

Computational complexity and simulability of non-Hermitian quantum dynamics

Brian Barch^{1,2,*} and Daniel Lidar^{1,2,3,4}¹Department of Physics and Astronomy, [University of Southern California](#), Los Angeles, California 90089, USA²Center for Quantum Information Science & Technology, [University of Southern California](#), Los Angeles, California 90089, USA³Department of Electrical and Computer Engineering, [University of Southern California](#), Los Angeles, California 90089, USA⁴Department of Chemistry, [University of Southern California](#), Los Angeles, California 90089, USA

(Received 13 June 2025; accepted 24 June 2026; published 21 July 2026)

Non-Hermitian (NH) quantum systems demonstrate striking differences from their Hermitian counterparts, leading to claims of NH advantage in areas ranging from metrology to entanglement generation. We show that in the context of quantum computation, any such NH advantage is unlikely to be scalable as an efficient computational resource: If coherent nonunitary evolution with renormalization could be realized with only polynomial overhead, then the resulting model could implement postselection, implying implausibly strong complexity-theoretic power under standard assumptions. We define $\text{NHBQP}(U)$ as the computational power of polynomial-size quantum circuits that, in addition to a standard universal unitary gate set, may apply a fixed gate U on $O(1)$ qubits that is not proportional to a unitary, with the state renormalized after each use of U . We prove that this model is powerful enough to decide every language in PostBQP (equivalently PP). Moreover, in the standard uniform circuit-family model, this characterization is tight: For any fixed such nonunitary gate U , $\text{NHBQP}(U) = \text{PostBQP} = \text{PP}$. PostBQP is believed intractable, so this suggests that any scalable NH computational advantage must come with a compensating cost limiting its efficiency. Additionally, we study simulation complexity of restricted classes of nonunitary systems by purifying them to postselected unitary evolution in a form preserving system locality. Using this framework, we show that unitary gates with postselection can simulate not only evolution under NH Hamiltonians but arbitrary quantum trajectories. Any NH model whose purification lies in a strongly simulable unitary family (e.g., Clifford, matchgate, or low-bond-dimension tensor-network circuits) remains efficiently classically simulable, provided the relevant postselected events occur with probability $\Omega(2^{-\text{poly}(n)})$. Thus, adding non-Hermiticity to a universal unitary system makes it infeasibly computationally powerful, while adding it to a strongly simulable system adds no computational power in this setting.

DOI: [10.1103/lhv6-6t1r](#)

I. INTRODUCTION

When a quantum system undergoes unitary dynamics interspersed with measurements, retaining the full measurement record leads to *conditional* evolution that is not captured by closed-system unitary dynamics, nor by unconditional open-system descriptions, which average over outcomes. In the limit of continuous weak measurement with minimally perturbing results, the combined effect of measurement backaction and Hamiltonian evolution can be expressed as coherent evolution under an effective non-Hermitian (NH) Hamiltonian [1,2]. NH Hamiltonians also arise in other contexts, including proposals to treat NH dynamics as fundamental via generalized inner products [3–6]. Experimentally,

NH behavior has been realized in a variety of platforms [7–10], with particularly direct implementations in optical settings with engineered loss [11–14].

Work on NH dynamics has revealed a range of phenomena, including measurement-induced phase transitions [15–25], violations or modifications of standard locality bounds [2,16,26], and changes in computational power relative to standard unitary models [27–29]. Motivated in part by these effects, a number of potential advantages over strictly Hermitian dynamics have been proposed, in the form of quantum speed limit violation [30,31], faster generation [32] and longer preservation [33,34] of entanglement, and enhanced quantum sensing near exceptional points [35–38]. A central practical question is whether such NH advantages can persist in scalable implementations, i.e., whether the required NH resources can be efficiently realized to achieve a task with bounded error. In this work, we address this question in the computational setting using tools from complexity theory.

The class of decision problems efficiently solvable with bounded error by a universal quantum computer is called BQP [39]. It has been shown that coherent nonunitary evolution under certain NH Hamiltonians, interpreted as state evolution via renormalization, can decide problems outside BQP [29,40,41]. Moreover, conditional NH evolution is commonly

*Contact author: barch@lanl.gov

†Present address: Quantum and Condensed Matter Physics Group (T-4), Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA.

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](#) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI.

expressed via a normalized (nonlinear) equation [1], which is operationally equivalent to renormalizing output probabilities during measurement. This is suggestive, as it has been shown that broad classes of nonlinear modifications of quantum mechanics would enable efficient solutions to problems believed to be intractable, including NP-complete and #P-hard problems [27].

Unlike earlier results that relate nonunitarity and postselection primarily at the level of computational power, we give an explicit construction that implements a postselection gadget from any fixed-support nonunitary gate [42]. Under additional implementability and gap assumptions, this extends to size-dependent families approaching unitarity. We then use $\text{NHBQP}(U)$ to denote the class of problems solvable by polynomial-size quantum circuits built from a universal unitary gate set together with a fixed nonunitary gate U on $O(1)$ qubits, where each nonunitary step is followed by explicit renormalization, equivalent to renormalizing during measurement. We show that this model can decide every language in $\text{PostBQP} = \text{PP}$ (defined in the next section). This bound is tight: For any fixed nonunitary gate U , $\text{NHBQP}(U) = \text{PostBQP} = \text{PP}$ (Theorem 2) [43].

As this level of computational power is widely regarded as implausible for physically realizable devices, we interpret our results as evidence that any scalable NH resource with the potential to enhance universal computation must be accompanied by a compensating cost—most naturally an exponentially small success probability—so that the overall implementation is not efficient. This motivates careful resource accounting when interpreting NH-enabled protocols as scalable computational primitives; that is, claims of computational advantage should explicitly include the physical overhead of realizing the conditioning/renormalization, rather than only the normalized (postselected) dynamics.

In the second half of the paper, we take a complementary approach and study when NH dynamics remains efficiently classically simulable. We do this by purifying nonunitary evolution into unitary evolution on a larger system followed by postselection. This yields a simple criterion: If the purified unitary circuit family is strongly simulable, then the induced nonunitary conditional dynamics is also strongly simulable, provided the postselected events occur with probability $\Omega(2^{-\text{poly}(n)})$. Using this framework, we bound the simulation complexity of nonunitary extensions of paradigmatic strongly simulable families (Clifford and matchgate circuits), and we extend the purification approach to Trotterized simulations of general quantum trajectories in a locality-preserving form relevant for tensor-network methods. These results help bound when non-Hermiticity is expected to increase computational power versus when it does not.

Recently, a similar proposal has been shown to decide P^{PP} , a superset of PP [41]. As discussed in Appendix A, this different computational complexity result comes from the assumption of oracular quantum circuit access made in Ref. [41], while in this work we assume a single quantum circuit instance.

We begin with background on NH quantum mechanics, computational complexity, and simulability in Sec. II. In Sec. III, we show how nonunitary gates implement postselection when supplemented by suitable unitaries, and we state

the corresponding hardness consequences. Extensions of simulability to NH systems and nonunitary circuits are discussed in Sec. IV. We conclude in Sec. V with implications and directions for future work. Technical calculations supporting several results are collected in the Appendixes B–F.

II. BACKGROUND

A. Non-Hermitian quantum mechanics

The paradigmatic conditional evolution motivating NH quantum mechanics is the *no-jump* trajectory. This describes evolution in a setting where quantum jumps are possible, but where one conditions on the event that no jump occurs, either by chance or by postselection [1,45]. Consider a state ρ evolving under the Lindblad equation [46,47]

$$\dot{\rho} = -i[H_0, \rho] + \sum_a \left(L_a \rho L_a^\dagger - \frac{1}{2} \{L_a^\dagger L_a, \rho\} \right). \quad (1)$$

In the quantum-trajectories picture, the jump term $L_a \rho L_a^\dagger$ transfers weight between different trajectories, while the remaining terms generate the drift evolution within a single trajectory [48]. Conditioning on the event that no jump occurs removes the jump term, yielding an unnormalized conditional evolution of the form

$$\dot{\rho} = -i(H\rho - \rho H^\dagger), \quad (2)$$

with the effective NH Hamiltonian

$$H = H_0 - \frac{i}{2} \sum_a L_a^\dagger L_a. \quad (3)$$

See, e.g., Eq. (56) and Appendix F for a concrete example. The (unnormalized) no-jump map over time t is $\rho \mapsto U \rho U^\dagger$ with $U = e^{-iHt}$. This map is completely positive (CP) but not trace preserving (TP), and it is coherent, in that it maps pure states to pure states up to normalization. In the trajectories interpretation, $\text{Tr}[U \rho U^\dagger]$ is the probability of the no-jump measurement record, so it must lie in $[0, 1]$.

Writing $H = H_0 - i\Gamma$ with $\Gamma = \frac{1}{2} \sum_a L_a^\dagger L_a \geq 0$, the trace of the unnormalized conditional state decreases according to

$$\partial_t \text{Tr}[\rho_t] = -2\text{Tr}[\Gamma \rho_t]. \quad (4)$$

Thus, $\Gamma \geq 0$ is necessary and sufficient to ensure that, for every initial state, the no-jump probability $\text{Tr}[\rho_t]$ is nonincreasing in time and hence never exceeds one. Later, we will frequently work with *normalized* conditional states, in which overall rescalings of U (equivalently, imaginary shifts of H by multiples of $i\mathbb{I}$) do not affect the dynamics.

There are two common approaches to studying NH dynamics: (1) normalized evolution of the conditional state, where H represents effective conditional dynamics and is directly physically applicable, or (2) the metric formalism underlying PT -symmetric (more generally quasi-Hermitian) systems, where H is treated as fundamental in a modified Hilbert space [1,5,6]. We primarily use the former approach, but will use techniques from the latter when it is convenient.

1. Normalized evolution

To interpret nonunitary evolution as mapping states to states, one can normalize its action on pure states,

$$|\psi_t\rangle \equiv \frac{U|\psi_0\rangle}{\|U|\psi_0\rangle\|}, \quad (5)$$

or trace normalize its action on mixed states,

$$\rho_t \equiv \frac{U\rho_0 U^\dagger}{\text{Tr}[U\rho_0 U^\dagger]}. \quad (6)$$

This is the usual conditional-state update: The denominator is the Born-rule probability of the no-jump measurement record and can be viewed as the normalization factor in Bayes rule [2]. The same normalized evolution is obtained whether one normalizes the state continuously in time, normalizes the state after each discrete nonunitary step, or normalizes the output probabilities during measurement. In the continuous picture, differentiating Eq. (6) yields a nonlinear (quadratic) equation of motion

$$\dot{\rho} = -i(H\rho - \rho H^\dagger) - i\rho \text{Tr}[\rho(H^\dagger - H)]. \quad (7)$$

Introducing a constant imaginary shift $H \mapsto H + ic\mathbb{I}$ rescales $U(t) = e^{-iHt}$ by e^{ct} and therefore has no effect on the normalized maps [Eqs. (5) and (6)]. Such a shift does, however, rescale the unnormalized no-jump probability $\text{Tr}[U(t)\rho U(t)^\dagger]$ by a factor e^{2ct} , and hence changes success probabilities (and physical overhead) whenever $U\rho U^\dagger$ is interpreted as a postselected/no-jump branch of an underlying trace-preserving evolution. Accordingly, when we focus solely on normalized trajectories or circuit models with explicit renormalization, we may choose c for convenience and need not impose $\Gamma \geq 0$ as a separate restriction. Whenever success probabilities are operationally relevant, however, the choice of c (and hence the absolute scale of Γ) must be tracked explicitly. Throughout this paper, when working in the normalized-evolution picture we will use “nonunitary” to mean “not proportional to a unitary,” since proportional operators are equivalent under normalization.

2. PT -symmetric systems

A Hamiltonian is said to be pseudo-Hermitian if there exists a Hermitian, invertible [49], generally nonunique, “metric” operator η such that $H^\dagger \eta = \eta H$ [3–5]. In this case, eigenvalues of H are either real or come in complex conjugate pairs. If, moreover, there exists a positive definite η , then H is said to be quasi-Hermitian; in the PT -symmetric literature, this corresponds to the unbroken PT -symmetry phase [50]. We use the PT -symmetric terminology (rather than quasi-Hermitian) because it is more common in the physics literature for describing such systems. In this case, one can write

$$H = S H_0 S^{-1} \quad (8)$$

for S and H_0 both Hermitian, where H_0 is isospectral to H under the similarity transform $S = \eta^{-1/2}$ (the positive square root of η^{-1}) [16,51]. The associated evolution operator decomposes as $U = S U_0 S^{-1}$ with unitary $U_0 = e^{-iH_0 t}$. While U is nonunitary in the standard inner product, one can use $\eta > 0$ to define an η inner product in which H is self-adjoint

and U is norm preserving. We will use this metric picture for geometric intuition, but otherwise work in the standard inner product. Additionally, notice that any diagonalizable pseudo-Hermitian $H = H_0 - i\Gamma$ necessarily has indefinite Γ , and so can only be physically generated up to an overall imaginary shift. This is shown in Appendix B 1. Intuitively, pseudo-Hermiticity forces the spectrum to be closed under complex conjugation, whereas a semidefinite Γ would force all eigenvalues to have imaginary parts of the same sign; the only consistent possibility for a diagonalizable pseudo-Hermitian Hamiltonian with semidefinite Γ is therefore $\Gamma = 0$.

3. Diagonalizing H

When H is diagonalizable but NH, it admits a biorthogonal spectral decomposition in terms of right and left eigenvectors $\{|r_i\rangle\}$ and $\{\langle l_i|\}$,

$$H = \sum_i v_i |r_i\rangle \langle l_i|, \quad (9)$$

with (generally complex) eigenvalues $\{v_i\}$. The eigenvectors can be chosen biorthonormal so that $\langle l_i|r_j\rangle = \delta_{ij}$. One may additionally normalize $\langle r_i|r_i\rangle = 1$, at the cost of leaving the left eigenvectors $\{\langle l_i|\}$ unnormalized. With this choice, the evolution operator is

$$U = \sum_i e^{-itv_i} |r_i\rangle \langle l_i|. \quad (10)$$

Real eigenvalues generate oscillatory behavior, while complex eigenvalues cause exponential growth or decay along the corresponding right eigenspaces.

When the imaginary parts are not all equal, any initial state undergoing normalized evolution will exponentially decay into subspace of right eigenspaces whose eigenvalues have maximal imaginary component and on which it has nonzero support [25]. That is, if $\mathcal{H}_\rho$ is the subspace where ρ has support, $v_{\max}^I$ is the maximal imaginary eigenvalue component within that subspace, and $\mathcal{H}_L$ the long-time subspace that ρ approaches, then

$$\begin{aligned} \mathcal{H}_\rho &= \text{span}\{|r_j\rangle \mid \langle l_j|\rho|l_j\rangle > 0\}, \\ \mathcal{H}_L &= \text{span}\{|r_j\rangle \in \mathcal{H}_\rho \mid \text{Im}(v_j) = v_{\max}^I\}. \end{aligned} \quad (11)$$

For nondegenerate H , this subspace will be one dimensional (1D), so such Hamiltonians are said to be in the “purifying” phase [15].

B. Computational complexity

Computational complexity theory provides a framework for classifying computational tasks according to the resources required to carry them out, such as time or memory [52–54]. Here, we primarily focus on decision problems, where (informally) an algorithm must output a single bit representing “yes” or “no” given an input string. More formally, the algorithm must determine whether a given string belongs to a specified set of strings, a task referred to as *deciding a language*. We also reference counting problems, where an algorithm must output the number of solutions associated with a given input, rather than determining if one exists. A complexity class is then defined as a collection of such

languages (informally, computational problems) that can be decided (solved) within a given resource bound by a specified computational model.

We call a family of algorithms $\{C_n\}$ efficient if C_n achieves a task using resources that scale polynomially with input size n . Moreover, the family is said to be uniform if there exists a classical algorithm that outputs a description of C_n in time $\text{poly}(n)$. The class polynomial time (P) consists of all languages efficiently decidable by a uniform family of deterministic classical algorithms. Bounded-error probabilistic polynomial time (BPP) extends this to classical randomized algorithms that output the correct answer with probability at least $2/3$, where any constant greater than $1/2$ can be amplified to $2/3$ by repetition and majority vote. The quantum analog, BQP, denotes the class of languages efficiently decidable with similar bounded error by a uniform family of quantum circuits. It is known that $P \subseteq BPP \subseteq BQP$.

Several complexity classes believed to be intractable in general, but useful as reference points, include PP, #P, and PostBQP. PP can be viewed as a relaxation of BPP in which the acceptance probability need only be greater than $1/2$, potentially by an arbitrarily small margin. Informally, #P is a class of counting problems: Given an input instance and an efficiently checkable yes/no condition, how many bitstrings satisfy that condition? [55] The two are related by the standard oracle equivalence $P^{\#P} = P^{PP}$ (see, e.g., Refs. [52,54]). As usual, the oracle notation P^C denotes the class solved by algorithms for P that may call a solver for C as an elementary subroutine. PostBQP describes BQP enhanced by postselection, defined below. It was shown that $\text{PostBQP} = \text{PP}$ [29], hence $P^{\text{PostBQP}} = P^{PP} = P^{\#P}$. Toda's theorem further shows that the entire polynomial hierarchy is contained in $P^{\#P}$ [44].

1. Postselection

Postselection is the hypothetical ability to condition on a desired measurement outcome after a computation has taken place [29,56–60]. More formally, PostBQP consists of decision problems solvable by a uniform family of polynomial-size quantum circuits with two designated output bits: an output bit and a postselection bit. The circuits must satisfy the following conditions on each input of size n : (1) the postselection bit equals 1 with probability at least $\Omega(2^{-\text{poly}(n)})$, and (2) conditioned on the postselection bit being 1, the output bit is correct with probability at least $2/3$ [29,57]. We use standard asymptotic notation: $P(s) \in \Omega(2^{-\text{poly}(n)})$ means $P(s) \geq 2^{-p(n)}$ for some polynomial $p(n)$ [61].

When a circuit has multiple postselection bits in parallel, these can be combined into a single postselection bit by taking the conjunction using reversible and gates; the postselection success probability then decreases multiplicatively, but remains at least $\Omega(2^{-\text{poly}(n)})$ if only polynomially many bits are used. Intermediate, even interleaved, postselection steps within a single quantum computation do not increase the power of PostBQP: One can coherently record each postselection condition in ancillas and postselect on their conjunction at the end of the computation [29]. In contrast, the oracle class $P^{\text{PostBQP}} = P^{PP}$ refers to classical polynomial-time computation with adaptive oracle access to a $\text{PostBQP} = \text{PP}$ subroutine (i.e., where later queries may depend on earlier

oracle answers); whether this collapses to PostBQP is not known (the adaptive case is subtle because postselection can amplify even exponentially small error probabilities), and we will not rely on any separation here.

In the quantum circuit model, postselection is equivalent to applying a projector onto the subspace where the postselection bit is |1>, followed by renormalization, prior to reading out the output bit. For our purposes, it will be important that exact postselection can be relaxed to an approximate version. Specifically, it suffices to implement a postselection gadget that is exponentially close to the ideal projector on instances where the postselected event has probability at least $\Omega(2^{-\text{poly}(n)})$. We will show that such postselection gadgets can be implemented using normalized evolution under NH Hamiltonians.

2. Simulability

Not all quantum circuit families realize the full computational power of BQP; many have output statistics that can be efficiently reproduced by classical algorithms, in which case the circuit family is considered simulable. Here, we are primarily concerned with circuit families that are strongly simulable [62–64]. Strong simulability requires a classical algorithm that can estimate marginal output probabilities to exponentially small additive error. We use the following strong-simulation notion [62], which is the one relevant for postselection.

Definition 1 (Strong simulability). Let $\{C_n\}_{n \geq 1}$ be a uniform family of quantum circuits with a measured output register X of length $\text{poly}(n)$. We say $\{C_n\}$ is *strongly simulable* if there exists a classical algorithm that, given n , a choice of indices $\mathcal{I}$ specifying any subset of the output bits, an assignment $x_{\mathcal{I}}$ to those bits, and an error parameter $\epsilon > 0$, outputs a probability estimate $\tilde{P}(X_{\mathcal{I}} = x_{\mathcal{I}})$ satisfying

$$|\tilde{P}(X_{\mathcal{I}} = x_{\mathcal{I}}) - P(X_{\mathcal{I}} = x_{\mathcal{I}})| \leq \epsilon \quad (12)$$

in time $\text{poly}(n, \ln(1/\epsilon))$.

The need for exponentially small additive error becomes essential once one conditions on events that may occur with probability as small as $2^{-\text{poly}(n)}$, as in postselection. The following proposition formalizes that strong simulability is preserved under such postselection.

Proposition 1 (Strong simulability is preserved under postselection). Let $\{C_n\}_{n \geq 1}$ be a family of circuits with measured output registers (X, S) , inducing a distribution $P_n(x, s)$ over bitstrings (x, s) , where S is a designated postselection register and X is a computational output register. Assume $\{C_n\}$ is strongly simulable in the sense of Definition 1.

Fix a postselection string s^* such that there exists a polynomial $q(n)$ with $P_n(s^*) \geq 2^{-q(n)}$ for all sufficiently large n . Then, any family of circuits that outputs the distribution

$$P_n^{\text{post}}(x) \equiv P_n(x | s^*) = \frac{P_n(x, s^*)}{P_n(s^*)} \quad (13)$$

is also strongly simulable. That is, for every $\epsilon > 0$, there is a classical algorithm that outputs $\tilde{P}_n^{\text{post}}(x)$ satisfying $|\tilde{P}_n^{\text{post}}(x) - P_n^{\text{post}}(x)| \leq \epsilon$ in time $\text{poly}(n, \ln(1/\epsilon))$.

A proof is provided in Appendix B 2. Note that the requirement of *strong* simulation is essential here: Conditioning

replaces probabilities by ratios such as $P_n(x, s^*)/P_n(s^*)$, and the denominator $P_n(s^*)$ may be as small as $2^{-\text{poly}(n)}$. A polynomial additive approximation to $P_n(x, s^*)$ and $P_n(s^*)$ is generally insufficient, as one needs additive accuracy on the scale of $P_n(s^*)$ in order to guarantee constant error in the conditional probability.

Tensor networks are natural candidates for extending simulability results to the NH setting, since their contraction rules do not require unitarity, and for many local quantum systems they yield efficient classical simulation. For example, dynamical tensor-network methods can simulate 1D adiabatic and short-time unitary evolution while keeping bond dimensions polynomial via Lieb-Robinson bounds [65,66]. In NH systems, however, the effective notion of locality can change [26], making it unclear which tensor-network techniques (and bond-dimension guarantees) generalize.

More importantly, for the present discussion, Proposition 1 highlights that accurate simulation of a postselected (and as we will later show, purified-NH) model requires exponentially small (or zero) additive error on the relevant marginals. Tensor networks can, in principle, provide such strong simulation in regimes with sufficiently low entanglement [64]. In contrast, most practical approximate tensor-network time-evolution algorithms yield only polynomially small additive errors for many outcome probabilities, which can be amplified under postselection and thus generally fall short of strong simulation. For this reason, tensor networks are primarily of theoretical interest for the strong-simulability questions we address, even though they remain extremely useful as practical tools.

3. Complexity of NH systems

Normalized evolution under certain NH Hamiltonians is known to solve decision problems beyond BQP [29]. This is striking, as NH Hamiltonians are often used as effective descriptions of open quantum systems, yet unconditional (e.g., Markovian) open-system evolution is described by CPTP maps and can be simulated by standard quantum circuits (and hence within the usual sampling model SampBQP, the class of distributions samplable by polynomial-size quantum circuits *without* postselection) [67,68].

The distinction is the conditioning step. Operationally, a nonunitary update arises as a Kraus branch $\rho \mapsto K\rho K^\dagger$ of a CPTP dilation, with success probability $p = \text{Tr}[K\rho K^\dagger]$ and postselected state $\rho' = K\rho K^\dagger/p$. Without conditioning (i.e., when the measurement outcome is not fixed), the overall evolution is CPTP and does not exceed the power of ordinary quantum computation. However, when the conditioned branch is treated as the output state, producing its output statistics amounts to sampling from a *postselected* circuit. Such postselected access can implement PostBQP = PP computations [29]. Accordingly, a generic polynomial-overhead simulation of these conditioned output distributions by ordinary SampBQP circuits would imply a collapse of postselection power (e.g., PostBQP = BQP), which is believed to be unlikely. Without such a collapse, one expects that reproducing the conditioned branch requires explicit postselection or potentially exponential repetition-until-success overhead.

Equivalently, the normalized nonunitary map is trace preserving but nonlinear in ρ [1,2]. It was shown that broad classes of nonlinear modifications of quantum dynamics enable polynomial-time algorithms for NP-complete decision problems and #P-complete counting problems [27]. This motivates investigating which normalized NH dynamics can effectively implement the kind of conditioning needed for postselection-type power.

In this work, we are specifically interested in coherent conditional circuits, which can be written in terms of nonunitary gates. Throughout, we work in the standard uniform quantum circuit model with polynomial-size circuits, renormalization after each nonunitary operation (equivalently, after conditioning on measurement outcomes), and standard gate-description assumptions (e.g., gate matrix entries that are efficiently approximable to inverse-exponential precision).

When gate entries are additionally assumed to be algebraic numbers [69], the set of languages decidable by polynomial-size nonunitary circuits with explicit renormalization is a subset of PP [29]. This can be seen by extending the proof that $\text{BQP} \subseteq \text{PP}$ to the present renormalized nonunitary setting [29,71]. As we show in Sec. IV, it can also be shown constructively: after a harmless overall rescaling that leaves the normalized map unchanged, each nonunitary step can be dilated into a unitary circuit on a larger system followed by postselection on a meter outcome, putting the resulting computational power within PostBQP. Additionally, one can also express the acceptance probability of such circuits in a form reducible to $\text{P}^{\#P}$ computation, yielding a (looser) counting-complexity upper bound [40].

III. HARDNESS OF NH HAMILTONIANS + BQP

Henceforth, we work in the normalized-evolution picture, following every nonunitary operation with renormalization. All results in this section are statements about this formal renormalized circuit model; they do not depend on a particular physical interpretation of the renormalization step (e.g., as a normalized map induced by postselection/no-jump conditioning, versus treating renormalization as an abstract normalization rule). Our goal in this section is to show that any fixed-support nonunitary gate can be used to implement a postselection gadget when supplemented by a universal set of unitary gates. This implies that NH dynamics supplemented with universal unitary gates has computational power far beyond what is believed feasible, and is therefore unlikely to be scalable without an accompanying unacceptable resource cost.

4. Single-qubit example

The basic $\text{NH} \Rightarrow$ postselection mechanism is already apparent for the one-qubit NH Hamiltonian $H = i\sigma^z$:

$$e^{\sigma^z t}(a|0\rangle + b|1\rangle) = ae^t|0\rangle + be^{-t}|1\rangle \sim |0\rangle + \frac{b}{a}e^{-2t}|1\rangle, \quad (14)$$

where $\sim$ denotes equality up to state renormalization. For t growing polynomially with the input size n , the $|1\rangle$ component is suppressed as $e^{-2t} = 2^{-\text{poly}(n)}$, so for inputs with non-negligible overlap on $|0\rangle$ ($|a|^2 \geq 2^{-\text{poly}(n)}$) the normalized

map approximates a projector onto $|0\rangle$. This implements a postselection-type primitive and, when combined with universal unitary gates to implement the surrounding computation, yields computational power beyond BQP (made precise below). This already suggests that if such evolution could be efficiently implemented for times $t \in \text{poly}(n)$, then it would enable postselection-type power and hence implausibly strong complexity-theoretic consequences.

A. Postselection and hardness for constant support-size circuits

We now demonstrate that this postselection mechanism is completely general: Any nonunitary gate yields a postselection gadget when surrounded by suitable unitaries. We begin by formally defining the complexity class associated with BQP enhanced with any nonunitary gate.

Definition 2 (NHBQP for a fixed nonunitary gate). Fix a universal set of unitary gates and a gate U acting on $m = O(1)$ qubits that is not proportional to a unitary. Define $\text{NHBQP}(U)$ as the class of languages decided with bounded error by a uniform family of polynomial-size circuits over this gate set, where each use of U is followed by renormalization.

We assume the fixed gate set has efficiently computable algebraic matrix entries, as is standard in quantum complexity theory. This avoids pathological encodings of nonuniform advice into arbitrary real constants and ensures that the counting-complexity characterizations used in Sec. IV apply cleanly. For physical gates whose exact matrix entries are not algebraic (e.g., $U = e^{-iHt}$ at generic real times), one can work within this framework by fixing an algebraic approximation of U to the required inverse-exponential precision.

We restrict attention to *well-formed* circuits, meaning that on every input the renormalization after each use of U is defined (equivalently, the postselected branch has nonzero norm at every such step). Circuits that violate this condition have undefined normalized evolution on that input and are not regarded as deciding a language with bounded error in this model.

Theorem 1 (Constant-support nonunitary gates enable postselection). Let U be a fixed nonunitary gate as in Definition 2, which can be implemented efficiently, and suppose each use of U is followed by renormalization. Then, together with any universal set of unitary gates, U allows efficient implementation of a postselection gadget with approximation error at most $2^{-\text{poly}(n)}$ [i.e., conditioned on successful postselection, the probability of the undesired postselected outcome is $O(2^{-\text{poly}(n)})$]. Consequently,

$$\text{PostBQP} = \text{PP} \subseteq \text{NHBQP}(U). \quad (15)$$

Proof. Write the singular value decomposition

$$U = VDW^\dagger, \quad (16)$$

where V, W are unitary and $D \geq 0$ is diagonal in a bitstring basis $\{|s_j\rangle\}$:

$$D = \sum_{j=1}^d \lambda_j |s_j\rangle\langle s_j|, \quad \lambda_1 \geq \dots \geq \lambda_d \geq 0, \quad d \equiv 2^m. \quad (17)$$

Since $m = O(1)$ is constant, $V^\dagger$ and W are fixed matrices independent of n . They can be precomputed once and

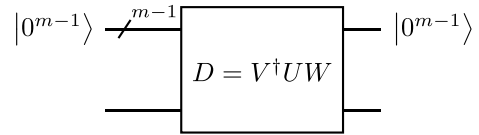


FIG. 1. A postselection gadget that uses an m -qubit nonunitary gate U together with unitaries V, W from its SVD to implement an effective diagonal nonunitary map $D = V^\dagger U W$ on a target qubit plus $m - 1$ ancillas initialized to $|0^{m-1}\rangle$. This implements postselection on the target (bottom) qubit. Since D is diagonal in the computational basis, the ancillas remain in the state $|0^{m-1}\rangle$.

approximated to accuracy $2^{-\text{poly}(n)}$ using $\text{poly}(n)$ unitary gates from any universal set [72]. Implementing $D = V^\dagger U W$ therefore incurs only polynomial overhead, even when the resulting gadget is repeated $r = \text{poly}(n)$ times.

Because U is not proportional to a unitary, it has at least two distinct singular values, so $\lambda_1 > \lambda_d$. By permuting the diagonal entries of D (absorbing the permutation into $V^\dagger, W$), we may assume the largest singular value corresponds to the desired state $|0^m\rangle$ and the smallest to the undesired state $|0^{m-1}1\rangle$. Applying D to the target qubit together with $m - 1$ ancillas initialized to $|0^{m-1}\rangle$ ensures the evolution begins and ends in the two-dimensional span of $|0^m\rangle$ and $|0^{m-1}1\rangle$ (see Fig. 1). In this block, for any amplitudes $a, b \in \mathbb{C}$ and target qubit initialized to $a|0\rangle + b|1\rangle$, we have

$$D^r(a|0^m\rangle + b|0^{m-1}1\rangle) = a\lambda_1^r|0^m\rangle + b\lambda_d^r|0^{m-1}1\rangle. \quad (18)$$

After renormalization, measuring the target qubit yields the undesired outcome 1 with probability

$$P(|0^{m-1}1\rangle) = \frac{|b|^2 \lambda_d^{2r}}{|a|^2 \lambda_1^{2r} + |b|^2 \lambda_d^{2r}} \leq \frac{1}{|a|^2} \left(\frac{\lambda_d}{\lambda_1} \right)^{2r}. \quad (19)$$

We take this failure probability as the postselection-gadget approximation error.

The definition of PostBQP assumes the outcome being postselected occurs with probability at least $2^{-\text{poly}(n)}$. In the present two-level reduction, this corresponds to assuming

$$|a|^2 \geq 2^{-k(n)} \quad (20)$$

for some polynomial $k(n)$. To ensure $P(|0^{m-1}1\rangle) \leq \varepsilon$ for a given ε uniformly over all inputs satisfying Eq. (20), it suffices to require

$$\left(\frac{\lambda_d}{\lambda_1} \right)^{2r} \leq \varepsilon 2^{-k(n)}. \quad (21)$$

Define the normalized singular radius

$$\Delta \equiv 1 - \frac{\lambda_d}{\lambda_1} \in (0, 1]. \quad (22)$$

Then, Eq. (21) is equivalent to

$$2r(-\ln(1 - \Delta)) \geq \ln(2^{k(n)}/\varepsilon). \quad (23)$$

Using the bound $-\ln(1 - \Delta) \geq \Delta$ [73], this is satisfied by any r such that

$$r \geq \frac{\ln(2^{k(n)}/\varepsilon)}{2\Delta}. \quad (24)$$

When $\varepsilon = 2^{-p(n)}$ for some polynomial $p(n)$, Eq. (24) can be satisfied by $r \in \Theta((p(n) + k(n))/\Delta) \in \text{poly}(n)$, where the second inclusion holds since Δ is a constant. Therefore, the implementation of a postselection gadget with the required error $2^{-\text{poly}(n)}$ is efficient.

Succeeding a polynomial-depth circuit of universal unitary gates with a single postselection gadget creates a circuit which can decide any language in PostBQP. Using $\text{PostBQP} = \text{PP}$ [29], we conclude $\text{PP} \subseteq \text{NHBQP}(U)$. ■

B. Postselection and hardness for variable support-size circuits

When the nonunitary gate varies with input size n , we need to redefine the NHBQP complexity class.

Definition 3 (NHBQP for a family of nonunitary gates). Fix a universal set of unitary gates. Let $\{U_n\}_{n \geq 1}$ be a family of nonunitary gates, where U_n acts on $m(n)$ qubits with $m(n) \in O(\text{poly}(n))$. Define $\text{NHBQP}(\{U_n\})$ as the class of languages decided with bounded error by a uniform family of polynomial-size circuits $\{C_n\}$ over this gate set, where C_n may use the gate U_n followed by renormalization.

Here, “uniform” means that there exists a classical polynomial-time algorithm that outputs a description of C_n . When we allow a size-dependent family of nonunitary gates $\{U_n\}_{n \geq 1}$, we assume it is specified uniformly and does not encode nonuniform advice. That is, that there exists a classical polynomial-time algorithm that, on input 1^n , outputs a description of U_n , and likewise that any auxiliary descriptions used below (e.g., circuit decompositions or the singular value transform (SVD) ingredients) must be generable uniformly in n . Without such a restriction, one can encode arbitrary truth tables into $\{U_n\}$ (or into $\{V_n^\dagger, W_n\}$), thereby trivializing the model [54]. Two independent issues now arise: (1) whether the SVD unitaries $V_n^\dagger, W_n$ are efficiently implementable, and (2) whether the normalized singular radius closes sufficiently slowly with n . The next corollary makes these requirements explicit and shows that, under them, the same postselection-gadget construction remains efficient.

Corollary 1 (Families of nonunitary gates). Let $\{U_n\}$ be a family of nonunitary gates, where U_n acts on $m(n)$ qubits and each use is followed by renormalization. Assume that for any n and any desired inverse-exponential accuracy $2^{-p(n)}$ for polynomial p , one can efficiently implement unitary circuits that approximate $V_n^\dagger$ and W_n from an SVD $U_n = V_n D_n W_n^\dagger$ to error at most $2^{-p(n)}$ (e.g., in operator norm). Let $\lambda_{1,n} \geq \dots \geq \lambda_{d,n}$ be the singular values of U_n , and define

$$\Delta_n \equiv 1 - \frac{\lambda_{d,n}}{\lambda_{1,n}}. \quad (25)$$

If $\Delta_n \in \Omega(1/\text{poly}(n))$, then $\{U_n\}$ together with universal unitary gates can efficiently implement a postselection gadget with error $2^{-\text{poly}(n)}$, and hence decide every language in PostBQP. Consequently,

$$\text{PostBQP} = \text{PP} \subseteq \text{NHBQP}(\{U_n\}). \quad (26)$$

Proof. The proof follows Theorem 1 with Δ replaced by Δ_n and with the compilation of $V^\dagger, W$ replaced by the assumed efficient implementations of $V_n^\dagger, W_n$. For target error $\varepsilon = 2^{-p(n)}$ and postselection-event probability at least $2^{-k(n)}$,

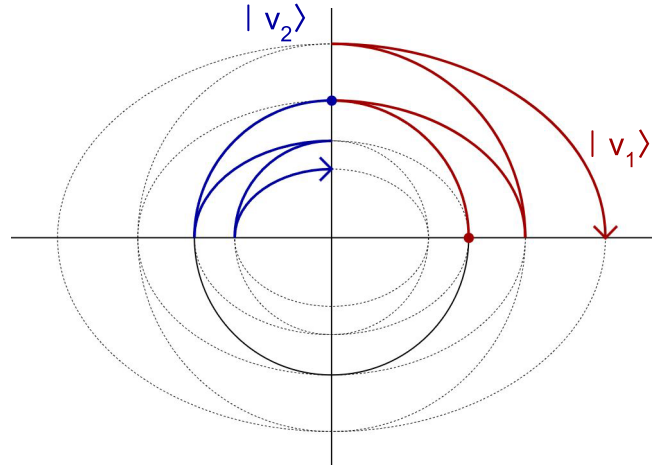


FIG. 2. Geometric representation of two rounds of unitary evolution followed by PT -symmetric evolution for a $d = 2$ system. For initial state $|v_1\rangle$ ($|v_2\rangle$), initialized along the major (minor) ellipse axis, the net effect is growth (decay) of the state norm. The unit circle is shown in solid black, while dashed lines describe the accessible circles and ellipses of other radii. When applied repeatedly to a superposition of the two states, this allows one to approximately project onto $|v_1\rangle$ up to normalization.

the same calculation yields the sufficient condition

$$r \geq \frac{\ln(2^{k(n)}/\varepsilon)}{2\Delta_n}. \quad (27)$$

If $\Delta_n \in \Omega(1/\text{poly}(n))$, then $r \in \text{poly}(n)$ is sufficient, and the postselection-gadget reduction remains efficient. ■

The normalized singular radius Δ_n has a “distance-from-unitarity” interpretation. Up to constant factors, Δ_n equals the minimum operator-norm distance between a rescaling of U_n and the unitary group. In Appendix C, we show that

$$\min_{\alpha > 0, T \in \mathcal{U}(d)} \|\alpha U_n - T\| = \frac{\lambda_{1,n} - \lambda_{d,n}}{\lambda_{1,n} + \lambda_{d,n}} = c_n \Delta_n, \quad (28)$$

where $c_n \equiv \lambda_{1,n}/(\lambda_{1,n} + \lambda_{d,n}) \in (1/2, 1]$. Thus, $\Delta_n \in \Omega(1/\text{poly}(n))$ is equivalent to requiring that no rescaling αU_n is superpolynomially close to the unitary group in operator norm.

The reason Theorem 1 applies even when $U = e^{-iHt}$ arises from PT -symmetric Hamiltonians, which exhibit quasiperiodic behavior, can be understood geometrically. Unitary evolution preserves the standard state norm and so rotates states on a hypersphere in state space. Similarly, PT -symmetric evolution preserves the norm with respect to some metric $\eta > 0$ [5,26] and so moves states on an ellipsoid whose principal axes are set by η . By surrounding the PT -symmetric evolution by suitable unitaries (as in the SVD-based gadget), one can cancel the unitary “rotational” component while retaining the anisotropic stretching responsible for the postselection mechanism, as depicted in Fig. 2.

When considering gates generated by NH Hamiltonians as $U = e^{-iHt}$, the opaque nature of the SVD can make it difficult to express the required unitaries V, W and repetition count r directly in terms of H . However, for certain important subclasses, one can solve for the relevant ingredients explicitly. In

particular, when H is diagonalizable with complex spectrum, or when H is single-qubit PT symmetric, one can directly construct the required basis changes and obtain tighter, more transparent bounds on the resources needed to implement a postselection gadget. We address these cases in the next two subsections.

C. Diagonalizable, complex spectrum

If H is diagonalizable and has at least two eigenvalues with different imaginary parts, then $U = e^{-iHt}$ produces differential growth or decay between eigenspaces without the need to intersperse additional unitaries. This allows us to replace the general SVD-based discussion by an explicit diagonalization-based postselection gadget. Examples of such H include PT -symmetric Hamiltonians in the broken-symmetry regime, such as the NH transverse-field Ising model with strong imaginary field [15,25]. Consider the spectral decomposition

$$H = \sum_{j=1}^d (h_j - i\gamma_j) |r_j\rangle\langle l_j|, \quad \gamma_1 \leq \gamma_2 \leq \dots \leq \gamma_d, \quad (29)$$

where $h_j, \gamma_j \in \mathbb{R}$, $\{|r_j\rangle\}$ and $\{|l_j\rangle\}$ are biorthonormal, and H is assumed diagonalizable. Then,

$$U = e^{-iHt} = \sum_{j=1}^d e^{-\gamma_j t} e^{-ih_j t} |r_j\rangle\langle l_j|, \quad (30)$$

so time evolution rescales the amplitude in each right eigenspace j by $e^{-\gamma_j t}$, up to normalization.

As shown in Appendix D, one can construct unitaries $\hat{V}^\dagger$ and $\hat{W}$ (distinct from the SVD unitaries) such that the effective map $\hat{D} \equiv \hat{V}^\dagger U \hat{W}$ has the two-dimensional block structure used below even when the extremal right eigenvectors $|r_1\rangle$ and $|r_d\rangle$ are nonorthogonal. In particular, restricted to the subspace spanned by $|0^m\rangle$ and $|0^{m-1}1\rangle$ (with $m-1$ ancilla qubits initialized to $|0^{m-1}\rangle$ as in Fig. 1), the effective map

$$\begin{aligned} \hat{D} \equiv \hat{V}^\dagger U \hat{W} &= a_r e^{-\gamma_1 t} |0^m\rangle\langle 0^m| + a_r e^{-\gamma_d t} G + \dots \\ &\sim |0^m\rangle\langle 0^m| + e^{-(\gamma_d - \gamma_1)t} G + \dots \end{aligned} \quad (31)$$

for a scalar a_r and a 2×2 matrix G (depending on H and t but bounded in norm) acting on $\text{span}\{|0^m\rangle, |0^{m-1}1\rangle\}$. Here, the ellipsis denotes terms that never contribute when the ancillas are initialized to $|0^{m-1}\rangle$.

Since $\gamma_d > \gamma_1$, the coefficient $e^{-(\gamma_d - \gamma_1)t}$ decays exponentially in t , so $\hat{D}$ approaches a projector onto $|0^m\rangle$ in the relevant two-dimensional block. Moreover, Appendix D shows that $\|G\| \leq \|G\|_F = 1$ (where the second norm is the Frobenius norm), so this suppression bound holds uniformly, independent of the angle between $|r_1\rangle$ and $|r_d\rangle$.

To bound the induced postselection-gadget error, consider an input state $a|0^m\rangle + b|0^{m-1}1\rangle$ with $|a|^2 \geq 2^{-k(n)}$ (the relevant regime for postselection). Using $\|G\| \leq 1$, the undesired outcome probability after normalized evolution is $O(e^{-2(\gamma_d - \gamma_1)t} / |a|^2)$. Thus, to achieve error at most ε , it suffices to choose t such that

$$e^{-2(\gamma_d - \gamma_1)t} \leq \varepsilon 2^{-k(n)}. \quad (32)$$

Equivalently,

$$t \geq \frac{\ln(2^{k(n)}/\varepsilon)}{2(\gamma_d - \gamma_1)}. \quad (33)$$

Taking $\varepsilon = 2^{-p(n)}$ yields $t \in O((p(n) + k(n))/(\gamma_d - \gamma_1))$, which is efficient both for fixed H and for n -dependent families H_n whenever $\gamma_d - \gamma_1 \in \Omega(1/\text{poly}(n))$. Conversely, if $\gamma_d - \gamma_1$ shrinks faster than any inverse polynomial, then (potentially after an irrelevant imaginary shift) the spectrum becomes nearly real and the induced evolution is too close to unitarity to yield an efficient postselection gadget.

D. Single-qubit PT symmetric

In the single-qubit PT -symmetric case (in the quasi-Hermitian sense of Sec. II A 2), the SVD ingredients can be obtained in closed form. As shown in Appendix E, after choosing an appropriate orthonormal basis and dropping terms proportional to the identity, any single-qubit PT -symmetric Hamiltonian can be written as

$$H = \begin{pmatrix} 0 & g - \gamma \\ g + \gamma & 0 \end{pmatrix}, \quad (34)$$

with $0 < \gamma < g$. The corresponding time-evolution operator is

$$U = \cos(\omega t) \mathbb{I} - \frac{i}{\omega} \sin(\omega t) H, \quad \omega \equiv \sqrt{g^2 - \gamma^2} \in \mathbb{R}. \quad (35)$$

At the special time $t = \pi/(2\omega)$, one has $U = -i\omega^{-1}H$. Let σ^x denote the bit-flip operator in this basis and define

$$\lambda_{\pm} \equiv \frac{g \pm \gamma}{\omega}. \quad (36)$$

Then,

$$\sigma^x U = -i \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}, \quad (37)$$

so $\lambda_{\pm}$ are the singular values of U and the effective diagonal gate in the SVD gadget can be taken as $D = i\sigma^x U$ (i.e., $W = \mathbb{I}$ and $V^\dagger = i\sigma^x$).

To suppress the $|1\rangle$ component to error $\varepsilon = 2^{-p(n)}$, it suffices to choose

$$r = \left\lceil \frac{\ln(1/\varepsilon)}{\ln(\lambda_+/\lambda_-)} \right\rceil = O\left(p(n) \frac{g}{\gamma}\right), \quad (38)$$

where the final scaling follows from the lower bound $\ln(\lambda_+/\lambda_-) = \ln((g + \gamma)/(g - \gamma)) \geq 2\gamma/g$ for $0 < \gamma < g$, so $1/\ln(\lambda_+/\lambda_-) = O(g/\gamma)$. Thus, $r \in O(\text{poly}(n))$ provided $\gamma/g \in \Omega(1/\text{poly}(n))$. In this case, the only additional unitary needed to realize the postselection gadget (up to an irrelevant phase) is the bit-flip σ^x , which is proportional to the Hermitian part of H , namely, $g\sigma^x$.

IV. UPPER BOUNDS VIA PURIFICATION

In this section, we obtain stricter upper bounds on NHBQP(U) and reduced nonunitary models by *purifying* the nonunitary evolution into unitary evolution on a larger system together with postselection on a designated meter register.

Theorem 1 establishes a lower bound on worst-case hardness only in the presence of a universal family of unitary gates, so it does not address nonunitary extensions of nonuniversal unitary models, such as Clifford circuits, matchgate circuits, or unitary dynamics efficiently simulable under tensor networks. Reducing nonunitary evolution to a purification gadget, composed of unitary evolution with postselection, allows us to transfer classical simulation results for the purified unitary family back to the original nonunitary dynamics. In particular, Proposition 1 implies that strong simulability is preserved under postselection on events occurring with probability $\Omega(2^{-\text{poly}(n)})$. Therefore, whenever a nonunitary model admits a purification whose underlying unitary circuit family is strongly simulable, the nonunitary model itself is strongly simulable.

A. Exact power of NHBQP(U)

We can determine the computational power of NHBQP(U) by combining Theorem 1 with a general upper bound, via a reduction to postselected unitary circuits in PostBQP.

Theorem 2 (Characterization of NHBQP(U)). Fix a universal finite set of unitary gates and a fixed m -qubit gate U with $m = O(1)$, which is not proportional to a unitary, and let all gates have efficiently specifiable algebraic gate entries. Consider a uniform circuit family generated by this gate set, where each use of U is followed by renormalization. Then, the class of languages decidable by this family is exactly PP. That is,

$$\text{NHBQP}(U) = \text{PostBQP} = \text{PP}. \quad (39)$$

Proof. We first establish the upper bound $\text{NHBQP}(U) \subseteq \text{PostBQP} = \text{PP}$ [29]. The inclusion into PP can be obtained by extending the standard proof that $\text{BQP} \subseteq \text{PP}$ to the present renormalized nonunitary setting [29,71]. Here, we give an alternative proof in terms of purification into postselected unitary circuits in PostBQP, as it motivates the ensuing simulability results for restricted systems.

Let $\{C_n\}$ be a uniform family of NHBQP(U) circuits deciding a language L with bounded error, and assume $\{C_n\}$ is well formed, meaning that every renormalization step is defined (equivalently, the postselected branch has nonzero norm at each such step).

We reduce $\{C_n\}$ to a postselected unitary circuit family. Since global rescalings of U do not affect the induced normalized evolution, fix any $\alpha > 0$ such that $K \equiv \alpha U$ satisfies $K^\dagger K \leq \mathbb{I}$ (e.g., $\alpha = 1/\|U\|$). By the polar decomposition, $K = QR$ with Q unitary and $R = (K^\dagger K)^{1/2} \geq 0$ satisfying $0 \leq R \leq \mathbb{I}$. R can be realized as a Kraus branch of a unitary U_R on the system A plus a single-qubit meter B of the form $\langle 0|U_R|0\rangle_B = R_A$, for

$$U_R = \begin{pmatrix} R_A & \sqrt{\mathbb{I}_A - R_A^2} \\ \sqrt{\mathbb{I}_A - R_A^2} & -R_A \end{pmatrix}_B \quad (40)$$

written in the computational basis of B . Therefore, K can be implemented (up to the unitary Q) by a unitary circuit followed by postselection on the meter outcome.

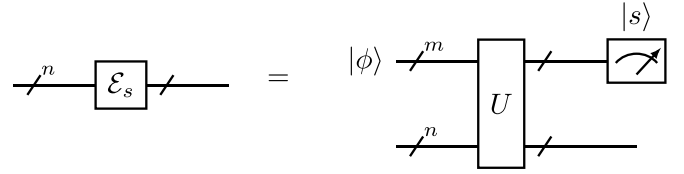


FIG. 3. Representation of a nonunitary (normalized) conditional map $\mathcal{E}_s$ as a purification gadget composed of a unitary circuit U on a system-meter hybrid, followed by postselection of the meter measurement outcome s .

Now, replace each use of U in C_n by the corresponding unitary dilation gadget for K and postselect the associated meter qubit on the designated outcome. Conditioned on all such postselections succeeding, the (renormalized) evolution of the system register is exactly the same as in the original NHBQP(U) circuit, since the overall scalar factor α cancels under renormalization at each step. Because U acts on $m = O(1)$ qubits, the required dilation unitaries are constant size $m + 1$ and can be approximated to inverse-exponential accuracy with only polynomial overhead using any universal unitary gate set. Finally, multiple, even interleaved, postselection conditions can be combined into a single final postselection by coherently recording each condition and postselecting on their conjunction (this does not increase the power of PostBQP) [29]. Thus, $L \in \text{PostBQP}$, and using $\text{PostBQP} = \text{PP}$ [29] gives $\text{NHBQP}(U) \subseteq \text{PP}$.

Next, we show the lower bound: $\text{PP} \subseteq \text{NHBQP}(U)$. By Theorem 1, the gate U together with universal unitary gates can efficiently implement a postselection gadget with error $2^{-\text{poly}(n)}$ in the standard regime where the postselected event has probability at least $2^{-\text{poly}(n)}$. Therefore, any PostBQP computation can be simulated within NHBQP(U); hence, we obtain $\text{PP} = \text{PostBQP} \subseteq \text{NHBQP}(U)$. ■

Note that we did not assume an external classical polynomial-time algorithm with adaptive oracle access to NHBQP(U) subroutines. Such oracle access would correspond to classes such as $\text{P}^{\text{PostBQP}} = \text{P}^{\text{PP}} = \text{P}^{\#P}$, which (unlike Refs. [40,41]) we do not claim here.

B. Purifying nonunitary gates into unitary circuits with postselection

The polar decomposition used in the proof of Theorem 2 is sufficient when $m = O(1)$ and one assumes access to universal unitary gates. However, it gives no bound on the resources required to implement Q and U_R , especially when the nonunitary operator U is allowed to depend on system size, and so generalizes poorly to nonuniversal settings. Thus, we will instead consider the following more general construction of a purification gadget, which enables more transparent decompositions. Consider a unitary circuit U acting on a system A and a meter B , where B is initialized in a pure state $|\phi\rangle$ and then measured in the computational basis, as in Fig. 3. Conditioning on the meter outcome s induces the (unnormalized) map

$$\tilde{\mathcal{E}}_s(\rho) \equiv \langle s|U(\rho \otimes |\phi\rangle\langle\phi|)U^\dagger|s\rangle_B, \quad (41)$$

and the corresponding normalized (generally nonlinear) conditional state-update map

$$\mathcal{E}_s(\rho) \equiv \frac{\tilde{\mathcal{E}}_s(\rho)}{\text{Tr}[\tilde{\mathcal{E}}_s(\rho)]}. \quad (42)$$

Defining the effective (generally nonunitary) operator on A by

$$C_s \equiv \langle s|_B U |\phi\rangle_B, \quad (43)$$

we have $\tilde{\mathcal{E}}_s(\rho) = C_s \rho C_s^\dagger$.

C. Strong simulability via postselected purification

If the unitary circuit U in a purification gadget is strongly simulable, then so are the induced postselected output probabilities (equivalently, the normalized conditional map $\mathcal{E}_s$), provided the postselection success probability is not too small. Indeed, if one can estimate the joint probability $P_U(x, s)$ and the marginal $P_U(s)$ to exponentially small additive error, then one can compute the conditional distribution

$$P_{\mathcal{E}_s}(x) = \langle x | \mathcal{E}_s(\rho) | x \rangle = \frac{P_U(x, s)}{P_U(s)} = P_U(x | s). \quad (44)$$

By Proposition 1, this computation remains efficient (in the strong-simulation sense) as long as

$$P_U(s) \geq 2^{-\text{poly}(n)}. \quad (45)$$

The same reasoning applies to conditional probabilities, e.g.,

$$P_{\mathcal{E}_s}(x_1 | x_2) = P_U(x_1 | x_2, s). \quad (46)$$

Thus, whenever an NH model admits a purification gadget whose underlying unitary circuit family is strongly simulable, the NH model is strongly simulable as well (in the postselected/conditional sense relevant here).

Below, we apply this framework to nonunitary extensions of paradigmatic strongly simulable unitary models: Clifford circuits and matchgate circuits. We then turn to tensor-network-motivated purifications that preserve locality.

D. Clifford circuits

Clifford circuits are generated by Hadamard, phase, and cnot gates, and form the normalizer of the Pauli group: For any Clifford circuit U and Pauli string σ , $U\sigma U^\dagger$ is another Pauli string. By the Gottesman-Knill theorem, Clifford circuits with computational-basis inputs and measurements are efficiently strongly simulable [74,75].

As a nonunitary building block, let U be an $(n+1)$ -qubit Clifford circuit acting on n system qubits and a one-qubit meter B , with B initialized and postselected in $|0\rangle$. The induced effective operation on the system is

$$C \equiv \langle 0|_B U |0\rangle_B. \quad (47)$$

One can also see explicitly how C acts on Pauli strings. For an n -qubit Pauli string σ ,

$$\begin{aligned} C\sigma C^\dagger &= \langle 0|_B U (\sigma \otimes |0\rangle\langle 0|) U^\dagger |0\rangle_B \\ &= \frac{1}{2} \langle 0|_B U [\sigma \otimes (\mathbb{I} + \sigma^z)] U^\dagger |0\rangle_B \\ &= \frac{1}{2} (\langle 0|\sigma_1|0\rangle_B + \langle 0|\sigma_2|0\rangle_B), \end{aligned} \quad (48)$$

where $\sigma_1 \equiv U(\sigma \otimes \mathbb{I})U^\dagger$ and $\sigma_2 \equiv U(\sigma \otimes \sigma^z)U^\dagger$ are $(n+1)$ -qubit Pauli strings. Each contraction $\langle 0|\sigma_k|0\rangle_B$ is either an n -qubit Pauli string (if the meter factor in σ_k is $\mathbb{I}$ or σ^z) or the zero operator (if it is σ^x or σ^y). Thus, C maps a single Pauli string to a linear combination of up to two Pauli strings, which suggests that Pauli-string tracking without meters can proliferate terms and become potentially inefficient. Nevertheless, the overall system plus meter dilation circuit remains a Clifford circuit with a postselection and therefore remains efficiently simulable via standard stabilizer-tableau updates conditioned on the desired measurement outcomes [76]. This highlights that including nonunitary non-Clifford operations (in this case C arising from postselected Clifford circuits) in an otherwise Clifford circuit need not preclude efficient classical simulation. This contrasts with the unitary case, where adding any non-Clifford *unitary* to the Clifford group yields a universal gate set for BQP and destroys efficient classical simulation.

As a simple example, take $U = \text{CNOT}_{12}$ with the meter as the target. Then, $C = |0\rangle\langle 0|$ is the projector onto $|0\rangle$ on the system qubit. Circuits generated by Clifford subcircuits and such projectors were studied in Ref. [77], where a range of distinct conditional-information phenomena were observed, including ultrafast spreading of conditional mutual information.

Finally, note that this nonunitary effect is specific to conditioning. If postselection is replaced by a partial trace, then for any Pauli string σ ,

$$\begin{aligned} \langle \sigma \rangle &= \text{Tr}[\sigma \text{Tr}_B[U(\rho \otimes |0\rangle\langle 0|)U^\dagger]] \\ &= \text{Tr}[(\sigma \otimes \mathbb{I}_B)U(\rho \otimes |0\rangle\langle 0|)U^\dagger] \\ &= \text{Tr}[(\rho \otimes |0\rangle\langle 0|)U^\dagger(\sigma \otimes \mathbb{I}_B)U] \\ &= \text{Tr}[\rho \langle 0|U^\dagger(\sigma \otimes \mathbb{I}_B)U|0\rangle_B], \end{aligned} \quad (49)$$

so the unconditional channel does not introduce the same potential for Pauli-string branching.

E. Matchgate circuits

Matchgate circuits form another paradigmatic strongly simulable family [78–81]. They are generated by two-qubit unitary gates of the form

$$U(F, G) = \begin{pmatrix} F_{11} & 0 & 0 & F_{12} \\ 0 & G_{11} & G_{12} & 0 \\ 0 & G_{21} & G_{22} & 0 \\ F_{21} & 0 & 0 & F_{22} \end{pmatrix}, \quad (50)$$

with $\det(F) = \det(G)$. Circuits composed of nearest-neighbor matchgates on a 1D chain, with product-state inputs and computational-basis measurements, can be strongly simulated efficiently via a mapping to free fermions (Jordan-Wigner transformation).

A simple nonunitary extension is obtained by coupling a system qubit to a one-qubit meter via a matchgate, then postselecting the meter. When the meter is initialized to $|+\rangle$ and postselected to $|0\rangle$, the induced single-qubit map on the system is

$$\langle 0|_B U(F, G) |+\rangle_B = \frac{1}{\sqrt{2}} \begin{pmatrix} F_{11} & F_{12} \\ G_{21} & G_{22} \end{pmatrix}. \quad (51)$$

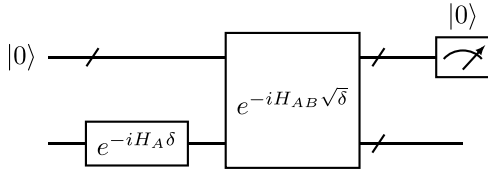


FIG. 4. A single time step of the purification gadget implementing no-jump NH evolution. The unitary $e^{-iH_A\delta}$ is block diagonal in the meter basis (a controlled evolution), while $e^{-iH_{AB}\sqrt{\delta}}$ generates $O(\sqrt{\delta})$ transitions in the meter. Postselecting on the meter reproduces the no-jump Kraus operator $C = \langle 0|U|0\rangle_B$ to $O(\delta)$, hence H_{eff}^0 to leading order.

The choice of F and G subject to matchgate constraints allows this to produce a broad family of single-qubit system operators, including arbitrary unitary gates under normalized evolution. The matchgate determinant condition $\det(F) = \det(G)$ can always be satisfied by adjusting phases on the unused rows of F and G , and it does not restrict the induced single-qubit map in Eq. (51).

Our purification-based simulation applies only when the enlarged unitary circuit (system plus meters) remains within the standard strongly simulable class of 1D nearest-neighbor matchgate circuits. Since each effective gate of the form in Eq. (51) requires coupling to a unique meter qubit, such gates cannot be inserted arbitrarily along the chain without leaving this class. In particular, to preserve a 1D nearest-neighbor geometry, they can be placed only at the ends of the chain, so at most two such gates can be included. This amounts to only a constant overhead even under nonpurified simulation. This restriction is in fact essential: If gates of the form in Eq. (51) could be applied freely as primitives in a matchgate circuit, then, since they realize arbitrary single-qubit unitaries under normalization, the resulting model would be universal for BQP. We do not attempt to characterize the most general class of postselected matchgate circuits, but simply take this case as an example.

F. Trotterized NH dynamics

The Trotterization framework used to simulate Lindblad evolution [82,83] can be adapted to conditional (trajectory) evolution by conditioning on specific meter outcomes (postselection), rather than averaging over them (partial trace). Operationally, one alternates short unitary interactions between a system and a meter with measurements of the meter (Fig. 4). Averaging over meter outcomes yields unconditional (Lindbladian) dynamics, while conditioning on the meter remaining in its initial state at every step produces the no-jump trajectory and evolution under an effective NH Hamiltonian on the system.

The construction below is a discrete-time analog of the standard no-jump derivation [1], formulated in such a way that the system-meter unitary can be chosen to preserve locality, as required by tensor-network simulations. A derivation that treats general trajectories including jump events and the unconditional Lindblad limit is given in Appendix F.

Fix a meter basis $\{|j\rangle\}$ and consider a system A coupled to a meter B with Hamiltonian

$$H \equiv H_A + \frac{1}{\sqrt{\delta}} H_{AB}, \quad (52)$$

where $\delta > 0$ is the time step, and

$$H_A \equiv \sum_j H_j \otimes |j\rangle\langle j|, \quad H_{AB} \equiv \sum_{j \neq k} L_{jk} \otimes |j\rangle\langle k|. \quad (53)$$

Hermiticity of H_{AB} implies $L_{jk} = L_{kj}^\dagger$, and the scaling $H_{AB}/\sqrt{\delta}$ is the standard choice that makes jump amplitudes $O(\sqrt{\delta})$ and jump probabilities $O(\delta)$ per time step.

For a single step, let

$$U \equiv e^{-iH\delta} = e^{-i(\delta H_A + \sqrt{\delta} H_{AB})}. \quad (54)$$

Expanding in $\sqrt{\delta}$ gives

$$U = \mathbb{I} - i\sqrt{\delta} H_{AB} - i\delta H_A - \frac{\delta}{2} H_{AB}^2 + O(\delta^{3/2}). \quad (55)$$

The $O(\sqrt{\delta})$ term is off block diagonal in the meter basis, since H_{AB} maps $|j\rangle$ to $|k \neq j\rangle$.

1. No-jump (drift) time step

Initialize the meter in $|0\rangle$ and postselect the measurement outcome $|0\rangle$ at the end of the step, as in Fig. 4. The induced (unnormalized) system map is the Kraus operator

$$C \equiv \langle 0|U|0\rangle_B. \quad (56)$$

Using the expansion above, the $\sqrt{\delta}$ term vanishes under $\langle 0|\cdot|0\rangle_B$, and one obtains

$$C = \mathbb{I}_A - i\delta \langle 0|H_A|0\rangle_B - \frac{\delta}{2} \langle 0|H_{AB}^2|0\rangle_B + O(\delta^{3/2}). \quad (57)$$

Writing Hermitian system Hamiltonian $\langle 0|H_A|0\rangle_B = H_0$ and using

$$\langle 0|H_{AB}^2|0\rangle_B = \sum_{j \neq 0} L_{0j} L_{j0} = \sum_{j \neq 0} L_{j0}^\dagger L_{j0}, \quad (58)$$

we obtain

$$\begin{aligned} C &= \mathbb{I}_A - i\delta \left(H_0 - \frac{i}{2} \sum_{j \neq 0} L_{j0}^\dagger L_{j0} \right) + O(\delta^{3/2}) \\ &= e^{-i\delta H_{\text{eff}}^0} + O(\delta^{3/2}), \end{aligned} \quad (59)$$

where the no-jump effective Hamiltonian is

$$H_{\text{eff}}^0 \equiv H_0 - \frac{i}{2} \sum_{j \neq 0} L_{j0}^\dagger L_{j0} = \langle 0|H_A|0\rangle_B - \frac{i}{2} \langle 0|H_{AB}^2|0\rangle_B. \quad (60)$$

Up to an overall imaginary shift (physically irrelevant under normalized evolution), this construction can generate an arbitrary NH Hamiltonian on A , since $\sum_j L_{j0}^\dagger L_{j0} \geq 0$ but is otherwise unconstrained.

The Trotter error can be reduced from $O(\delta^{3/2})$ to $O(\delta^2)$ by picking H_{AB} to only include L_{jk} terms with $j = 0$ or $k = 0$. Under this choice, odd powers of H_{AB} are off diagonal in the meter basis, causing the $O(\delta^{3/2})$ term to drop out of Eq. (57).

While this choice would not affect H_{eff}^0 , it would complicate the construction of other H_{eff}^k for $k \neq 0$.

2. Other no-jump sectors and jump events

If instead the meter is initialized and postselected in $|k\rangle$, the same derivation yields a (generally different) no-jump Hamiltonian

$$H_{\text{eff}}^k \equiv H_k - \frac{i}{2} \sum_{j \neq k} L_{jk}^\dagger L_{jk}. \quad (61)$$

Additionally, if the meter begins in $|j\rangle$ and is postselected in $|k \neq j\rangle$, the leading contribution comes from the $O(\sqrt{\delta})$ part of U and produces a jump map. As shown in Appendix F, the induced unnormalized update is

$$\tilde{\mathcal{E}}_{k \leftarrow j}(\rho) = \delta L_{kj} \rho L_{kj}^\dagger + O(\delta^{3/2}), \quad (62)$$

which becomes the corresponding quantum-jump update after renormalization. Together, the family of no-jump maps and jump maps realizes a discrete-time quantum trajectories model specified by $\{H_k\}_k$ and $\{L_{kj}\}_{k \neq j}$.

3. A locality-preserving choice

A key advantage of this dynamical purification gadget is that H can be chosen so that locality in the system is preserved by the system-meter unitary U , and hence by tensor-network representations thereof. Suppose the desired jump operators are local, with $L_{j0} = L_{jA}$ acting on a localized subregion j_A of A . Introduce a meter B with local subregions j_B paired to j_A , and define

$$H_A \equiv H_0 \otimes \mathbb{I}_B, \quad H_{AB} \equiv \sum_j (L_{jA} \otimes \sigma_{jB}^+ + L_{jA}^\dagger \otimes \sigma_{jB}^-). \quad (63)$$

Taking the meter input and postselection state to be $|0\rangle_B \equiv |0^m\rangle_B$, we obtain

$$\begin{aligned} H_{\text{eff}}^0 &= H_0 - \frac{i}{2} \sum_{jk} L_{jA}^\dagger L_{kA} \langle 0^m | \sigma_{jB}^- \sigma_{kB}^+ | 0^m \rangle_B \\ &= H_0 - \frac{i}{2} \sum_j L_{jA}^\dagger L_{jA}. \end{aligned} \quad (64)$$

Because the expectation $\langle 0^m | \sigma_{jB}^- \sigma_{kB}^+ | 0^m \rangle_B$ vanishes unless $j = k$, the anti-Hermitian correction is a sum of local terms $L_{jA}^\dagger L_{jA}$. In particular, the interaction graph induced by H_{eff}^0 is determined by H_0 together with said local terms, so it preserves the same notion of locality as the underlying unitary system dynamics. A representative 1D example is shown in Fig. 5.

This same construction yields the jump terms $\{L_{jA} \rho L_{jA}^\dagger\}_j$ by postselecting the corresponding meter excitations, and yields the full Lindblad evolution if meter outcomes are traced rather than postselected, as reviewed in Appendix F. Multiple jump operators per site can be included by taking j_B to be a small register and coupling each local operator to a distinct raising operator within that register.

When extending from a single system Hamiltonian H_0 to a family $\{H_k\}_k$ controlled by meter outcomes, locality requires that changes in the meter state only induce local changes in the system Hamiltonian. One sufficient condition is that H_A

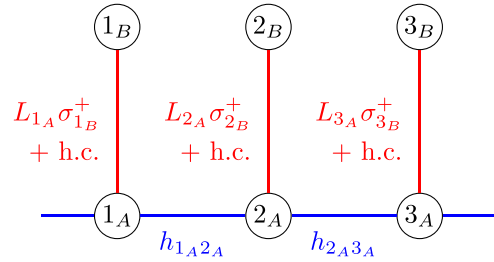


FIG. 5. Locality-preserving couplings in Eq. (63) for a 1D chain system Hamiltonian $H_0 = \sum_{(i,j)} h_{ij}$ on A with single-qubit L_{jA} . The system-system couplings (blue) and system-meter couplings (red) together define a local system-meter Hamiltonian H whose no-jump sector yields H_{eff}^0 in Eq. (64).

decomposes into locally controlled terms,

$$H_A = \sum_j H_{jAB}, \quad H_{jAB} \equiv \sum_i h_{jA}^i \otimes |i\rangle\langle i|_{jB}, \quad (65)$$

where each h_{jA}^i acts only on a neighborhood of A around j_A . More general trajectory models can also be implemented by allowing feedback between steps (choosing the next H conditioned on measured meter outcomes), at the cost of stronger control assumptions.

Finally, this locality-preserving purification gadget construction connects directly to simulability. If the unitary system-meter circuit U is strongly simulable (for example, via an exact or exponentially accurate tensor-network contraction), then by Proposition 1 the induced conditional dynamics are also strongly simulable, provided the postselection success probabilities remain at least $2^{-\text{poly}(n)}$ for the trajectories of interest.

G. PT-symmetric dynamics

When the NH Hamiltonian H is quasi-Hermitian (equivalently, PT symmetric in the unbroken phase), questions about $U = e^{-iHt}$ can be reduced to the corresponding questions about a unitary evolution. As discussed in Sec. II A 2, in this regime there exists a positive definite metric $\eta > 0$ such that $H^\dagger \eta = \eta H$ [84]. Defining $S \equiv \eta^{-1/2} > 0$, one obtains a Hermitian Hamiltonian $H_0 \equiv S^{-1} H S$, and hence the similarity transform $H = S H_0 S^{-1}$, with $U = S U_0 S^{-1}$ and $U_0 \equiv e^{-iH_0 t}$.

If U_0 admits an efficient tensor-network representation (and contraction algorithm) with bond dimension $f(n)$, and if both S and S^{-1} admit matrix product operator (MPO) representations of bond dimension $q(n) \in O(\text{poly}(n))$, then $U = S U_0 S^{-1}$ has an MPO representation of bond dimension at most $f(n) q^2(n)$. Thus, in this regime, tensor-network simulability of U follows from that of U_0 with only polynomial overhead. An example of such a system is the 1D NH transverse-field Ising model [15,25], for which S is a product operator. For general S , the required bond dimension may be exponential, in which case the map $U = S U_0 S^{-1}$ need not preserve efficient tensor-network simulability.

It is also useful to represent the similarity transform as a purification gadget [51]. The one-qubit purification in Eq. (40) applies to any positive operator that is a contraction.

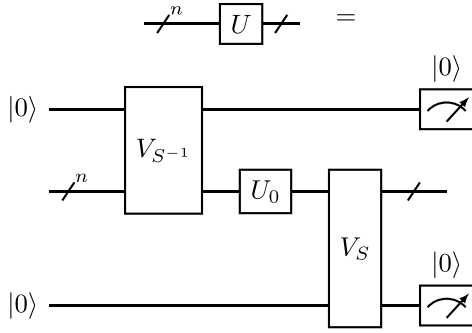


FIG. 6. Two-meter purification gadget for $U = SU_0 S^{-1}$ for PT -symmetric dynamics. The unitaries V_S and $V_{S^{-1}}$ satisfy $\langle 0|V_S|0\rangle = \alpha S$ and $\langle 0|V_{S^{-1}}|0\rangle = \beta S^{-1}$ for suitable scalars $\alpha, \beta > 0$ that make the implemented operators contractions. Postselecting both meters in $|0\rangle$ yields $(\alpha\beta)U$.

Accordingly, pick constants $\alpha, \beta > 0$ such that $\alpha S \leq \mathbb{I}$ and $\beta S^{-1} \leq \mathbb{I}$. Then, there exist unitaries V_S on $A \otimes B$ and $V_{S^{-1}}$ on $A \otimes B'$ (with single-qubit meters B, B') such that

$$\langle 0|V_S|0\rangle_B = \alpha S, \quad \langle 0|V_{S^{-1}}|0\rangle_{B'} = \beta S^{-1}. \quad (66)$$

With two-meter qubits initialized and postselected in $|0\rangle$ as in Fig. 6, the effective operator is

$$\langle 00|V_S U_0 V_{S^{-1}}|00\rangle_{BB'} = \alpha\beta S U_0 S^{-1} = \alpha\beta U, \quad (67)$$

which equals U up to an overall scalar. This scalar does not affect normalized evolution or strong-simulation statements, but it does affect physical success probabilities of postselection.

When S factorizes across local subregions, the purification gadget preserves locality in a particularly transparent way. For example, if $S = \bigotimes_j S_{j_A}$ is a product operator on local subregions j_A of A , then one can choose V_S as a product of two-qubit gates $V_S = \bigotimes_j V_{j_A j_B}$ coupling each j_A to a local meter qubit j_B , with

$$\langle 0|V_{j_A j_B}|0\rangle_{j_B} = \alpha_j S_{j_A} \quad (68)$$

for appropriate local rescalings α_j . In this case, both V_S and $V_{S^{-1}}$ can be expressed as unitary MPOs of constant bond dimension, and the unitary dilation introduces only local system-meter couplings, as depicted in Fig. 7.

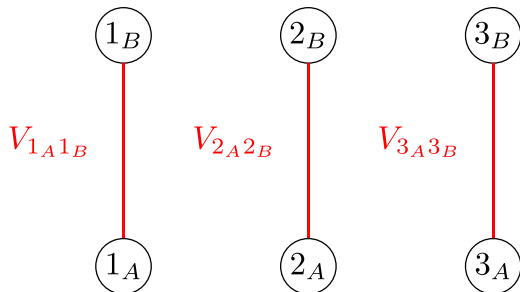


FIG. 7. Product structure of the unitary V_S when S is a product operator, yielding a locality-preserving purification gadget.

V. DISCUSSION

This work establishes a bidirectional relationship between normalized NH dynamics and postselection. On one hand, Sec. III approximates an explicit postselection gadget to exponentially small error, built from normalized evolution under a coherent non-unitary gate together with suitable surrounding unitaries (in our construction, $V^\dagger$ and W arising from an SVD of the nonunitary gate). For fixed-support gates, this applies to any gate not proportional to a unitary, and for size-dependent gate families it applies under the explicit uniform implementability and inverse-polynomial singular-value-gap assumptions stated there. This leads to our central complexity-theoretic implication (Theorem 1): Universal unitary gates supplemented with any fixed gate U that is not proportional to a unitary, together with renormalization after each use of U (equivalently, renormalizing the output probabilities during measurement), can implement a postselection gadget and therefore decide every language in PostBQP. Moreover, Theorem 2 shows this is tight in the uniform circuit-family model: $\text{NHBQP}(U) = \text{PostBQP} = \text{PP}$ for any fixed such gate U .

On the other hand, Sec. IV shows that a broad class of nonunitary evolutions can be *purified* into unitary evolution on a larger system followed by postselection on a meter register, thereby inheriting strong simulability from the unitary supersystem where applicable. As a special case, a single time step of a general quantum-trajectory update can be realized as unitary evolution on a larger system followed by postselection on a meter register, with a per-time-step error $O(\delta^{3/2})$ for time step δ .

A key message is that the above reductions identify *sufficient* conditions for equivalence, but they are not obviously *necessary*. For example, the SVD-based postselection gadget in Sec. III makes the role of a singular-value gap explicit, but it hides the dependence of the surrounding unitaries on microscopic descriptions such as a local Hamiltonian H , which could allow one to better bound the cost of implementing them. Likewise, the purification gadget constructions provide a route to simulation and upper bounds, but their quantitative performance is tied to the chosen dilation and to discretization error [e.g., for the Trotterized approach, $O(\delta^{3/2})$ per step]. It would be valuable to develop postselection gadgets that can be implemented more directly from a local NH generator, and to sharpen the resource analysis (gate complexity, success probability, and accumulated error over long times) for physically motivated families of NH systems.

Our simulability results in Secs. II and IV are inherently *strong-simulation* statements, in that they rely on the ability to estimate relevant probabilities to exponentially small additive error, which is what is needed to remain stable under conditioning. If the purified unitary model is, for example, only *weakly* simulable (sampling without probability estimation), then Proposition 1 does not apply, and postselection can in general increase classical complexity substantially. Understanding when sub-strong-simulation results can be transferred through purification and what sampling complexity classes are natural for restricted NH models remains an important open direction.

Although NH evolution is unlikely to be efficiently scalable as a computational primitive in the universal-unitary

setting suggested by Theorem 1, NH physics has already been realized experimentally at modest sizes, particularly in photonics where postselection and loss engineering are natural ingredients. Relatedly, postselection on “no error detected” is implicit in successful runs of error-detection protocols, so the corresponding conditional evolution is effectively nonunitary. Even if such procedures do not scale efficiently in the worst case, they can serve as platforms for probing NH phenomena and for demonstrating postselection-gadget behavior at small sizes in settings where conditioning is already part of the experimental setup. It is also interesting to explore proposals in which NH dynamics might emerge without explicit postselection, for example via special relativity as discussed in Ref. [85]. If physically realized in a controllable way, such effects would have striking complexity-theoretic implications.

Finally, the no-jump trajectory generating NH systems makes up a small subset of all possible conditional evolutions. More general conditional dynamics are described by quantum trajectories: NH drift interspersed with stochastic jumps. It remains an open question how robust NH-induced dynamical signatures (such as locality breakdown) are under a limited number of jumps. For example, Ref. [2] gives evidence for persistent Lieb-Robinson bound violations with a strength that decays rapidly with the number of jumps. While the ensemble-averaged evolution is linear and CPTP, the state conditioned on an observed measurement record is trace normalized and therefore evolves via a nonlinear Bayesian update rule. However, in the standard circuit model, stochastic trajectories together with classical feedforward are already encompassed by BQP; achieving postselection-like computational power would require conditioning on *particular* (potentially exponentially unlikely) records, i.e., explicit postselection. An interesting open direction is to characterize the complexity of restricted measurement-and-feedback/trajectory models and to quantify when, and at what cost in success probability or resources, trajectory-level conditioning can emulate the postselection gadgets studied here.

ACKNOWLEDGMENTS

This material is based upon work supported by, or in part by, the U.S. Army Research Laboratory and the U.S. Army Research Office under Contract/Grant No. W911NF2310255, and by the U.S. Department of Energy, Office of Advanced Scientific Computing Research, Accelerated Research for Quantum Computing program, Fundamental Algorithmic Research towards Quantum Utility (FAR-Qu) project. We are grateful to Qi Zhang and Biao Wu for insightful discussions about computational complexity.

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: RELATION TO THE BNQP MODEL [41]

In a similar work, Zhang and Wu [41] introduce a model of nonunitary quantum computing that augments a standard

universal unitary gate set with a fixed non-unitary single-qubit gate $G = \text{diag}(g, g^{-1})$, for $g > 0$, $g \neq 1$. They define the corresponding bounded-error class bounded-error nonlinear quantum polynomial-time (BNQP) and argue that $\text{BNQP} = \text{P}^{\#P}$.

At the level of normalized output statistics without adaptive classical access to a PP routine, their fixed-gate model coincides with the special case of our framework $\text{NHBQP}(U)$ with $U = G$: Since renormalization only multiplies the state by an overall scalar, it commutes through subsequent linear gates and cancels upon final normalization, and their definition does not allow intermediate measurements followed by further gates. In our uniform circuit-family setting (polynomial-size circuits with renormalization and no external oracle access), and using Theorem 2, we thus obtain $\text{BNQP} = \text{NHBQP}(G) = \text{PostBQP} = \text{PP}$. Accordingly, oracle-class expressions such as $\text{P}^{\#P} = \text{P}^{\text{PP}}$ must involve an additional assumption of classical polynomial-time computation with adaptive oracle access to a PP (equivalently PostBQP) subroutine, which we do not include in the definition of $\text{NHBQP}(U)$.

To pinpoint where the stronger $\text{BNQP} = \text{P}^{\#P}$ -type conclusion can enter, note that the route in Ref. [41] is explicitly motivated by their earlier Lorentz-quantum-computer construction [40], where they first propose a bounded-error procedure for a PP-complete language (MAJSAT), and then invoke the standard identity $\text{P}^{\text{PP}} = \text{P}^{\#P}$.

Specifically, in Ref. [40] the authors discuss a problem they call MAX- k -INDEPENDENT-SET (IS) as an example of a $\text{P}^{\#P}$ (equivalently P^{PP}) task, and give a procedure that uses polynomially many oracular calls to their PP subroutine for MAJSAT. This is consistent with P^{PP} , but it relies on an additional oracle assumption (adaptive classical access to a PP subroutine) that is not part of the standard uniform circuit-family definition of BNQP [nor of our $\text{NHBQP}(U)$]. Note that while $\text{MAX-}k\text{-IS} \in \text{P}^{\#P}$, we are not aware of a proof that the particular MAX- k -IS formulation used there is $\text{P}^{\#P}$ -complete or $\text{P}^{\#P}$ -hard.

Since the subsequent $\text{P}^{\#P}$ discussion in Ref. [41] is built on this PP subroutine and on oracle-class reasoning, we view $\text{P}^{\#P}$ as reflecting either (1) an oracle-augmented model (adaptive classical access to a PP routine, absent in our work) or (2) an implicit promise/extra assumption not present in the standard uniform circuit-family definition of BNQP.

In additional contrast, our results treat arbitrary nonunitary gates (and size-dependent families $\{U_n\}$ under explicit uniformity/implementability assumptions), and we further develop a purification-based simulability framework for identifying efficiently simulable NH dynamics. Zhang and Wu [41] also analyze physical schemes for implementing G and argue that the required physical resources scale exponentially, supporting the view that the apparent complexity-theoretic power of fixed nonunitary gates is unlikely to be scalable in a universal quantum-computing setting.

APPENDIX B: PROOFS FROM BACKGROUND

1. Pseudo-Hermitian H has indefinite Γ

Here, we prove the statement in Sec. II A that any diagonalizable pseudo-Hermitian Hamiltonian has indefinite anti-Hermitian part.

Proof. Suppose $H = H_0 - i\Gamma$ with $H_0 = H_0^\dagger$ and $\Gamma = \Gamma^\dagger \geq 0$, and let $H|r_j\rangle = v_j|r_j\rangle$. Normalizing $|r_j\rangle$ so that $\langle r_j|r_j\rangle = 1$ gives

$$v_j = \langle r_j|H|r_j\rangle = \langle r_j|H_0|r_j\rangle - i\langle r_j|\Gamma|r_j\rangle, \quad (\text{B1})$$

so $\text{Im}(v_j) = -\langle r_j|\Gamma|r_j\rangle \leq 0$. Pseudo-Hermiticity implies the spectrum is closed under complex conjugation: If v is an eigenvalue, then v^* is also an eigenvalue. Together with $\text{Im}(v_j) \leq 0$ for all j , this forces $\text{Im}(v_j) = 0$ for all j (otherwise the conjugate eigenvalue would have positive imaginary part). Hence, every v_j is real and thus $\langle r_j|\Gamma|r_j\rangle = 0$ for all j . Since $\Gamma \geq 0$, this forces $\Gamma|r_j\rangle = 0$ for all j . If H is diagonalizable, the right eigenvectors span the space, so $\Gamma = 0$. ■

2. Closure of strong simulability under postselection

Here, we provide the proof of Proposition 1, showing closure of strong simulability under postselection.

Proof. Fix n and abbreviate

$$A \equiv P_n(x, s^*), \quad B \equiv P_n(s^*), \quad (\text{B2})$$

so that $0 \leq A \leq B$ and $P_n^{\text{post}}(x) = A/B$. Let $\tilde{A}$ and $\tilde{B}$ be estimates of A and B with additive error at most ϵ' :

$$|\tilde{A} - A| \leq \epsilon', \quad |\tilde{B} - B| \leq \epsilon'. \quad (\text{B3})$$

Strong simulability of $P_n(x, s^*)$ implies that $\tilde{A}$ and $\tilde{B}$ can be classically calculated in time $\text{poly}(n, \ln(1/\epsilon'))$. Assume $\epsilon' \leq B/2$. Then,

$$\tilde{B} \geq B - \epsilon' \geq B/2 > 0, \quad (\text{B4})$$

so the denominator $\tilde{B}$ is strictly positive. Define the estimate

$$\tilde{P}_n^{\text{post}}(x) \equiv \frac{\tilde{A}}{\tilde{B}}. \quad (\text{B5})$$

Its error obeys

$$\begin{aligned} \left| \frac{\tilde{A}}{\tilde{B}} - \frac{A}{B} \right| &= \left| \frac{\tilde{A}B - A\tilde{B}}{\tilde{B}B} \right| = \left| \frac{B(\tilde{A} - A) + A(B - \tilde{B})}{\tilde{B}B} \right| \\ &\leq \frac{B|\tilde{A} - A| + A|B - \tilde{B}|}{|\tilde{B}|B} \leq \frac{B\epsilon' + A\epsilon'}{(B/2)B} \leq \frac{4\epsilon'}{B}. \end{aligned} \quad (\text{B6})$$

Therefore, to guarantee conditional additive error at most ϵ , we may assume without loss of generality that $0 < \epsilon \leq 1$ (otherwise set $\epsilon \equiv 1$), and choose

$$\epsilon' \equiv \frac{\epsilon}{4} 2^{-q(n)}. \quad (\text{B7})$$

Since $B \geq 2^{-q(n)}$, this choice ensures both $\epsilon' \leq (\epsilon/4)B \leq B/4$ (hence, $\epsilon' \leq B/2$) and

$$\frac{4\epsilon'}{B} \leq \epsilon, \quad (\text{B8})$$

so the ratio estimate satisfies $|\tilde{P}_n^{\text{post}}(x) - P_n^{\text{post}}(x)| \leq \epsilon$.

Finally, with this choice,

$$\ln(1/\epsilon') = \ln(1/\epsilon) + O(q(n)), \quad (\text{B9})$$

and since $q(n)$ is polynomial in n , a run-time $\text{poly}(n, \ln(1/\epsilon'))$ is also $\text{poly}(n, \ln(1/\epsilon))$. ■

APPENDIX C: DISTANCE FROM UNITARITY

Let $U \in \mathbb{C}^{d \times d}$ be nonzero and not proportional to a unitary, with singular values $\lambda_1 \geq \dots \geq \lambda_d \geq 0$ (so $\lambda_1 > 0$), and recall the normalized singular radius $\Delta \equiv 1 - \frac{\lambda_d}{\lambda_1} \in (0, 1]$. Throughout this Appendix, $\|\cdot\|$ denotes the operator norm, i.e., the largest singular value.

Lemma 1 (Distance to the unitary group up to rescaling). Let $U = VDW^\dagger$ be a singular value decomposition, with V, W unitary and $D = \text{diag}(\lambda_1, \dots, \lambda_d) \geq 0$. Then,

$$\min_{\alpha > 0, T \in \mathcal{U}(d)} \|\alpha U - T\| = \frac{\lambda_1 - \lambda_d}{\lambda_1 + \lambda_d} = c\Delta, \quad (\text{C1})$$

$$c \equiv \frac{\lambda_1}{\lambda_1 + \lambda_d} \in \left(\frac{1}{2}, 1\right].$$

Equivalently,

$$\Delta = \frac{1}{c} \min_{\alpha > 0, T \in \mathcal{U}(d)} \|\alpha U - T\|. \quad (\text{C2})$$

Proof. Fix $\alpha > 0$. By Fan's theorem [86], among all unitaries T the minimizer of $\|\alpha U - T\|$ is the polar factor of αU , which for an SVD is $VW^\dagger$. Thus,

$$\min_{T \in \mathcal{U}(d)} \|\alpha U - T\| = \|\alpha U - VW^\dagger\|. \quad (\text{C3})$$

Using unitary invariance of the operator norm,

$$\|\alpha U - VW^\dagger\| = \|V(\alpha D - \mathbb{I})W^\dagger\| = \|\alpha D - \mathbb{I}\|. \quad (\text{C4})$$

Since $\alpha D - \mathbb{I}$ is diagonal, and since $\lambda_d \leq \lambda_j \leq \lambda_1$,

$$\|\alpha D - \mathbb{I}\| = \max_j |\alpha \lambda_j - 1| = \max\{|\alpha \lambda_1 - 1|, |\alpha \lambda_d - 1|\}. \quad (\text{C5})$$

The function $f(\alpha) = \max\{|\alpha \lambda_1 - 1|, |\alpha \lambda_d - 1|\}$ is minimized when $\alpha \lambda_1 - 1 = 1 - \alpha \lambda_d$, so the minimizing value of α is

$$\alpha^* = \frac{2}{\lambda_1 + \lambda_d}. \quad (\text{C6})$$

At this value,

$$f(\alpha^*) = \left| \frac{2\lambda_1}{\lambda_1 + \lambda_d} - 1 \right| = \frac{\lambda_1 - \lambda_d}{\lambda_1 + \lambda_d}. \quad (\text{C7})$$

This proves the claimed formula and the identification with $c\Delta$. ■

APPENDIX D: CONSTRUCTING THE UNITARIES

$\hat{V}, \hat{W}$ IN EQ. (31)

Here, we show how to construct unitaries $\hat{V}^\dagger$ and $\hat{W}$ such that $\hat{V}^\dagger U \hat{W}$ has the two-dimensional block structure used in Eq. (31), for a diagonalizable NH evolution with complex spectrum.

Let H be diagonalizable with right and left eigenvectors $\{|r_j\rangle\}$ and $\{|l_j\rangle\}$, biorthonormalized so that $\langle l_i|r_j\rangle = \delta_{ij}$. Write H as in Eq. (29), and normalize the right eigenvectors so that $\langle r_j|r_j\rangle = 1$ for all j . Then, the time evolution operator $U = e^{-iHt}$ is given by Eq. (30).

Fix right eigenvectors $|r_1\rangle$ and $|r_d\rangle$ corresponding to extremal decay rates γ_1 and γ_d [as in Eq. (29)], and define their

overlap $b_r \equiv \langle r_d | r_1 \rangle$. Since $|r_1\rangle$ and $|r_d\rangle$ are linearly independent and normalized, $|b_r| < 1$. Define normalized vectors in $\text{span}\{|r_1\rangle, |r_d\rangle\}$ orthogonal to $|r_d\rangle$ and $|r_1\rangle$, respectively:

$$|r_1^\perp\rangle \equiv a_r(|r_1\rangle - b_r|r_d\rangle), \quad |r_d^\perp\rangle \equiv a_r(|r_d\rangle - b_r^*|r_1\rangle), \quad (\text{D1})$$

where $a_r \equiv \frac{1}{\sqrt{1-|b_r|^2}}$. Then, $\{|r_1^\perp\rangle, |r_d^\perp\rangle\}$ and $\{|r_1\rangle, |r_d\rangle\}$ are orthonormal pairs and can be completed to orthonormal bases of the full Hilbert space.

Using these pairs, define unitaries $\hat{W}$ and $\hat{V}^\dagger$ by their action on the two computational basis states of interest:

$$\begin{aligned} \hat{W} &\equiv e^{i h_1 t} |r_1^\perp\rangle \langle 0^m| + e^{i h_d t} |r_d^\perp\rangle \langle 0^{m-1} 1| + \dots, \\ \hat{V}^\dagger &\equiv |0^m\rangle \langle r_1| + |0^{m-1} 1\rangle \langle r_d| + \dots, \end{aligned} \quad (\text{D2})$$

where the ellipses denote any extension to a full unitary on the orthogonal complements. The explicit choice of the extensions does not affect the 2×2 block in the relevant subspace.

A direct expansion gives

$$\begin{aligned} U\hat{W} &= a_r e^{-\gamma_1 t} |r_1\rangle \langle 0^m| + e^{-\gamma_d t} |r_d\rangle \langle 0^{m-1} 1| \\ &\quad - a_r b_r e^{-\gamma_d t} e^{i(h_1 - h_d)t} |r_d\rangle \langle 0^m| + \dots, \end{aligned} \quad (\text{D3})$$

and hence

$$\hat{V}^\dagger U \hat{W} = a_r e^{-\gamma_1 t} |0^m\rangle \langle 0^m| + a_r e^{-\gamma_d t} G + \dots, \quad (\text{D4})$$

where G is supported on $\text{span}\{|0^m\rangle, |0^{m-1} 1\rangle\}$ and, in the ordered basis $\{|0^m\rangle, |0^{m-1} 1\rangle\}$, is

$$G = \begin{pmatrix} -|b_r|^2 e^{i(h_1 - h_d)t} & b_r^*/a_r \\ -(b_r/a_r) e^{i(h_1 - h_d)t} & 1 - |b_r|^2 \end{pmatrix}. \quad (\text{D5})$$

Finally,

$$\begin{aligned} \|G\|_F^2 &= \sum_{i,j} |G_{ij}|^2 = |b_r|^4 + 2|b_r|^2(1 - |b_r|^2) + (1 - |b_r|^2)^2 \\ &= 1, \end{aligned} \quad (\text{D6})$$

so $\|G\| \leq \|G\|_F = 1$, independent of the angle between $|r_1\rangle$ and $|r_d\rangle$. This is the bound used in Sec. III to control the postselection error in the complex-spectrum case.

APPENDIX E: SINGLE-QUBIT PT -SYMMETRIC H

1. PT -symmetric H

Here, we show that any single-qubit quasi-Hermitian Hamiltonian (i.e., one for which there exists a positive definite metric $\eta > 0$ satisfying $H^\dagger \eta = \eta H$, corresponding to the unbroken PT -symmetric phase) is unitarily equivalent, up to an additive scalar and a choice of orthonormal basis, to the canonical form used in Sec. III D.

First, write the traceless part of H as

$$H \equiv H_0 - i\Gamma, \quad (\text{E1})$$

where H_0 and Γ are Hermitian. Parametrize them as

$$H_0 \equiv g\vec{n} \cdot \vec{\sigma}, \quad \Gamma \equiv \gamma\vec{m} \cdot \vec{\sigma}, \quad (\text{E2})$$

where $g, \gamma > 0$, $\vec{n}, \vec{m}$ are unit vectors, and $\vec{\sigma}$ is the Pauli vector. We can assume $\vec{n}$ and $\vec{m}$ are not parallel. If $\vec{m} = \pm\vec{n}$, then $H = (g \mp i\gamma)\vec{n} \cdot \vec{\sigma}$ has eigenvalues $\pm(g \mp i\gamma)$, which are nonreal for $\gamma > 0$. Hence, such H is not quasi-Hermitian.

As recalled above, quasi-Hermiticity means that there exists $\eta > 0$ such that $H^\dagger \eta = \eta H$. Write η in Bloch form as

$$\eta \equiv r_0 \mathbb{I} + \vec{r} \cdot \vec{\sigma}, \quad (\text{E3})$$

with $r_0 \in \mathbb{R}$, and $\vec{r} \in \mathbb{R}^3$ since η is Hermitian. Using $H^\dagger = H_0 + i\Gamma$, the condition becomes

$$0 = H^\dagger \eta - \eta H = [H_0, \eta] + i[\Gamma, \eta]. \quad (\text{E4})$$

Using the Pauli identities

$$[\vec{a} \cdot \vec{\sigma}, \vec{b} \cdot \vec{\sigma}] = 2i(\vec{a} \times \vec{b}) \cdot \vec{\sigma}, \quad \{\vec{a} \cdot \vec{\sigma}, \vec{b} \cdot \vec{\sigma}\} = 2(\vec{a} \cdot \vec{b})\mathbb{I}, \quad (\text{E5})$$

and the fact that $[\vec{a} \cdot \vec{\sigma}, \mathbb{I}] = 0$ and $\{\vec{a} \cdot \vec{\sigma}, \mathbb{I}\} = 2\vec{a} \cdot \vec{\sigma}$, we obtain

$$0 = g(\vec{n} \times \vec{r}) \cdot \vec{\sigma} + \gamma r_0 \vec{m} \cdot \vec{\sigma} + \gamma(\vec{m} \cdot \vec{r})\mathbb{I}. \quad (\text{E6})$$

Now, choose an orthonormal basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ of $\mathbb{R}^3$ such that $\vec{n} = \vec{v}_1$ and $\vec{m} = m_1 \vec{v}_1 + m_2 \vec{v}_2$ (this is always possible since $\vec{m}$ is not parallel to $\vec{n}$). Let $\sigma_i \equiv \vec{v}_i \cdot \vec{\sigma}$ and write $\vec{r} = r_1 \vec{v}_1 + r_2 \vec{v}_2 + r_3 \vec{v}_3$, making $\vec{n} \times \vec{r} = r_2 \vec{v}_3 - r_3 \vec{v}_2$. Equating coefficients of the Pauli basis in Eq. (E6) gives three equations from the Pauli components and one from the identity component:

$$\gamma r_0 m_1 = 0, \quad (\text{E7a})$$

$$-gr_3 + \gamma r_0 m_2 = 0, \quad (\text{E7b})$$

$$gr_2 = 0, \quad (\text{E7c})$$

$$\vec{m} \cdot \vec{r} = m_1 r_1 + m_2 r_2 = 0. \quad (\text{E7d})$$

Since $g, \gamma > 0$, we have $r_2 = 0$. If $m_1 \neq 0$, then the first, second, and fourth equations, respectively, force $r_0 = r_3 = r_1 = 0$, giving $\eta = 0$, a contradiction. Therefore, $m_1 = 0$, so $\vec{m} = \vec{v}_2$ and $m_2 = 1$. In this basis,

$$H = g\sigma_1 - i\gamma\sigma_2, \quad \eta = r_0 \left(\mathbb{I} + \frac{\gamma}{g} \sigma_3 \right) + r_1 \sigma_1. \quad (\text{E8})$$

The eigenvalues of η are $r_0 \pm \sqrt{r_1^2 + (r_0\gamma/g)^2}$, so $\eta > 0$ implies $r_0 > 0$ and

$$\sqrt{\frac{\gamma^2}{g^2} + \frac{r_1^2}{r_0^2}} < 1, \quad (\text{E9})$$

which is only satisfiable when $\gamma < g$.

Finally, writing H in the σ_3 eigenbasis yields Eq. (34), the canonical single-qubit PT -symmetric form used in Sec. III D.

2. Ellipsoidal periodic U

Since σ_1 and σ_2 anticommute, we have $H^2 = (g^2 - \gamma^2)\mathbb{I}$. Define $\omega \equiv \sqrt{g^2 - \gamma^2}$, which is strictly positive in the unbroken PT -symmetry regime $\gamma < g$. Then,

$$U = e^{-iHt} = \cos(\omega t)\mathbb{I} - \frac{i}{\omega} \sin(\omega t)H. \quad (\text{E10})$$

At $t = \pi/(2\omega)$, this becomes $U = -(i/\omega)H$. Geometrically, this periodic behavior reflects the fact that PT -symmetric evolution preserves the norm induced by the metric $\eta > 0$, corresponding to rotations on the associated ellipsoid in state space. At the exceptional point $\gamma = g$, $\omega \rightarrow 0$, signaling the

breakdown of periodic evolution. Additionally, when $\gamma > g$, one has $\omega = i\sqrt{\gamma^2 - g^2}$; the same closed form continues analytically and yields hyperbolic functions and exponential amplification/decay as one expects in the PT -broken phase.

APPENDIX F: TRAJECTORIES DERIVATION

Here, we show how the Hamiltonian in Eq. (53) generates quantum jumps and Lindblad evolution, in addition to the no-jump trajectories discussed in the main text.

Recall that H acts on system A and meter B as

$$H = H_A + \frac{1}{\sqrt{\delta}} H_{AB}, \quad (F1)$$

and the time-step unitary is $U \equiv e^{-iH\delta}$. Expanding to the required order gives

$$U = \mathbb{I} - i\sqrt{\delta} H_{AB} - i\delta H_A - \frac{\delta}{2} H_{AB}^2 + O(\delta^{3/2}). \quad (F2)$$

1. Quantum jumps

Initialize the meter in $|j\rangle$ and postselect on $|k\rangle$ with $k \neq j$. Define the effective (unnormalized) Kraus operator

$$C_{k \leftarrow j} \equiv \langle k|U|j\rangle_B. \quad (F3)$$

Because $\langle k|\mathbb{I}|j\rangle = \langle k|H_A|j\rangle = 0$ for $k \neq j$, the leading contribution comes from H_{AB} :

$$C_{k \leftarrow j} = -i\sqrt{\delta} \langle k|H_{AB}|j\rangle_B + O(\delta). \quad (F4)$$

Using $H_{AB} = \sum_{a \neq b} L_{ab} \otimes |a\rangle\langle b|$, we have $\langle k|H_{AB}|j\rangle_B = L_{kj}$, hence

$$C_{k \leftarrow j} = -i\sqrt{\delta} L_{kj} + O(\delta). \quad (F5)$$

Therefore, the postselected (unnormalized) state after one step is

$$\tilde{\mathcal{E}}_{k \leftarrow j}(\rho) \equiv C_{k \leftarrow j} \rho C_{k \leftarrow j}^\dagger = \delta L_{kj} \rho L_{kj}^\dagger + O(\delta^{3/2}), \quad (F6)$$

which is the jump form stated in Eq. (62). The scaling $H_{AB}/\sqrt{\delta}$ ensures that the jump probability per step, $\text{Tr}[\tilde{\mathcal{E}}_{k \leftarrow j}(\rho)]$, is $O(\delta)$, matching the usual quantum-trajectories limit.

2. Lindblad evolution from tracing out the meter

Now, replace postselection by a partial trace over the meter:

$$\rho(\delta) \equiv \mathcal{E}(\rho) = \text{Tr}_B[U(\rho \otimes |j\rangle\langle j|)U^\dagger]. \quad (F7)$$

Since this operation is trace preserving, $\tilde{\mathcal{E}} = \mathcal{E}$, and the normalization step can be dropped. Decompose the trace over the meter basis:

$$\begin{aligned} \mathcal{E}(\rho) &= \langle j|U(\rho \otimes |j\rangle\langle j|)U^\dagger|j\rangle_B \\ &+ \sum_{k \neq j} \langle k|U(\rho \otimes |j\rangle\langle j|)U^\dagger|k\rangle_B. \end{aligned} \quad (F8)$$

The $k \neq j$ terms reproduce the jump contributions $\delta L_{kj} \rho L_{kj}^\dagger$ to leading order as above, while the $j \rightarrow j$ block produces the coherent drift and the anticommutator term from H_{AB}^2 , described in Eqs. (60) and (61). Collecting terms yields

$$\begin{aligned} \rho(\delta) &= \rho - i\delta[H_j, \rho] - \frac{\delta}{2} \sum_{k \neq j} \{L_{kj}^\dagger L_{kj}, \rho\} + \delta \sum_{k \neq j} L_{kj} \rho L_{kj}^\dagger \\ &+ O(\delta^{3/2}), \end{aligned} \quad (F9)$$

which in differential form gives the Lindblad equation with system Hamiltonian H_j in the limit $\delta \rightarrow 0$.

-
- [1] Y. Ashida, Z. Gong, and M. Ueda, Non-Hermitian physics, *Adv. Phys.* **69**, 249 (2020).
 - [2] Y. Ashida and M. Ueda, Full-counting many-particle dynamics: Nonlocal and chiral propagation of correlations, *Phys. Rev. Lett.* **120**, 185301 (2018).
 - [3] C. M. Bender and S. Boettcher, Real spectra in non-Hermitian Hamiltonians having $\mathcal{PT}$ symmetry, *Phys. Rev. Lett.* **80**, 5243 (1998).
 - [4] C. M. Bender, D. C. Brody, and H. F. Jones, Complex extension of quantum mechanics, *Phys. Rev. Lett.* **89**, 270401 (2002).
 - [5] A. Mostafazadeh, Pseudo-Hermiticity versus PT symmetry: The necessary condition for the reality of the spectrum of a non-Hermitian Hamiltonian, *J. Math. Phys.* **43**, 205 (2002).
 - [6] A. Mostafazadeh, Is pseudo-Hermitian quantum mechanics an indefinite-metric quantum theory? *Czech. J. Phys.* **53**, 1079 (2003).
 - [7] Y. Wu, W. Liu, J. Geng, X. Song, X. Ye, C. Yu, X. Peng, and J. Du, Observation of parity–time symmetry breaking in a single-spin system, *Science* **364**, 878 (2019).
 - [8] J. Li, A. K. Harter, J. Liu, L. de Melo, Y. N. Joglekar, and L. Luo, Observation of parity–time symmetry breaking transitions in a dissipative Floquet system of ultracold atoms, *Nat. Commun.* **10**, 855 (2019).
 - [9] M. Naghiloo, M. Abbasi, Y. N. Joglekar, and K. W. Murch, Quantum state tomography across the exceptional point in a single dissipative qubit, *Nat. Phys.* **15**, 1232 (2019).
 - [10] Y. Zhang, J. Carrasquilla, and Y. B. Kim, Observation of a non-Hermitian supersonic mode on a trapped-ion quantum computer, *Nat. Commun.* **16**, 3286 (2025).
 - [11] C. E. Rüter, K. G. Makris, R. El-Ganainy, D. N. Christodoulides, M. Segev, and D. Kip, Observation of parity–time symmetry in optics, *Nat. Phys.* **6**, 192 (2010).
 - [12] A. Regensburger, C. Bersch, M.-A. Miri, G. Onishchukov, D. N. Christodoulides, and U. Peschel, Parity–time synthetic photonic lattices, *Nature (London)* **488**, 167 (2012).
 - [13] J. M. Zeuner, M. C. Rechtsman, Y. Plotnik, Y. Lumer, S. Nolte, M. S. Rudner, M. Segev, and A. Szameit, Observation of a topological transition in the bulk of a non-Hermitian system, *Phys. Rev. Lett.* **115**, 040402 (2015).
 - [14] H. Hodaei, M.-A. Miri, A. U. Hassan, W. Hayenga, M. Heinrich, D. N. Christodoulides, and M. Khajavikhan, Tunable parity–time-symmetric microring lasers, *Opt. Lett.* **40**, 4955 (2015).
 - [15] S. Gopalakrishnan and M. J. Gullans, Entanglement and purification transitions in non-Hermitian quantum mechanics, *Phys. Rev. Lett.* **126**, 170503 (2021).

- [16] N. Matsumoto, K. Kawabata, Y. Ashida, S. Furukawa, and M. Ueda, Continuous phase transition without gap closing in non-Hermitian quantum many-body systems, *Phys. Rev. Lett.* **125**, 260601 (2020).
- [17] Y. Li, X. Chen, and M. P. A. Fisher, Quantum Zeno effect and the many-body entanglement transition, *Phys. Rev. B* **98**, 205136 (2018).
- [18] B. Skinner, J. Ruhman, and A. Nahum, Measurement-induced phase transitions in the dynamics of entanglement, *Phys. Rev. X* **9**, 031009 (2019).
- [19] A. Chan, R. M. Nandkishore, M. Pretko, and G. Smith, Unitary-projective entanglement dynamics, *Phys. Rev. B* **99**, 224307 (2019).
- [20] Y. Li, X. Chen, and M. P. A. Fisher, Measurement-driven entanglement transition in hybrid quantum circuits, *Phys. Rev. B* **100**, 134306 (2019).
- [21] M. J. Gullans and D. A. Huse, Dynamical purification phase transition induced by quantum measurements, *Phys. Rev. X* **10**, 041020 (2020).
- [22] S. Choi, Y. Bao, X.-L. Qi, and E. Altman, Quantum error correction in scrambling dynamics and measurement-induced phase transition, *Phys. Rev. Lett.* **125**, 030505 (2020).
- [23] M. Ippoliti, M. J. Gullans, S. Gopalakrishnan, D. A. Huse, and V. Khemani, Entanglement phase transitions in measurement-only dynamics, *Phys. Rev. X* **11**, 011030 (2021).
- [24] K. Das Agarwal, T. K. Konar, L. G. C. Lakkaraju, and A. Sen(De), Recognizing critical lines via entanglement in non-Hermitian systems, *Phys. Rev. A* **113**, 022201 (2026).
- [25] B. Barch, N. Anand, J. Marshall, E. Rieffel, and P. Zanardi, Scrambling and operator entanglement in local non-Hermitian quantum systems, *Phys. Rev. B* **108**, 134305 (2023).
- [26] B. Barch, Locality, correlations, information, and non-Hermitian quantum systems, *Phys. Rev. B* **110**, 094307 (2024).
- [27] D. S. Abrams and S. Lloyd, Nonlinear quantum mechanics implies polynomial-time solution for NP -complete and $\#P$ problems, *Phys. Rev. Lett.* **81**, 3992 (1998).
- [28] K. Mochizuki and R. Hamazaki, Distinguishability transitions in nonunitary boson-sampling dynamics, *Phys. Rev. Res.* **5**, 013177 (2023).
- [29] S. Aaronson, Quantum computing, postselection, and probabilistic polynomial-time, *Proc. R. Soc. A* **461**, 3473 (2005).
- [30] C. M. Bender, D. C. Brody, and H. F. Jones, Faster than Hermitian quantum mechanics, *Phys. Rev. Lett.* **98**, 040403 (2007).
- [31] P. Lu, T. Liu, Y. Liu, X. Rao, Q. Lao, H. Wu, F. Zhu, and L. Luo, Realizing quantum speed limit in open system with a PT -symmetric trapped-ion qubit, [arXiv:2206.00940](https://arxiv.org/abs/2206.00940).
- [32] Z.-Z. Li, W. Chen, M. Abbasi, K. W. Murch, and K. B. Whaley, Speeding up entanglement generation by proximity to higher-order exceptional points, *Phys. Rev. Lett.* **131**, 100202 (2023).
- [33] S. Chakraborty and A. K. Sarma, Delayed sudden death of entanglement at exceptional points, *Phys. Rev. A* **100**, 063846 (2019).
- [34] A. Fring and T. Frith, Eternal life of entropy in non-Hermitian quantum systems, *Phys. Rev. A* **100**, 010102 (2019).
- [35] H. Hodaei, A. U. Hassan, S. Wittek, H. Garcia-Gracia, R. El-Ganainy, D. N. Christodoulides, and M. Khajavikhan, Enhanced sensitivity at higher-order exceptional points, *Nature (London)* **548**, 187 (2017).
- [36] W. Chen, Ş. K. Özdemir, G. Zhao, J. Wiersig, and L. Yang, Exceptional points enhance sensing in an optical microcavity, *Nature (London)* **548**, 192 (2017).
- [37] M. Parto, C. Leefmans, J. Williams, R. M. Gray, and A. Marandi, Enhanced sensitivity via non-Hermitian topology, *Light: Sci. Appl.* **14**, 6 (2025).
- [38] Q.-C. Wu, Y.-H. Zhou, T. Liu, Y.-H. Kang, Q.-P. Su, and C.-P. Yang, Enhanced quantum sensing in time-modulated non-Hermitian systems, *Chin. J. Phys.* **98**, 1116 (2025).
- [39] E. Bernstein and U. Vazirani, Quantum complexity theory, *SIAM J. Comput.* **26**, 1411 (1997).
- [40] Q. Zhang and B. Wu, The power of the Lorentz quantum computer, *Entropy* **28**, 266 (2026).
- [41] Q. Zhang and B. Wu, Physics and computation: An insight from non-Hermitian quantum computing, *Phys. Rev. A* **113**, 042410 (2026).
- [42] Specifically, not *proportional* to a unitary.
- [43] If one additionally allows a classical polynomial-time controller with adaptive oracle access to such postselected subroutines, the corresponding oracle class is $P^{PP} = P^{\#P}$, which contains the polynomial hierarchy by Toda's theorem [44]. We do not assume such oracle access in the definition of $NHBQP(U)$.
- [44] S. Toda, PP is as hard as the polynomial-time hierarchy, *SIAM J. Comput.* **20**, 865 (1991).
- [45] T. A. Brun, A simple model of quantum trajectories, *Am. J. Phys.* **70**, 719 (2002).
- [46] G. Lindblad, On the generators of quantum dynamical semigroups, *Commun. Math. Phys.* **48**, 119 (1976).
- [47] R. Alicki and K. Lendi, *Quantum Dynamical Semigroups and Applications*, Lecture Notes in Physics Vol. 717 (Springer, Berlin, 2007).
- [48] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, Oxford, 2002).
- [49] Generalizations that allow singular η can arise for nondiagonalizable Hamiltonians at exceptional points, but for our purposes it is sufficient to restrict to invertible η and (when needed) diagonalizable H .
- [50] Strictly, unbroken PT symmetry is a sufficient condition for quasi-Hermiticity but not an equivalent definition.
- [51] S. Karuvade, A. Alase, and B. C. Sanders, Observing a changing Hilbert-space inner product, *Phys. Rev. Res.* **4**, 013016 (2022).
- [52] C. Papadimitriou, *Computational Complexity* (Addison Wesley Longman, Reading, MA, 1995).
- [53] More precisely, a function f is in $\#P$ if there exists a polynomial q and a polynomial-time computable predicate $R(x, w)$ such that $f(x) = |\{w \in \{0, 1\}^{q(|x|)} : R(x, w) = 1\}|$.
- [54] A. Yu. Kitaev, A. H. Shen, and M. N. Vyalyi, *Classical and Quantum Computation, Graduate Studies in Mathematics* (American Mathematical Society, Providence, RI, 2000), Vol. 47.
- [55] S. Arora and B. Barak, *Computational Complexity: A Modern Approach* (Cambridge University Press, Cambridge, 2009).
- [56] S. Aaronson and A. Arkhipov, The computational complexity of linear optics, in *Proceedings of the 43rd Annual ACM Symposium on Theory of Computing, STOC'11* (ACM Press, New York, NY, USA, 2011), pp. 333–342.

- [57] S. Aaronson, PostBQP postscripts: A confession of mathematical errors, blog post on Shtetl-Optimized, 2014, <https://scottaaronson.blog/?p=2072>.
- [58] S. Aaronson and A. Ambainis, Forrelation: A problem that optimally separates quantum from classical computing, in *Proceedings of the 47th Annual ACM Symposium on Theory of Computing, STOC'15* (ACM Press, New York, NY, USA, 2015), pp. 307–316.
- [59] Specifically, $f(n) \in O(g(n))$ [$\Omega(g(n))$] if there exist positive constants c and n_0 such that for all $n \geq n_0$: $0 \leq f(n) \leq c \cdot g(n)$ [$0 \leq c \cdot g(n) \leq f(n)$]. Also, $f(n) \in \Theta(g(n))$ if and only if $f(n) \in O(g(n))$ and $f(n) \in \Omega(g(n))$.
- [60] S. Aaronson and L. Chen, Complexity-theoretic foundations of quantum supremacy experiments, in *Proceedings of the 32nd Computational Complexity Conference, CCC'17* (Schloss Dagstuhl—Leibniz-Zentrum fuer Informatik, Dagstuhl, DEU, 2017), p. 2.
- [61] L. Chen, A note on oracle separations for BQP, [arXiv:1605.00619](https://arxiv.org/abs/1605.00619).
- [62] M. J. Bremner, R. Jozsa, and D. J. Shepherd, Classical simulation of commuting quantum computations implies collapse of the polynomial hierarchy, *Proc. R. Soc. A* **467**, 459 (2011).
- [63] M. Van den Nest, Classical simulation of quantum computation, the Gottesman-Knill theorem, and slightly beyond, *Quantum Inf. Comput.* **10**, 258 (2010).
- [64] T. H. Johnson, J. D. Biamonte, S. R. Clark, and D. Jaksch, Solving search problems by strongly simulating quantum circuits, *Sci. Rep.* **3**, 1235 (2013).
- [65] T. J. Osborne, Efficient approximation of the dynamics of one-dimensional quantum spin systems, *Phys. Rev. Lett.* **97**, 157202 (2006).
- [66] M. B. Hastings, Quantum adiabatic computation with a constant gap is not useful in one dimension, *Phys. Rev. Lett.* **103**, 050502 (2009).
- [67] More generally, one may assume a standard “tameness”/approximability condition ensuring that nonzero postselection probabilities are at least $2^{-\text{poly}(n)}$ (see, e.g., Ref. [70] for a detailed discussion in the PostBQP setting).
- [68] I. Aloisio, G. White, C. Hill, and K. Modi, Sampling complexity of open quantum systems, *PRX Quantum* **4**, 020310 (2023).
- [69] S. Aaronson, The equivalence of sampling and searching, in *Computer Science—Theory and Applications*, edited by A. Kulikov and N. Vereshchagin (Springer, Berlin, 2011), pp. 1–14.
- [70] G. Kuperberg, How hard is it to approximate the Jones polynomial? *Theory Comput.* **11**, 183 (2009).
- [71] This bound holds for $\Delta \in (0, 1)$, but if $\Delta = 1$ then $\lambda_d = 0$ and the gadget is exact for $r = 1$.
- [72] L. M. Adleman, J. DeMarrais, and M.-D. A. Huang, Quantum computability, *SIAM J. Comput.* **26**, 1524 (1997).
- [73] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, 10th ed (Cambridge University Press, Cambridge, 2010).
- [74] D. Gottesman, The Heisenberg representation of quantum computers, [arXiv:quant-ph/9807006](https://arxiv.org/abs/quant-ph/9807006).
- [75] R. Jozsa and M. Nest, Classical simulation complexity of extended Clifford circuits, *Quantum Inf. Comput.* **14**, 633 (2014).
- [76] S. Aaronson and D. Gottesman, Improved simulation of stabilizer circuits, *Phys. Rev. A* **70**, 052328 (2004).
- [77] S. Sang, Z. Li, T. H. Hsieh, and B. Yoshida, Ultrafast entanglement dynamics in monitored quantum circuits, *PRX Quantum* **4**, 040332 (2023).
- [78] L. G. Valiant, Quantum circuits that can be simulated classically in polynomial time, *SIAM J. Comput.* **31**, 1229 (2002).
- [79] R. Jozsa and A. Miyake, Matchgates and classical simulation of quantum circuits, *Proc. R. Soc. A* **464**, 3089 (2008).
- [80] O. Shtanko, A. Deshpande, P. S. Julienne, and A. V. Gorshkov, Complexity of fermionic dissipative interactions and applications to quantum computing, *PRX Quantum* **2**, 030350 (2021).
- [81] A. M. Projansky, J. Necaie, and J. D. Whitfield, Gaussianity and simulability of Cliffords and matchgates, *J. Phys. A: Math. Theor.* **58**, 195302 (2025).
- [82] In the broken PT -symmetric phase, no such $\eta > 0$ exists, and the reduction below does not apply.
- [83] R. Cleve and C. Wang, Efficient quantum algorithms for simulating Lindblad evolution, in *Proceedings of the 44th International Colloquium on Automata, Languages, and Programming (ICALP 2017)*, Leibniz International Proceedings in Informatics (LIPIcs), edited by I. Chatzigiannakis, P. Indyk, F. Kuhn, and A. Muscholl (Schloss Dagstuhl—Leibniz-Zentrum fuer Informatik, Dagstuhl, Germany, 2017), Vol. 80, pp. 17:1–17:14.
- [84] Z. Ding, X. Li, and L. Lin, Simulating open quantum systems using Hamiltonian simulations, *PRX Quantum* **5**, 020332 (2024).
- [85] I. L. Paiva, A. Te'eni, B. Y. Peled, E. Cohen, and Y. Aharonov, Non-inertial quantum clock frames lead to non-Hermitian dynamics, *Commun. Phys.* **5**, 298 (2022).
- [86] K. Fan and A. J. Hoffman, Some metric inequalities in the space of matrices, *Proc. Am. Math. Soc.* **6**, 111 (1955).