

Symmetry and self-bound droplets in dipolar molecular gases

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Recent experiments with degenerate molecular gases dressed by elliptically polarized microwave fields have enabled new control of dipolar interactions via engineered anisotropy. We reveal a symmetry structure of the dipolar interaction that generates degeneracies among the interaction parameters, enabling a classification of spatial symmetries and equilibrium shapes of the gases. Exploiting these symmetries, we analyze solutions including beyond-mean-field quantum fluctuations in the dilute gas regime and develop a complementary variational theory. We map out the phase diagram of self-bound droplets and characterize their widths, energies, and densities.

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I. INTRODUCTION

Gases of ultracold alkali molecules have long held the promise of strong long-range and anisotropic dipolar interactions. Early progress with fermions [1] was hindered by losses due to chemical reactions [2,3], although deep Fermi degeneracy was ultimately achieved [4]. By using microwave shielding [5–7] to limit losses, bosonic molecules were finally condensed in 2024 [8,9] (see also Refs. [10–12]).

The observation in magnetic atoms of several long-lived localized droplets arising from a trapped dipolar condensate [13] and macrodroplets [14] led to the realization of a self-bound dipolar droplet [15], stable in free space without external confinement, stabilized by quantum fluctuations [16] (see also Refs. [17–23]). Self-bound droplets in two-component atomic condensates have since been observed [24,25].

Experiments now bring this physics to molecular gases with self-bound droplets recently reported [26], which demonstrated several important advances. First, higher dipolar interaction strengths than can be achieved with atomic magnetic gases. Second, the sign of the effective dipolar interaction can be changed for molecular gases. The change of dipolar interaction sign was proposed for *antidipolar* atomic gases by fast rotation of the magnetic field [27,28], and modification of the interaction by rotation was realized experimentally for trapped atoms [29]. Theory showed that such trapped gases prefer a pancake shape, and with confinement perpendicular to the dipole orientation can result in a stack of “pancakelets” [30] (see also Refs. [31,32]). However, lifetimes of atomic gases with rapidly rotating magnetic fields have been limited, whereas molecular gases show impressive life-

times [26]. Finally, with interactions controlled by elliptically polarized microwave fields demonstrated in Ref. [26], interactions of molecular gases have increased anisotropy beyond the cylindrical symmetry of the atomic dipolar interaction.

Existing theory for molecular Bose gases has investigated molecular condensates [33], droplets [34–36], and layering [37], and has been limited to the cylindrically symmetric interaction; only very recent work considers elliptically controlled anisotropy in the context of supersolids [38,39].

In this work, we uncover a symmetry framework for the microwave-dressed dipolar interaction, showing that each combination of interaction parameters belongs to a sextet of degenerate configurations related by coordinate permutations. This structure organizes the interaction landscape, revealing where cylindrical symmetry is preserved or broken, and classifying the equilibrium shapes of the cloud. This symmetry analysis follows directly from the form of the interaction, and is therefore not specific to the later theoretical treatment. We then apply the symmetries to the extended Gross-Pitaevskii equation (eGPE) including beyond mean-field effects and we develop variational solutions that capture the key anisotropy effects. The eGPE approach is not valid when $a_s < 0$ or in the strongly interacting regime. Focusing on self-bound droplets, we derive the phase diagram, first in the thermodynamic limit and then for finite molecule number. We demonstrate the properties of self-bound droplets and compare the energy and peak density to the thermodynamic limit scaling.

II. INTERACTION

We consider a system with effective intermolecular dipole-dipole potential $U(\mathbf{r})$ and s -wave interactions with interaction strength $g_s = 4\pi\hbar^2 a_s/m$ for molecules with s -wave scattering length a_s . We assume the molecules are shielded by a microwave field [39–43] with ellipticity ξ , giving [44]

$$U(\mathbf{r}) = \frac{3|g_s|}{4\pi r^3} [\epsilon_0(1 - 3\cos^2\theta) + \sqrt{3}\epsilon_2 \sin^2\theta \cos 2\phi], \quad (1)$$

where θ is the angle between $\mathbf{r}$ and the z axis, and ϕ is the angle between $\boldsymbol{\rho} = (x, y)$ and the x axis. The first dipolar

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interaction term is the usual ϕ -independent term Y_2^0 with relative dipole interaction strength ϵ_0 [45], and the second $Y_2^2 + Y_2^{-2}$ term depends on both θ and ϕ , with relative dipole interaction strength $\epsilon_2 \propto \sin 2\xi$. The contribution from $Y_2^{\pm 1}$ is generally small [41]. As a wide range of positive and negative dipolar interactions is possible experimentally [26,41,46], we allow for positive, zero, and negative values of ϵ_0 and ϵ_2 . With no ellipticity of the microwave field, $\xi = 0$ so that $\epsilon_2 = 0$, results are equivalent to those for dipolar atoms when $\epsilon_0 > 0$ and antidipolar atoms when $\epsilon_0 < 0$. The dependence of results on (ϵ_0, ϵ_2) is via functions of $U(\mathbf{r})$, which can be written as

$$U(\mathbf{r}) = \frac{3|g_s|}{4\pi r^5} [(\epsilon_0 + \sqrt{3}\epsilon_2)x^2 + (\epsilon_0 - \sqrt{3}\epsilon_2)y^2 - 2\epsilon_0 z^2], \quad (2)$$

so the ground state and its energy are unchanged by those changes to ϵ_0 and ϵ_2 that correspond to permutations of the coordinates $\{x, y, z\}$, for example, states with ϵ_0, ϵ_2 are equivalent to those with $\epsilon_0, -\epsilon_2$, exchanging the roles of x and y .

That is, each state belongs to a sextet with the same energy and wavefunction related by coordinate permutations. The transformation between the parameter sets is given by $\epsilon' = T\epsilon$, where $\epsilon = (\epsilon_0, \epsilon_2)^T$ and T is any product of the reflections $T_n = \begin{pmatrix} \cos \vartheta & \sin \vartheta \\ \sin \vartheta & -\cos \vartheta \end{pmatrix}$ and rotations $R_n = \begin{pmatrix} \cos \vartheta & -\sin \vartheta \\ \sin \vartheta & \cos \vartheta \end{pmatrix}$ in parameter space, for $\vartheta = 2\pi n/3$ with $n = 0, 1, 2$, i.e., T is from the dihedral group D_3 .

Equation (2) also shows when the system has cylindrical symmetry, assuming the trap, if any, has the same symmetry: When $\epsilon_2 = 0$, they have x, y symmetry; when $\epsilon_0 = \epsilon_2/\sqrt{3}$, they have y, z symmetry; and when $\epsilon_0 = -\epsilon_2/\sqrt{3}$, they have x, z symmetry, i.e., the symmetry axes of the D_3 group. On a symmetry axis, the sextet collapses to three distinct states due to degeneracy.

This also shows that near the $\epsilon_0 > 0$ axis and near $\epsilon_0 = -\epsilon_2/\sqrt{3}$ the gas is prolate, whereas near the $\epsilon_0 < 0$ axis and near $\epsilon_0 = \epsilon_2/\sqrt{3}$ the cloud is oblate, subject to any modification by a trap.

Finally, Eq. (2) establishes the relative widths of the gas, up to potential modifications by a trap. For example, when $\epsilon_0 > \epsilon_2/\sqrt{3} > 0$ the coefficients are in decreasing order of (x, y, z) , so the real-space widths increase in that order to minimize energy.

In what follows, we particularly focus on three cases: prolate symmetry ($\epsilon_0 > 0, \epsilon_2 = 0$), oblate symmetry ($\epsilon_0 < 0, \epsilon_2 = 0$), and asymmetric, midway between the symmetry axes ($\epsilon_0 = 0$). After transformations, these three cases cover 12 lines, every 30° , in parameter space.

The above discussion on the symmetry and lengths is summarized in Fig. 1. The blue dots, linked by blue lines, are equivalent states. As we omit $\epsilon_2 < 0$, states are in groups of two if on a symmetry axis, otherwise groups of three. The D_3 symmetries tile the plane: Each of the three colored regions has all of the different possible states, subject to permutation of the coordinates.

III. eGPE

As an application, we consider the ground-state wavefunction, $\psi(\mathbf{x})$, given by the solution to the time-independent

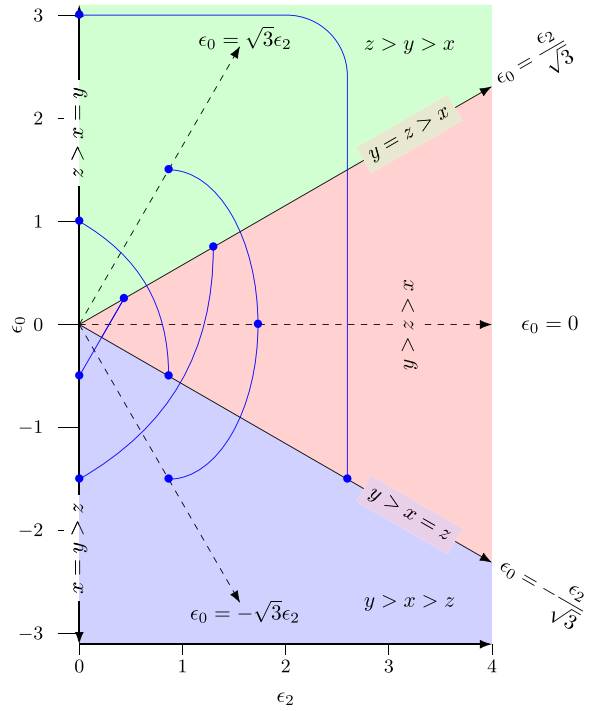


FIG. 1. Symmetry of dipolar molecular gases and relative cloud widths. The blue dots show equivalent states under permutations of the coordinates (the shape of the connecting blue lines is arbitrary). There is also simple reflection symmetry about the line $\epsilon_2 = 0$: The $\epsilon_2 < 0$ results and their connection to the dots are not shown (if three dots are linked above, there are another three for $\epsilon_2 < 0$, and if two are shown there is another one for $\epsilon_2 < 0$). The three colored regions (six including $\epsilon_2 < 0$) demarcate domains with different relative widths. The notation $x = y > z$ means that the cloud has x, y symmetry, and the x, y width is greater than the z width, unless broken by a confining potential. The dashed lines indicate $\epsilon_0 = 0$ and other members of its sextet.

eGPE

$$\left[-\frac{\hbar^2 \nabla^2}{2m} + V(\mathbf{x}) + \Phi(\mathbf{x}) + \gamma_{\text{QF}} |\psi|^3 \right] \psi = \mu \psi, \quad (3)$$

where μ is the chemical potential, $V(\mathbf{x})$ is the trap, and ψ is normalized to the total number of molecules $\int d\mathbf{x} n(\mathbf{x}) = N$, with density $n(\mathbf{x}) = |\psi(\mathbf{x})|^2$.

The effective two-body interaction potential is $\Phi(\mathbf{x}) = g_s |\psi(\mathbf{x})|^2 + \int d\mathbf{x}' U(\mathbf{x} - \mathbf{x}') |\psi(\mathbf{x}')|^2$. As ϵ_0 or ϵ_2 can be zero, the dipole lengths $\epsilon_0 a_s$ and $\epsilon_2 a_s$ are not useful lengths for all possible parameters, and we use the s -wave scattering length, a_s , as our length unit, which we take to be positive, and $E_0 = \hbar^2 / ma_s^2$ as our energy unit. The interaction in these units is then completely parametrized by ϵ_0 and ϵ_2 .

Quantum fluctuations are included in the local density approximation with $\gamma_{\text{QF}} = 32g_s a_s^{3/2} \mathcal{Q}_5(\epsilon_0, \epsilon_2) / 3\sqrt{\pi}$, where

$$\mathcal{Q}_5(\epsilon_0, \epsilon_2) = \int \frac{d\Omega_k}{4\pi} [1 + \bar{U}(\mathbf{k})]^{5/2}, \quad (4)$$

and $g_s \bar{U}(\mathbf{k})$ is the Fourier transform of $U(\mathbf{r})$ [see Eq. (A3)]. There is an attractive component to the interaction potential,

i.e., $\bar{U}(\mathbf{k}) < -1$, when

$$\epsilon_0 + \sqrt{3}|\epsilon_2| > 1 \text{ or } \epsilon_0 < -\frac{1}{2}, \quad (5)$$

and then Q_5 has an imaginary part due to instabilities of the homogeneous dipolar gas, which we discard [21,22]; we show the extent of the imaginary part in our results [see also Eq. (B6)]. For certain parameters Q_5 can be evaluated, including $Q_5(\epsilon_0, 0)$ for $\epsilon_0 > 0$ [47,48], $\text{Re}\{Q_5(\epsilon_0, 0)\} = 5\pi(1 - \epsilon_0)^3/(32\sqrt{3}|\epsilon_0|)$ for $\epsilon_0 < -\frac{1}{2}$, and $Q_5(0, \epsilon_2)$ given by Eq. (B5). Otherwise, we integrate over one angle numerically using Eq. (B2).

The eGPE depends on ϵ_0 and ϵ_2 , both in $\Phi(\mathbf{x})$ and in Q_5 , only via $U(\mathbf{r})$, so the symmetries of the previous section apply, subject to the trap. For example, the wavefunction with $(\epsilon_0 < 0, \epsilon_2 = 0)$ and trap $V(\mathbf{x}) = \frac{1}{2}m[\omega_\rho^2(x^2 + y^2) + \omega_z^2 z^2]$ is equal to that with $(\epsilon'_0 = |\epsilon_0|/2, \epsilon'_2 = \sqrt{3}|\epsilon_0|/2)$ and $\omega'_y = \omega'_z = \omega_\rho, \omega'_x = \omega_z$ after permutation of coordinates.

In the following, we will use the energy to compare solutions, which is given by

$$E = \int d\mathbf{x} \psi^* \left[-\frac{\hbar^2 \nabla^2}{2m} + V(\mathbf{x}) + \frac{1}{2}\Phi(\mathbf{x}) + \frac{2}{5}\gamma_{\text{QF}}|\psi|^3 \right] \psi. \quad (6)$$

IV. VARIATIONAL

We consider a large class of ansätze where the density $n(\mathbf{x})$ depends on $\mathbf{x}$ in the scaling form $n(\mathbf{x}) = n_s(s)$ for some function n_s , with $s^2 = x^2/l_x^2 + y^2/l_y^2 + z^2/l_z^2$. The variational Gaussian we consider below is an example, as is Thomas-Fermi. Then $\int d\mathbf{x} \Phi(\mathbf{x})n(\mathbf{x}) = g_s(1 - \epsilon_0 f - \sqrt{3}\epsilon_2 f_2) \int d\mathbf{x} [n(\mathbf{x})]^2$ [49], where the functions $f(l_x/l_z, l_y/l_z)$ and $f_2(l_x/l_z, l_y/l_z)$ are defined in Appendix C. The function $f(x, y)$ is given in terms of incomplete elliptic integrals in Refs. [50–52]. After the transformations $\epsilon' = \mathbf{T}\epsilon$, $\epsilon_0 f(l_x/l_z, l_y/l_z) + \sqrt{3}\epsilon_2 f_2(l_x/l_z, l_y/l_z) = \epsilon'_0 f(l'_x/l'_z, l'_y/l'_z) + \sqrt{3}\epsilon'_2 f_2(l'_x/l'_z, l'_y/l'_z)$, where l'_i is the transformed length. For example, after $\mathbf{T}_0$ that exchanges x and y , $l'_x/l'_z = l_y/l_z$, which gives us the relations $f(y, x) = f(x, y)$ and $f_2(y, x) = -f_2(x, y)$, so $f_2(x, x) = 0$. After $\mathbf{T}_1$, we usefully get the functional form of f_2 in terms of the well-known f :

$$f_2(x, y) = \frac{1}{3}[f(1/y, x/y) - f(1/x, y/x)], \quad (7)$$

with the limits $f_2(x, \infty) = 1/(1+x)$ and $f_2(x, 0) = -1$ for $x \neq 0$ [53]. The range is $-1 \leq f_2(x, y) \leq 1$, and $f_2(x, 1) = f(x, 1) = -\frac{1}{2}f_s(1/x)$, with $f_s(x) \equiv f(x, x)$ [54].

In the following, we specialize to a variational Gaussian ansatz $\psi_v(\mathbf{x}) = \sqrt{\frac{N}{\pi^{3/2}l_x l_y l_z}} e^{-\frac{1}{2}(x^2/l_x^2 + y^2/l_y^2 + z^2/l_z^2)}$.

V. RESULTS

We consider self-bound droplets of dipolar molecules, i.e., $V(\mathbf{x}) = 0$. The domain of self-bound droplets in the thermodynamic limit, $N \rightarrow \infty$, is given by Eq. (5). The formation of self-bound droplets also requires a sufficient number of molecules. The phase boundary between self-bound droplets and the dispersed gas is found by varying N , ϵ_0 , ϵ_2 so that $E = 0$, using stationary states which solve Eq. (3). The phase

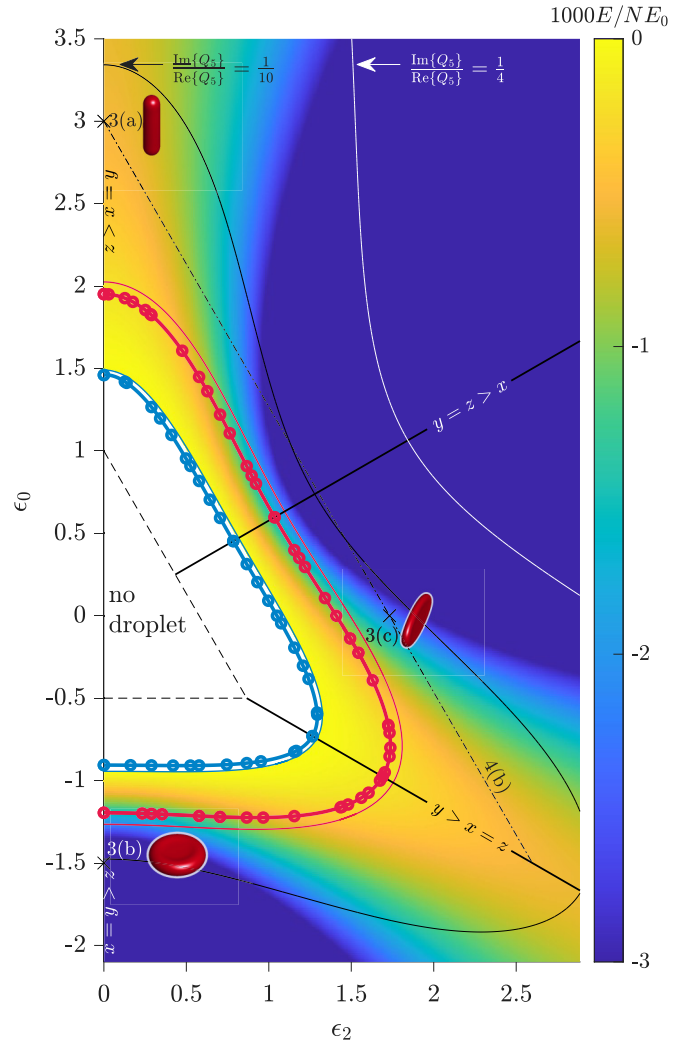


FIG. 2. Phase diagram for droplets of dipolar molecular gases, with no droplet in the white region near the origin, and stable self-bound droplets outside this region. The energetic stability boundary, $E = 0$, is for $N = 1000$ using the eGPE (red circles and thick curve) and variational (thin red curve), $N = 5000$ using the eGPE (blue circles and thick curve) and variational (thin blue curve), and the thermodynamic limit, Eq. (5) (dashed lines). Also shown is the variational energy for $N = 5000$ (background color), contours where $\text{Im}\{Q_5\} = \text{Re}\{Q_5\}/10$ (thin black curve) and $\text{Im}\{Q_5\} = \text{Re}\{Q_5\}/4$ (thin white curve) [see also Eq. (B6)], and the symmetry axes from Fig. 1 (black lines at $\epsilon_0 = \pm\epsilon_2/\sqrt{3}$). The labels on the symmetry axes show the relative lengths from Fig. 1. The cases from Fig. 3 are marked with $\times$ and the case from Fig. 4(b) is marked with a dash-dotted line.

diagram for $N = 1000$ (red) and $N = 5000$ (blue) is given in Fig. 2, showing that the boundary approaches the thermodynamic limit (5) as N increases. The variational energies are also shown (background color). The boundary from eGPE calculations (circles and thick curve) extends to slightly closer to the thermodynamic limit than the variational Gaussian (thin curve), as the eGPE gives lower-energy states.

Compared to the $(\epsilon_0 > 0, \epsilon_2 = 0)$ case, the lower energies for the other two cases, $(\epsilon_0 < 0, \epsilon_2 = 0)$ and $\epsilon_0 = 0$, are striking. To understand this, we consider the large N limit (see

Appendix D),

$$\frac{E}{NE_0} = -\frac{50\pi^2(\epsilon_{\text{lim}} - 1)^3}{3[64Q_5(\epsilon_0, \epsilon_2)]^2}, \quad (8)$$

where $\epsilon_{\text{lim}} = -\min_{\mathbf{k}} \bar{U}(\mathbf{k})$, i.e.,

$$\epsilon_{\text{lim}} = \begin{cases} \epsilon_0 + \sqrt{3}|\epsilon_2|, & \epsilon_0 \geq -|\epsilon_2|/\sqrt{3}, \\ 2|\epsilon_0|, & \epsilon_0 \leq -|\epsilon_2|/\sqrt{3}. \end{cases} \quad (9)$$

The energy (8) is achieved with peak density (see also Refs. [55,56])

$$n_{\text{peak}} a_s^3 = \frac{25\pi(\epsilon_{\text{lim}} - 1)^2}{[64Q_5(\epsilon_0, \epsilon_2)]^2}, \quad (10)$$

i.e., independent of N like a liquid. The three cases marked in Fig. 2, all have $\epsilon_{\text{lim}} = 3$, but $[Q_5(3, 0)]^2 \approx 10[Q_5(0, \sqrt{3})]^2 \approx 20[Q_5(-1.5, 0)]^2$, which explains the bulk of the relative energy differences.

As expected from the symmetry, ascending near $\epsilon_0 = \epsilon_2/\sqrt{3}$ gives rapidly decreasing energy, equivalent to descending near the $\epsilon_0 < 0$ axis, whereas descending near $\epsilon_0 = -\epsilon_2/\sqrt{3}$ the decrease in energy is slower, equivalent to the $\epsilon_0 > 0$ axis.

Two contours of the imaginary part of the quantum fluctuation term are shown (note that these are independent of N), revealing an appreciable region where droplets are formed, before the imaginary part becomes significant.

Example profiles of droplets are given in Fig. 3. The case $\epsilon_0 = 3, \epsilon_2 = 0$ is shown in Fig. 3(a) with a prolate profile, and the characteristic liquid-like plateau in density in the bulk along z . The case $\epsilon_0 = -1.5, \epsilon_2 = 0$ is shown in Fig. 3(b) with an oblate profile. Figure 3(c) shows the profile on the ϵ_2 axis for $\epsilon_2 = \sqrt{3}$ showing distinct profiles on the three axes.

For $\epsilon_0 = 0$, we minimize the variational energy to find that the ratio $\lambda \equiv l_y/l_z$ is constant in the limit $l_x \ll l_z$, given by $(\lambda^2 - 5\lambda^4)K(1 - \lambda^2) = (2 - 5\lambda^2 - \lambda^4)E(1 - \lambda^2)$, with E and K the complete elliptic integrals, i.e., $\lambda \approx 1.964$. We also find that all our eGPE results for $\epsilon_0 = 0$, which we have computed for $1.25 \leq \epsilon_2 \leq 6$ and $10^2 \leq N \leq 10^5$, have $l_y/l_z \in 2 \pm 0.1$. Properties of the droplets are shown in Fig. 4. The long width(s) increase significantly from $N = 1000$ (red) to $N = 5000$ (blue), whereas the shortest width(s) show a smaller increase and sometimes decrease. Figure 4(b) shows the widths starting from a cylindrically symmetric ($x = y$) prolate droplet on the left at $\epsilon_0 = 3, \epsilon_2 = 0$ progressing to a cylindrically symmetric ($y = z$) oblate droplet in the middle at $\epsilon_0 = 3/4, \epsilon_2 = 3\sqrt{3}/4$, and ending on a cylindrically symmetric ($x = z$) prolate droplet on the right at $\epsilon_0 = -3/2, \epsilon_2 = 3\sqrt{3}/2$. The minimum width in our results is the axial width of the oblate droplet for $\epsilon_0 \approx -2$ with $\sqrt{\langle z^2 \rangle} \approx 5a_s$. The possibility of a self-bound droplet entering a crossover two-dimensional regime is of interest.

Figure 4(c) shows the energy per particle, which decreases sharply from $E = 0$ at the phase boundary before quantum fluctuations dominate; i.e., Q_5 dominates over ϵ_{lim} in Eq. (8). In Figs. 4(b), 4(d), and 4(f), $\epsilon_{\text{lim}} = 3$ is constant, so the variation in Eqs. (8) and (10) is entirely due to quantum fluctuations. Figure 4(e) shows the peak density increase steeply as $|\epsilon_0|$ increases away from the critical point. However, for larger $|\epsilon_0| \gtrsim 2$ the peak density decreases, due again to quantum

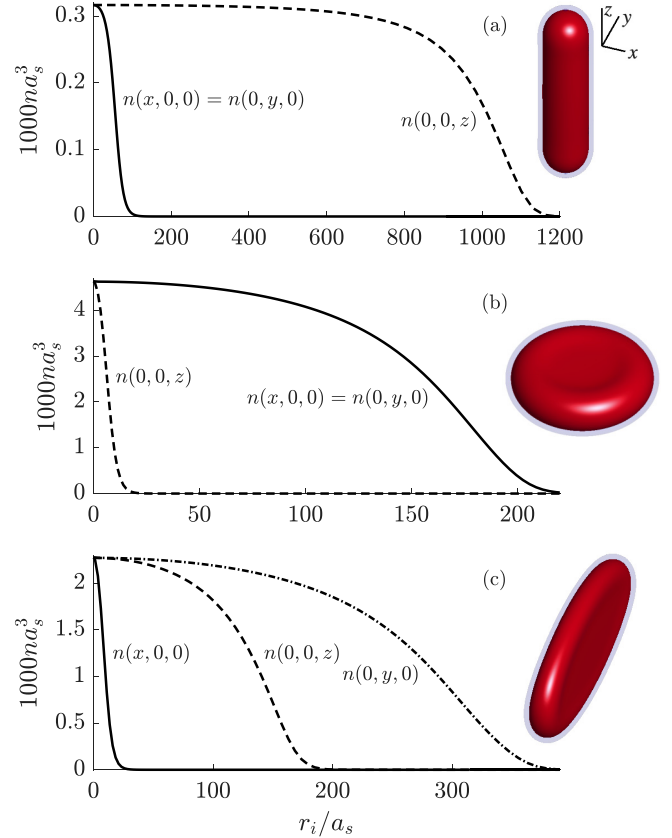


FIG. 3. Profiles for $N = 5000$ and (a) $\epsilon_0 = 3, \epsilon_2 = 0$, (b) $\epsilon_0 = -1.5, \epsilon_2 = 0$, and (c) $\epsilon_0 = 0, \epsilon_2 = \sqrt{3}$ for the x axis (solid), y axis (dash-dotted), and z axis (dashed). On the right (not to scale) are isodensity surfaces at 90% (red) and 5% (blue) of peak density.

fluctuations. In Fig. 4(f), the peak density is largest for a cylindrically symmetric oblate droplet. Figures 4(e) and 4(f) show that our results are in the dilute gas regime as $n_{\text{peak}} |a_s \epsilon_i|^3 \ll 1$.

VI. CONCLUSIONS

We analyzed the symmetry landscape of degenerate molecular gases dressed by elliptically polarized microwave fields and showed that states form sextets under D_3 symmetry that tile the interaction plane, representing the same state with permuted coordinates, capturing the degeneracies of the dipolar interaction landscape. We identified where the system has cylindrical symmetry, and when such a gas is prolate or oblate, and partitioned parameter space by the ordering of the principal widths of the cloud. We identified three axes in parameter space (prolate, oblate, and asymmetric), which together represent every 30° in parameter space, due to the symmetry. We used the symmetries to classify the solutions of the eGPE, and to derive the analytic variational interaction energy. Then, focusing on self-bound droplets with $a_s > 0$ and in the dilute gas regime, we identified the phase diagram in the thermodynamic limit, and computed the finite N phase boundary. We found a limiting form of the energy and peak density of a self-bound droplet, and compared it to numerical results at finite number. We found the asymptotic aspect ratio

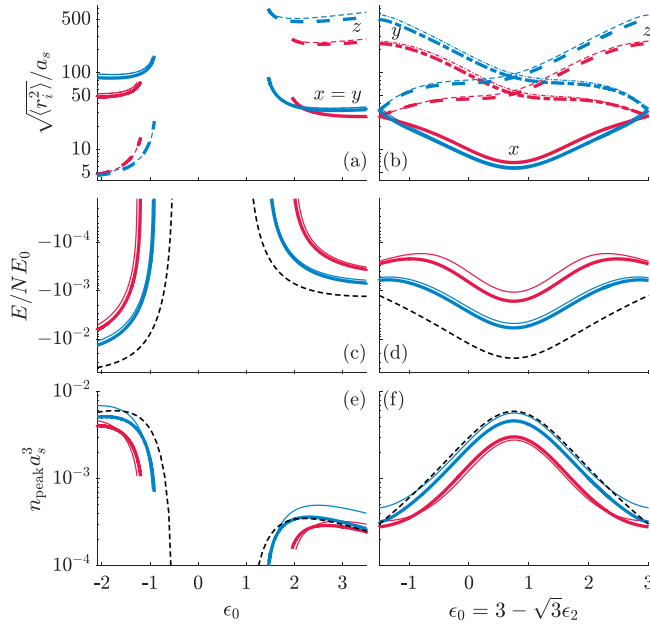


FIG. 4. Properties of droplets for $N = 1000$ (red) and $N = 5000$ (blue); thick curves are eGPE and thin curves are variational. (a), (b) RMS widths along the x (solid), y (dash dotted), and z (dashed) axes [in panel (a) $\sqrt{\langle y^2 \rangle}$ is equal to and obscured by $\sqrt{\langle x^2 \rangle}$; for variational $\sqrt{\langle r^2 \rangle} = l_i/\sqrt{2}$]. (c), (d) Energy per particle, also showing Eq. (8) (black dashed). (e), (f) Peak density, also showing Eq. (10) (black dashed). In panels (a), (c), and (e), results are for $\epsilon_2 = 0$, and panels (b), (d), and (f) for $\epsilon_0 + \sqrt{3}\epsilon_2 = 3$.

of the two long widths of our key asymmetric case, $\epsilon_0 = 0$. We illustrated the profiles of self-bound droplets and their widths, energy, and peak density.

The tiling of the interaction plane invites direct experimental verification by scanning (ϵ_0, ϵ_2) , relating self-bound profiles at different elements of the sextet, and/or by interchanging trap frequencies relative to the polarization direction for trapped gases. Further work will investigate the excitation spectra [57] of self-bound droplets of molecules.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [58].

APPENDIX A: MOMENTUM-SPACE INTERACTION

We use a grid that is shaped to amply cover our density, which is generally a different size in each direction, and transform onto a zero-padded grid in momentum space to use a spherical cutoff for the dipolar interaction. The cutoff Fourier transform is found using the spherical wave expansion of the

plane wave [59]:

$$\int_{r < R} d\mathbf{r} e^{-i\mathbf{k} \cdot \mathbf{r}} \frac{3}{4\pi r^3} \sum_{m=-2}^2 c_m Y_2^m(\theta, \phi) = -3 \sum_{m=-2}^2 c_m Y_2^{m*}(\theta_k, \phi_k) \int_0^R dr r^{-1} j_2(kr) \quad (\text{A1})$$

$$= -s(kR) \sum_{m=-2}^2 c_m Y_2^{m*}(\theta_k, \phi_k) \quad (\text{A2})$$

where $s(\kappa) = 1 + 3\kappa^{-2} \cos \kappa - 3\kappa^{-3} \sin \kappa$, θ_k is the angle between $\mathbf{k}$ and k_z , and ϕ_k is the angle between (k_x, k_y) and the k_x axis. Using c_m for our interaction gives [38,60]

$$\frac{\bar{U}^R(\mathbf{k})}{s(kR)} = \epsilon_0(3 \cos^2 \theta_k - 1) - \sqrt{3}\epsilon_2 \sin^2 \theta_k \cos 2\phi_k. \quad (\text{A3})$$

For $R \rightarrow \infty$, $s(kR) \rightarrow 1$ and $\bar{U}^R(\mathbf{k}) \rightarrow \bar{U}(\mathbf{k})$.

APPENDIX B: QUANTUM FLUCTUATION COEFFICIENT

The function $\bar{U}(\mathbf{k})$ is real, so for a given $\mathbf{k}$, the principal root $[1 + \bar{U}(\mathbf{k})]^{5/2}$ is positive real or positive imaginary, and $Q_5(\epsilon_0, \epsilon_2)$ is complex with positive real and imaginary parts if in the region (5), and real otherwise. First, we consider $\epsilon_2 = 0$:

$$Q_5(x, 0) = \frac{5(x-1)^3}{16\sqrt{3x}} \left[\ln(\sqrt{1+2x} - \sqrt{3x}) - \frac{\ln(1-x)}{2} \right] + \frac{1}{16} (11 + 4x + 9x^2) \sqrt{1+2x}. \quad (\text{B1})$$

For $\epsilon_2 \neq 0$, we pull out a ϕ_k dependent factor that is imaginary for some parameters, before integrating over θ_k , so we must take the absolute value of the real part before numerically integrating over ϕ_k :

$$\text{Re}\{Q_5(\epsilon_0, \epsilon_2)\} = \frac{2}{\pi} \int_0^{\pi/2} d\phi_k |\text{Re}\{I(\phi_k)\}|, \quad (\text{B2})$$

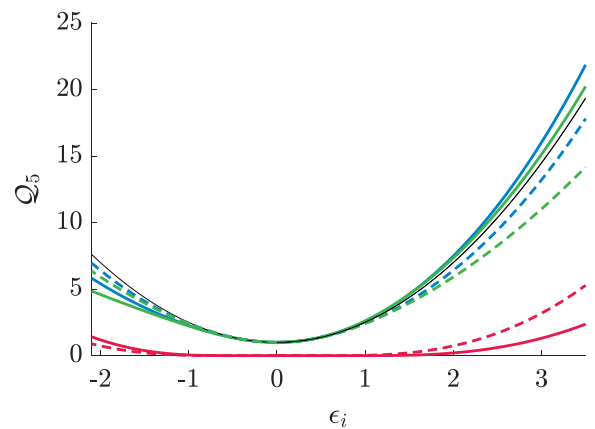


FIG. 5. Quantum fluctuations coefficient along $\epsilon_2 = 0$ (solid) and $\epsilon_0 = 0$ (dashed). Shown are $\text{Re}\{Q_5\}$ (blue), $\text{Im}\{Q_5\}$ (red), Q'_5 (B6) (green), and the small parameter approximation (B7) (thin black).

$$\text{Im}\{Q_5(\epsilon_0, \epsilon_2)\} = \frac{2}{\pi} \int_0^{\pi/2} d\phi_k \text{Im}\{I(\phi_k)\}, \quad (\text{B3})$$

$$I(\phi_k) = \left(1 - \frac{2}{\sqrt{3}}\epsilon_2 \cos 2\phi_k\right)^{5/2} Q_5\left(\frac{\sqrt{3}\epsilon_0 + \epsilon_2 \cos 2\phi_k}{\sqrt{3} - 2\epsilon_2 \cos 2\phi_k}, 0\right). \quad (\text{B4})$$

The integral can be evaluated for these two cases:

$$Q_5(\epsilon_0, \epsilon_2) = \begin{cases} {}_3F_2\left(-\frac{5}{4}, -\frac{3}{4}, \frac{1}{2}; \frac{3}{4}, \frac{5}{4}; 3\epsilon_2^2\right), & \epsilon_0 = 0, \\ \frac{5\pi}{32}\left(\frac{3}{2} + \sqrt{3}|\epsilon_2|\right)^{5/2} {}_2F_1\left(-\frac{5}{2}, \frac{1}{2}; 1; \frac{4|\epsilon_2|}{2|\epsilon_2| + \sqrt{3}}\right), & \epsilon_0 = -\frac{1}{2}. \end{cases} \quad (\text{B5})$$

The real part of Q_5 retains the real terms in the integrand of Eq. (A2) of Ref. [21] even where the Bogoliubov energy term is imaginary. An alternative is to exclude the entire contribution of unstable modes to that integral [61], i.e., to use Cutoff III of Ref. [21] generalized to include ϵ_2 (evaluated using Eq. (A10) of Ref. [21], noting [62]). This defines Q'_5 , which we evaluate analytically. With $\alpha \equiv \text{Im}\{Q_5\}/\text{Re}\{Q_5\}$,

$$\frac{Q'_5(\epsilon_0, \epsilon_2)}{\text{Re}\{Q_5(\epsilon_0, \epsilon_2)\}} = 1 - \frac{11}{16}\alpha \approx \begin{cases} 0.93, & \alpha = \frac{1}{10}, \\ 0.83, & \alpha = \frac{1}{4}. \end{cases} \quad (\text{B6})$$

The results for Q_5 along the axes are shown in Fig. 5, along with the small ϵ_0 and ϵ_2 limit [63]:

$$Q_5(\epsilon_0, \epsilon_2) \approx 1 + \frac{3}{2}(\epsilon_0^2 + \epsilon_2^2). \quad (\text{B7})$$

APPENDIX C: ANISOTROPY FUNCTIONS

These are given by

$$f(x, y) = - \int \frac{d\Omega_k}{4\pi} \left(\frac{3 \cos^2 \theta_k}{\phi_+ \sin^2 \theta_k + \cos^2 \theta_k} - 1 \right), \quad (\text{C1})$$

$$f_2(x, y) = \int \frac{d\Omega_k}{4\pi} \frac{\phi_- \sin^2 \theta_k}{\phi_+ \sin^2 \theta_k + \cos^2 \theta_k}, \quad (\text{C2})$$

where $\phi_{\pm} \equiv x^{-2} \cos^2 \phi_k \pm y^{-2} \sin^2 \phi_k$.

APPENDIX D: LARGE N LIMIT

The droplet profile becomes increasingly flat topped so that $\int d\mathbf{x} |\psi|^p \rightarrow n_{\text{peak}}^{p/2-1} N$, kinetic energy becomes insignificant, and we find n_{peak} that minimizes

$$\frac{E}{N} \rightarrow \frac{1}{2} g_s (1 - \epsilon_{\text{lim}}) n_{\text{peak}} + \frac{2}{5} \gamma_{\text{QF}} n_{\text{peak}}^{3/2}. \quad (\text{D1})$$

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- [1] K. K. Ni, S. Ospelkaus, M. H. G. de Miranda, A. Pe'er, B. Neyenhuis, J. J. Zirbel, S. Kotochigova, P. S. Julienne, D. S. Jin, and J. Ye, A high phase-space-density gas of polar molecules, *Science* **322**, 231 (2008).
- [2] S. Ospelkaus, K.-K. Ni, D. Wang, M. H. G. de Miranda, B. Neyenhuis, G. Quémener, P. S. Julienne, J. L. Bohn, D. S. Jin, and J. Ye, Quantum-state controlled chemical reactions of ultracold potassium-rubidium molecules, *Science* **327**, 853 (2010).
- [3] K.-K. Ni, S. Ospelkaus, D. Wang, G. Quemener, B. Neyenhuis, M. H. G. de Miranda, J. L. Bohn, J. Ye, and D. S. Jin, Dipolar collisions of polar molecules in the quantum regime, *Nature (London)* **464**, 1324 (2010).
- [4] L. De Marco, G. Valtolina, K. Matsuda, W. G. Tobias, J. P. Covey, and J. Ye, A degenerate Fermi gas of polar molecules, *Science* **363**, 853 (2019).
- [5] T. Karman and J. M. Hutson, Microwave shielding of ultracold polar molecules, *Phys. Rev. Lett.* **121**, 163401 (2018).
- [6] L. Lassablière and G. Quémener, Controlling the scattering length of ultracold dipolar molecules, *Phys. Rev. Lett.* **121**, 163402 (2018).
- [7] L. Anderegg, S. Burchesky, Y. Bao, S. S. Yu, T. Karman, E. Chae, K.-K. Ni, W. Ketterle, and J. M. Doyle, Observation of microwave shielding of ultracold molecules, *Science* **373**, 779 (2021).
- [8] N. Bigagli, C. Warner, W. Yuan, S. Zhang, I. Stevenson, T. Karman, and S. Will, Collisionally stable gas of bosonic dipolar ground-state molecules, *Nat. Phys.* **19**, 1579 (2023).
- [9] N. Bigagli, W. Yuan, S. Zhang, B. Bulatovic, T. Karman, I. Stevenson, and S. Will, Observation of Bose-Einstein condensation of dipolar molecules, *Nature (London)* **631**, 289 (2024).
- [10] T. Langen, G. Valtolina, D. Wang, and J. Ye, Quantum state manipulation and cooling of ultracold molecules, *Nat. Phys.* **20**, 702 (2024).
- [11] J. Lin, G. Chen, M. Jin, Z. Shi, F. Deng, W. Zhang, G. Quémener, T. Shi, S. Yi, and D. Wang, Microwave shielding of bosonic NaRb molecules, *Phys. Rev. X* **13**, 031032 (2023).
- [12] Z. Shi, Z. Huang, F. Deng, W.-J. Jin, S. Yi, T. Shi, and D. Wang, Bose-Einstein condensate of ultracold sodium-rubidium molecules with tunable dipolar interactions, *arXiv:2508.20518*.
- [13] H. Kadau, M. Schmitt, M. Wenzel, C. Wink, T. Maier, I. Ferrier-Barbut, and T. Pfau, Observing the Rosensweig instability of a quantum ferrofluid, *Nature (London)* **530**, 194 (2016).
- [14] L. Chomaz, S. Baier, D. Petter, M. J. Mark, F. Wächtler, L. Santos, and F. Ferlaino, Quantum-fluctuation-driven crossover from a dilute Bose-Einstein condensate to a macrodroplet in a dipolar quantum fluid, *Phys. Rev. X* **6**, 041039 (2016).
- [15] M. Schmitt, M. Wenzel, F. Böttcher, I. Ferrier-Barbut, and T. Pfau, Self-bound droplets of a dilute magnetic quantum liquid, *Nature (London)* **539**, 259 (2016).
- [16] D. S. Petrov, Quantum mechanical stabilization of a collapsing Bose-Bose mixture, *Phys. Rev. Lett.* **115**, 155302 (2015).
- [17] T. Lahaye, C. Menotti, L. Santos, M. Lewenstein, and T. Pfau, The physics of dipolar bosonic quantum gases, *Rep. Prog. Phys.* **72**, 126401 (2009).
- [18] L. Chomaz, I. Ferrier-Barbut, F. Ferlaino, B. Laburthe-Tolra, B. L. Lev, and T. Pfau, Dipolar physics: A review of experiments with magnetic quantum gases, *Rep. Prog. Phys.* **86**, 026401 (2023).

- [19] L. Chomaz, Quantum-stabilized states in magnetic dipolar quantum gases, *Annu. Rev. Condens. Matter Phys.* **17**, 47 (2026).
- [20] D. Baillie, R. M. Wilson, R. N. Bisset, and P. B. Blakie, Self-bound dipolar droplet: A localized matter wave in free space, *Phys. Rev. A* **94**, 021602(R) (2016).
- [21] R. N. Bisset, R. M. Wilson, D. Baillie, and P. B. Blakie, Ground-state phase diagram of a dipolar condensate with quantum fluctuations, *Phys. Rev. A* **94**, 033619 (2016).
- [22] F. Wächtler and L. Santos, Quantum filaments in dipolar Bose-Einstein condensates, *Phys. Rev. A* **93**, 061603(R) (2016).
- [23] F. Wächtler and L. Santos, Ground-state properties and elementary excitations of quantum droplets in dipolar Bose-Einstein condensates, *Phys. Rev. A* **94**, 043618 (2016).
- [24] C. R. Cabrera, L. Tanzi, J. Sanz, B. Naylor, P. Thomas, P. Cheiney, and L. Tarruell, Quantum liquid droplets in a mixture of Bose-Einstein condensates, *Science* **359**, 301 (2018).
- [25] G. Semeghini, G. Ferioli, L. Masi, C. Mazzinghi, L. Wolswijk, F. Minardi, M. Modugno, G. Modugno, M. Inguscio, and M. Fattori, Self-bound quantum droplets of atomic mixtures in free space, *Phys. Rev. Lett.* **120**, 235301 (2018).
- [26] S. Zhang, W. Yuan, N. Bigagli, H. Kwak, T. Karman, I. Stevenson, and S. Will, Observation of self-bound droplets of ultracold dipolar molecules, *Nature (London)* **651**, 601 (2026).
- [27] S. Giovanazzi, A. Görlitz, and T. Pfau, Tuning the dipolar interaction in quantum gases, *Phys. Rev. Lett.* **89**, 130401 (2002).
- [28] D. Baillie and P. B. Blakie, Rotational tuning of the dipole-dipole interaction in a Bose gas of magnetic atoms, *Phys. Rev. A* **101**, 043606 (2020).
- [29] Y. Tang, W. Kao, K.-Y. Li, and B. L. Lev, Tuning the dipole-dipole interaction in a quantum gas with a rotating magnetic field, *Phys. Rev. Lett.* **120**, 230401 (2018).
- [30] M. Wenzel, Macroscopic states of dipolar quantum gases, Ph.D. thesis, University of Stuttgart, Germany, 2018.
- [31] K. Mukherjee, T. A. Cardinale, L. Chergui, P. Stürmer, and S. M. Reimann, Droplets and supersolids in ultra-cold atomic quantum gases, *Eur. Phys. J. Spec. Top.* **232**, 3417 (2023).
- [32] K. Mukherjee, M. N. Tengstrand, T. A. Cardinale, and S. M. Reimann, Supersolid stacks in antidipolar Bose-Einstein condensates, *Phys. Rev. A* **108**, 023302 (2023).
- [33] W.-J. Jin, F. Deng, S. Yi, and T. Shi, Bose-Einstein condensates of microwave-shielded polar molecules, *Phys. Rev. Lett.* **134**, 233003 (2025).
- [34] W. Zhang, K. Chen, S. Yi, and T. Shi, Quantum phases for finite-temperature gases of bosonic polar molecules shielded by dual microwaves, *PRX Quantum* **6**, 040307 (2025).
- [35] T. Langen, J. Boronat, J. Sánchez-Baena, R. Bombín, T. Karman, and F. Mazzanti, Dipolar droplets of strongly interacting molecules, *Phys. Rev. Lett.* **134**, 053001 (2025).
- [36] M. Ciardi, K. R. Pedersen, T. Langen, and T. Pohl, Self-bound superfluid membranes and monolayer crystals of ultracold polar molecules, *Phys. Rev. Lett.* **135**, 153401 (2025).
- [37] C. J. Polteraer and R. E. Zillich, Phases of a Bose-Einstein condensate of microwave-shielded dipolar molecules, [arXiv:2507.18986](https://arxiv.org/abs/2507.18986).
- [38] T. A. Cardinale, T. Bland, and S. M. Reimann, Exploring molecular supersolidity via exact and mean-field theories: Single microwave shielding, [arXiv:2509.18051](https://arxiv.org/abs/2509.18051).
- [39] W. Zhang, H. Liu, F. Deng, K. Chen, S. Yi, and T. Shi, Supersolid phases in ultracold gases of microwave shielded polar molecules, [arXiv:2506.23820](https://arxiv.org/abs/2506.23820).
- [40] F. Deng, X.-Y. Chen, X.-Y. Luo, W. Zhang, S. Yi, and T. Shi, Effective potential and superfluidity of microwave-shielded polar molecules, *Phys. Rev. Lett.* **130**, 183001 (2023).
- [41] T. Karman, N. Bigagli, W. Yuan, S. Zhang, I. Stevenson, and S. Will, Double microwave shielding, *PRX Quantum* **6**, 020358 (2025).
- [42] F. Deng, X. Hu, W.-J. Jin, S. Yi, and T. Shi, Two- and many-body physics of ultracold molecules dressed by dual microwave fields, *Nat. Commun.* **16**, 11219 (2025).
- [43] A. Schindewolf, J. Hertkorn, I. Stevenson, M. Ciardi, P. Gross, D. Wang, T. Karman, G. Quemener, S. Will, T. Pohl, and T. Langen, *Colloquium: Strongly dipolar molecular Bose-Einstein condensates: From few- to many-body physics*, *Rev. Mod. Phys.* (2026), <https://doi.org/10.1103/rf55l-f93m>.
- [44] Using a_{dm} defined in Ref. [26], our $\epsilon_m = a_{dm}/3a_s$ for $m = 0, 2$. Using $\epsilon_{3,2}$ defined in Ref. [39], our $\epsilon_2 = \epsilon_{3,2}/\sqrt{3}$.
- [45] Not to be confused with the vacuum permittivity.
- [46] W. Yuan, S. Zhang, N. Bigagli, H. Kwak, C. Warner, T. Karman, I. Stevenson, and S. Will, Extreme loss suppression and wide tunability of dipolar interactions in an ultracold molecular gas, [arXiv:2505.08773](https://arxiv.org/abs/2505.08773).
- [47] A. R. P. Lima and A. Pelster, Quantum fluctuations in dipolar Bose gases, *Phys. Rev. A* **84**, 041604(R) (2011).
- [48] A. R. P. Lima and A. Pelster, Beyond mean-field low-lying excitations of dipolar Bose gases, *Phys. Rev. A* **86**, 063609 (2012).
- [49] By taking the Fourier transform, changing variables to $u_i = l_i k_i$, integrating the angular variables, and changing variables back to k_i .
- [50] S. Giovanazzi, P. Pedri, L. Santos, A. Griesmaier, M. Fattori, T. Koch, J. Stuhler, and T. Pfau, Expansion dynamics of a dipolar Bose-Einstein condensate, *Phys. Rev. A* **74**, 013621 (2006).
- [51] K. Glaum and A. Pelster, Bose-Einstein condensation temperature of dipolar gas in anisotropic harmonic trap, *Phys. Rev. A* **76**, 023604 (2007).
- [52] We align our dipoles along z , whereas Ref. [50] aligns along x , but our function $f(x, y)$ is exactly the same as theirs.
- [53] For $0 < y \ll 1$, as x is reduced $f_2(x, y)$ changes sharply from ≈ -1 for $x \gg y$, through zero at $x = y$, to 1 at $x = 0$.
- [54] L. Santos, G. V. Shlyapnikov, P. Zoller, and M. Lewenstein, Bose-Einstein condensation in trapped dipolar gases, *Phys. Rev. Lett.* **85**, 1791 (2000).
- [55] I. Ferrier-Barbut, H. Kadau, M. Schmitt, M. Wenzel, and T. Pfau, Observation of quantum droplets in a strongly dipolar Bose gas, *Phys. Rev. Lett.* **116**, 215301 (2016).
- [56] J. Dalibard, Magnetic interactions between cold atoms: Quantum droplets and supersolid states, Lecture series, Collège de France, 2024, https://pro.college-de-france.fr/jean.dalibard/CdF/2024/total_en_2024.pdf.
- [57] D. Baillie, R. M. Wilson, and P. B. Blakie, Collective excitations of self-bound droplets of a dipolar quantum fluid, *Phys. Rev. Lett.* **119**, 255302 (2017).

- [58] D. Baillie, Symmetry and self-bound droplets in dipolar molecular gases [data set], Zenodo (2026), <https://doi.org/10.5281/zenodo.20078499>.
- [59] J. D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, New York, 1999).
- [60] S. Ronen, D. C. E. Bortolotti, and J. L. Bohn, Bogoliubov modes of a dipolar condensate in a cylindrical trap, *Phys. Rev. A* **74**, 013623 (2006).
- [61] H. Saito, Path-integral Monte Carlo study on a droplet of a dipolar Bose-Einstein condensate stabilized by quantum fluctuation, *J. Phys. Soc. Jpn.* **85**, 053001 (2016).
- [62] The prefactor of Eq. (A10) of Ref. [21] should be $1/128$, not $1/64$.
- [63] For larger parameters Q_5 increases like $|\epsilon_0|^{5/2}$ or $|\epsilon_2|^{5/2}$ [see Eq. (4)], but the imaginary part of Q_5 will then be significant.