can be calculated by considering the four terms in Eq. (82). For the first term in this case,

$$v_g \langle 1, g | a_{\text{in}}^\dagger(t) a_{\text{in}}(t) | 1, g \rangle = \frac{1}{t_p}, \quad t \leq t_p, \quad v_g \langle 1, g | a_{\text{in}}^\dagger(t) a_{\text{in}}(t) | 1, g \rangle = 0, \quad t > t_p, \quad (95)$$

while for second term,

$$g_R \langle a_{\text{in}}^\dagger(t) \sigma^-(t) \rangle = \frac{-2}{t_p} [e^{-\gamma t/2} - 1], \quad t \leq t_p, \quad g_R \langle a_{\text{in}}^\dagger(t) \sigma^-(t) \rangle = 0, \quad t > t_p. \quad (96)$$

For the third term,

$$g_R \langle \sigma^+(t) a_{\text{in}}(t) \rangle = \frac{2}{t_p} [1 - e^{-\gamma t/2}], \quad t \leq t_p, \quad g_R \langle \sigma^+(t) a_{\text{in}}(t) \rangle = 0, \quad t > t_p, \quad (97)$$

and for the last term, we can use the solution of the chiral TLS population, so that

$$\frac{|g_R|^2}{v_g} \langle 1, g | \sigma^+(t)\sigma^-(t) | 1, g \rangle = \frac{4}{t_p} [e^{-\gamma t/2} - 1]^2, \quad t \leq t_p, \quad \frac{|g_R|^2}{v_g} \langle 1, g | \sigma^+(t)\sigma^-(t) | 1, g \rangle = \frac{4}{t_p} [e^{\gamma t_p/2} - 1]^2 e^{-\gamma t}, \quad t > t_p. \quad (98)$$

Putting everything together, we obtain the desired solution for the transmitted photon flux for a chiral TLS interaction:

$$v_g \langle a_R^\dagger(t) a_R(t) \rangle = \frac{1}{t_p} + \frac{4}{t_p} [e^{-\gamma t/2} - 1] + \frac{4}{t_p} [e^{-\gamma t/2} - 1]^2, \quad t \leq t_p, \quad v_g \langle \hat{a}_R^\dagger(t) \hat{a}_R(t) \rangle = \frac{4}{t_p} [e^{\gamma t_p/2} - 1]^2 e^{-\gamma t}, \quad t > t_p. \quad (99)$$

We note that for $t > t_p$, the only term contributing is the TLS population one.

The symmetric system can be derived in a similar way, which yields both transmitted and reflected fields.

B. Derivation of the transmitted first-order photon correlation function (chiral TLS)

Considering the same chiral system as above, we now seek to calculate $\langle a_R^\dagger(t) a_R(t + \tau) \rangle$, since this is needed to connect to the time-dependent spectra. This two-time correlation function is obtained from

$$\begin{aligned} v_g \langle a_R^\dagger(t) a_R(t + \tau) \rangle &= v_g \left\langle \left(a_{\text{in}}^\dagger(t) - \frac{g_R}{v_g} \sigma^+(t) \right) \left(a_{\text{in}}(t + \tau) - \frac{g_R}{v_g} \sigma^-(t + \tau) \right) \right\rangle \\ &= v_g \langle a_{\text{in}}^\dagger(t) a_{\text{in}}(t + \tau) \rangle - g_R \langle a_{\text{in}}^\dagger(t) \sigma^-(t + \tau) \rangle - g_R \langle \sigma^+(t) a_{\text{in}}(t + \tau) \rangle + \frac{|g_R|^2}{v_g} \langle \sigma^+(t) \sigma^-(t + \tau) \rangle \\ &\equiv C_1(t, t + \tau) + C_2(t, t + \tau) + C_3(t, t + \tau) + C_4(t, t + \tau), \end{aligned} \quad (100)$$

which we have split into four parts, to help with the analysis.

The first term in Eq. (100) is

$$v_g \langle a_{\text{in}}^\dagger(t) a_{\text{in}}(t + \tau) \rangle = f^*(t) f(t + \tau), \quad (101)$$

and the second term is

$$-g_R \langle a_{\text{in}}^\dagger(t) \sigma^-(t + \tau) \rangle = \sqrt{\gamma} f^*(t) \langle 0, g | \sigma^-(t + \tau) | 1, g \rangle. \quad (102)$$

To solve these equations, we can first use Eq. (42) for any general quantum pulse shape:

$$\langle 0, g | \frac{d\sigma^-(t)}{dt} | 1, g \rangle = -\frac{\gamma}{2} \langle 0, g | \sigma^-(t) | 1, g \rangle - \sqrt{\gamma} f(t), \quad (103)$$

which gives a formal solution

$$\langle 0, g | \sigma^-(t) | 1, g \rangle = -\sqrt{\gamma} e^{(i\delta - \gamma/2)t} \int_{-\infty}^t f(t') e^{(-i\delta + \gamma/2)t'} dt'. \quad (104)$$

Introducing this in Eq. (102), gives

$$-g_R \langle a_{\text{in}}^\dagger(t) \sigma^-(t + \tau) \rangle = -\gamma f^*(t) e^{(i\delta - \gamma/2)(t + \tau)} \int_{-\infty}^{(t + \tau)} f(t') e^{(-i\delta + \gamma/2)t'} dt', \quad (105)$$

which will depend on the pulse profile, $f(t)$, used in each case.

The third term in Eq. (100) can be derived in a similar way,

$$g_R \langle \sigma^+(t) a_{\text{in}}(t + \tau) \rangle = \gamma f(t + \tau) e^{(-i\delta - \gamma/2)t} \int_{-\infty}^t f^*(t') e^{(i\delta + \gamma/2)t'} dt'. \quad (106)$$

Finally, to calculate the last term of Eq. (100), we use the quantum regression theorem,

$$\frac{d}{d\tau} \langle \sigma^+(t) \sigma^-(t + \tau) \rangle = -\gamma \langle \sigma^+(t) \sigma^-(t + \tau) \rangle - \sqrt{\gamma} [f(t + \tau) \langle \sigma^+(t) \rangle], \quad (107)$$

where one uses $\langle \sigma^+(t) \sigma^-(t) \rangle$ as the initial condition, so clearly the time-dependent TLS population plays a significant role here.

For a rectangular pulse shape, the previous terms can be derived analytically,

$$\begin{aligned} C_1(t, t + \tau) &= \frac{1}{t_p}, \quad 0 < t \leq t_p \text{ and } 0 < t + \tau \leq t_p, \\ C_1(t, t + \tau) &= 0, \quad t \leq t_p \text{ and } t + \tau > t_p, \quad C_1(t, t + \tau) = 0, \quad t > t_p. \end{aligned} \quad (108)$$

The second term, in this case, gives

$$\begin{aligned} C_2(t, t + \tau) &= \frac{2}{t_p} (e^{-\gamma(t + \tau)/2} - 1), \quad 0 < t \leq t_p \text{ and } 0 < t + \tau \leq t_p, \\ C_2(t, t + \tau) &= \frac{2}{t_p} (1 - e^{\gamma t_p/2}) e^{-\gamma(t + \tau)/2}, \quad 0 < t \leq t_p \text{ and } t + \tau > t_p, \quad C_2(t, t + \tau) = 0, \quad t > t_p. \end{aligned} \quad (109)$$

While the third term becomes

$$\begin{aligned} C_3(t, t + \tau) &= \frac{2}{t_p} (e^{-\gamma t/2} - 1), \quad 0 < t \leq t_p \text{ and } 0 < t + \tau \leq t_p, \quad C_3(t, t + \tau) = 0, \quad 0 < t \leq t_p \text{ and } t + \tau > t_p, \\ C_3(t, t + \tau) &= 0, \quad t > t_p. \end{aligned} \quad (110)$$

Finally, the fourth term is

$$\begin{aligned} C_4(t, t + \tau) &= \frac{4}{t_p} (1 - e^{-\gamma t/2} + e^{-\gamma(t + \tau)/2} - e^{-\gamma(t + \tau)/2}), \quad 0 < t \leq t_p \text{ and } 0 < t + \tau \leq t_p, \\ C_4(t, t + \tau) &= \frac{4}{t_p} e^{-\gamma \tau/2} (-e^{\gamma(t_p/2 - t)} + e^{\gamma(t_p - t)/2} + e^{-\gamma t} - e^{-\gamma t/2}), \quad 0 < t \leq t_p \text{ and } t + \tau > t_p, \\ C_4(t, t + \tau) &= \frac{4}{t_p} (e^{\gamma t_p/2} - 1)^2 e^{-\gamma(t + \tau)/2}, \quad t > t_p. \end{aligned} \quad (111)$$

We highlight that this last term is the only term contributing to the correlation function when $t > t_p$, showing a clear influence from the TLS population effects, as a function of time.

C. Time dependent spectra and spectral intensity as a function of frequency

In a rotating wave approximation, the time-dependent spectrum is obtained from the two-time first-order correlation functions calculated in Sec. VB, using [63]

$$S(\omega, t) = \text{Re} \left[\int_0^t dt' \int_0^{t-t'} d\tau v_g \langle a_R^\dagger(t') a_R(t' + \tau) \rangle e^{i\Delta_{\omega p} \tau} \right], \quad (112)$$

where $\Delta_{\omega p} = \omega - \omega_p$.

The time-dependent spectral intensity can also be calculated from the same two-time correlation functions:

$$I(\omega, t) = \text{Re} \left[\int_0^\infty d\tau v_g \langle a_R^\dagger(t) a_R(t + \tau) \rangle e^{i\Delta_{\omega p} \tau} \right]. \quad (113)$$

It is important to note that the integration of Eq. (113) over time, $I(\omega) = \int_0^\infty I(\omega, t) dt$, agrees with the stationary spectra, i.e., $I(\omega) = S(\omega)$, so the time-integrated spectral intensity is equivalent to the long-time spectra, even though $S(\omega, t)$ and $I(\omega, t)$ contain very different dynamical signatures.

We stress that these spectral functions are quite general and can be applied to the input and output fields, with or without the symmetric or chiral TLS.

VI. MATRIX PRODUCT STATES APPROACH

In order to extend our analytical results, we will also make use of MPS, which is an exact numerical approach based on tensor network theories [61,62].

Using the MPS approach, the complete system, which contains both the TLS and the waveguide, is discretized in time. The state is decomposed in a tensor product by performing Schmidt decompositions (or singular value decompositions). Each of the tensors represents a ‘bin’ or bins, with the TLS part represented as the ‘TLS bin’ and the photonic part represented as the ‘time bins’ [73,74].

The *quantum pulse* can be contained in the photonic part of the initial state [58]. If we have an initial state,

$$|\psi_0\rangle = |i\rangle_s \otimes |\phi_0\rangle, \quad (114)$$

where $|i\rangle_s$ corresponds to the TLS part and $|\phi_0\rangle$ to the photonic part, and each time bin contains a tensor product of the right and left channels at the corresponding time step.

Thus a one-photon pulse can be introduced from

$$|\phi_0\rangle = b_{\text{in}}^\dagger |0, \dots, 0\rangle, \quad (115)$$

with

$$b_{\text{in}}^\dagger = \int dt f(t) b_\alpha^\dagger(t), \quad (116)$$

and $f(t)$ is normalized as before, where

$$b_\alpha(t) = \frac{1}{\sqrt{2\pi}} \int d\omega b_\alpha(\omega) e^{-i(\omega - \omega_0)t}, \quad (117)$$

is a quantum noise creation operator in the time domain with the commutation relations $[b_\alpha(t), b_{\alpha'}(t')] = \delta_{\alpha, \alpha'} \delta(t - t')$, where $\alpha = L, R$ correspond to the left and right channel,

respectively [55]. In this case, for a right propagating pulse, the input pulse is introduced in the right channel,

$$b_{\text{in}}^\dagger = \int dt f(t) b_R^\dagger(t). \quad (118)$$

Writing this input state in the MPS description, we have

$$\begin{aligned} |\phi_0\rangle_N &= \sum_{k=1}^m f_k \Delta B_k^\dagger / \sqrt{\Delta t} |0, \dots, 0\rangle \\ &= \sum_{k=1}^m f_k \Delta B_k^\dagger / \sqrt{\Delta t} |0, \dots, 0\rangle_{1, \dots, m} \otimes |0, \dots, 0\rangle_{m+1, \dots, N} \\ &= \sum_{k=1}^m A_{a_1}^{(i_1)} A_{a_1, a_2}^{(i_2)} \dots A_{a_{m-2}, a_{m-1}}^{(i_{m-1})} A_{a_{m-1}}^{(i_m)} |0, \dots, 0\rangle_{1, \dots, m} \otimes \\ &\quad \times |0, \dots, 0\rangle_{m+1, \dots, N}, \end{aligned} \quad (119)$$

where f_k is the discretized version of $f(t)$ which contains the shape of the pulse at each time bin, and

$$\Delta B_k^\dagger = \int_{t_k}^{t_{k+1}} dt' b_R^\dagger(t'), \quad (120)$$

is the quantum noise creation matrix product operator (MPO), with the commutator $[\Delta B_k, \Delta B_{k'}^\dagger] = \Delta t \delta_{k, k'}$, and Δt is a time step.

It is important to note that ΔB_k has units of $[\sqrt{s}]$, hence, to calculate the photon flux as in Eq. (81) with the MPS approach,

$$n_{R/L}^{\text{out}}(t_k) = \frac{\langle \Delta B_k^\dagger \Delta B_k \rangle}{(\Delta t)^2}. \quad (121)$$

The $A_k^{(i_k)}$ terms can be written as follows:

if $k = 1$:

$$A_1^{(0)} = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad A_1^{(1)} = \begin{pmatrix} 0 & f_1 \end{pmatrix}, \quad (122)$$

if $1 < k < m$:

$$A_k^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_k^{(1)} = \begin{pmatrix} 0 & f_k \\ 0 & 0 \end{pmatrix}, \quad (123)$$

if $k = m$:

$$A_m^{(0)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad A_m^{(1)} = \begin{pmatrix} f_m \\ 0 \end{pmatrix}, \quad (124)$$

where, in our present case, each i_k has two dimensions corresponding to 0 and 1 excitations.

For a two-photon pulse, we have

$$\begin{aligned} |\phi_0\rangle_m &= \frac{1}{\Delta t \sqrt{2}} \left(\sum_{k=1}^m (f_k \Delta B_k^\dagger)^2 + 2 \sum_{k=1}^m \sum_{l=k+1}^m f_k \Delta B_k^\dagger f_l \Delta B_l^\dagger \right) \\ &\quad \times |0, \dots, 0\rangle, \end{aligned} \quad (125)$$

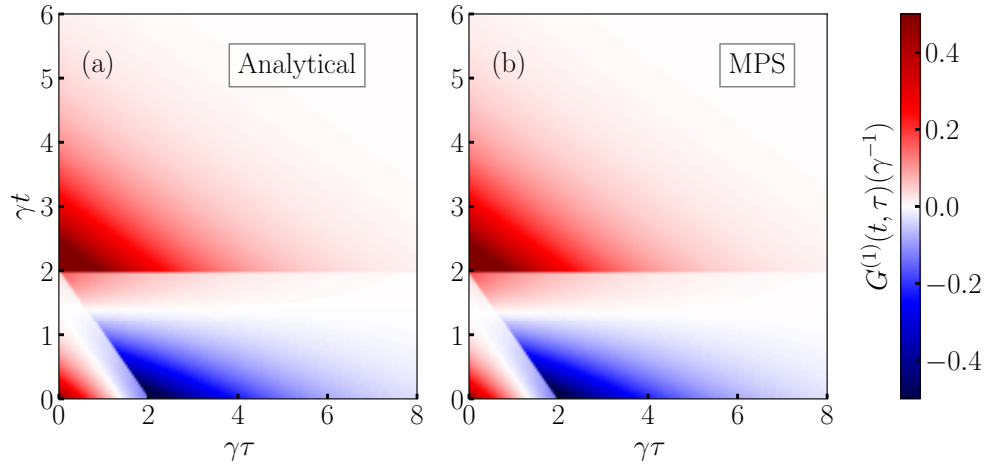


FIG. 2. Chiral TLS results for the first-order photon correlation function. Correlation function, $G^{(1)}(t, \tau)$, for a chiral TLS system with a rectangular 1-photon quantum pulse with a pulse duration of $t_p = 2\gamma^{-1}$, solved analytically in (a) using Eq. (100), and using MPS theory in (b).

and the tensors now contain another dimension to count for the two excitations:

if $k = 1$:

$$\begin{aligned} A_1^{(0)} &= (1 \quad 0 \quad 0), \quad A_1^{(1)} = (0 \quad \sqrt{2}f_1 \quad 0), \\ A_1^{(2)} &= (0 \quad 0 \quad \sqrt{2}f_1), \end{aligned} \quad (126)$$

if $1 < k < m$:

$$\begin{aligned} A_k^{(0)} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_k^{(1)} = \begin{pmatrix} 0 & \sqrt{2}f_k & 0 \\ 0 & 0 & \sqrt{2}f_k \\ 0 & 0 & 0 \end{pmatrix}, \\ A_k^{(2)} &= \begin{pmatrix} 0 & 0 & \sqrt{2}f_k \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned} \quad (127)$$

if $k = m$:

$$A_m^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad A_m^{(1)} = \begin{pmatrix} 0 \\ \sqrt{2}f_m \\ 0 \end{pmatrix}, \quad A_m^{(2)} = \begin{pmatrix} \sqrt{2}f_m \\ 0 \\ 0 \end{pmatrix}. \quad (128)$$

This is then transformed to have the required canonical shape, where we have the Orthogonality Center (OC) [73] in the present time bin and the rest of the pulse time bins are left normalized, in order to combine them with the TLS bin, and solve the time dynamics. This method can easily be extended to pulses containing higher numbers of photons.

Figure 2 shows the first-order correlation function, $G^{(1)}(t, \tau)$, from both MPS and the derived analytical expressions (Sec. VB), and they clearly agree.

VII. CHIRAL EMITTER POPULATIONS AND STATIONARY SPECTRA FOR SHORT PULSES

For our first applications example, we will benchmark our MPS results versus our analytical results, and consider a chiral TLS excited with a short top-hat quantum pulse, containing one and two photons. We will show both the TLS and photon

populations, as well as the input and transmitted stationary spectra.

In Fig. 3, we study an incident rectangular pulse with a width of $t_p = 2\gamma^{-1}$ (shaded region), containing either one photon [Fig. 3(a)] or two photons [Fig. 3(b)], interacting with the chiral TLS. The TLS population dynamics are calculated analytically (blue circles) and using MPS (solid blue lines), which show perfect agreement. The normalized photon population or photon flux (dashed red lines), propagating to the right is obtained from Eq. (121).

For these relatively short pulses, we observe a significant TLS population as the pulse interacts with the TLS, which then exponentially decays. These results completely deviate from the weak excitation regime, even in the one-photon case, consistent with [51]. Furthermore, the TLS population is higher in the single photon case. This is due to a nonlinear feature in the two-photon pulse solution, highlighting the complex nonlinear population saturation interactions with two photons. These population effects are further studied below, where the effects of pulse duration are investigated in detail, showing significant TLS populations, greater than 0.1, even for quantum pulses as long as $100\gamma^{-1}$.

Figure 3(c) shows the input and transmitted one-photon stationary spectrum for the same pulse duration as in (a) ($t_p = 2\gamma^{-1}$) and for a shorter pulse duration of $t_p = 0.5\gamma^{-1}$. For both pulse durations, all the population effects we observed in the TLS dynamical figures perfectly cancel out, i.e., for one-photon pulses, the stationary spectrum is completely insensitive to population dynamics. This is why a WEA solution can happen to give the same result as a full TLS solution with dynamical excitation, even though a WEA assumption for the population is clearly *not* valid. However, this correspondence is not the case when the input pulse contains two photons, as shown in Fig. 3(d), where the transmitted spectra are calculated again for a pulse durations of $t_p = 2\gamma^{-1}$ and $t_p = 0.5\gamma^{-1}$, and compared with the input pulse. Here, a clear nonlinear feature can be observed, where an additional peak with bandwidth γ appears at the central frequency (green solid line), qualitatively differing from the input field spectrum.

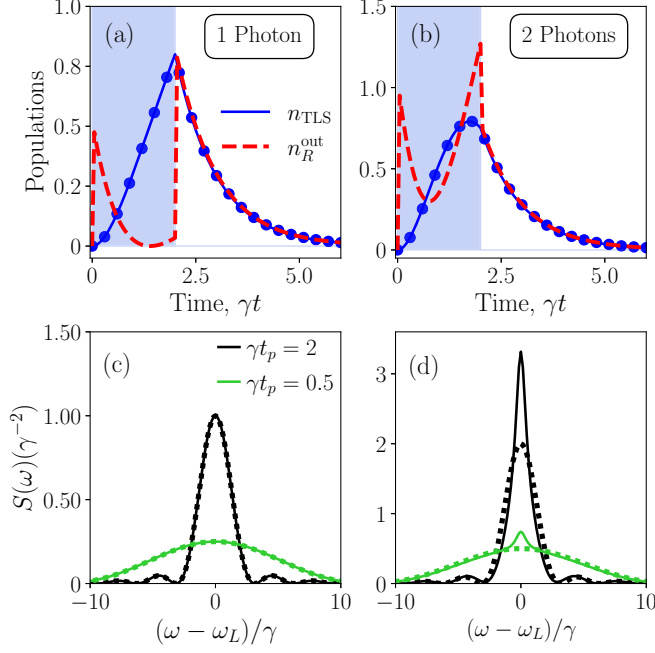


FIG. 3. Chiral emitter populations, photon populations and stationary spectra. [(a) and (b)] Quantum population dynamics using an incident rectangular quantum pulse (with duration $t_p = 2\gamma^{-1}$, cyan shaded region) containing: (a) one photon, and (b) two photons; both panels show the TLS population (solid blue) calculated using MPS, with the corresponding analytical solution (blue circles), and the normalized photon population of the right moving photons (dashed red). (c) Stationary spectra, $S(\omega)$, for one-photon with $t_p = 2\gamma^{-1}$ (black solid line) and $t_p = 0.5\gamma^{-1}$ (green solid line), and the corresponding solutions without the TLS interaction (dashed lines). (d) Stationary spectra, $S(\omega)$, for a two-photon pulse, with similar labels to (c), where dotted lines show solutions without the TLS interaction, corresponding to the input pulse.

VIII. PHOTON AND EMITTER POPULATIONS FOR BOTH CHIRAL AND SYMMETRIC TWO-LEVEL SYSTEMS

A. Symmetrical versus chiral populations for short pulses

In Fig. 3, we showed example results for the chiral TLS. For completeness, here we also compare chiral solutions with

the results for the symmetrical TLS in Fig. 4. For these simulations, we again consider an incident pulse with a width $t_p = 2\gamma^{-1}$, containing either 1 photon [Figs. 4(a) and 4(b)] or 2 photons [Figs. 4(c) and 4(d)], which interacts with a chiral TLS with $\gamma_R = \gamma$, $\gamma_L = 0$ (in green), and with a symmetrical TLS with $\gamma_R = \gamma_L = \gamma/2$ (in magenta).

The observables are calculated using MPS (solid lines), and the TLS populations $[n_{\text{TLS}}(t)]$ are also compared with the analytical solution (circles) derived in Secs. II and III, which once more agree perfectly.

In the one-photon solution, it can be seen that the TLS population of the symmetrical emitter is exactly half the population observed with a chiral TLS. This agrees with the analytical results from the previous sections. However, the two-photon solution deviates from this, where saturation effects become important. Additionally, the transmitted and reflected fluxes are shown in dash-dotted and dotted lines, respectively, where the nonlinearities of the two-photon can again be observed.

The emitted photon flux from Eq. (81) can be integrated in time, yielding the total output photon quanta

$$N_{R/L}^{\text{out}}(t) = \int_0^t dt' v_s \langle a_{R/L}^\dagger(t') a_{R/L}(t') \rangle. \quad (129)$$

Equivalently, $n_{R/L}^{\text{out}}$ can be calculated using MPS, and integrated following $N_{R/L}^{\text{out}} = \int_0^t n_{R/L}^{\text{out}}(t') dt'$. When this photon population is added to the TLS population, we recover the total number of quantum excitations in the system. For example, for a chiral emitter, this total is

$$N_{\text{total}}^{\text{chiral}}(t) = N_R^{\text{out}}(t) + n_{\text{TLS}}(t), \quad (130)$$

and for a symmetrical case, we need to consider both the transmitted and reflected flux, so that

$$N_{\text{total}}^{\text{sym}}(t) = N_R^{\text{out}}(t) + N_L^{\text{out}}(t) + n_{\text{TLS}}(t), \quad (131)$$

where $n_{\text{TLS}}(t)$ is the time-dependent TLS population. In both cases, $N_{\text{total}}^{\text{sym/chiral}}(t) = 1$ for a 1-photon pulse, and $N_{\text{total}}^{\text{sym/chiral}}(t) = 2$ for the two-photon one, so the total number of quanta is conserved.

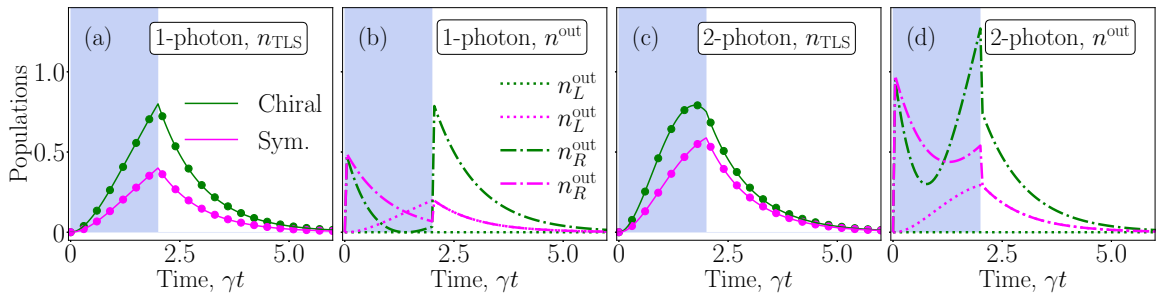


FIG. 4. Population dynamics for both chiral and symmetrical emitters. (a) and (b) One-photon pulse results. Quantum dynamics for a single TLS-waveguide system, using an incident rectangular quantum pulse (with pulse duration $t_p = 2\gamma^{-1}$, cyan shaded region) for a chiral (green) and symmetrical (magenta) TLS. (a) shows the TLS population (solid), with the corresponding analytical solutions (circles), and (b) shows the normalized photon population of the right (dash-dot) and left (dotted) moving photons. [(c) and (d)] Two-photon pulse results, where now (c) shows the TLS population (solid), with the corresponding analytical solutions (circles), and (d) the normalized photon population of the right (dash-dot) and left (dotted) moving photons.

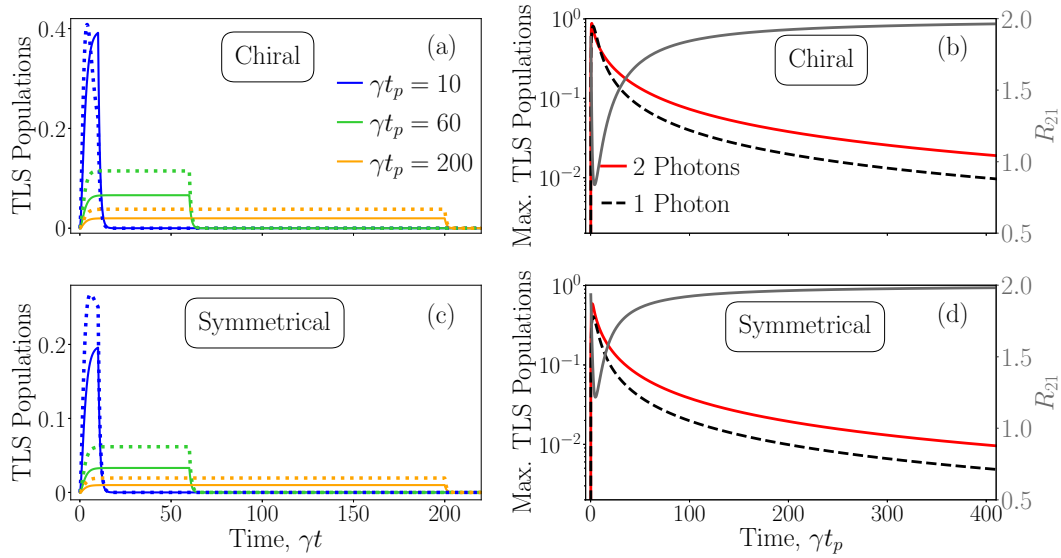


FIG. 5. Chiral and symmetric populations for different pulse durations. [(a) and (b)] Chiral TLS results. (a) Quantum pulse excited TLS populations for pulse lengths of $t_p = 10\gamma^{-1}$ (blue), $60\gamma^{-1}$ (green), and $200\gamma^{-1}$ (orange) for a one-photon pulse (solid line) and a two-photon pulse (dotted lines). (b) Maximum value of the TLS population versus pulse length for a one-photon pulse (dashed black) and a two-photon pulse (red). The ratio between the two and one-photon solutions (R_{21}) is shown in the grey line. [(c) and (d)] Symmetrical TLS results. (c) Quantum pulse excited TLS populations for the same pulse lengths as in (a). (d) Maximum value of the TLS population versus pulse length.

B. Influence of pulse duration

To better understand the emitter population time dynamics as a function of pulse duration, in Fig. 5, we study the TLS population for different pulse lengths.

First, in Figs. 5(a) and 5(c), the TLS populations are shown for one-photon pulses (solid lines) and two-photon pulses (dotted lines) with three different pulse lengths: $t_p = 10\gamma^{-1}$ (blue), $60\gamma^{-1}$ (green), and $200\gamma^{-1}$ (orange). Figure 5(a) shows the chiral solution while Fig. 5(c) is the symmetrical solution. We observe a clear deviation from the WEA approximation (where no population dynamics are taken into account) in every solution with significant population dynamics in all the results. Indeed, the population of the TLS is still considerable even for pulses as long as $t_p = 200\gamma^{-1}$. In addition, we see how the population nonlinearities between the one-photon and two-photon solutions appear more in the shorter pulses, where they both have highly different dynamics, but when the pulses get sufficiently long, they reach a plateau, and the two-photon solution tends to just double the single photon one.

In order to see the trend of two-photon to one-photon population ratios more clearly, in Figs. 5(b) and 5(d), the maximum values of the TLS population are shown versus the pulse duration for the one-photon pulses (solid red lines) and two-photon pulses (dashed black lines), where again Fig. 5(b) is the chiral solution and Fig. 5(d) is the symmetrical one. Additionally, the ratio between both solutions is computed from

$$R_{21} = \frac{\max[n_{\text{TLS}}^{(2)}]}{\max[n_{\text{TLS}}^{(1)}]}. \quad (132)$$

Here, we highlight two important effects: (i) it can be seen that the TLS population does not tend to zero until remarkably long pulses ($t_p \approx 500\gamma^{-1}$), even in the one-photon solution,

showing that the WEA does not hold for these longer pulses; and (ii), the ratio between the maximum population for the two-photon solution and the one-photon tends to two only for sufficiently long pulses, which indicates that there is an enhanced population excitation in the two-photon solution, even for these relatively long pulses. This is important as often a WEA is assumed in models of quantum pulses interacting with TLSs in waveguides, and our theory allows one to quantify whether this is valid or not.

IX. STATIONARY AND TIME-DEPENDENT SPECTRAL RESULTS

For the limit of one-photon excitation, we found the interesting result that the time-dependent spectrum evolves to exactly the same spectrum as the input pulse when working with a chiral system. To better understand this, consider the earlier derived long-time transmitted spectrum, $S(\omega)$ defined in Eq. (80) for a chiral pulse,

$$S(\omega) = |f(\omega)|^2,$$

which is a consequence of the fact that the TLS population effects completely cancel out in the long time limit.

Clearly, this is different in the time-dependent case $[S(\omega, t)]$, when time t is now finite. This can be recognized if we study $\langle a_R^\dagger(t)a_R(t+\tau) \rangle$ in the region where $t > t_p$, where the only term remaining is the fourth term of Eq. (100), $C_4(t, t+\tau)$, therefore dominating the dynamics in this region:

$$v_g \langle a_R^\dagger(t)a_R(t+\tau) \rangle = \frac{4}{t_p} (e^{\gamma t_p/2} - 1)^2 e^{-\gamma(t+\tau/2)}, \quad (133)$$

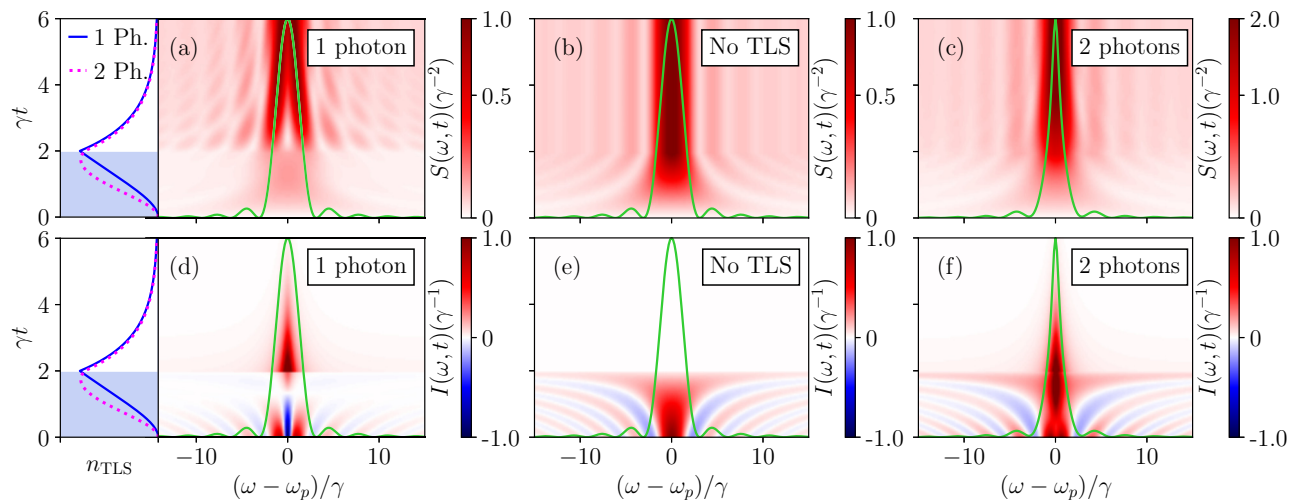


FIG. 6. Chiral TLS results, with pulse length $t_p = 2\gamma^{-1}$. [(a)–(c)] Time-dependent spectrum for the transmitted pulse, $S(\omega, t)$ [computed from Eq. (112)], for (a) a one-photon with the chiral TLS interaction, (b) a one-photon without the chiral TLS interaction, and (c) a two-photon pulse interacting with the chiral TLS. In case (b), this corresponds to only considering the first term $C_1(t, t + \tau)$ [Eq. (108)], and it yields an identical solution to the input field. [(d)–(f)] Time-dependent spectral intensities, $I(\omega, t)$ [computed from Eq. (113)], for the cases shown in (a)–(c). The corresponding long time limit $S(\omega)$, is plotted in all cases with the green solid curve. On the left panels, the TLS population (n_{TLS} , which has a maximum value of 1 on the graph scale) is plotted for both the interaction with a one-photon pulse (solid blue curve) and a two-photon pulse (dashed magenta curve) and the input pulse profile are shown with a blue shaded region.

which requires as the initial condition,

$$v_g \langle a_R^\dagger(t) a_R(t) \rangle = \frac{4}{t_p} [e^{\gamma t_p/2} - 1]^2 e^{-\gamma t}. \quad (134)$$

Equation (134) resembles closely Eq. (48), i.e., the population dynamics, showing that TLS population effects directly affect both $S(\omega, t)$ and $I(\omega, t)$.

For a pulse length of $t_p = 2\gamma^{-1}$, Fig. 6 shows the time-dependent spectrum, $S(\omega, t)$, and spectral intensity, $I(\omega, t)$, for a one-photon [Fig. 6(a)] and two-photon [Fig. 6(c)] rectangular quantum pulse with $t_p = 2\gamma^{-1}$ interacting with a chiral TLS. The one photon case can also be solved analytically and we obtain perfect agreement with the MPS solutions.

Figure 6(b) represents the solution without the TLS for reference, which obviously has no population effects. Note that neglecting TLS population effects, i.e., the WEA, in the dynamical case leads to unphysical time-dependent spectra, $S(\omega, t)$, (which is negative, when neglecting the term C_4), but this still produces identical *stationary spectra*. This aspect was not observed before but opens the road to see single photon population effects in experimental observables. As a guide to the eye, the TLS population is shown on the left panels to better understand the correlation with the population dynamics. *In this dynamical case, we can appreciate the time-dependent TLS population effects on the spectrum, even in the one-photon solution, both during the pulse and after the pulse.* Additionally, we observe a more pronounced center peak in the two-photon solution (compared to the one-photon case), manifested by nonlinear effects.

The time-dependent spectral intensity results, in Figs. 6(d)–6(f), show a qualitatively different profile, as a function of time, as we now also probe part of the dynamics of the two-time photon correlation function. For instance, we can have positive and negative values, though in practice,

actual measurements will yield a net positive quantity when using spectral filters [63]. Nevertheless, we again see clear TLS population effects in both the one- and two-photon solutions, with an enhancement of the central peak in the latter case.

Clearly, this population behavior is not exclusively imprinted on the chiral solution, and in Fig. 7, we show the time-dependent spectral results when considering a symmetrical emitter. The spectrum $S(\omega, t)$ and the spectral intensity, $I(\omega, t)$, are calculated using MPS, where we use again a rectangular pulse of pulse length $t_p = 2\gamma^{-1}$ containing one photon, in Figs. 7(a) and 7(d). For comparison, the same observables are calculated without a TLS interaction in Figs. 7(b) and 7(e). We observe how, in the stationary spectra, where we recover the harmonic oscillator solution again. However, population effects are clearly evident in both the dynamical spectrum and the spectral intensity, with results that highly differ from the case with no TLS. For instance, in the spectral intensity [Fig. 7(d)], we observe positive values at times longer than the pulse length (for $t_p > 2\gamma^{-1}$), while these effects are (obviously) not visible when there is not a TLS interacting with the pulse [Fig. 7(e)].

Further, we have also calculated the solution for a two-photon pulse with the same pulse length ($t_p = 2\gamma^{-1}$) interacting with the symmetrical emitter [Figs. 7(c) and 7(f)], where we again observe similar nonlinear population effects. However, in this case, we can also observe the population effects in the stationary spectrum (represented by the green solid line).

Influence of pulse duration on the transmitted spectra

As previously seen in Sec. VIII B, the TLS population dynamics depend on the pulse duration, with results that deviate

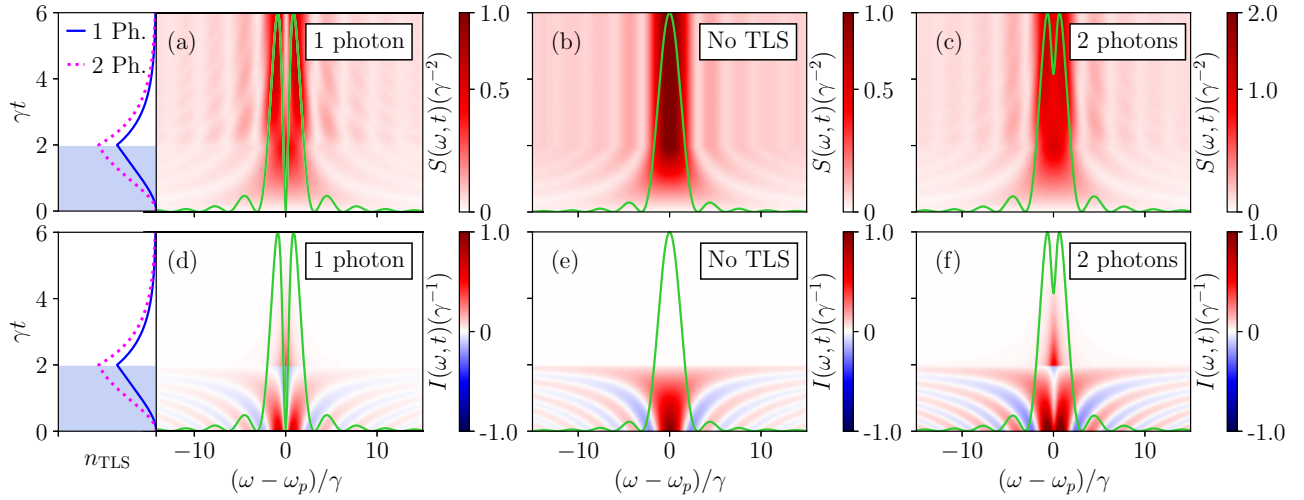


FIG. 7. Symmetric TLS results, with pulse length $t_p = 2\gamma^{-1}$. Graphs and curves are labeled the same as in Fig. 6 (chiral TLS case), with (a)–(c) showing the time-dependent spectrum, $S(\omega, t)$ and (d)–(f) showing the time-dependent spectral intensities, $I(\omega, t)$. On the left panels, the TLS population (n_{TLS} , which has a maximum value of 1 on the graph scale) is plotted for both the interaction with a one-photon pulse (solid blue curve) and a two-photon pulse (dashed magenta curve).

from the WEA approximation and visible nonlinear effects in the case of a two-photon pulse.

We next examine the role of using a longer pulse on the emitted spectra. In Fig. 8, the transmitted spectra and spectral intensity are calculated for a chiral TLS interacting with a rectangular pulse of length $t_p = 10\gamma^{-1}$, containing one photon in Figs. 8(a) and 8(d) and two photons in Figs. 8(c) and 8(f). The pulse results with no interaction are shown for reference in Figs. 8(b) and 8(e). Here, we observe how the TLS effects can still be appreciable in both cases, with solutions that deviate from the cases with no TLS interaction. In the one-photon solutions, we observe how the presence of the TLS effects starts to become more subtle. However, in the

two-photon cases, the nonlinearities mentioned above are still present, where we can even see a stronger breakup of the pulse in frequency. Similar pulse breakup effects have been found in Ref. [37] using ensembles of chiral emitters.

In Fig. 9, we show results similar to the previous ones, but with an even longer pulse ($t_p = 50\gamma^{-1}$). In this case, we observe solutions that are much closer to the case with no TLS. Specifically, in the one-photon solutions, some small dynamical effects can still be seen when the pulse turns on and off, with the populations reaching a quasi-steady-state during most of the pulse. The two-photon solution still has some significant deviations from the case without the TLS interaction, but shows no visible

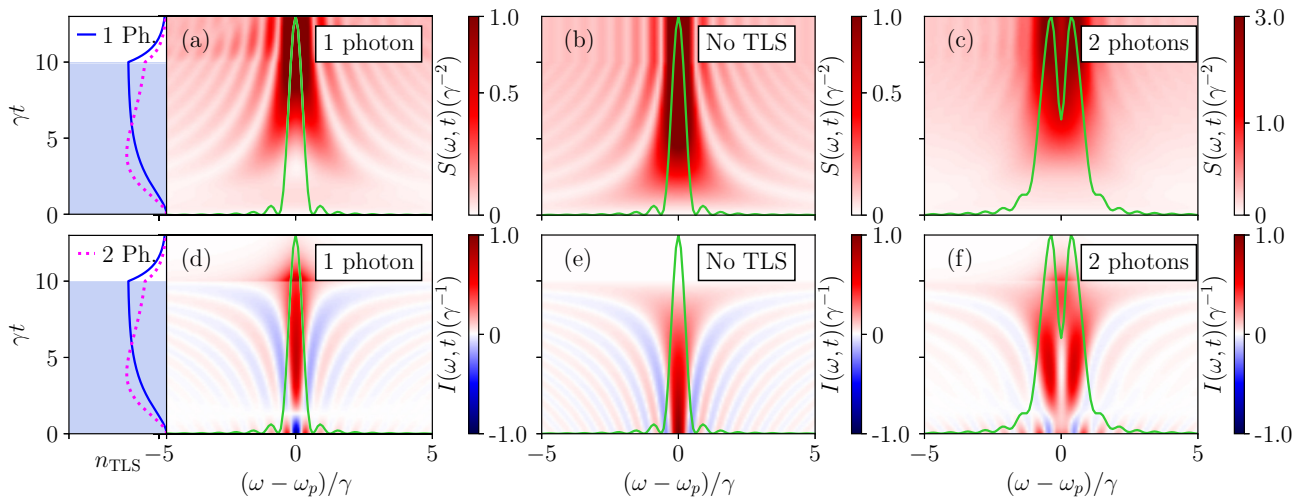


FIG. 8. Chiral TLS results, with pulse length $t_p = 10\gamma^{-1}$. Graphs and curves are labeled the same as in Fig. 6, with (a)–(c) showing the time-dependent spectrum, $S(\omega, t)$ and (d)–(f) showing the time-dependent spectral intensities, $I(\omega, t)$. On the left panels, the TLS population (n_{TLS} , which has a maximum value of 1 on the graph scale) is plotted for both the interaction with a one-photon pulse (solid blue curve) and a two-photon pulse (dashed magenta curve).

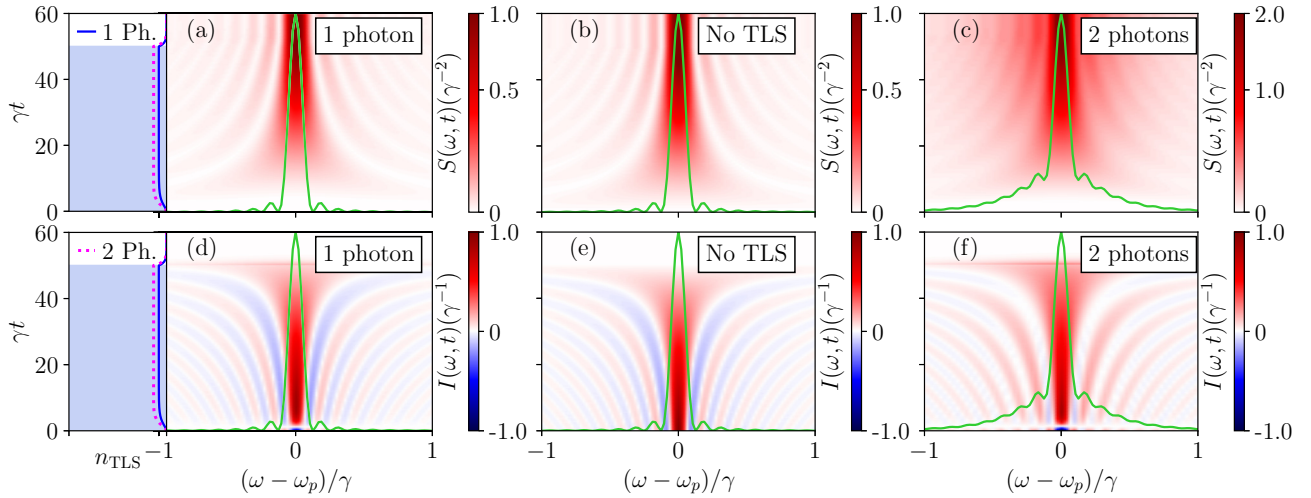


FIG. 9. Chiral TLS results, with pulse length $\gamma t_p = 50\gamma^{-1}$. Graphs and curves are labeled the same as in Fig. 6, with (a)–(c) showing the time-dependent spectrum, $S(\omega, t)$ and (d)–(f) showing the time-dependent spectral intensities, $I(\omega, t)$. On the left panels, the TLS population (n_{TLS} , which has a maximum value of 1 on the graph scale) is plotted for both the interaction with a one-photon pulse (solid blue curve) and a two-photon pulse (dashed magenta curve).

pulse breakup effects (which is caused by TLS saturation effects).

These results confirm that the dynamical effects seen in the spectral results are directly related to the TLS population effects and that they persist as long as there is a reasonable TLS population.

X. GAUSSIAN SHAPED QUANTUM PULSES

For completeness, we next show results for few-photon excitations with Gaussian pulses, since the MPS results can use arbitrary waveforms. In this case, the temporal pulse has the following normalized shape,

$$f(t) = \frac{1}{\sqrt{2\pi}t_p} \exp\left\{-\frac{(t-t_c)^2}{2t_p^2}\right\}, \quad (135)$$

where t_c is the center of the pulse and t_p is its width. Figure 10 shows the various populations, for a 1-photon pulse, with a width of $t_p = 1\gamma^{-1}$ and centered at $t_c = 3\gamma^{-1}$, interacting with a chiral emitter. Here again, the shaded blue area represents the incident pulse, the solid blue line is the TLS population, and the red dashed line represents the normalized transmitted photon population. Interestingly, there is an instant when the photon population is zero in this case (i.e., just after the center of the pulse). Additionally, the left photon populations stay at zero at all times since this is a chiral system.

The corresponding spectral functions $S(\omega, t)$ and $I(\omega, t)$, are shown in Fig. 11. To better appreciate the influence of population effects on the dynamical spectrum, in Fig. 11, again we show both the population decay and the dynamic spectra side by side. As also seen in Fig. 10, one can clearly observe in Fig. 11(a) the effects of the TLS population dynamics, which causes a spectral hole and additional interferences in the transmitted time-dependent spectra. This can be appreciated by comparing these solutions with the corresponding

$S(\omega, t)$ without the TLS, shown in Fig. 11(b). However, as in the case of the rectangular pulse, in the long time limit, all the TLS population effects cancel out and we recover the same stationary spectra in both cases (green solid lines).

In addition, rich features are observed in the spectral intensity $I(\omega, t)$ in Fig. 11(d), where again, by comparison with the case without the TLS [Fig. 11(e)], we see additional interferences during the pulse, and intensity positive values after the pulse while the TLS is decaying, that cannot be observed without this interaction. Nevertheless, if we integrate the spectral intensity in both cases, we again recover the same solution, which agrees with the stationary spectra (green solid lines).

Finally, Figs. 11(c) and 11(f) show the same spectral observables, $S(\omega, t)$ and $I(\omega, t)$ respectively [Eqs. (112) and (113)], when using a similar Gaussian pulse but now con-

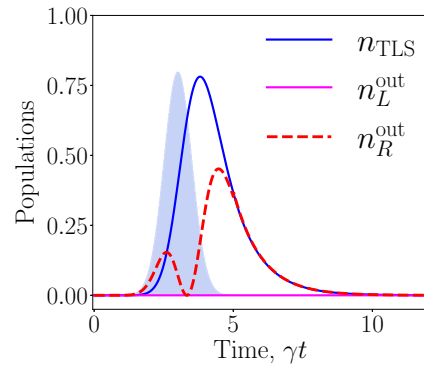


FIG. 10. Chiral TLS results with a one-photon Gaussian pulse, with $t_p = 1\gamma^{-1}$. Quantum population dynamics for a chiral single TLS-waveguide system, using an incident Gaussian quantum pulse (cyan shaded region) containing one photon. The figure shows the TLS population (solid blue), and the normalized photon population of the right (dashed red) and left (magenta) moving photons.

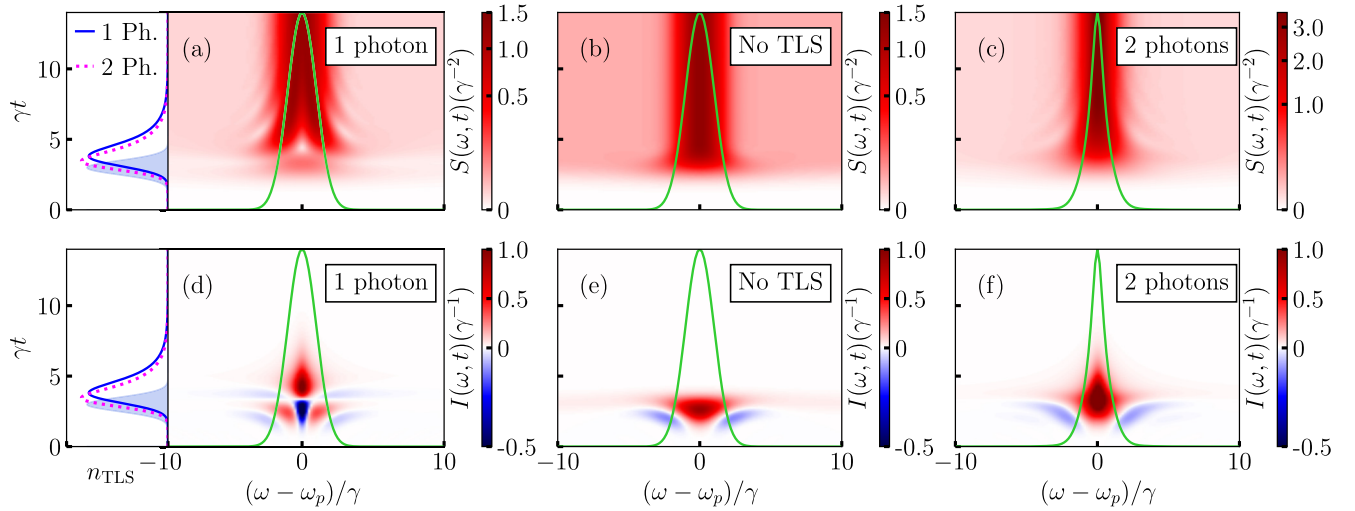


FIG. 11. Chiral TLS results, with pulse length $t_p = 1\gamma^{-1}$, using a Gaussian pulse shape. Graphs and curves are labeled the same as in Fig. 6, with (a)–(c) showing the time-dependent spectrum, $S(\omega, t)$ and (d)–(f) showing the time-dependent spectral intensities, $I(\omega, t)$. On the left panels, the TLS population (n_{TLS} , which has a maximum value of 1 on the graph scale) is plotted for both the interaction with a one-photon pulse (solid blue curve) and a two-photon pulse (dashed magenta curve), and the Gaussian pulse profile is shown with the blue shaded region.

taining two photons. Here we see again the TLS dynamical effects in both cases, and, in agreement with what we saw with rectangular pulses, the stationary spectrum now shows the nonlinear excitation, with a narrower width and a more pronounced central peak showing for the transmitted pulse.

These results show further details about how population dynamics influence the $S(\omega, t)$ spectral plots, and also which ones are due to the specific shape of the pulse used as well (e.g., showing convolution effects with a sinc function or a Gaussian function). Note also, that an analytical solution for the one-photon excitation is possible in terms of the error function (or via a numerical integration), but may not be so easy to obtain for the two-photon case.

XI. CONCLUSIONS

We have presented a detailed theoretical analysis of few-photon quantum pulses interacting with a single TLS in a waveguide, using both chiral and symmetric TLS emitters.

For the theory, we have shown two approaches to compute the TLS population information, visible both in the time dynamics and in certain spectral observables. We started by deriving the analytical solution of the TLS population by solving the corresponding equations of motion in the Heisenberg picture. We presented the general differential equations for a pulse containing one photon (a Fock state), and then solved for a rectangular pulse. Here, we have also shown both solutions with a symmetrical emitter or a chiral emitter that couples only to the right photon channel, and then we have repeated the same procedure for a rectangular pulse containing a two photon Fock state.

Next, we used the same method to derive the analytical solution for the one-photon stationary spectra (i.e., in the long time limit). Here, we demonstrated how all the population effects that were observed in the time dynamics perfectly cancel out, and we recover the same *stationary spectra* with or without the interaction with the TLS (for this observable), for

a chiral emitter. For a symmetric emitter, we obtain the same result as a WEA, for this observable, essentially recovering elastic scattering theory. However, a WEA is in general not valid, as we show.

In order to better understand these effects and to connect temporal population effects to experimental observables, we then introduced a derivation for a time-dependent spectrum, starting with the derivation of the transmitted photon population, and then the transmitted first-order photon correlation functions, and we have shown how these relate to the time-dependent spectra and a spectral intensity function.

Our analytical results were then benchmarked, checked, and extended by using a second theoretical method, based on MPS, which we have introduced after the analytical derivations. This MPS approach is a numerically exact tensor network approach which both helped us to verify the analytical solutions, and extend our simulations to more efficient calculations with pulses of different shapes and containing more photons. For future work, it can also easily include a wide range of different types of quantum pulses, including low photon number coherent states.

After presenting our waveguide-QED models, application solutions with a rectangular incident pulse are shown, starting with the population results, where a comparison between symmetrical and chiral emitters is shown, for both one and two-photon pulses. In the general cases of short pulses (comparable to the inverse TLS decay times), we have observed high values of TLS populations making clear a deviation from the weak excitation limit. Furthermore, this feature has been investigated in detail with a study of the maximum TLS population as a function of pulse length, and we have seen how the weak excitation approximation is not valid until very long pulses, on the order of $t_p \sim 500\gamma^{-1}$. We have also observed in these cases how nonlinear Pauli blocking effects are amplified in the two-photon cases until similar pulse lengths. In addition, we showed how the time-dependent spectral profiles depend on the length of the pulse, where eventually the

population effects visible for one-photon pulses become less and less important, and we showed the clear difference between one-photon and two-photon pulses for the observables.

The transmitted spectral results have shown us how population effects influence both the dynamical spectra and the spectral intensity, even though they completely cancel out in the one-photon stationary solutions. We have also seen that this is not the case when considering a two-photon pulse, where the nonlinear effects are also seen in the stationary limit, yielding a narrowing of the central pulse peak for a chiral TLS (or even a pulse break up), and in general a breakup for a symmetric TLS.

Finally, we have calculated similar observables for a Gaussian incident quantum pulse, both containing one and two photons, interacting with a chiral TLS. In all cases, we observe similar behaviors as in the case of the rectangular pulse (which permit simpler analytical solutions). These results confirm that our findings in this paper are not particular to the shape of the pulse, and can be extended to other pulse shapes as well.

Although we have used scaled units in terms of the nominal decay rate, γ , various experimental systems can be used to explore the waveguide-QED effects we show here with embedded quantum emitters [1,36]. For example, at optical frequencies, one can use quantum dot nanowires and SiN waveguides [67], or quantum dots in photonic crystal wave-

guides [16,28], where the decay rates are around $\gamma = 1 \text{ ns}^{-1}$. These rates scale proportionally with the group index of the waveguide mode [27]. Experiments with such systems have already measured the few-photon nonlinear scattering matrix [75]. Moreover, recent experiments with quantum dots in cavities have shown how one can measure the dynamical resonance fluorescence, showing how pulsed excitation leads to features beyond the usual stationary spectra [63,64]. In the microwave regime, one can use circuit QED systems [14,23], though achieving chiral emission control is more difficult, as is the generation and measurement of Fock states. This work further motivates the use of pulsed resonance fluorescence schemes, especially in the context of waveguide QED systems.

ACKNOWLEDGMENTS

This work was supported by the Natural Sciences and Engineering Research Council of Canada (NSERC) [Discovery and Quantum Alliance Grants], the National Research Council of Canada (NRC), the Canadian Foundation for Innovation (CFI), Queen's University, Canada, and the Alexander von Humboldt Foundation through a Humboldt Award. We also thank Kisa Barkemeyer, Jacob Ewaniuk, and Nir Rotenberg for valuable contributions and discussions.

-
- [1] A. S. Sheremet, M. I. Petrov, I. V. Iorsh, A. V. Poshakinskiy, and A. N. Poddubny, Waveguide quantum electrodynamics: Collective radiance and photon-photon correlations, *Rev. Mod. Phys.* **95**, 015002 (2023).
 - [2] H. Zheng, D. J. Gauthier, and H. U. Baranger, Waveguide QED: Many-body bound-state effects in coherent and Fock-state scattering from a two-level system, *Phys. Rev. A* **82**, 063816 (2010).
 - [3] D. Witthaut and A. S. Sørensen, Photon scattering by a three-level emitter in a one-dimensional waveguide, *New J. Phys.* **12**, 043052 (2010).
 - [4] D. Roy, Two-photon scattering by a driven three-level emitter in a one-dimensional waveguide and electromagnetically induced transparency, *Phys. Rev. Lett.* **106**, 053601 (2011).
 - [5] G. Crowder, J. H. Carmichael, and S. Hughes, Quantum trajectory theory of few-photon cavity-QED systems with a time-delayed coherent feedback, *Phys. Rev. A* **101**, 023807 (2020).
 - [6] G. Crowder, L. Ramunno, and S. Hughes, Quantum trajectory theory and simulations of nonlinear spectra and multiphoton effects in waveguide-QED systems with a time-delayed coherent feedback, *Phys. Rev. A* **106**, 013714 (2022).
 - [7] J. T. Shen and S. Fan, Coherent photon transport from spontaneous emission in one-dimensional waveguides, *Opt. Lett.* **30**, 2001 (2005).
 - [8] H. Le Jeannic, A. Tiranov, J. Carolan, T. Ramos, Y. Wang, M. H. Appel, S. Scholz, A. D. Wieck, A. Ludwig, N. Rotenberg, L. Midolo, J. J. García-Ripoll, A. S. Sørensen, and P. Lodahl, Dynamical photon-photon interaction mediated by a quantum emitter, *Nat. Phys.* **18**, 1191 (2022).
 - [9] P. Longo, P. Schmitteckert, and K. Busch, Few-photon transport in low-dimensional systems, *Phys. Rev. A* **83**, 063828 (2011).
 - [10] G. Crowder, L. Ramunno, and S. Hughes, Improving on-demand single-photon-source coherence and indistinguishability through a time-delayed coherent feedback, *Phys. Rev. A* **110**, L031703 (2024).
 - [11] P. Türschmann, H. L. Jeannic, S. F. Simonsen, H. R. Haakh, S. Götzinger, V. Sandoghdar, P. Lodahl, and N. Rotenberg, Coherent nonlinear optics of quantum emitters in nanophotonic waveguides, *Nanophotonics* **8**, 1641 (2019).
 - [12] T. Pregnolato, X.-L. Chu, T. Schröder, R. Schott, A. D. Wieck, A. Ludwig, P. Lodahl, and N. Rotenberg, Deterministic positioning of nanophotonic waveguides around single self-assembled quantum dots, *APL Photonics* **5**, 086101 (2020).
 - [13] T. Lund-Hansen, S. Stobbe, B. Julsgaard, H. Thyrrstrup, T. Sünner, M. Kamp, A. Forchel, and P. Lodahl, Experimental realization of highly efficient broadband coupling of single quantum dots to a photonic crystal waveguide, *Phys. Rev. Lett.* **101**, 113903 (2008).
 - [14] M. Mirhosseini, E. Kim, X. Zhang, A. Sipahigil, P. B. Dieterle, A. J. Keller, A. Asenjo-Garcia, D. E. Chang, and O. Painter, Cavity quantum electrodynamics with atom-like mirrors, *Nature (London)* **569**, 692 (2019).
 - [15] A. González-Tudela, V. Paulisch, D. E. Chang, H. J. Kimble, and J. I. Cirac, Deterministic generation of arbitrary photonic states assisted by dissipation, *Phys. Rev. Lett.* **115**, 163603 (2015).
 - [16] A. Laucht, S. Pütz, T. Günthner, N. Hauke, R. Saive, S. Frédéric, M. Bichler, M.-C. Amann, A. W. Holleitner, M. Kaniber, and J. J. Finley, A waveguide-coupled on-chip single-photon source, *Phys. Rev. X* **2**, 011014 (2012).
 - [17] W. Nie, T. Shi, Y.-x. Liu, and F. Nori, Non-Hermitian waveguide cavity QED with tunable atomic mirrors, *Phys. Rev. Lett.* **131**, 103602 (2023).

- [18] X. Li and L. F. Wei, Designable single-photon quantum routings with atomic mirrors, *Phys. Rev. A* **92**, 063836 (2015).
- [19] A. Asenjo-Garcia, M. Moreno-Cardoner, A. Albrecht, H. J. Kimble, and D. E. Chang, Exponential improvement in photon storage fidelities using subradiance and “selective radiance” in atomic arrays, *Phys. Rev. X* **7**, 031024 (2017).
- [20] A. F. Kockum, G. Johansson, and F. Nori, Decoherence-free interaction between giant atoms in waveguide quantum electrodynamics, *Phys. Rev. Lett.* **120**, 140404 (2018).
- [21] X.-L. Chu, C. Papon, N. Bart, A. D. Wieck, A. Ludwig, L. Midolo, N. Rotenberg, and P. Lodahl, Independent electrical control of two quantum dots coupled through a photonic-crystal waveguide, *Phys. Rev. Lett.* **131**, 033606 (2023).
- [22] S. J. Masson and A. Asenjo-Garcia, Atomic-waveguide quantum electrodynamics, *Phys. Rev. Res.* **2**, 043213 (2020).
- [23] B. Kannan, M. J. Ruckriegel, D. L. Campbell, A. Frisk Kockum, J. Braumüller, D. K. Kim, M. Kjaergaard, P. Krantz, A. Melville, B. M. Niedzielski, A. Vepsäläinen, R. Winik, J. L. Yoder, F. Nori, T. P. Orlando, S. Gustavsson, and W. D. Oliver, Waveguide quantum electrodynamics with superconducting artificial giant atoms, *Nature (London)* **583**, 775 (2020).
- [24] U. Alushi, T. Ramos, J. J. García-Ripoll, R. Di Candia, and S. Felicetti, Waveguide QED with quadratic light-matter interactions, *PRX Quantum* **4**, 030326 (2023).
- [25] X. Wang, H.-B. Zhu, T. Liu, and F. Nori, Realizing quantum optics in structured environments with giant atoms, *Phys. Rev. Res.* **6**, 013279 (2024).
- [26] X. Wang and H.-R. Li, Chiral quantum network with giant atoms, *Quantum Sci. Technol.* **7**, 035007 (2022).
- [27] V. S. C. Manga Rao and S. Hughes, Single quantum-dot Purcell factor and β factor in a photonic crystal waveguide, *Phys. Rev. B* **75**, 205437 (2007).
- [28] M. Arcari, I. Söllner, A. Javadi, S. Lindskov Hansen, S. Mahmoodian, J. Liu, H. Thyrestrup, E. H. Lee, J. D. Song, S. Stobbe, and P. Lodahl, Near-unity coupling efficiency of a quantum emitter to a photonic crystal waveguide, *Phys. Rev. Lett.* **113**, 093603 (2014).
- [29] S. Paesani, Y. Ding, R. Santagati, L. Chakhmakchyan, C. Vigliar, K. Rottwitt, L. K. Oxenløwe, J. Wang, M. G. Thompson, and A. Laing, Generation and sampling of quantum states of light in a silicon chip, *Nat. Phys.* **15**, 925 (2019).
- [30] B. le Feber, N. Rotenberg, and L. Kuipers, Nanophotonic control of circular dipole emission, *Nat. Commun.* **6**, 6695 (2015).
- [31] A. B. Young, A. C. T. Thijssen, D. M. Beggs, P. Androvitsaneas, L. Kuipers, J. G. Rarity, S. Hughes, and R. Oulton, Polarization engineering in photonic crystal waveguides for spin-photon entanglers, *Phys. Rev. Lett.* **115**, 153901 (2015).
- [32] I. Söllner, S. Mahmoodian, S. L. Hansen, L. Midolo, A. Javadi, G. Kiršanskė, T. Pregolato, H. El-Ella, E. H. Lee, J. D. Song, S. Stobbe, and P. Lodahl, Deterministic photon-emitter coupling in chiral photonic circuits, *Nat. Nanotechnol.* **10**, 775 (2015).
- [33] M. Bello, G. Platero, J. I. Cirac, and A. González-Tudela, Unconventional quantum optics in topological waveguide QED, *Sci. Adv.* **5**, eaaw0297 (2019).
- [34] N. V. Hauff, H. Le Jeannic, P. Lodahl, S. Hughes, and N. Rotenberg, Chiral quantum optics in broken-symmetry and topological photonic crystal waveguides, *Phys. Rev. Res.* **4**, 023082 (2022).
- [35] N. Liu, X. Wang, X. Wang, X.-S. Ma, and M.-T. Cheng, Tunable single photon nonreciprocal scattering based on giant atom-waveguide chiral couplings, *Opt. Express* **30**, 23428 (2022).
- [36] S. Mahmoodian, P. Lodahl, and A. S. Sørensen, Quantum networks with chiral-light-matter interaction in waveguides, *Phys. Rev. Lett.* **117**, 240501 (2016).
- [37] S. Mahmoodian, G. Calajó, D. E. Chang, K. Hammerer, and A. S. Sørensen, Dynamics of many-body photon bound states in chiral waveguide QED, *Phys. Rev. X* **10**, 031011 (2020).
- [38] A. Asenjo-Garcia, J. D. Hood, D. E. Chang, and H. J. Kimble, Atom-light interactions in quasi-one-dimensional nanostructures: A Green’s-function perspective, *Phys. Rev. A* **95**, 033818 (2017).
- [39] A. Frisk Kockum, A. Miranowicz, S. De Liberato, S. Savasta, and F. Nori, Ultrastrong coupling between light and matter, *Nat. Rev. Phys.* **1**, 19 (2019).
- [40] W. Qin, A. F. Kockum, C. S. Muñoz, A. Miranowicz, and F. Nori, Quantum amplification and simulation of strong and ultrastrong coupling of light and matter, *Phys. Rep.* **1078**, 1 (2024), Quantum amplification and simulation of strong and ultrastrong coupling of light and matter.
- [41] P. Forn-Díaz, L. Lamata, E. Rico, J. Kono, and E. Solano, Ultrastrong coupling regimes of light-matter interaction, *Rev. Mod. Phys.* **91**, 025005 (2019).
- [42] B. Peropadre, D. Zueco, D. Porras, and J. J. García-Ripoll, Nonequilibrium and nonperturbative dynamics of ultrastrong coupling in open lines, *Phys. Rev. Lett.* **111**, 243602 (2013).
- [43] N. Gheeraert, X. H. H. Zhang, T. Sépulcre, S. Bera, N. Roch, H. U. Baranger, and S. Florens, Particle production in ultrastrong-coupling waveguide QED, *Phys. Rev. A* **98**, 043816 (2018).
- [44] E. Sanchez-Burillo, D. Zueco, J. J. García-Ripoll, and L. Martin-Moreno, Scattering in the ultrastrong regime: Nonlinear optics with one photon, *Phys. Rev. Lett.* **113**, 263604 (2014).
- [45] C. A. González-Gutiérrez, J. Román-Roche, and D. Zueco, Distant emitters in ultrastrong waveguide QED: Ground-state properties and non-Markovian dynamics, *Phys. Rev. A* **104**, 053701 (2021).
- [46] D. E. Chang, L. Jiang, A. V. Gorshkov, and H. J. Kimble, Cavity QED with atomic mirrors, *New J. Phys.* **14**, 063003 (2012).
- [47] S. Fan, S. E. Kocabaş, and J.-T. Shen, Input-output formalism for few-photon transport in one-dimensional nanophotonic waveguides coupled to a qubit, *Phys. Rev. A* **82**, 063821 (2010).
- [48] J.-T. Shen and S. Fan, Strongly correlated two-photon transport in a one-dimensional waveguide coupled to a two-level system, *Phys. Rev. Lett.* **98**, 153003 (2007).
- [49] J.-T. Shen and S. Fan, Strongly correlated multiparticle transport in one dimension through a quantum impurity, *Phys. Rev. A* **76**, 062709 (2007).
- [50] E. Rephaeli and S. Fan, Few-photon single-atom cavity QED with input-output formalism in fock space, *IEEE J. Sel. Top. Quantum Electron.* **18**, 1754 (2012).
- [51] E. Rephaeli, J.-T. Shen, and S. Fan, Full inversion of a two-level atom with a single-photon pulse in one-dimensional geometries, *Phys. Rev. A* **82**, 033804 (2010).
- [52] A. Nysteen, P. T. Kristensen, D. P. S. McCutcheon, P. Kaer, and J. Mørk, Scattering of two photons on a quantum emitter in a one-dimensional waveguide: exact dynamics and induced correlations, *New J. Phys.* **17**, 023030 (2015).

- [53] Y. Chen, M. Wubs, J. Mørk, and A. F. Koenderink, Coherent single-photon absorption by single emitters coupled to one-dimensional nanophotonic waveguides, *New J. Phys.* **13**, 103010 (2011).
- [54] B. Q. Baragiola, R. L. Cook, A. M. Brańczyk, and J. Combes, n -photon wave packets interacting with an arbitrary quantum system, *Phys. Rev. A* **86**, 013811 (2012).
- [55] C. W. Gardiner and P. Zoller, *Quantum Noise: A Handbook of Markovian and NonMarkovian Quantum Stochastic Methods with Applications to Quantum Optics* (Springer, Berlin, 2010).
- [56] K. Barkemeyer, A. Knorr, and A. Carmele, Heisenberg treatment of multiphoton pulses in waveguide QED with time-delayed feedback, *Phys. Rev. A* **106**, 023708 (2022).
- [57] S. Iblisdir, R. Orús, and J. I. Latorre, Matrix product states algorithms and continuous systems, *Phys. Rev. B* **75**, 104305 (2007).
- [58] P.-O. Guimond, M. Pletyukhov, H. Pichler, and P. Zoller, Delayed coherent quantum feedback from a scattering theory and a matrix product state perspective, *Quantum Sci. Technol.* **2**, 044012 (2017).
- [59] K. Barkemeyer, A. Knorr, and A. Carmele, Strongly entangled system-reservoir dynamics with multiphoton pulses beyond the two-excitation limit: Exciting the atom-photon bound state, *Phys. Rev. A* **103**, 033704 (2021).
- [60] L. Vanderstraeten, J. Haegeman, and F. Verstraete, Tangent-space methods for uniform matrix product states, *SciPost Phys. Lect. Notes* **7** (2019).
- [61] R. Orús, A practical introduction to tensor networks: Matrix product states and projected entangled pair states, *Ann. Phys.* **349**, 117 (2014).
- [62] C. Yang, F. C. Binder, V. Narasimhachar, and M. Gu, Matrix product states for quantum stochastic modeling, *Phys. Rev. Lett.* **121**, 260602 (2018).
- [63] S. Liu, C. Gustin, H. Liu, X. Li, Y. Yu, H. Ni, Z. Niu, S. Hughes, X. Wang, and J. Liu, Dynamic resonance fluorescence in solid-state cavity quantum electrodynamics, *Nat. Photonics* **18**, 318 (2024).
- [64] K. Boos, S. K. Kim, T. Bracht, F. Sbresny, J. M. Kaspari, M. Cygorek, H. Riedl, F. W. Bopp, W. Rauhaus, C. Calcagno, J. J. Finley, D. E. Reiter, and K. Müller, Signatures of dynamically dressed states, *Phys. Rev. Lett.* **132**, 053602 (2024).
- [65] M. J. Mehrabad, A. P. Foster, R. Dost, E. Clarke, P. K. Patil, A. M. Fox, M. S. Skolnick, and L. R. Wilson, Chiral topological photonics with an embedded quantum emitter, *Optica* **7**, 1690 (2020).
- [66] P. Lodahl, S. Mahmoodian, and S. Stobbe, Interfacing single photons and single quantum dots with photonic nanostructures, *Rev. Mod. Phys.* **87**, 347 (2015).
- [67] K. Mnaymneh, D. Dalacu, J. McKee, J. Lapointe, S. Haffouz, J. F. Weber, D. B. Northeast, P. J. Poole, G. C. Aers, and R. L. Williams, On-chip integration of single photon sources via evanescent coupling of tapered nanowires to SiN waveguides, *Adv. Quantum Technol.* **3**, 1900021 (2020).
- [68] P. Yao, V. S. C. Manga Rao, and S. Hughes, On-chip single photon sources using planar photonic crystals and single quantum dots, *Laser Photonics Rev.* **4**, 499 (2010).
- [69] P. Laferrière, E. Yeung, L. Giner, S. Haffouz, J. Lapointe, G. C. Aers, P. J. Poole, R. L. Williams, and D. Dalacu, Multiplexed single-photon source based on multiple quantum dots embedded within a single nanowire, *Nano Lett.* **20**, 3688 (2020).
- [70] C. Gardiner and P. Zoller, *The Quantum World of Ultra-Cold Atoms and Light - Book 1: Foundations of Quantum Optics* (Imperial College Press, London, 2014).
- [71] T. Stroucken, A. Knorr, P. Thomas, and S. W. Koch, Coherent dynamics of radiatively coupled quantum-well excitons, *Phys. Rev. B* **53**, 2026 (1996).
- [72] Y. Wang, Y. Zhang, Q. Zhang, B. Zou, and U. Schwingenschlogl, Dynamics of single photon transport in a one-dimensional waveguide two-point coupled with a jaynes-cummings system, *Sci. Rep.* **6**, 33867 (2016).
- [73] S. Arranz Regidor, G. Crowder, H. Carmichael, and S. Hughes, Modeling quantum light-matter interactions in waveguide QED with retardation, nonlinear interactions, and a time-delayed feedback: Matrix product states versus a space-discretized waveguide model, *Phys. Rev. Res.* **3**, 023030 (2021).
- [74] H. Pichler and P. Zoller, Photonic circuits with time delays and quantum feedback, *Phys. Rev. Lett.* **116**, 093601 (2016).
- [75] H. Le Jeannic, T. Ramos, S. F. Simonsen, T. Pregnolato, Z. Liu, R. Schott, A. D. Wieck, A. Ludwig, N. Rotenberg, J. J. García-Ripoll, and P. Lodahl, Experimental reconstruction of the few-photon nonlinear scattering matrix from a single quantum dot in a nanophotonic waveguide, *Phys. Rev. Lett.* **126**, 023603 (2021).