

Stroboscopic saturation of multiparameter quantum limits in distributed quantum sensing

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A naturally multiparameter sensing scenario arises in spatially distributed networks of quantum sensors, where each node probes a distinct parameter of interest. In such settings, the central challenges are to achieve quantum-enhanced sensitivity under well-defined resource constraints and to explicitly identify measurement strategies that attain the ultimate precision limits. Here, we analytically demonstrate quantum-enhanced sensitivity for a broad class of distributed quantum probes designed to simultaneously estimate a complete set of unknown reference-relative phase differences encoded locally. Although such locally encoded parameters can, in principle, be optimally estimated using separable strategies, our distributed protocol offers distinct advantages. In particular, for fixed network size and identical bare probe Hamiltonians, it is energetically more efficient and exhibits stroboscopic disentanglement at specific evolution times, enabling the accumulated pairwise information to be coherently mapped onto a specific subsystem. Consequently, all set of pairwise reference-relative phase differences can be accessed through a single collective measurement without requiring independent local estimation. We further construct the corresponding optimal measurement strategies and show that, within this class of probes, the ultimate multiparameter precision limits—the Holevo and quantum Cramér-Rao bounds—are saturable. We illustrate the framework with two concrete applications: simultaneous gravimetry across spatially separated locations and estimation of distributed coupling strengths. Feasibility analyses indicate that the proposed distributed quantum-enhanced sensing protocols operate within experimentally accessible regimes.

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I. INTRODUCTION

Quantum sensing is a key pillar of quantum technologies, with rapidly growing theoretical development and experimental realizations [1–5]. At the heart of quantum sensing lies the exploitation of uniquely quantum effects to achieve sensing precision beyond the standard quantum limit, relative to a well-defined sensing resource [6–10]. Deep within the field, strong connections emerge with quantum information theory [11], geometric approaches [12,13], and quantum estimation theory [14–18]. The latter provides a natural framework to classify quantum sensing protocols into two broad classes: single-parameter estimation and multiparameter estimation.

In the single-parameter case, the ultimate precision bounds are well established and known to be saturable with optimal measurement strategies [18]. Indeed, for a single parameter encoded through a unitary evolution, the maximal

attainable quantum Fisher information—optimized over all probe states and measurements—is determined solely by the spectral seminorm of the Hermitian generator of the parameter dependence, i.e., by the difference between its largest and smallest eigenvalues. Consequently, the ultimate quantum Cramér-Rao precision limit depends only on this generator and is asymptotically saturable by an optimal probe state given by an equal coherent superposition of the eigenvectors associated with these extremal eigenvalues [19].

In contrast, the multiparameter setting introduces additional layers of complexity. Even when each parameter is individually associated with a well-defined generator and corresponding optimal bound, the probe states that maximize sensitivity for different parameters may be incompatible [20,21]. That is, the optimal probe for one parameter need not be optimal for another [22]. This implies that a single probe state cannot, in general, simultaneously saturate the optimal precision bounds for all parameters [23,24]. Beyond probe incompatibility, measurement incompatibility further constrains multiparameter estimation. In general, the symmetric logarithmic derivatives (SLDs) corresponding to different parameters may not commute [24–26]. This noncommutativity prevents the existence of a single measurement that is simultaneously optimal for all parameters. That is, except for special cases [27], one cannot construct a measurement basis that is both independent of the unknown parameters

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and capable of saturating the theoretical multiparameter lower bound on estimation precision. Consequently, although ultimate precision bounds—such as the Holevo or multiparameter quantum Cramér-Rao bounds—are formally established [24], they are not guaranteed to be saturable in general.

It is important to emphasize that the multiparameter quantum Cramér-Rao bound coincides with the Holevo bound only in special cases, such as when the SLD operators commute on average with respect to the probe state [24,26]. In general, however, the Holevo and the multiparameter quantum Cramér-Rao bound do not coincide [28], and as such the quantum Fisher information matrix alone does not represent an attainable precision limit. Moreover, in realistic sensing implementations, decoherence during state preparation and evolution inevitably leads to mixed probe states. In such cases, the attainability of the Holevo bound becomes more subtle and is, in general, guaranteed only asymptotically in the number of experimental repetitions [29]. For finite data and realistic measurement strategies, alternative Cramér-Rao-type bounds may become operationally relevant [30,31]. These considerations highlight the importance of identifying physically realizable sensing protocols for which the ultimate multiparameter precision limits are demonstrably saturable under well-defined resource constraints.

The formalism of multiparameter quantum estimation naturally extends to scenarios involving spatially distributed networks of quantum sensors, giving rise to the framework of distributed quantum sensing [29,32]. In this setting, the goal is to enable multiple sensors to collectively exploit shared entangled states in order to enhance the precision of estimating global properties of the system. Distributed quantum sensing has captured significant attention in recent years—developed extensively both theoretically [33–45] and experimentally [46–55]. It has also been proposed for novel applications such as privacy-preserving quantum tasks [56,57] and for probing quantum theory in curved spacetime [58]. While the ultimate bounds on sensing precision in such distributed schemes are well established, distributed sensing inherits the fundamental challenges common to general multiparameter estimation problems: probe incompatibility, measurement incompatibility, and the subtle attainability of the Holevo bound. These challenges are further compounded by the spatial separation of the quantum probes, which introduces additional constraints on entanglement distribution and measurement implementation. This raises several critical questions: (1) Can one theoretically identify a broad and physically feasible class of quantum probes for which the ultimate lower bound is saturable? (2) Is it possible to construct a locally optimal parameter-independent measurement basis that saturates around the operating point? (3) Can such protocols be realized within experimentally realistic architectures?

In this work, we provide explicit answers to all three questions. (1) We analytically demonstrate quantum-enhanced sensitivity for a broad class of conditionally displaced quantum probes. We report cases where the precision scales quadratically or quartically with the sensing resources. These probes consist of quantized bosonic modes (here, mechanical oscillators prepared in thermal or coherent states) whose quadrature positions are conditionally displaced based on the state of an interacting subsystem. Remarkably, this class

includes nonlinear quantum optomechanical systems [59,60], solid-state spin systems [61,62], and hybrid dispersive tripartite architectures [63,64]—making our analysis both general and experimentally relevant. The key mechanism enabling quantum-enhanced sensitivity is the disentanglement of the mechanical oscillators at full evolution cycles, which transfers all parameter-dependent information to the coupled subsystem. As a result, all global information is transferred to the subsystems, ensuring that optimal measurements can be performed exclusively on them—eliminating the need to measure the full composite system. Our stroboscopic disentanglement at specific evolution times bears conceptual similarity to metrological Loschmidt-echo schemes. However, it does not involve the explicit time-reversal operation required in standard Loschmidt-echo protocols [65–67]. (2) Building on this, we show that the problem maps onto the estimation of a complete set of $N - 1$ pairwise reference-relative phase differences [33]. This mapping enables us to analytically construct an adaptive family of collective projective measurements via the Gram-Schmidt procedure [27]. The validity of this construction is guaranteed by the weak commutativity condition, which we prove rigorously [24,26,27]. Note that although the optimal estimation of pairwise parameters encoded locally can, in principle, be achieved using optimal separable strategies [37], our distributed protocol offers distinct advantages. Under identical probe Hamiltonians and fixed network size, we show that the entangled strategy is energetically more efficient, as the relevant sensing resource is coherently delocalized across the network. Moreover, as stressed above, the probe dynamics exhibit stroboscopic disentanglement at specific evolution times, enabling the accumulated pairwise information to be coherently mapped onto the spin (or optical) degrees of freedom in spin-(opto)mechanical systems. Consequently, all set of pairwise reference-relative phase differences can be accessed through a collective measurement without requiring independent local estimation. (3) Finally, we propose two concrete physical implementations: the simultaneous estimation of multiple gravitational accelerations (quantum gravimetry) and the estimation of coupling strengths across spatially separated locations. We also provide quantitative feasibility analyses demonstrating that these protocols are within reach of current experimental capabilities.

II. QUANTUM ESTIMATION BACKGROUND

Consider a quantum state $\rho(\theta)$ that encodes d unknown parameters $\theta = (\theta_1, \dots, \theta_d)$. The fundamental lower bound on the precision of simultaneously estimating these parameters is given by the multiparameter quantum Cramér-Rao inequality $\text{Cov}[\hat{\theta}] \geq [\mu F(\theta)]^{-1} \geq [\mu Q(\theta)]^{-1}$ [23,24], where $\text{Cov}[\hat{\theta}]_{ij} = \langle \hat{\theta}_i \hat{\theta}_j \rangle - \langle \hat{\theta}_i \rangle \langle \hat{\theta}_j \rangle$ are the entries of the covariance matrix of an unbiased estimator $\hat{\theta}$ of θ , μ denotes the number of independent repetitions (shots) of the full probe state used in the estimation procedure, $F(\theta)$ is the classical Fisher information matrix (CFIM), and $Q(\theta)$ is the quantum Fisher information matrix (QFIM). Both Fisher matrices will be given explicitly later. Instead of working directly with matrix inequalities, in multiparameter quantum sensing it is often convenient to express the bounds in scalar form. To this end, one introduces a positive semidefinite weight matrix

$\mathcal{W}$, leading to the scalar multiparameter Cramér-Rao bounds $\text{Tr}[\mathcal{W}\text{Cov}[\check{\Theta}]] \geq \frac{1}{\mu} \text{Tr}[\mathcal{W}F(\theta)^{-1}] \geq \frac{1}{\mu} \text{Tr}[\mathcal{W}Q(\theta)^{-1}]$. In this work, for simplicity, we take $\mathcal{W}$ to be the identity matrix, which assigns uniform weight to all uncertainties in θ . Under this choice, the scalar bound becomes [23,24]

$$\text{Tr}[\text{Cov}[\check{\Theta}]] \geq \frac{1}{\mu} \text{Tr}[F(\theta)^{-1}] \geq \frac{1}{\mu} \text{Tr}[Q(\theta)^{-1}], \quad (1)$$

where the CFIM has entries [24]

$$[F(\theta)]_{ij} = \int \frac{\partial_i p(x|\theta) \partial_j p(x|\theta)}{p(x|\theta)} dx, \quad \partial_i \equiv \partial/\partial\theta_i, \quad (2)$$

with $p(x|\theta) = \text{Tr}[\rho(\theta)\Pi_x]$ accounting for the conditional probability of obtaining outcome x when measuring the state $\rho(\theta)$ with a positive operator-valued measure $\{\Pi_x\}$ [18]. This means that the CFIM determines the sensing precision for a given measurement strategy. On the other hand, the QFIM has entries [13,24]

$$[Q(\theta)]_{ij} = 2 \sum_{\substack{m,n \\ \varepsilon_m + \varepsilon_n \neq 0}} \text{Re} \left[\frac{\langle \varepsilon_m | \partial_i \rho(\theta) | \varepsilon_n \rangle \langle \varepsilon_n | \partial_j \rho(\theta) | \varepsilon_m \rangle}{\varepsilon_m + \varepsilon_n} \right], \quad (3)$$

where $\rho(\theta) = \sum_i \varepsilon_i(\theta) |\varepsilon_i(\theta)\rangle \langle \varepsilon_i(\theta)|$ is expressed in spectral decomposition with eigenvalues $\varepsilon_i(\theta) \equiv \varepsilon_i$ and eigenvectors $|\varepsilon_i(\theta)\rangle \equiv |\varepsilon_i\rangle$. As an important case, for pure states $\rho(\theta) = |\psi(\theta)\rangle \langle \psi(\theta)|$, Eq. (3) simplifies to [24,68]

$$[Q(\theta)]_{ij} = 4 \text{Re}(\langle \partial_i \psi(\theta) | \partial_j \psi(\theta) \rangle - \langle \partial_i \psi(\theta) | \psi(\theta) \rangle \langle \psi(\theta) | \partial_j \psi(\theta) \rangle). \quad (4)$$

Ubiquitous to any quantum sensing protocol are two essential goals: (1) demonstrate that $\text{Tr}[\text{Cov}[\check{\Theta}]] \propto N_{\text{exc}}^{-r}$, with $r > 1$. This scaling signifies quantum-enhanced sensitivity [2, 7–9], which is the total estimation uncertainty, quantified by the trace of the covariance matrix of the unbiased estimators $\check{\Theta}$, decreases superlinearly with the sensing resource—here, the number of initial excitations (spins or spectral gap) N_{exc} . Strictly speaking, our scaling is with respect to the generator seminorm, in line with Ref. [19]. This means that by increasing the number of sensing resources, the estimation uncertainty decreases faster than the standard quantum limit $r = 1$ [1,2]. This behavior is a hallmark of quantum metrology, where quantum features are used to surpass the standard quantum limit $r = 1$ and achieve quantum-enhanced scaling $r > 1$. In the context of distributed sensing, quantum-enhanced sensitivity should be demonstrated for a fixed network size while varying the sensing resources. (2) identifying the optimal measurement strategy that extracts the maximum precision allowed by the quantum probe, i.e., $\text{Tr}[F(\theta)^{-1}] = \text{Tr}[Q(\theta)^{-1}]$ in the multiparameter case, and $F(\theta) = Q(\theta)$ in the single-parameter case. The existence of a measurement that attains the quantum lower bound is what defines saturability. It is worth emphasizing that, unlike the single-parameter case where $\theta = \theta$, in the multiparameter scenario the quantum Cramér-Rao bound is generally not saturable [23]. This limitation arises because the optimal measurements corresponding to different parameters may be incompatible [26]. As a result, one cannot, in general, simultaneously achieve the precision limits for all parameters at once. This fundamental issue,

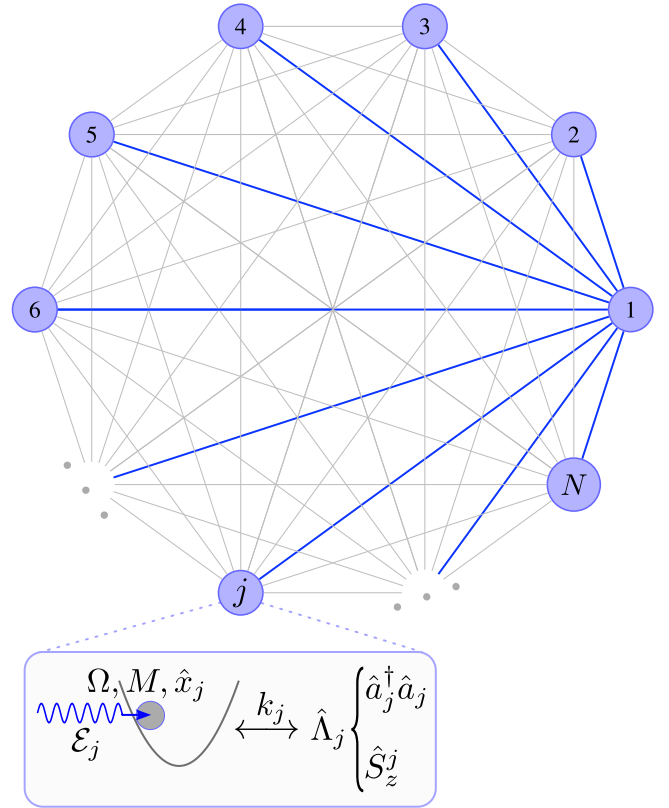


FIG. 1. Distributed quantum sensing model. We consider a network of N nodes. At each node j , the system consists of a mechanical oscillator of mass M and frequency Ω , driven with amplitude $\mathcal{E}_j$. Each node j is described by the composite Hilbert space $\mathcal{H}_j = \mathcal{H}_{\hat{\Lambda}_j} \otimes \mathcal{H}_{\hat{h}_j}$, corresponding to the $\hat{\Lambda}_j$ quantum system and its associated harmonic oscillator mode. This oscillator interacts with a general quantum system $\hat{\Lambda}_j$. We propose two physical implementations in which $\hat{\Lambda}_j$ can be either the number operator $\hat{a}_j^\dagger \hat{a}_j$ or the higher-spin collective operator $\hat{S}_z^j$ (see the main text for details). The sensing task is to simultaneously estimate the $N - 1$ unknown parameters k_j or $\mathcal{E}_j$ across the network.

stemming from noncommuting SLDs, places intrinsic constraints on the simultaneous estimation of multiple parameters [13,21,26,27,69,70].

III. THE MODEL

We consider a network of N quantum nodes. Each node, labeled by index j , is governed by the Hamiltonian

$$\hat{H}_j = \hat{H}_{\Lambda,j}^{(0)} + \frac{1}{2M} \hat{p}_j^2 + \frac{M\Omega^2}{2} \hat{x}_j^2 + \hbar \frac{\tilde{k}_j}{x_0} \hat{\Lambda}_j \hat{x}_j + \hbar \frac{\tilde{\mathcal{E}}_j}{x_0} \hat{x}_j, \quad (5)$$

which describes a driven mechanical harmonic oscillator coupled to a general quantum system (see Fig. 1). Specifically, the mechanical degree of freedom, with mass M , frequency Ω , and driven amplitude $\tilde{\mathcal{E}}_j$, is described by canonical position and momentum operators $\hat{x}_j$ and $\hat{p}_j$, satisfying the commutation relation $[\hat{x}_j, \hat{p}_{j'}] = i\hbar \delta_{j,j'}$. $\hat{H}_{\Lambda,j}^{(0)}$ accounts for the bare local Hamiltonian of the degree of freedom associated with $\hat{\Lambda}_j$, assumed identical at every sensing node, obeying $[\hat{H}_{\Lambda,j}^{(0)}, \hat{\Lambda}_j] = 0$. The quantity $x_0 = \sqrt{\hbar/(2M\Omega)}$ denotes the

zero-point fluctuation amplitude. The term proportional to $\tilde{k}_j$ captures a dispersive interaction between the mechanical position $\hat{x}_j$ and a general operator $\hat{\Lambda}_j$, obeying

$$\hat{\Lambda}_j |\lambda_n\rangle_{j=\lambda_{n,j}} |\lambda_n\rangle_j, \quad {}_j\langle\lambda_n|\lambda_{n'}\rangle_{j'} = \delta_{n,n'} \delta_{j,j'}, \quad [\hat{\Lambda}_j, \hat{\Lambda}_{j'}] = 0, \quad (6)$$

where the index n labels both the n th eigenvalue and its corresponding n th eigenvector $|\lambda_n\rangle_j$, which are associated with node j . Hence, the index n runs over the spectrum of $\hat{\Lambda}_j$, whereas $1 \leq j \leq N$. By introducing bosonic annihilation and creation operators defined as $\hat{b}_j = \frac{1}{2}(\frac{\hat{x}_j}{x_0} + i\frac{\hat{p}_j}{p_0})$, $\hat{b}_j^\dagger = \frac{1}{2}(\frac{\hat{x}_j}{x_0} - i\frac{\hat{p}_j}{p_0})$, where $p_0 = \sqrt{\hbar M \Omega/2}$ and $[\hat{b}_j, \hat{b}_{j'}^\dagger] = \delta_{j,j'}$, the Hamiltonian in Eq. (5) becomes (in the interaction picture with respect to $\hat{H}_{\Lambda,j}^{(0)}$)

$$\frac{\hat{H}_j}{\Omega \hbar} = \hat{b}_j^\dagger \hat{b}_j + (k_j \hat{\Lambda}_j + \mathcal{E}_j)(\hat{b}_j^\dagger + \hat{b}_j), \quad (7)$$

where we have defined the scaled quantities $k_j \equiv \frac{\tilde{k}_j}{\Omega}$ and $\mathcal{E}_j \equiv \frac{\tilde{\mathcal{E}}_j}{\Omega}$. Note that each node j is described by the composite Hilbert space $\mathcal{H}_j = \mathcal{H}_{\hat{\Lambda}_j} \otimes \mathcal{H}_{\hat{b}_j}$, corresponding to the $\hat{\Lambda}_j$ quantum system and its associated harmonic oscillator mode. Moreover, the total Hamiltonian of the network (all nodes) is $\hat{H} = \sum_{j=1}^N \hat{H}_j$. The corresponding unitary time-evolution operator for the j th node can then be expressed in closed form as [60,71,72]

$$\hat{u}_j(\tau) = e^{i(k_j \hat{\Lambda}_j + \mathcal{E}_j)^2 (\tau - \sin \tau)} e^{-(k_j \hat{\Lambda}_j + \mathcal{E}_j)(\eta(\tau) \hat{b}_j^\dagger - \eta^*(\tau) \hat{b}_j)} e^{-i \hat{b}_j^\dagger \hat{b}_j \tau}. \quad (8)$$

In the above expression, $\eta(\tau) \equiv 1 - e^{-i\tau}$, and $\tau \equiv \Omega t$ is the dimensionless time. Since $[\hat{H}_j, \hat{H}_{j'}] = 0$ for all $1 \leq (j, j') \leq N$, the total unitary operator for the network is given by the product

$$\hat{U}(\tau) = \prod_{j=1}^N \hat{u}_j(\tau). \quad (9)$$

In this work, we focus on a specific class of initial states that enable quantum-enhanced sensitivity across the entire network [73]. Specifically, we consider states of the form

$$|\psi(0)\rangle = |W_N\rangle (|\alpha\rangle^{\otimes N}), \quad (10)$$

where $|\alpha\rangle^{\otimes N}$ is a collection of N coherent states with amplitudes α and $|W_N\rangle = \frac{1}{\sqrt{N}} \sum_{j=1}^N |j\rangle$ is the W state [74,75]. Each basis state $|j\rangle$ is defined as

$$|j\rangle = |\lambda'\rangle_1 |\lambda'\rangle_2 \cdots |\lambda\rangle_j \cdots |\lambda'\rangle_{N-1} |\lambda'\rangle_N, \quad (11)$$

representing the configuration in which the eigenvalue λ appears at site j , while all other sites are in the eigenstate $|\lambda'\rangle$. Recall that λ and λ' in the W state must satisfy Eq. (6).

With the above definitions, it is straightforward to obtain the full wavefunction of the entire network at arbitrary time τ , which takes the form

$$|\psi(\tau)\rangle = \frac{1}{\sqrt{N}} \left(|1\rangle |\alpha_1(\tau)\rangle + \sum_{j=2}^N e^{\xi_j(\tau)} |j\rangle |\alpha_j(\tau)\rangle \right), \quad (12)$$

where $\xi_j(\tau) \in \mathbb{C}$ is

$$\begin{aligned} \xi_j(\tau) = & \frac{1}{2} e^{-i\tau} (\lambda - \lambda') [\alpha (e^{2i\tau} - 1) (k_1 - k_j) \\ & + 2[2\mathcal{E}_1 k_1 - 2\mathcal{E}_j k_j \\ & + (k_1 - k_j)(k_1 + k_j)(\lambda + \lambda')] \\ & \times (\tau - \sin \tau)(\sin \tau - i \cos \tau)], \end{aligned} \quad (13)$$

and $|\alpha_j(\tau)\rangle = |\alpha e^{-i\tau} - (k_1 \lambda' + \mathcal{E}_1) \eta(\tau)\rangle \cdots |\alpha e^{-i\tau} - (k_j \lambda + \mathcal{E}_j) \eta(\tau)\rangle \cdots |\alpha e^{-i\tau} - (k_N \lambda' + \mathcal{E}_N) \eta(\tau)\rangle$ denotes the j th position of the eigenvalue λ within a set of N coherent states. See the detailed derivation in Appendix A.

IV. QUANTUM SENSING RESOURCES

The central sensing resource in our distributed quantum sensing scheme is characterized by the pair of eigenvalues (λ_j, λ'_j) of the j th local operator $\hat{\Lambda}_j$, and in particular—as will be computed explicitly later—by their difference. For simplicity, in this section, we assume identical sensing subspaces at all nodes, so that $\lambda_j = \lambda$ and $\lambda'_j = \lambda'$. We then define

$$\lambda - \lambda' \equiv N_{\text{exc}}, \quad (14)$$

thereby identifying N_{exc} as the relevant metrological resource. In the concrete physical realizations discussed later, N_{exc} corresponds either directly to the number of excitations or, in the spin case, to a quantity proportional to it through the chosen sensing subspace.

While the quantity $\lambda - \lambda' = N_{\text{exc}}$ quantifies the metrological resource in our work, it is also important to compare the energetic cost of the distributed entangled probe with that of the corresponding separable benchmark built from the locally optimal single-node probes. To do so, we define the relevant energetic benchmark from the bare preparation energy of the $\hat{\Lambda}_j$ probes within the two-level sensing subspace selected by the protocol. We thus restrict attention to the local sensing subspace

$$\mathfrak{S}_j = \text{span}\{|\lambda'\rangle_j, |\lambda\rangle_j\}, \quad (15)$$

which is precisely the subspace used to encode the metrological resource. Let $\hat{H}_{\Lambda,j}^{(0)}$ denote the bare local Hamiltonian of the degree of freedom associated with $\hat{\Lambda}_j$, assumed identical at every sensing node. Let $E_{\lambda'}$ and E_λ be the corresponding bare energies of $\hat{H}_{\Lambda,j}^{(0)}$, and define the local energy gap

$$\Delta_\Lambda \equiv E_\lambda - E_{\lambda'} > 0. \quad (16)$$

Since an overall energy offset is irrelevant for the comparison, we define the shifted bare Hamiltonian on $\mathfrak{S}_j$ as

$$\tilde{H}_{\Lambda,j}^{(0)} \equiv \hat{H}_{\Lambda,j}^{(0)} - E_{\lambda'} \mathbb{I}_j. \quad (17)$$

Because $\hat{\Lambda}_j |\lambda\rangle_j = \lambda |\lambda\rangle_j$ and $\hat{\Lambda}_j |\lambda'\rangle_j = \lambda' |\lambda'\rangle_j$, the projector onto the upper sensing level can be written, within $\mathfrak{S}_j$, as

$$|\lambda\rangle_j \langle\lambda| = \frac{\hat{\Lambda}_j - \lambda' \mathbb{I}_j}{\lambda - \lambda'}. \quad (18)$$

Hence, the shifted bare Hamiltonian takes the form

$$\tilde{H}_{\Lambda,j}^{(0)} = \Delta_\Lambda \frac{\hat{\Lambda}_j - \lambda' \mathbb{I}_j}{\lambda - \lambda'} = \Delta_\Lambda |\lambda\rangle_j \langle\lambda|, \quad \text{on } \mathfrak{S}_j. \quad (19)$$

This identifies the corresponding local excitation operator

$$\hat{N}_j^{(\Lambda)} \equiv \frac{\hat{\Lambda}_j - \lambda' \mathbb{I}_j}{\lambda - \lambda'}, \quad (20)$$

which has eigenvalue 0 on $|\lambda'\rangle_j$ and eigenvalue 1 on $|\lambda\rangle_j$. The total shifted bare preparation energy of the sensing probe is then

$$\tilde{H}_\Lambda^{(0)} = \sum_{j=1}^N \tilde{H}_{\Lambda,j}^{(0)} = \Delta_\Lambda \sum_{j=1}^N \hat{N}_j^{(\Lambda)}. \quad (21)$$

We now compare Eq. (21) for the two probe strategies. For the distributed entangled protocol, the sensing degree of freedom is prepared in the W -type state. Since this state contains exactly one occupation of the upper sensing level delocalized across the entire network, it follows immediately that

$$\langle \tilde{H}_\Lambda^{(0)} \rangle_{\text{ent}} = \Delta_\Lambda. \quad (22)$$

Importantly, this result is independent of the network size N , namely, the sensing excitation is not replicated across the network, but coherently delocalized over all nodes.

The corresponding separable benchmark is the product state built from the locally optimal single-node probes (justified later in this work),

$$|\psi(0)\rangle_{\text{sep}} = \bigotimes_{j=1}^N \frac{|\lambda\rangle_j + |\lambda'\rangle_j}{\sqrt{2}}. \quad (23)$$

In this case, each node has upper-level population $1/2$, and therefore

$$\langle \tilde{H}_\Lambda^{(0)} \rangle_{\text{sep}} = \frac{N}{2} \Delta_\Lambda. \quad (24)$$

Subtracting Eq. (22) from Eq. (24), one obtains

$$\langle \tilde{H}_\Lambda^{(0)} \rangle_{\text{sep}} - \langle \tilde{H}_\Lambda^{(0)} \rangle_{\text{ent}} = \Delta_\Lambda \left(\frac{N}{2} - 1 \right). \quad (25)$$

This quantity is strictly positive for all $N \geq 3$, namely,

$$\langle \tilde{H}_\Lambda^{(0)} \rangle_{\text{sep}} > \langle \tilde{H}_\Lambda^{(0)} \rangle_{\text{ent}}, \quad N \geq 3. \quad (26)$$

Hence, under identical local probe Hamiltonians and for the same sensing subspace $\mathcal{S}_j$, the distributed entangled probe is strictly more energy efficient than the corresponding separable benchmark. This is because the W -type state carries a single delocalized occupation of the upper sensing level across the whole network, whereas the separable probe carries $N/2$ such occupations on average. We illustrate this energy advantage explicitly for the two physical realizations considered in this work in Secs. VII and VIII.

V. DYNAMICS OF QUANTUM ENTANGLEMENT

To further characterize the quantum dynamics described by Eq. (12), we evaluate the bipartite entanglement between the set of general systems $\{\hat{\Lambda}_j\}$ and the collection of mechanical oscillators. We quantify this entanglement using the linear entropy S_L . Since the total global state remains pure under unitary evolution, the linear entropy of the reduced state,

$$S_L = 1 - \text{Tr}(\rho^2), \quad \text{where} \quad \rho = \text{Tr}_{\text{fields}}[|\psi(\tau)\rangle\langle\psi(\tau)|], \quad (27)$$

provides a valid bipartite entanglement monotone [71,76]. In particular, it is a symmetric concave function of the Schmidt coefficients, vanishes for product states, and is nonincreasing on average under local operations and classical communication for pure states [77]. Moreover, the linear entropy is considerably simpler to evaluate than the von Neumann entropy, as it does not require diagonalization of the reduced density matrix. Its explicit calculation yields (see Appendix B for the derivation)

$$S_L = \frac{2}{N^2} \left[\frac{N(N-1)}{2} - \sum_{1 \leq i < j \leq N} e^{2(k_i^2 + k_j^2)(\lambda - \lambda')^2 [\cos \tau - 1]} \right]. \quad (28)$$

Three key insights follow from the above results. First, the linear entropy S_L is independent of both the initial mechanical coherent amplitude α and the mechanical driving amplitude $\mathcal{E}$. Second, at $\tau = \pi$, the system reaches a point of maximal correlation between its components. As a result, information is generally not confined to the $\hat{\Lambda}$ subsystem alone; therefore, measurements restricted to $\hat{\Lambda}$ are generally not sufficient to access the full information available in the total state. Third, at $\tau = 2\pi$, these correlations vanish, and all parameter dependence is confined to the $\hat{\Lambda}_j$ sector, so measurements on the mechanical subsystem are unnecessary. The saturating measurement can then be implemented as a collective measurement on the $\hat{\Lambda}_j$ subsystem. Indeed, at the evolution time $\tau = 2\pi$ (and at integer multiples thereof), the unitary time-evolution operator reduces to

$$\hat{U}_j(\tau = 2\pi) = e^{2\pi i(k_j \hat{\Lambda}_j + \mathcal{E}_j)^2} e^{-2\pi i \hat{b}_j^\dagger \hat{b}_j}, \quad (29)$$

from which the full wavefunction of the entire network at $\tau = 2\pi$ takes the form

$$|\psi(\tau = 2\pi)\rangle = \frac{1}{\sqrt{N}} \left(|1\rangle + \sum_{j=2}^N e^{i\beta_j \Phi_j} |j\rangle \right) (|\alpha\rangle^{\otimes N}). \quad (30)$$

In the above, β_j denotes a multiplicative prefactor, while Φ_j represents the relative phase encoding the unknown parameters of interest. Note that the phases Φ_j are defined relative to a reference node $j = 1$, such that they correspond to two-point phase differences, namely, $\Phi_j = \mathcal{E}_1 - \mathcal{E}_j$ or equivalently $\Phi_j = k_1 - k_j$. These relative phases are illustrated by the solid blue lines in Fig. 1. To provide β_j and Φ_j explicit expressions, we distinguish between two distinct scenarios.

In Case 1, all mechanical driving amplitudes are unknown and different, i.e., $\mathcal{E}_j \neq \mathcal{E}_{j'} \neq 0$, while all coupling strengths are equal, $k_j = k$. In this situation, the relative phases in Eq. (30) simplify to $\beta_j = -4\pi k(\lambda - \lambda')$, which is the same for all nodes (i.e., node independent). The corresponding phase factors are $\Phi_j = \mathcal{E}_1 - \mathcal{E}_j$. Thus, only the differences between $\mathcal{E}_1$ and each $\mathcal{E}_j$ appear explicitly. In Case 1, the estimation task is precisely to determine these relative unknown driving amplitudes $\mathcal{E}_j$.

In Case 2, all mechanical drivings are set to zero, $\mathcal{E}_j = 0$, while the coupling strengths are different and nonzero, $k_j \neq k_{j'} \neq 0$. In this case, the relative phases in Eq. (30) reduce to $\beta_j = -2\pi(k_1 + k_j)(\lambda - \lambda')(\lambda + \lambda')$. The corresponding phase factors are $\Phi_j = k_j - k_1$. Therefore, only the

TABLE I. Distributed quantum sensing cases. Case 1: The estimation task is to determine $\Phi_j = \mathcal{E}_j^- = \mathcal{E}_1 - \mathcal{E}_j$, under the conditions that all coupling strengths are identical, $k_j = k$ for all $1 \leq j \leq N$, and all mechanical drivings are different and nonzero, $\mathcal{E}_j \neq \mathcal{E}_{j'} \neq 0$ for all $1 \leq (j, j') \leq N$. Case 2: The estimation task is to determine $\Phi_j = k_j^- = k_1 - k_j$, under the conditions that all mechanical drivings vanish, $\mathcal{E}_j = 0$ for all $1 \leq j \leq N$, while the coupling strengths are different and nonzero, $k_j \neq k_{j'} \neq 0$ for all $1 \leq (j, j') \leq N$. We define $k_j^+ = k_1 + k_j$.

Case	Φ_j	β_j	Conditions
1	$\mathcal{E}_j^- = \mathcal{E}_1 - \mathcal{E}_j$	$-4\pi k(\lambda - \lambda')$	$k_j = k$ and $\mathcal{E}_j \neq \mathcal{E}_{j'} \neq 0$.
2	$k_j^- = k_1 - k_j$	$-2\pi k_j^+(\lambda - \lambda')(\lambda + \lambda')$	$k_j \neq k_{j'} \neq 0$ and $\mathcal{E}_j = 0$.

differences between k_1 and each k_j appear explicitly. The estimation task in this Case 2 is to determine these relative unknown coupling strengths k_j^- . A concise overview of these two scenarios is provided in Table I.

As a final observation, note that all mechanical oscillators return to their initial states $|\alpha\rangle$ at multiples of $\tau = 2\pi$ [see Eq. (30)], while the subsystems associated with the operators $\{\hat{\Lambda}_j\}$ acquire distinct relative phases at each site j . This behavior suggests that the unknown relative parameters can be estimated solely by probing the subsystem $\hat{\Lambda}_j$, namely, measurements on $\{\hat{\Lambda}_j\}$. Indeed, the mechanical oscillators dynamically transfer the full information content to the general systems $\{\hat{\Lambda}_j\}$ at these specific times. At other times, e.g., at multiples of $\tau = \pi$, the system is maximally correlated, and hence a joint measurement across all modes and parties is likely required to extract the ultimate sensing capability of the network. Note that this disentangling behavior also holds when the mechanical oscillator is initially in a thermal state, thus avoiding initial ground-state preparation of the mechanical object [78]. Indeed, a thermal state can be written in the Glauber-Sudarshan P representation as $\rho_{\text{th}} = \int d^2\alpha P_{\text{th}}(\alpha) |\alpha\rangle\langle\alpha|$ [79]. When this thermal state evolves under the action of $\hat{U}(\tau)$ of Eq. (9), the result is, in general, a conditionally displaced state of the form $\rho_{\text{th}} \rightarrow \int d^2\alpha P_{\text{th}}(\alpha) |\alpha_j(\tau)\rangle\langle\alpha_{j'}(\tau)|$, with displacements as in Eq. (12). These displacements vanish at $\tau = 2\pi$ since $\eta(\tau = 2\pi) = 0$. As a result, the mechanical party returns to its initial state ρ_{th} at each full oscillation period. This mechanical disentanglement, followed by the transfer of information to all systems $\hat{\Lambda}_j$, constitutes the key dynamical feature that enables quantum-enhanced sensing across all nodes.

As a final remark, our stroboscopic protocol shares conceptual similarities with metrological Loschmidt-echo schemes [65–67], but it does not constitute a strict Loschmidt echo, as it does not involve an explicit time-reversal operation of the sensing Hamiltonian. Instead, at integer multiples of 2π , the bosonic mode undergoes intrinsic periodic reinitialization under the sensing dynamics, stroboscopically disentangling from the $\hat{\Lambda}_j$ degrees of freedom and leaving only parameter-dependent collective relative phases imprinted on the $\hat{\Lambda}_j$ probes.

VI. DISTRIBUTED QUANTUM-ENHANCED ESTIMATION

With the quantum state of Eq. (30) at hand, we now address the task of simultaneous multiparameter estimation across a network of size N . To this end, using Eq. (4), we compute the trace of the inverse of the QFIM as (see details in

Appendix C):

$$\text{Tr}[Q^{-1}] = \frac{N}{2} \sum_{j=2}^N \frac{1}{\beta_j^2}. \quad (31)$$

The above expression demonstrates the achievement of quantum-enhanced sensitivity, as $\text{Tr}[Q^{-1}]$ decreases quadratically with β_j , which itself scales linearly with the initial excitation amplitude $\beta_j \propto (\lambda - \lambda')$ in Case 1. Remarkably, in Case 2, the dependence becomes $\beta_j \propto (\lambda - \lambda')(\lambda + \lambda')$, leading to a quartic scaling of sensitivity with the initial excitations. To the best of our knowledge, this scaling has not been previously reported in the context of distributed quantum sensing schemes. Note that quartic scaling is a direct consequence of the seminorm of the generator associated with nonlinear Hamiltonian encodings, as in our present case [19]. This indicates that, in Case 2, the ultimate estimation precision improves at an even faster rate with increasing resources. The inverse-square and quartic scalings of the sensing uncertainty with the number of initial excitations (the sensing resource) provide a clear signature of quantum-enhanced sensing [7–9]. This is particularly evident in Case 1 of Table I, where

$$\text{Tr}[Q^{-1}] = \frac{N(N-1)}{32\pi^2 k^2 (\lambda - \lambda')^2}, \quad \text{Case 1.} \quad (32)$$

Note that the lower bound is independent of the estimated parameters $\mathcal{E}_j^-$. Consequently, by imposing the condition $\text{Tr}[Q^{-1}] < 1$, one obtains the following requirement for the considered sensing resources $(\lambda - \lambda') > \sqrt{\frac{N(N-1)}{32\pi^2 k^2}}$.

Remarkably, in Case 2 of Table I, one finds $\beta_j \propto (\lambda - \lambda')(\lambda + \lambda')$. Hence, in principle, a quartic scaling of $\text{Tr}[Q^{-1}]$ of Eq. (31) can be achieved when $\lambda' = 0$, that is,

$$\text{Tr}[Q^{-1}] = \frac{N}{8\pi^2 \lambda^4} \sum_{j=2}^N (k_j^+)^{-2}, \quad \text{Case 2.} \quad (33)$$

In this case, the lower bound depends on the combinations $k_j^+ = k_1 + k_j$, which are generally unknown. Accordingly, Eq. (33) should be understood as a conditional scalar precision bound, valid for fixed values of k_j^+ . In the present work, we adopt the conditional-precision viewpoint and assume that k_j^+ are either known from prior calibration or obtained from a preliminary coarse-estimation stage.

While we have already demonstrated that our distributed scheme is energetically more efficient than an optimal separable strategy, it is useful to further clarify the comparison between our distributed entangled protocol and an optimal separable (local) sensing approach. Since the parameters k_j and $\mathcal{E}_j$ are encoded in independent local Hamiltonians, one

can indeed construct a separable strategy in which each node is probed independently and the parameters are estimated locally. For independent local estimators arising from separable probing, one has $\text{Cov}(k_1, k_j) = \text{Cov}(\mathcal{E}_1, \mathcal{E}_j) = 0$. Hence,

$$\begin{aligned}\text{Var}(\mathcal{E}_1 - \mathcal{E}_j) &= \text{Var}(\mathcal{E}_1) + \text{Var}(\mathcal{E}_j), \\ \text{Var}(k_1 - k_j) &= \text{Var}(k_1) + \text{Var}(k_j).\end{aligned}$$

For a single site prepared in the product state

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|\lambda\rangle_j + |\lambda'\rangle_j) \otimes |\alpha\rangle_j$$

and evolved for $\tau = 2\pi$, the single-parameter QFIs are

$$Q^{\text{single}}(k_j) = 16\pi^2 k_j^2 (\lambda^2 - \lambda'^2) \quad (34)$$

for estimating k_j with $\mathcal{E}_j = 0$, and

$$Q^{\text{single}}(\mathcal{E}_j) = 16\pi^2 k^2 (\lambda - \lambda')^2 \quad (35)$$

for estimating $\mathcal{E}_j$ with $k_j = k \neq 0$. Note that the optimal local probe $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|\lambda\rangle_j + |\lambda'\rangle_j) \otimes |\alpha\rangle_j$ can straightforwardly be constructed from the propagator $u_j(\tau = 2\pi) = e^{2\pi i(k_j \hat{\Lambda}_j + \mathcal{E}_j)^2}$ for a single site [see Eq. (29)]. By defining $G_j \equiv (k_j \hat{\Lambda}_j + \mathcal{E}_j)^2$, the stroboscopic evolution at time $\tau = 2\pi$ is $u_j(\tau = 2\pi) = e^{2\pi i G_j}$. For each parameter $\chi \in \{k_j, \mathcal{E}_j\}$, the corresponding local generator is $K_\chi = i[\partial_\chi u_j(\tau = 2\pi)]u_j^\dagger(\tau = 2\pi)$, which is a function of $\hat{\Lambda}_j$ and hence diagonal in the $\{|\lambda\rangle_j, |\lambda'\rangle_j\}$ basis. Restricted to this two-level subspace, the equal superposition state $(|\lambda\rangle_j + |\lambda'\rangle_j)/\sqrt{2}$ maximizes the variance $\Delta^2 K_\chi$ and thus the QFI $\sim \Delta^2 K_\chi$. Consequently, for both k_j and $\mathcal{E}_j$, our probe saturates the generator seminorm bound and achieves the ultimate single-parameter quantum-limited sensitivity allowed by the underlying unitary dynamics [19]. Equations (34) and (35) coincide with the scaling obtained in our distributed protocol for the corresponding parameter differences.

Having established this single-parameter optimality, it is important to emphasize that, under a comparison in which each node is allocated an independent optimal probe with identical interrogation time and local resources, there is no enhancement in the fundamental quantum Fisher scaling due to entanglement itself. This is consistent with the fact that the Hamiltonians are local and mutually commuting [37]. For explicit demonstrations of the advantages of simultaneous multiparameter estimation of global field parameters in three-dimensional magnetometry, see Refs. [22,80,81]. Specifically, in Ref. [80], a suitably entangled probe for a uniform magnetic field yields up to a threefold reduction in the total estimation variance compared with optimal individual-estimation strategies using the same number of particles. In contrast, here we focus on estimating site-dependent relative couplings $(k_1 - k_j)$ [or $(\mathcal{E}_1 - \mathcal{E}_j)$], where each parameter acts only locally on a single node rather than globally on all probes. Note that the advantage of the present distributed entangled protocol is therefore not a scaling improvement, but rather a resource-architectural one. By delocalizing the excitation across the network in a W -type state, the physically relevant quantities—namely, the complete set of pairwise $N - 1$ reference-relative phase differences—are encoded directly as collective relative phases within a single global probe. This eliminates

the need to independently estimate each local parameter and subsequently combine them via classical postprocessing, and allows the estimation task to be performed through a single joint measurement. Consequently, the protocol preserves the same generator-based scaling with the sensing resource as the optimal local separable strategy, while encoding the complete set of $N - 1$ reference-relative phase differences into a single global probe state. The advantage of the present distributed protocol is not a universal reduction of the total estimation variance, but rather its energetic efficiency and resource-architectural character; namely, a single delocalized probe encodes the full set of reference-relative differences and can be read out collectively.

In the following, we propose and discuss the fundamental multiparameter precision limits of two physical implementations for realizing distributed quantum-enhanced sensing.

VII. DISTRIBUTED QUANTUM GRAVIMETRY: CASE 1 PROPOSAL

To illustrate the fundamental distributed precision limits relevant to Case 1 (see Table I), we consider a network of N nonlinear quantum optomechanical systems designed to estimate relative differences in gravitational acceleration encoded at spatially separated sites. This information is captured by the relative phases $\Phi_j = \mathcal{E}_1 - \mathcal{E}_j$, where $\mathcal{E}_j$ will be shown to explicitly depend on the local gravitational acceleration at site j . The aim, therefore, is to estimate these differences across the whole network at arbitrarily distances, thereby enabling distributed quantum-enhanced gravimetry. It is worth emphasizing that the estimation of a single gravitational acceleration parameter using nonlinear optomechanical interactions—known as gravimetry—has been recently explored [60,82]. Optomechanical systems, which arise from the radiation-pressure interaction between a quantized electromagnetic field (hereafter referred to as the light field) and a mechanical oscillator, have been extensively studied both theoretically and experimentally [59], with applications ranging from quantum information tasks [83,84] to high-precision sensing [60].

Following our previous definitions, we describe a quantum optomechanical gravimetry network with N nodes via the Hamiltonian at site j as

$$\hat{H}_j = \frac{\hat{p}_j^2}{2M} + \frac{M\Omega^2}{2}\hat{x}_j^2 + \hbar \frac{\tilde{k}}{x_0} \hat{a}_j^\dagger \hat{a}_j \hat{x}_j + \underbrace{Mg_j \cos(v_j)\hat{x}_j}_{\Delta U_j^g}, \quad (36)$$

where $\hat{a}_j$ ($\hat{a}_j^\dagger$) denotes the annihilation (creation) operator of the light field, ΔU_j^g is the gravitational potential energy, v_j is the rotation angle with respect to the free-fall direction (without loss of generality we take $v_j = 0$, i.e., a vertically oriented gravimeter), and g_j is the unknown gravitational acceleration. By comparing Eq. (36) with Eq. (5), one directly identifies

$$\hat{\Lambda}_j = \hat{a}_j^\dagger \hat{a}_j, \quad (37)$$

which satisfies all the conditions specified in Eq. (6). We consider the initial state

$$|\psi(0)\rangle = |W_N\rangle(|\alpha\rangle^{\otimes N}), \quad (38)$$

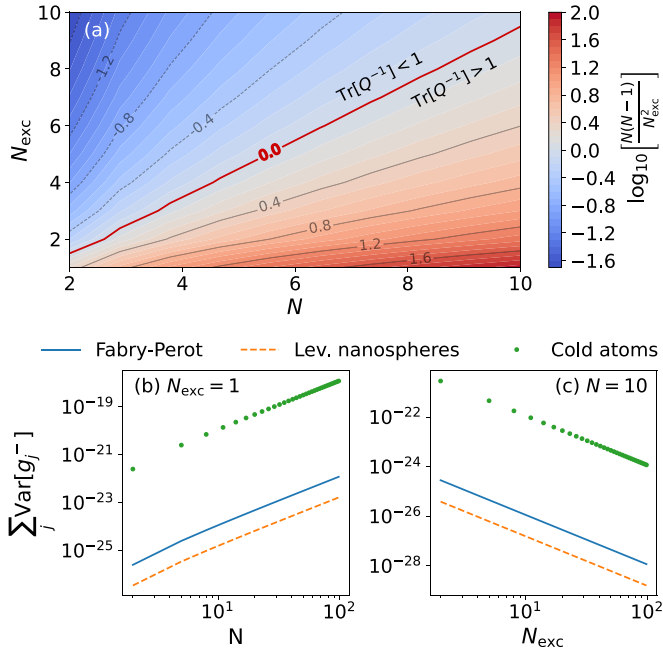


FIG. 2. Distributed quantum-enhanced sensing for Case 1. (a) $\frac{N(N-1)}{N_{\text{exc}}^2}$ as functions of network size N and number of initial excitations N_{exc} . (b) Sensitivity as a function of network size N for $N_{\text{exc}} = 1$. (c) Sensitivity as a function of the number of excitations N_{exc} for a fixed network size $N = 10$. We consider $\mu = 10^4$.

where $|W_N\rangle = \frac{1}{\sqrt{N}}(|N_{\text{exc}}, 0, \dots, 0\rangle + \dots + |0, \dots, 0, N_{\text{exc}}\rangle)$ is the N -partite W state with N_{exc} excitations. In this case, it is straightforward to identify $\lambda = N_{\text{exc}}$ and $\lambda' = 0$. Since $\mathcal{E}_j = \frac{x_0 M g_j}{\Omega \hbar}$, then a simple use of chain rule yields the following QFIM in the International System of Units:

$$\text{Tr}[Q^{-1}] = \frac{N(N-1)}{32\pi^2 k^2 N_{\text{exc}}^2} \left(\frac{\Omega \hbar}{x_0 M} \right)^2. \quad (39)$$

Note that $k = \tilde{k}/\Omega$ is dimensionless, and $[\frac{\Omega \hbar}{x_0 M}] = \frac{m}{s^2}$. Hence, the units of $\text{Tr}[Q^{-1}]$ are consistent with the covariance of the estimator. The final estimation results are

$$\text{Tr}[\text{Cov}[\mathbf{g}^{-}]] = \sum_{j=2}^N \text{Var}[g_j^{-}] \geq \mu^{-1} \frac{N(N-1)}{32\pi^2 k^2 N_{\text{exc}}^2} \left(\frac{\Omega \hbar}{x_0 M} \right)^2. \quad (40)$$

Note that in Eq. (40), the parameters satisfy $N \geq 2$ and $N_{\text{exc}} \geq 1$, while Ω , M , and k are device dependent. For comparison, we consider three optomechanical architectures reported in Ref. [60]: (1) a Fabry-Perot configuration with $\Omega = 10^3$ Hz, $M = 10^{-6}$ kg, and $k = 2.3$ [71,85]; (2) a levitated micro-object confined in an ion trap coupled to an optical cavity with $\Omega = 10^2$ Hz, $M = 10^{-14}$ kg, and $k = 1963$ [86,87]; and (3) cold atoms in a cavity with $\Omega = 10^2$ Hz, $M = 10^{-25}$ kg, and $k = 2.3 \times 10^6$ [88].

To further illustrate the distributed quantum-enhanced sensitivity in Case 1, in Fig. 2(a) we plot the quantity $\frac{N(N-1)}{N_{\text{exc}}^2}$ from Eq. (39) as functions of the network size N and the number of excitations N_{exc} . As the figure illustrates, the inverse-square dependence on N_{exc} competes directly with the N growth of

the network. Consequently, achieving lower sensing uncertainties imposes choosing N_{exc} at least on the order of N , that is, $N_{\text{exc}} \sim N$. Note that, for a given network size N , one indeed achieves the quadratic scaling of the QFIM with respect to our sensing resource, that is, the initial probe excitations N_{exc} .

In Fig. 2(b), we show the distributed sensitivity as a function of N for $N_{\text{exc}} = 1$. As seen in the figure, the sensitivity across the network increases quadratically with its size N for a fixed N_{exc} , with the levitated optomechanical system exhibiting the lowest overall uncertainties. This is readily evident from Eq. (40). Moreover, the sensitivity (sum of variances) ranges from $\sim 10^{-27} \text{ m}^2/\text{s}^4$ for the levitated microsphere system to $\sim 10^{-18} \text{ m}^2/\text{s}^4$ for the cold-atom platform. For comparison with the single-parameter case, we define the root-mean-square total estimation error as $\Delta_{\text{rms}} \equiv \sqrt{\text{Tr}[\text{Cov}[\hat{\Theta}]]}$, which results in $10^{-15} \text{ m/s}^2 \leq \Delta_{\text{rms}}^{\text{Case 1}} \lesssim 10^{-13} \text{ m/s}^2$ for the levitated optomechanical system, comparable to the minimal theoretical prediction for the standard deviation of $\Delta g \sim 10^{-15} \text{ m/s}^2$ [60].

In Fig. 2(c), we show the distributed quantum-enhanced sensitivity as a function of the number of excitations N_{exc} for a fixed network size $N = 10$. The figure illustrates that the hierarchy of optomechanical systems remains the same, with the overall sensitivity decreasing quadratically with N_{exc} .

In terms of energy cost, the relevant probe degree of freedom in this realization is the cavity/light mode, for which

$$\hat{L}_j = \hat{a}_j^\dagger \hat{a}_j. \quad (41)$$

The natural bare local probe Hamiltonian is therefore the free photonic Hamiltonian

$$\hat{H}_{a,j}^{(0)} = \hbar \omega_c \hat{a}_j^\dagger \hat{a}_j, \quad (42)$$

where ω_c is the cavity mode frequency. In this case, the sensing subspace is

$$\mathfrak{S}_j^{(a)} = \text{span}\{|\lambda'\rangle_j, |\lambda\rangle_j\}, \quad \lambda' = 0, \quad \lambda = N_{\text{exc}}, \quad (43)$$

so that the bare energy gap between the two sensing levels is

$$\Delta_a = \hbar \omega_c (\lambda - \lambda') = \hbar \omega_c N_{\text{exc}}. \quad (44)$$

Accordingly, the shifted bare Hamiltonian introduced above reduces, on $\mathfrak{S}_j^{(a)}$, to

$$\tilde{H}_{a,j}^{(0)} = \Delta_a |N_{\text{exc}}\rangle_j \langle N_{\text{exc}}|. \quad (45)$$

Hence, the distributed W -type probe carries exactly one occupation of the upper sensing level, delocalized across the entire network, and therefore,

$$\langle \tilde{H}_a^{(0)} \rangle_{\text{ent}} = \hbar \omega_c N_{\text{exc}}. \quad (46)$$

In contrast, the corresponding separable benchmark built from the locally optimal single-node probes has upper-level population $1/2$ at each node, which gives

$$\langle \tilde{H}_a^{(0)} \rangle_{\text{sep}} = \frac{N}{2} \hbar \omega_c N_{\text{exc}}. \quad (47)$$

Thus,

$$\langle \tilde{H}_a^{(0)} \rangle_{\text{sep}} - \langle \tilde{H}_a^{(0)} \rangle_{\text{ent}} = \hbar \omega_c N_{\text{exc}} \left(\frac{N}{2} - 1 \right), \quad (48)$$

which is strictly positive for all $N \geq 3$. Therefore, in the photon-number realization, the distributed entangled probe is strictly more energy efficient than the corresponding separable benchmark: It carries the energy of a single delocalized N_{exc} -photon excitation, whereas the separable probe carries $N/2$ such excitations on average. Importantly, this comparison is independent of the initial coherent amplitude of the mechanical modes and follows solely from the delocalized encoding of the photonic sensing resource.

Saturability of the ultimate quantum precision bound

As discussed in Sec. II, the saturation of the quantum Cramér-Rao bound in multiparameter estimation is generally obstructed by the noncommutativity of the SLDs [24,26]. Hence, even if a multiparameter sensing scheme exhibits quantum-enhanced sensitivity, the absence of a measurement capable of saturating the bound renders the scheme fundamentally impractical. Remarkably, we show that our scheme saturates the quantum-enhanced precision bound when an appropriate projective measurement is applied. This demonstrates that the ultimate sensing limit can, in fact, be achieved through optimal measurements.

The key observation is that our pure state probe satisfies the weak commutativity condition for SLD compatibility, namely, $\langle \psi | [\mathcal{L}_k, \mathcal{L}_{k'}] | \psi \rangle = 0$ for all k, k' ; see Appendix D for a detailed proof. Under this condition, the SLD quantum Cramér-Rao bound coincides with the attainable Holevo bound and is locally saturable [21,89]. For pure states, this weak commutativity condition is both necessary and sufficient for local attainability of the SLD quantum Cramér-Rao bound [21,89]. Building on this, Pezzè *et al.* [27] derived necessary and sufficient conditions for projective measurements to saturate the QFIM in pure state models and provided an explicit Gram-Schmidt construction of the optimal projectors from the subspace spanned by the SLDs. We adopt that construction directly.

To illustrate the optimal measurement construction, consider the first nontrivial network $N = 3$ (the generalization to larger N is direct) with projectors $\{\Pi_0, \Pi_1, \Pi_2\}$ onto the orthonormal kets with controllable known phases $\{\vartheta_2, \vartheta_3\}$

$$\begin{aligned} |\Sigma\rangle &= \frac{1}{\sqrt{3}}(|1\rangle + e^{i\beta_2\vartheta_2}|2\rangle + e^{i\beta_3\vartheta_3}|3\rangle), \\ |u_1\rangle &= \frac{1}{\sqrt{2}}(-|1\rangle + e^{i\beta_2\vartheta_2}|2\rangle), \\ |u_2\rangle &= \frac{1}{\sqrt{6}}(-|1\rangle - e^{i\beta_2\vartheta_2}|2\rangle + 2e^{i\beta_3\vartheta_3}|3\rangle), \end{aligned} \quad (49)$$

with $\Pi_0 = |\Sigma\rangle\langle\Sigma|$, $\Pi_1 = |u_1\rangle\langle u_1|$, and $\Pi_2 = |u_2\rangle\langle u_2|$. For this measurement basis, one computes the CFIM $\mathcal{F}(\vartheta_2, \vartheta_3)$ as a function of the controllable reference phases $\{\vartheta_2, \vartheta_3\}$. When the references are tuned to the true values $\{\Phi_2, \Phi_3\}$ (e.g., via adaptive procedures that first coarsely estimate the phases and then measures in the locally optimal basis [27,90–92]), the CFIM equals the QFIM:

$$\lim_{\vartheta_2, \vartheta_3 \rightarrow \Phi_2, \Phi_3} \mathcal{F}(\vartheta_2, \vartheta_3) = \mathcal{Q}(\Phi_2, \Phi_3) = \frac{4}{9} \begin{pmatrix} 2\beta_2^2 & -\beta_2\beta_3 \\ -\beta_2\beta_3 & 2\beta_3^2 \end{pmatrix}. \quad (50)$$

This is precisely the saturability of the SLD-quantum Cramér-Rao bound. The same Gram-Schmidt construction of the optimal measurement basis can be generalized immediately to any number of network size N [27].

VIII. DISTRIBUTED COUPLING-STRENGTH ESTIMATION: CASE 2 PROPOSAL

We now consider the fundamental distributed precision limits for Case 2 (see Table I), where all coupling strengths are unknown and the mechanical oscillators are undriven. For this scenario, we propose two physical implementations: (1) the optomechanical architectures described above and (2) conditionally displaced spin-mechanical systems [61]. Conditional displacement Hamiltonians describe interactions in which the energy separation between the emitter's internal states (e.g., a spin system) is strongly influenced by local electrostatic or magnetic fields [61]. These local fields can be engineered using, for example, oscillating electrodes or vibrating magnetic tips attached to a mechanical oscillator. This implies, equivalently, that the displacement of the mechanical oscillator becomes conditional on the emitter's quantum state [61]. Such Hamiltonians have been extensively studied both theoretically [62,93–106] and experimentally [61,66]. The paradigmatic form is given by

$$\hat{H}_j = \frac{\hat{p}_j^2}{2M} + \frac{M\Omega^2}{2}\hat{x}_j^2 + \hbar \frac{\tilde{k}_j}{x_0} \hat{S}_z^j \hat{x}_j, \quad (51)$$

where $\tilde{k}_j$ is the unknown coupling strength at site j . $\hat{S}_\beta^j = \sum_{i=1}^{N_{\text{exc}}} \frac{\hat{\sigma}_{i,j}^\beta}{2}$, with $(\beta = x, y, z)$, denotes the collective spin operator at site j , composed of N_{exc} spin- $\frac{1}{2}$ particles indexed by i , where $\hat{\sigma}_{i,j}^\beta$ are the Pauli matrices along the β direction. Note that the coupling constant $\tilde{k}_j$ is system dependent and can encode multiple unknown physical properties of the system, including magnetic fields, magnetic field gradients, circulating currents, deformation potential constants, gate capacitances, and gate charges (see Ref. [61] for a detailed list of relevant coupling constants). Consequently, estimating these distributed coupling constants across the sensor network enables one to infer valuable information about the physical properties of disparate systems. Furthermore, for consistency with Case 1, we denote by N_{exc} the total number of individual spins coupled to the mechanical oscillator. The operators $\hat{S}_\beta^j$ satisfy the usual commutation relations

$$[\hat{S}_n^j, \hat{S}_m^{j'}] = i\varepsilon_{nm\ell} \delta_{j,j'} \hat{S}_\ell^j, \quad (52)$$

$$(\hat{S}^j)^2 |s, m\rangle = s(s+1) |s, m\rangle, \quad (53)$$

$$\hat{S}_z^j |s, m\rangle = m |s, m\rangle, \quad (54)$$

where $\varepsilon_{nm\ell}$ is the Levi-Civita symbol, $s \leq N_{\text{exc}}/2$ is the spin quantum number, and $-s \leq m \leq s$ labels the angular momentum projection along the z axis. We focus on the symmetric subspace $2s = N_{\text{exc}}$, with N_{exc} assumed even. Analogously to the previous case, we consider the initial state

$$|\psi(0)\rangle = |W_N\rangle(|\alpha\rangle^{\otimes N}), \quad (55)$$

where

$$|W_N\rangle = \frac{1}{\sqrt{N}} \left(\left| \pm \frac{N_{\text{exc}}}{2}, 0, \dots, 0 \right\rangle + \dots + \left| 0, \dots, 0, \pm \frac{N_{\text{exc}}}{2} \right\rangle \right), \quad (56)$$

for $N_{\text{exc}} \geq 2$, is the N -partite W state. In the above, $\pm$ accounts for chosen values $m = \pm s$. Note that for spin-1/2, that is, $N_{\text{exc}} = 1$, implies that $\lambda = \pm \frac{N_{\text{exc}}}{2}$ and $\lambda' = \mp \frac{N_{\text{exc}}}{2}$. Thus, $\beta_j \propto (\lambda - \lambda')(\lambda + \lambda') = 0$, resulting in singular Fisher matrix [see Eq. (31)]. For the case of $N_{\text{exc}} \geq 2$, the parameters are properly identified as $\lambda = \pm \frac{N_{\text{exc}}}{2}$ and $\lambda' = 0$. While other combinations are possible, maximal sensitivity is achieved when λ takes the value $m = \pm s = \pm \frac{N_{\text{exc}}}{2}$ and λ' takes the value $m = 0$. Using the chain rule as before, one gets

$$\sum_{j=2}^N \text{Var}[\tilde{k}_j^-] \geq \frac{1}{\mu} \frac{2\Omega^2 N}{\pi^2 N_{\text{exc}}^4} \sum_{j=2}^N (k_j^+)^{-2}. \quad (57)$$

As seen from the above expression, the additional terms effectively act as nuisance parameters in the sensing protocol, in the sense that they contribute unwanted dependencies that degrade the estimation precision [23]. To further simplify the above expression, we assume that the effective couplings satisfy $k_j^+ \sim 2k$, meaning they are all of the same order of magnitude. This places us in the scenario where no coupling dominates and sensitivity is evenly distributed across the parameters. Equivalently, one can adopt a stepwise estimation protocol. First performing a coarse calibration of the nuisance parameters k_j^+ , followed by a refined estimation of the target parameters k_j^- [107]. With this simplification, one gets

$$\sum_{j=2}^N \text{Var}[\tilde{k}_j^-] \geq \frac{1}{\mu} \frac{2\Omega^2 N}{\pi^2 N_{\text{exc}}^4} \sum_{j=2}^N (k_j^+)^{-2} \sim \frac{1}{\mu} \frac{\Omega^2 N(N-1)}{2\pi^2 k^2 N_{\text{exc}}^4}. \quad (58)$$

To illustrate the distributed quantum-enhanced sensitivity in Case 2, Fig. 3(a) shows the dependence of the sensing precision on both the network size N and the number of initial probe excitations N_{exc} as given by the right-hand side of Eq. (58). In contrast to Case 1, here, the inverse-quartic scaling N_{exc}^4 competes with the N^2 scaling of the network size. As a result, achieving lower sensing uncertainty requires only $N_{\text{exc}} \sim \sqrt{N}$, marking a favorable resource trade-off in distributed quantum-enhanced metrology.

In Fig. 3(b), we show the distributed sensitivity as a function of N for fixed $N_{\text{exc}} = 2$. In contrast to the previous case, the hierarchy of optomechanical sensitivities is inverted, with the cold-atom platform achieving the lowest distributed multiparameter uncertainty for estimating the relative interaction coupling strengths—here, $10^{-9} \text{ Hz} \lesssim \Delta_{\text{rms}}^{\text{Case 2}} \lesssim 10^{-7} \text{ Hz}$. As predicted by Eq. (58), the overall uncertainty increases quadratically with N for a given N_{exc} . Moreover, the Fabry-Perot configuration and the spin-mechanical system exhibit the worst scenarios with the highest overall uncertainties. For reference, the black dashed line in Fig. 3(b) indicates $\sum_j \text{Var}[\tilde{k}_j^-] = 1$. For the spin-mechanical system, we consider mechanical frequencies as low as $\Omega \sim 10^3 \text{ Hz}$, with higher frequencies further degrading the sensitivity.

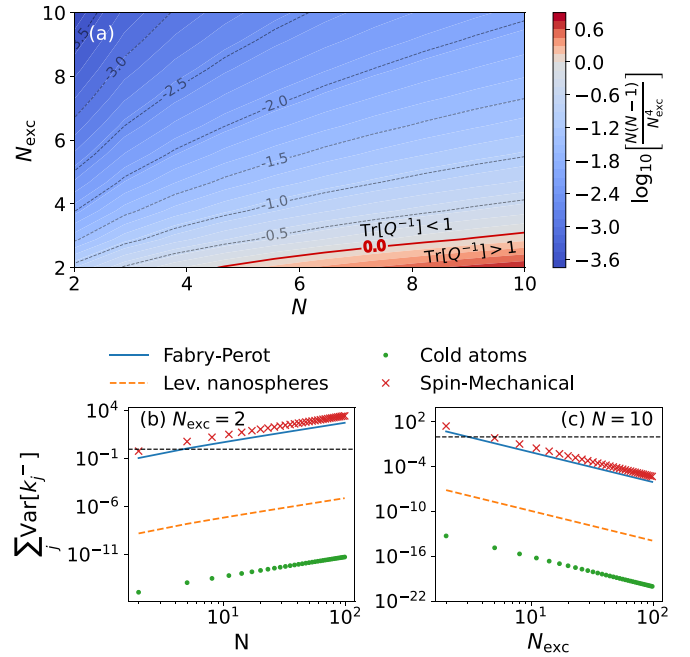


FIG. 3. Distributed quantum-enhanced sensing for Case 2. (a) Dependence of the network size N and the number of excitations N_{exc} in the right-hand side of Eq. (58). (b) Sensitivity as a function of network size N for a single excitation $N_{\text{exc}} = 2$. (c) Sensitivity as a function of the number of excitations N_{exc} for a fixed network size $N = 10$. We consider $\mu = 10^4$.

In Fig. 3(c), we show the distributed quantum-enhanced sensitivity as a function of the number of excitations N_{exc} for a fixed network size $N = 10$. As the figure shows, the cold-atom system achieves the lowest distributed uncertainty $10^{-12} \text{ Hz} \lesssim \Delta_{\text{rms}}^{\text{Case 2}} \lesssim 10^{-9} \text{ Hz}$. Furthermore, a higher number of excitations (or spin number) is required to reach lower sensitivities.

Similarly, in the collective-spin realization the relevant probe degree of freedom is the spin ensemble. In the symmetric subspace, one has $s = N_{\text{exc}}/2$, and we choose the sensing subspace

$$\mathfrak{S}_j^{(s)} = \text{span}\{|s, 0\rangle_j, |s, s\rangle_j\}, \quad (59)$$

which corresponds to

$$\lambda' = 0, \quad \lambda = s = \frac{N_{\text{exc}}}{2}. \quad (60)$$

A natural bare local probe Hamiltonian is then a Zeeman-type Hamiltonian with spin-splitting frequency ω_s , so that the bare energy gap between the two sensing levels is

$$\Delta_s = \hbar\omega_s s = \hbar\omega_s \frac{N_{\text{exc}}}{2}. \quad (61)$$

Accordingly, within $\mathfrak{S}_j^{(s)}$ the shifted bare Hamiltonian reduces to

$$\tilde{H}_{s,j}^{(0)} = \Delta_s |s, s\rangle_j \langle s, s|. \quad (62)$$

Hence, the distributed W -type probe carries exactly one occupation of the upper sensing level, delocalized across the

entire network, and therefore

$$\langle \tilde{H}_s^{(0)} \rangle_{\text{ent}} = \hbar \omega_s \frac{N_{\text{exc}}}{2}. \quad (63)$$

In contrast, the corresponding separable benchmark built from the locally optimal single-node probes has upper-level population $1/2$ at each node, which gives

$$\langle \tilde{H}_s^{(0)} \rangle_{\text{sep}} = \frac{N}{2} \hbar \omega_s \frac{N_{\text{exc}}}{2}. \quad (64)$$

Thus,

$$\langle \tilde{H}_s^{(0)} \rangle_{\text{sep}} - \langle \tilde{H}_s^{(0)} \rangle_{\text{ent}} = \hbar \omega_s \frac{N_{\text{exc}}}{2} \left(\frac{N}{2} - 1 \right), \quad (65)$$

which is strictly positive for all $N \geq 3$. Therefore, in the collective-spin realization, the distributed entangled probe is strictly more energy efficient than the corresponding separable benchmark; that is, it carries the energy of a single delocalized fully polarized Dicke excitation, whereas the separable probe carries $N/2$ such excitations on average.

IX. DISTRIBUTED QUANTUM-ENHANCED SINGLE-PARAMETER ESTIMATION

A particularly important case arises when only one relative parameter Φ_j remains unknown, with all others assumed known. This defines a distributed single-parameter scenario. Within our general multiparameter framework, the single-parameter precision limits are computed directly from the (j, j) entry of the QFIM, that is, $[Q]_{jj} \equiv Q^{\text{single}}(\Phi_j)$, corresponding to the precision limits of the unknown parameter difference between node 1 and node j ; see Appendix C for details:

$$Q^{\text{single}}(\Phi_j) = \begin{cases} \sim \frac{(N-1)}{N^2} k^2 N_{\text{exc}}^2, & \text{for Case 1,} \\ \sim \frac{(N-1)}{N^2} k_j^{+2} N_{\text{exc}}^4, & \text{for Case 2.} \end{cases} \quad (66)$$

This reveals a clear competition between the network size N and the number of initial excitations N_{exc} , analogous to the case where all Φ_j are unknown throughout the network.

The central importance of single-parameter quantum estimation rests on three key advantages. First, in the limit of large datasets, it is possible to construct unbiased and efficient estimators that asymptotically saturate the quantum Cramér-Rao bound [68]. Second, an optimal measurement that achieves the ultimate sensitivity always exists for single-parameter sensing scenarios [18]. Third, because such an optimal measurement is guaranteed, the quantum Cramér-Rao bound serves as a rigorous and operationally meaningful benchmark for experiments, against which any practical (nonoptimal) measurement strategy can be directly compared [2,18]. The optimal measurement basis is obtained from the eigenbasis of the SLD operator $\mathcal{L}(\theta)$, which is defined for pure states as

$$\mathcal{L}(\theta) = 2(|\partial_\theta \psi_\theta\rangle\langle\psi_\theta| + |\psi_\theta\rangle\langle\partial_\theta \psi_\theta|). \quad (67)$$

Using the expression for the SLD given in Eq. (67), we can explicitly construct the optimal measurement basis for the quantum probe state defined in Eq. (30) for estimating Φ_k as (see Appendix E for a detailed derivation)

$$\Pi_{\pm}^{(k)} = |v_k^{\pm}\rangle\langle v_k^{\pm}|, \quad (68)$$

where

$$|v_k^{\pm}\rangle = \frac{1}{\sqrt{2}} \left[\left(1 \mp \frac{i}{\sqrt{N-1}} \right) |\psi\rangle \pm i \sqrt{\frac{N}{N-1}} e^{i\beta_k \Phi_k} |k\rangle \right]. \quad (69)$$

Finally, another important case occurs when one aims to estimate a single-parameter Φ_j , while all the other parameters $\{\Phi_k\}_{k \neq j}$ are also unknown. In this nuisance scenario [23], the precision is bounded by $\text{Var}[\Phi_j] \geq \frac{1}{\mu} (Q^{-1})_{jj} = \frac{1}{\mu} \frac{N}{2\beta_j^2}$. As expected, the precision is reduced by a factor of $\frac{2(N-1)}{N}$ compared to the case shown in Eq. (66), where all other parameters are known.

X. DISCUSSION

Our proposal has both advantages and limitations.

On the advantages side: First, a key strength is that we provide a fully analytical derivation for estimating multiple unknown parameters distributed over N spatially separated nodes, applicable to a broad and physically relevant class of quantum systems. Second, the quantum-enhanced sensitivity in our protocol relies crucially on stroboscopic disentangling generated by the conditional-displacement interaction. At times $\Omega t = 2\pi$, all parameter dependence is guaranteed to be imprinted solely on the probe (e.g., light modes or collective spin systems), while the mechanical oscillator returns to—and decouples in—its initial state. As a result, all relevant information about the unknown parameters can be extracted through measurements on the probe only, using an optimally constructed measurement basis [27]. Furthermore, since the probe state is a W -type state, the measurement yields a discrete set of outcomes. This structure enables the use of efficient estimators, such as maximum-likelihood and Bayesian inference methods, which can be practically employed to saturate the precision bound [108,109]. Third, the protocol exhibits quadratic and quartic scaling of the Fisher information with respect to the number of initial probe excitations—a hallmark of quantum-enhanced performance. Fourth, the sensing nodes can be realized across a wide range of opto- and spin-mechanical platforms. In all case studies considered, the overall root-mean-square error closely tracks the single-node quantum limit (see Cases 1 and 2) [60,110]. This shows that the distributed protocol does not incur a significant penalty compared to the best single-node strategy, while enabling spatially distributed field sensing. Fifth, promising physical implementations include levitated micro- and nanospheres for gravimetry, and cold-atom optomechanical architectures for estimating distributed coupling strengths. More broadly, a nonlinear optomechanical interaction and a conditional-displacement spin-mechanical coupling can be combined into a hybrid interaction of the form

$$\hat{H}_{\text{hybrid}} \propto (\hat{a}^\dagger \hat{a} + \tilde{\Omega} \hat{S}_z^j)(\hat{b}^\dagger + \hat{b}), \quad (70)$$

where $\tilde{\Omega}$ is an auxiliary parameter. This effective interaction maps directly onto our model and, as a result, inherits the same precision guarantees [61,63,111]. This provides a concrete and scalable route to extend the present scheme to a variety of hybrid quantum architectures.

Three practical challenges merit closer examination: First, preparing and verifying the required entangled resource states at scale is nontrivial [112,113]. Nevertheless, the generation of W states has been extensively studied—both theoretically [74,75,114–116] and experimentally [117–121]. In the specific nonlinear optomechanical setup used in Case 1, the relevant entangled resource reduces to a multipartite N00N-type state. Here too, high- N N00N states have been experimentally realized [52,113,122–124] and theoretically proposed [125]. These developments collectively point to viable routes for both state preparation and readout. However, the associated overheads—especially as the number of nodes N increases—remain a key bottleneck. Second, to keep our results analytically transparent and broadly accessible, we assume ideal unitary dynamics and do not fully model decoherence channels. However, an analytical approximation of the conditional-displacement dynamics under dissipation can, in principle, be incorporated by following the approach of Ref. [71]. This method alternates between unitary evolution and nonunitary damping steps, resulting in a probe state with explicitly damped off-diagonal complex coefficients. This effectively captures the impact of decoherence without resorting to a full quantum master equation treatment. In addition, the effects of noise and imperfections on distributed quantum sensing have recently been explored both theoretically and experimentally. On the theory side, it has been demonstrated that noisy distributed quantum sensors retain quantum advantages compared to strategies that lack entanglement between different sites [126]. Moreover, noisy distributed quantum-enhanced sensors are found to be inherently more resilient to imperfections in the quantum network than to local operational errors. Importantly, these conclusions remain valid even when explicitly accounting for dephasing during the sensing evolution [127]. The robustness of quantum networks comprising typical noisy quantum-repeater nodes has been quantified by determining the critical repeater efficiency required to maintain network connectivity under realistic imperfections [128]. Experimentally, recent studies using noisy distributed trapped-ion sensors have successfully demonstrated recovery of optimal sensing bounds despite the presence of noise [54]. Third, the scheme relies on operating at stroboscopic times $\Omega t = 2\pi$. Deviations from these times lead to residual entanglement between the probe and the mechanical oscillator, which degrades decoupling and thus reduces sensing performance. This sensitivity to timing errors is documented in stroboscopic gravimetric opto- and spin-mechanical protocols [110]. While such imperfections can partially degrade the ideal quantum-enhanced scaling, the protocol can still maintain an advantage over classical sensing strategies under experimentally realistic levels of noise and timing deviations.

XI. CONCLUSIONS

In this work, we have presented four main contributions. First, we analytically derived the ultimate multiparameter precision limits for a broad and physically relevant class of quantum probes, in which each probe consists of a mechanical mode conditionally displaced by a general system. The

probe dynamics exhibit stroboscopic disentanglement at specific evolution times, enabling the accumulated pairwise set of $N - 1$ reference-relative phase differences to be coherently mapped onto the degrees of freedom of the general system. As a result, the relevant parameter information becomes independent of the mechanical modes of the network. This mechanism shares conceptual similarities with echo-based metrology, while not relying on an explicit time-reversal operation. Our results demonstrate quantum-enhanced sensitivity with respect to the excitation (number of spins) resource N_{exc} . In particular, we show that the achievable precision scales quadratically or quartically. This enhancement arises from the structure of the generator and the nonlinear encoding [19]. It is important to emphasize that the unknown parameters are encoded in local Hamiltonians, and therefore optimal local (separable) strategies can, in principle, reach the same fundamental scaling behaviors. Our bounds remain fully consistent with established theorems for locally generated parameters [37]. However, for fixed network size and identical bare probe Hamiltonians, we demonstrate that our distributed protocol is strictly more energy efficient than the corresponding optimal local strategy for networks with more than two sites. The advantage thus lies not in surpassing the fundamental local precision limits, nor in a universal constant-factor improvement of the total variance, but in energetic efficiency and in the collective encoding of the complete set of $N - 1$ reference-relative differences within a single distributed probe.

Second, we rigorously prove that our class of quantum probes satisfies the weak commutativity condition. Within this class, this ensures that both the Holevo bound and the quantum Cramér-Rao bound coincide and are locally saturable around the operating point, guaranteeing that the ultimate multiparameter precision limits can, in principle, be achieved. Third, building on this result, we construct a family of projective measurements, parametrized by known reference phases, that can be made locally optimal via adaptive tuning.

Fourth, we apply our framework to two physically realizable systems: (1) nonlinear cavity quantum optomechanical platforms and (2) solid-state spin systems coupled to mechanical oscillators, both extendable to hybrid spin-optomechanical architectures [64]. The first enables simultaneous estimation of gravitational accelerations at distinct spatial locations, while the second addresses distributed coupling-strength estimation. Finally, we provide experimentally realistic parameter regimes indicating that these protocols are within reach of current technological capabilities.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: ANALYTICAL DERIVATION OF THE NETWORKED QUANTUM PROBE STATE

This section clarifies how the general wavefunction of Eq. (30) is obtained in the main text.

Recall the Hamiltonian

$$\frac{\hat{H}_j}{\Omega\hbar} = \hat{b}_j^\dagger \hat{b}_j + (k_j \hat{\Lambda}_j + \mathcal{E}_j)(\hat{b}_j^\dagger + \hat{b}_j), \quad (\text{A1})$$

with unitary operator

$$\hat{u}_j(\tau) = e^{i(k_j \hat{\Lambda}_j + \mathcal{E}_j)^2(\tau - \sin \tau)} e^{-(k_j \hat{\Lambda}_j + \mathcal{E}_j)(\eta(\tau) \hat{b}_j^\dagger - \eta^*(\tau) \hat{b}_j)} e^{-i \hat{b}_j^\dagger \hat{b}_j \tau}, \quad (\text{A2})$$

which can be written as

$$\hat{u}_j(\tau) = e^{i(k_j \hat{\Lambda}_j + \mathcal{E}_j)^2(\tau - \sin \tau)} D_j[-(k_j \hat{\Lambda}_j + \mathcal{E}_j)\eta(\tau)] e^{-i \hat{b}_j^\dagger \hat{b}_j \tau}, \quad (\text{A3})$$

where the displacement operator is defined as

$$D_j[z] \equiv e^{z \hat{b}_j^\dagger - z^* \hat{b}_j}, \quad (\text{A4})$$

with $\eta(\tau) \equiv 1 - e^{-i\tau}$ and $\tau \equiv \Omega t$. Since $[\hat{H}_j, \hat{H}_{j'}] = 0$ for all $1 \leq (j, j') \leq N$, the total unitary operator for the network is given by the product

$$\hat{U}(\tau) = \prod_{j=1}^N \hat{u}_j(\tau). \quad (\text{A5})$$

We considered the initial state

$$|\psi(0)\rangle = |W_N\rangle(|\alpha\rangle^{\otimes N}). \quad (\text{A6})$$

For simplicity, and without loss of generality, let us consider the first nontrivial network, namely, $N = 3$. In this case,

$$|\psi(0)\rangle = |W_3\rangle(|\alpha\rangle^{\otimes 3}) = \frac{1}{\sqrt{3}}(|\lambda, \lambda', \lambda'\rangle + |\lambda', \lambda, \lambda'\rangle + |\lambda', \lambda', \lambda\rangle)|\alpha\rangle|\alpha\rangle|\alpha\rangle. \quad (\text{A7})$$

The corresponding evolved wavefunction reads

$$\begin{aligned} |\psi(\tau)\rangle = & \left(\frac{1}{\sqrt{3}} e^{i(k_1 \lambda + \mathcal{E}_1)^2(\tau - \sin \tau)} D_1[-(k_1 \lambda + \mathcal{E}_1)\eta(\tau)] D_1[\alpha e^{-i\tau}] \right. \\ & \times e^{i(k_2 \lambda' + \mathcal{E}_2)^2(\tau - \sin \tau)} D_2[-(k_2 \lambda' + \mathcal{E}_2)\eta(\tau)] D_2[\alpha e^{-i\tau}] e^{i(k_3 \lambda' + \mathcal{E}_3)^2(\tau - \sin \tau)} D_3 \\ & \times [-(k_3 \lambda' + \mathcal{E}_3)\eta(\tau)] D_3[\alpha e^{-i\tau}] |\lambda, \lambda', \lambda'\rangle |\text{vac}\rangle |\text{vac}\rangle |\text{vac}\rangle \Big) \\ & + \left(\frac{1}{\sqrt{3}} e^{i(k_1 \lambda' + \mathcal{E}_1)^2(\tau - \sin \tau)} D_1[-(k_1 \lambda' + \mathcal{E}_1)\eta(\tau)] D_1[\alpha e^{-i\tau}] \right. \\ & \times e^{i(k_2 \lambda + \mathcal{E}_2)^2(\tau - \sin \tau)} D_2[-(k_2 \lambda + \mathcal{E}_2)\eta(\tau)] D_2[\alpha e^{-i\tau}] e^{i(k_3 \lambda' + \mathcal{E}_3)^2(\tau - \sin \tau)} D_3 \\ & \times [-(k_3 \lambda' + \mathcal{E}_3)\eta(\tau)] D_3[\alpha e^{-i\tau}] |\lambda', \lambda, \lambda'\rangle |\text{vac}\rangle |\text{vac}\rangle |\text{vac}\rangle \Big) \\ & + \left(\frac{1}{\sqrt{3}} e^{i(k_1 \lambda' + \mathcal{E}_1)^2(\tau - \sin \tau)} D_1[-(k_1 \lambda' + \mathcal{E}_1)\eta(\tau)] D_1[\alpha e^{-i\tau}] \right. \\ & \times e^{i(k_2 \lambda' + \mathcal{E}_2)^2(\tau - \sin \tau)} D_2[-(k_2 \lambda' + \mathcal{E}_2)\eta(\tau)] D_2[\alpha e^{-i\tau}] e^{i(k_3 \lambda + \mathcal{E}_3)^2(\tau - \sin \tau)} D_3 \\ & \times [-(k_3 \lambda + \mathcal{E}_3)\eta(\tau)] D_3[\alpha e^{-i\tau}] |\lambda', \lambda', \lambda\rangle |\text{vac}\rangle |\text{vac}\rangle |\text{vac}\rangle \Big). \end{aligned} \quad (\text{A8})$$

In the above, $|\text{vac}\rangle$ denotes the field vacuum state. When acted upon by the product of two displacement operators on the same mode j , one obtains

$$D_j[z_1] D_j[z_2] |\text{vac}\rangle = e^{\frac{1}{2}(z_1 z_2^* - z_1^* z_2)} |z_1 + z_2\rangle, \quad (\text{A9})$$

where $|z_1 + z_2\rangle$ is the coherent state with complex amplitude $z_1 + z_2$. Equation (A8) simplifies to

$$\begin{aligned}
 |\psi(\tau)\rangle = & \left(\frac{1}{\sqrt{3}} e^{i(k_1\lambda + \mathcal{E}_1)^2(\tau - \sin \tau)} e^{i(k_2\lambda' + \mathcal{E}_2)^2(\tau - \sin \tau)} e^{i(k_3\lambda' + \mathcal{E}_3)^2(\tau - \sin \tau)} e^{-i\alpha(\mathcal{E}_1 + k_1\lambda) \sin \tau} e^{-i\alpha(\mathcal{E}_2 + k_2\lambda') \sin \tau} e^{-i\alpha(\mathcal{E}_3 + k_3\lambda') \sin \tau} \right. \\
 & \times |\lambda, \lambda', \lambda'\rangle |\alpha e^{-i\tau} - (k_1\lambda + \mathcal{E}_1)\eta(\tau)\rangle |\alpha e^{-i\tau} - (k_2\lambda' + \mathcal{E}_2)\eta(\tau)\rangle |\alpha e^{-i\tau} - (k_3\lambda' + \mathcal{E}_3)\eta(\tau)\rangle \Big) \\
 & + \left(\frac{1}{\sqrt{3}} e^{i(k_1\lambda' + \mathcal{E}_1)^2(\tau - \sin \tau)} e^{i(k_2\lambda + \mathcal{E}_2)^2(\tau - \sin \tau)} e^{i(k_3\lambda' + \mathcal{E}_3)^2(\tau - \sin \tau)} e^{-i\alpha(\mathcal{E}_1 + k_1\lambda') \sin \tau} e^{-i\alpha(\mathcal{E}_2 + k_2\lambda) \sin \tau} e^{-i\alpha(\mathcal{E}_3 + k_3\lambda') \sin \tau} \right. \\
 & \times |\lambda', \lambda, \lambda'\rangle |\alpha e^{-i\tau} - (k_1\lambda' + \mathcal{E}_1)\eta(\tau)\rangle |\alpha e^{-i\tau} - (k_2\lambda + \mathcal{E}_2)\eta(\tau)\rangle |\alpha e^{-i\tau} - (k_3\lambda' + \mathcal{E}_3)\eta(\tau)\rangle \Big) \\
 & + \left(\frac{1}{\sqrt{3}} e^{i(k_1\lambda' + \mathcal{E}_1)^2(\tau - \sin \tau)} e^{i(k_2\lambda' + \mathcal{E}_2)^2(\tau - \sin \tau)} e^{i(k_3\lambda + \mathcal{E}_3)^2(\tau - \sin \tau)} e^{-i\alpha(\mathcal{E}_1 + k_1\lambda') \sin \tau} e^{-i\alpha(\mathcal{E}_2 + k_2\lambda') \sin \tau} e^{-i\alpha(\mathcal{E}_3 + k_3\lambda) \sin \tau} \right. \\
 & \times |\lambda', \lambda', \lambda\rangle |\alpha e^{-i\tau} - (k_1\lambda' + \mathcal{E}_1)\eta(\tau)\rangle |\alpha e^{-i\tau} - (k_2\lambda' + \mathcal{E}_2)\eta(\tau)\rangle |\alpha e^{-i\tau} - (k_3\lambda + \mathcal{E}_3)\eta(\tau)\rangle \Big). \quad (\text{A10})
 \end{aligned}$$

Without loss of generality, we fix the global phase at the first site $j = 1$, so that the wavefunction can be further simplified as

$$\begin{aligned}
 |\psi(\tau)\rangle = & \frac{1}{\sqrt{3}} |\lambda, \lambda', \lambda'\rangle |\alpha_\lambda\rangle_1 |\alpha_{\lambda'}\rangle_2 |\alpha_{\lambda'}\rangle_3 \\
 & + \frac{1}{\sqrt{3}} e^{-i(\lambda - \lambda')[2\mathcal{E}_1 k_1 - 2\mathcal{E}_2 k_2 + (k_1 - k_2)(k_1 + k_2)(\lambda + \lambda')]\tau + (-2\mathcal{E}_1 k_1 - \alpha(k_1 - k_2) + 2\mathcal{E}_2 k_2 - (k_1 - k_2)(k_1 + k_2)(\lambda + \lambda')) \sin \tau} |\lambda', \lambda, \lambda'\rangle |\alpha_{\lambda'}\rangle_1 |\alpha_\lambda\rangle_2 |\alpha_{\lambda'}\rangle_3 \\
 & + \frac{1}{\sqrt{3}} e^{-i(\lambda - \lambda')[2\mathcal{E}_1 k_1 - 2\mathcal{E}_3 k_3 + (k_1 - k_3)(k_1 + k_3)(\lambda + \lambda')]\tau + (-2\mathcal{E}_1 k_1 - \alpha(k_1 - k_3) + 2\mathcal{E}_3 k_3 - (k_1 - k_3)(k_1 + k_3)(\lambda + \lambda')) \sin \tau} |\lambda', \lambda', \lambda\rangle |\alpha_{\lambda'}\rangle_1 |\alpha_{\lambda'}\rangle_2 |\alpha_\lambda\rangle_3, \quad (\text{A11})
 \end{aligned}$$

where we have defined the coherent amplitudes as

$$|\alpha_\Xi\rangle_j = |\alpha e^{-i\tau} - (k_j \Xi + \mathcal{E}_j)\eta(\tau)\rangle, \quad (\text{A12})$$

where Ξ is a scalar. In the above, Ξ takes the values $\Xi = (\lambda, \lambda')$. We can further simplify the notation by writing

$$|\alpha_j\rangle = |\alpha_{\lambda'}\rangle_1 \cdots |\alpha_\lambda\rangle_j \cdots |\alpha_{\lambda'}\rangle_N, \quad (\text{A13})$$

which represents the tensor product of N coherent states, with the λ -dependent displacement appearing at site j —All the remaining states are displaced states whose displacement explicitly depends on λ' . This is the notation used throughout the main text.

While Eq. (A11) gives the wavefunction at arbitrary times, in our analysis we focus primarily on the dynamics at $\tau = 2\pi$ (see main text). At this specific time, Eq. (A11) reduces to

$$\begin{aligned}
 |\psi(\tau = 2\pi)\rangle = & \frac{1}{\sqrt{3}} (|1\rangle + e^{-2\pi i(\lambda - \lambda')[2\mathcal{E}_1 k_1 - 2\mathcal{E}_2 k_2 + (k_1 - k_2)(k_1 + k_2)(\lambda + \lambda')]} |2\rangle \\
 & + e^{-2\pi i(\lambda - \lambda')[2\mathcal{E}_1 k_1 - 2\mathcal{E}_3 k_3 + (k_1 - k_3)(k_1 + k_3)(\lambda + \lambda')]} |3\rangle) |\alpha\rangle_1 |\alpha\rangle_2 |\alpha\rangle_3, \quad (\text{A14})
 \end{aligned}$$

where all mechanical states return to their initial state $(|\alpha\rangle)^{\otimes 3}$ and

$$|1\rangle = |\lambda\rangle_1 |\lambda'\rangle_2 |\lambda'\rangle_3, \quad (\text{A15})$$

$$|2\rangle = |\lambda'\rangle_1 |\lambda\rangle_2 |\lambda'\rangle_3, \quad (\text{A16})$$

$$|3\rangle = |\lambda'\rangle_1 |\lambda'\rangle_2 |\lambda\rangle_3, \quad (\text{A17})$$

or, in general,

$$|j\rangle = |\lambda'\rangle_1 |\lambda'\rangle_2 \cdots |\lambda\rangle_j \cdots |\lambda'\rangle_{N-1} |\lambda'\rangle_N. \quad (\text{A18})$$

Case 1: In Eq. (A14), if we set $\mathcal{E}_j \neq \mathcal{E}_{j'} \neq 0$ while keeping $k_j = k$ for all j , we immediately obtain

$$|\psi(\tau = 2\pi)\rangle = \frac{1}{\sqrt{3}} (|1\rangle + e^{-4\pi i(\lambda - \lambda')k(\mathcal{E}_1 - \mathcal{E}_2)} |2\rangle + e^{-4\pi i(\lambda - \lambda')k(\mathcal{E}_1 - \mathcal{E}_3)} |3\rangle) |\alpha\rangle_1 |\alpha\rangle_2 |\alpha\rangle_3. \quad (\text{A19})$$

Case 2: In Eq. (A14), if we set $\mathcal{E}_j = 0$ and $k_j \neq k_{j'} \neq 0$, we get

$$|\psi(\tau = 2\pi)\rangle = \frac{1}{\sqrt{3}}(|1\rangle + e^{-2\pi i(\lambda - \lambda')(k_1 - k_2)(k_1 + k_2)(\lambda + \lambda')}|2\rangle + e^{-2\pi i(\lambda - \lambda')(k_1 - k_3)(k_1 + k_3)(\lambda + \lambda')}|3\rangle)|\alpha\rangle_1|\alpha\rangle_2|\alpha\rangle_3. \quad (\text{A20})$$

Equations (A19) and (A20) correspond to the cases analyzed in detail in the main text and match the scenarios presented in Table I.

From the above expressions, it is clear that we can ignore the coherent amplitudes and focus in the $\{|j\rangle\}$ subsystems. For completeness, we will compute the QFIM for Eq. (A19). Using Eq. (4), the QFIM reads as

$$Q = \frac{64\pi^2(\lambda - \lambda')^2 k^2}{9} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad (\text{A21})$$

with the trace of the inverse as

$$\text{Tr}[Q^{-1}] = \frac{3}{[4\pi(\lambda - \lambda')k]^2}. \quad (\text{A22})$$

APPENDIX B: LINEAR ENTROPY OF THE REDUCED PROBE STATE ACROSS THE NETWORK

Recall the general pure state shown in Eq. (12):

$$|\psi(\tau)\rangle = \frac{1}{\sqrt{N}} \left(|1\rangle|\alpha_1(\tau)\rangle + \sum_{j=2}^N e^{i\xi_j(\tau)} |j\rangle|\alpha_j(\tau)\rangle \right), \quad (\text{B1})$$

where

$$\xi_j(\tau) = \frac{1}{2} e^{-i\tau} (\lambda - \lambda') [\alpha(e^{2i\tau} - 1)(k_1 - k_j) + 2[2\mathcal{E}_1 k_1 - 2\mathcal{E}_j k_j + (k_1 - k_j)(k_1 + k_j)(\lambda + \lambda')](\tau - \sin \tau)(\sin \tau - i \cos \tau)]. \quad (\text{B2})$$

We aim to evaluate the linear entropy of the reduced probe state, defined as

$$S_L = 1 - \text{Tr}[\rho_{\text{probe}}^2], \quad (\text{B3})$$

where ρ_{probe} is the reduced density matrix obtained by tracing out the mechanical (field) degrees of freedom from the full pure state, i.e.,

$$\rho_{\text{probe}} = \text{Tr}_{\text{fields}}[|\psi(\tau)\rangle\langle\psi(\tau)|]. \quad (\text{B4})$$

To compute S_L , we require the trace of the product of partial states involving overlaps of mechanical coherent states. Specifically, for any two coherent states $|\Upsilon_1\rangle$ and $|\Upsilon_2\rangle$, the trace over the mechanical subsystem reduces to the inner product:

$$\text{Tr}_{\text{fields}}(|\Upsilon_1\rangle\langle\Upsilon_2|) = \langle\Upsilon_2|\Upsilon_1\rangle = \exp\left(-\frac{1}{2}|\Upsilon_1|^2 - \frac{1}{2}|\Upsilon_2|^2 + \Upsilon_2^* \Upsilon_1\right). \quad (\text{B5})$$

Below, we provide explicit analytical expressions for the linear entropy for small network sizes $N = 2, 3, 4, 5$.

For $N = 2$,

$$\mathcal{S}_L^{(2)} = \frac{1}{2} [1 - \exp(2(k_1^2 + k_2^2)(\lambda - \lambda')^2 [\cos \tau - 1])]. \quad (\text{B6})$$

For $N = 3$,

$$\mathcal{S}_L^{(3)} = \frac{2}{9} [3 - \exp(2(k_1^2 + k_2^2)(\lambda - \lambda')^2 [\cos \tau - 1]) - \exp(2(k_1^2 + k_3^2)(\lambda - \lambda')^2 [\cos \tau - 1]) - \exp(2(k_2^2 + k_3^2)(\lambda - \lambda')^2 [\cos \tau - 1])]. \quad (\text{B7})$$

For $N = 4$,

$$\mathcal{S}_L^{(4)} = \frac{1}{8} [6 - \exp(2(k_1^2 + k_2^2)(\lambda - \lambda')^2 [\cos \tau - 1]) - \exp(2(k_1^2 + k_3^2)(\lambda - \lambda')^2 [\cos \tau - 1]) - \exp(2(k_2^2 + k_3^2)(\lambda - \lambda')^2 [\cos \tau - 1]) - \exp(2(k_1^2 + k_4^2)(\lambda - \lambda')^2 [\cos \tau - 1]) - \exp(2(k_2^2 + k_4^2)(\lambda - \lambda')^2 [\cos \tau - 1]) - \exp(2(k_3^2 + k_4^2)(\lambda - \lambda')^2 [\cos \tau - 1])]. \quad (\text{B8})$$

For $N = 5$,

$$\mathcal{S}_L^{(5)} = \frac{2}{25} [10 - \exp(2(k_1^2 + k_2^2)(\lambda - \lambda')^2 [\cos \tau - 1]) - \exp(2(k_1^2 + k_3^2)(\lambda - \lambda')^2 [\cos \tau - 1]) - \exp(2(k_2^2 + k_3^2)(\lambda - \lambda')^2 [\cos \tau - 1]) - \exp(2(k_1^2 + k_4^2)(\lambda - \lambda')^2 [\cos \tau - 1]) - \exp(2(k_2^2 + k_4^2)(\lambda - \lambda')^2 [\cos \tau - 1]) - \exp(2(k_3^2 + k_4^2)(\lambda - \lambda')^2 [\cos \tau - 1]) - \exp(2(k_1^2 + k_5^2)(\lambda - \lambda')^2 [\cos \tau - 1]) - \exp(2(k_2^2 + k_5^2)(\lambda - \lambda')^2 [\cos \tau - 1]) - \exp(2(k_3^2 + k_5^2)(\lambda - \lambda')^2 [\cos \tau - 1]) - \exp(2(k_4^2 + k_5^2)(\lambda - \lambda')^2 [\cos \tau - 1])]. \quad (\text{B9})$$

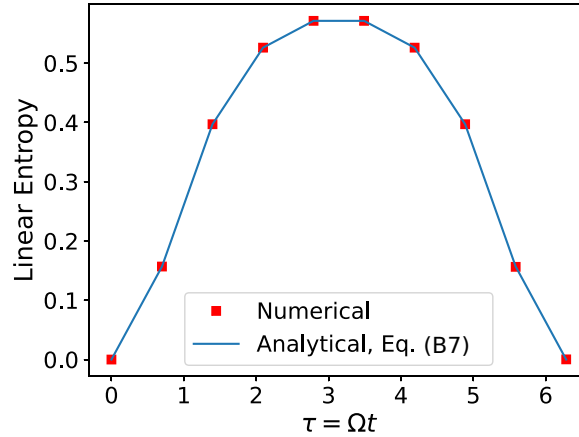


FIG. 4. The figure compares the analytical prediction Eq. (B7) with the numerical evaluation of the linear entropy obtained from the full Hamiltonian evolution. We considered $k_1 = 0.05$, $k_2 = 0.10$, $k_3 = 0.15$, driving amplitudes $\mathcal{E}_1 = 0.25$, $\mathcal{E}_2 = 0.33$, $\mathcal{E}_3 = 0.18$, initial mechanical coherent states $\alpha_1 = \alpha_2 = \alpha_3 = 1$, and $\lambda' = 0$ and $\lambda = 5$.

From the explicit expressions derived for small network sizes, the structure of the linear entropy reveals a consistent and systematic pattern. This allows for a straightforward generalization to an arbitrary number of nodes N :

$$\mathcal{S}_L^{(N)} = 1 - \frac{1}{N^2} \left[N + 2 \sum_{1 \leq i < j \leq N} \exp(2(k_i^2 + k_j^2)(\lambda - \lambda')^2 [\cos \tau - 1]) \right] \quad (\text{B10})$$

$$= \frac{2}{N^2} \left[\frac{N(N-1)}{2} - \sum_{1 \leq i < j \leq N} \exp(2(k_i^2 + k_j^2)(\lambda - \lambda')^2 [\cos \tau - 1]) \right], \quad (\text{B11})$$

which gives a closed-form expression valid for any network size.

The above closed-form expression for the linear entropy can, in fact, be established rigorously by mathematical induction, owing to the recursive structure that the reduced $\hat{\Lambda}_j$ -density matrix acquires as the number of nodes increases. As an independent sanity check of the analytical result, we have also performed a full numerical simulation for the optomechanical dynamics, that is, $\hat{\Lambda}_j = \hat{a}_j^\dagger \hat{a}_j$, for a three-node network with parameters $k_1 = 0.05$, $k_2 = 0.10$, $k_3 = 0.15$, driving amplitudes $\mathcal{E}_1 = 0.25$, $\mathcal{E}_2 = 0.33$, $\mathcal{E}_3 = 0.18$, initial mechanical coherent states $\alpha_1 = \alpha_2 = \alpha_3 = 1$, and $\lambda' = 0$, $\lambda = 5$. In Fig. 4, we compare the analytical prediction with the numerical evaluation of the linear entropy obtained from the full Hamiltonian evolution. The agreement is exact within numerical precision. Furthermore, we have explicitly verified that varying the coherent amplitudes α_i and the driving strengths $\mathcal{E}_i$ leaves the linear entropy unchanged, thereby confirming that its independence from these parameters is a structural property of the model rather than a consequence of a particular choice of values.

APPENDIX C: QUANTUM FISHER INFORMATION MATRIX FOR GENERAL DISTRIBUTED QUANTUM-ENHANCED SENSING

Consider the general state at $\tau = 2\pi$ from Eq. (30), derived as

$$|\psi\rangle = \frac{1}{\sqrt{N}} \left(|1\rangle + \sum_{j=2}^N e^{i\beta_j \Phi_j} |j\rangle \right), \quad (\text{C1})$$

where we have omitted the disentangled mechanical states $|\alpha\rangle^{\otimes N}$ for simplicity. To compute the QFIM entries using Eq. (4), we first evaluate the partial derivatives

$$|\partial_j \psi\rangle = \frac{i\beta_j}{\sqrt{N}} e^{i\beta_j \Phi_j} |j\rangle, \quad (\text{C2})$$

the inner products

$$\langle \partial_j \psi | \partial_{j'} \psi \rangle = \frac{\beta_j^2}{N} \delta_{j,j'} \quad (\text{C3})$$

and the overlaps

$$\langle \psi | \partial_j \psi \rangle = \frac{1}{N} i\beta_j. \quad (\text{C4})$$

Therefore, the QFIM is given by

$$\mathcal{Q} = \frac{4}{N} \text{diag}(\beta^2) - \frac{4}{N^2} \beta \beta^T, \quad (\text{C5})$$

or in terms of its matrix entries

$$[\mathcal{Q}]_{ij} = \frac{4}{N} \delta_{ij} \beta_i^2 - \frac{4}{N^2} \beta_i \beta_j, \quad (\text{C6})$$

where δ_{ij} is the Kronecker delta, $\beta = (\beta_2, \dots, \beta_N)^T$ (of length $N-1$), and let $\text{diag}(\beta^2)$ denote the diagonal matrix with entries β_j^2 .

Equation (C5) can be rewritten as

$$\mathcal{Q} = A - uv^T, \quad (\text{C7})$$

where

$$A \equiv \frac{4}{N} \text{diag}(\beta^2), \quad u \equiv \frac{2}{N} \beta, \quad v \equiv \frac{2}{N} \beta. \quad (\text{C8})$$

We now use the Sherman-Morrison identity to obtain the inverse of the QFIM in Eq. (C5). This identity reads as

$$(A - uv^T)^{-1} = A^{-1} + \frac{A^{-1}uv^TA^{-1}}{1 - v^TA^{-1}u}, \quad (\text{C9})$$

which holds for invertible A , i.e., $\beta_j \neq 0$ for all j , and when the scalar $(1 - v^TA^{-1}u) \neq 0$, i.e., $v^TA^{-1}u \neq 1$. The above terms explicitly are

$$A^{-1} = \frac{N}{4} \text{diag}(\beta^{-2}), \quad (\text{C10})$$

$$v^TA^{-1}u = \frac{1}{N} \sum_{j=2}^N \beta_j \beta_j^{-1} = \frac{N-1}{N} \neq 1 \quad \forall N, \quad (\text{C11})$$

$$A^{-1}uv^TA^{-1} = \left(\frac{1}{2}\beta^{-1}\right)\left(\frac{1}{2}\beta^{-1}\right)^T. \quad (\text{C12})$$

Replacing the above terms into Sherman-Morrison formula, one gets

$$Q^{-1} = \frac{N}{4} [\text{diag}(\beta^{-2}) + \beta^{-1}(\beta^{-1})^T], \quad (\text{C13})$$

or, similarly as before, in terms of its entry elements

$$[Q^{-1}]_{ij} = \frac{N}{4} \left(\frac{\delta_{ij}}{\beta_i^2} + \frac{1}{\beta_i \beta_j} \right), \quad (\text{C14})$$

which is a particularly useful representation for single-parameter distributed sensing and nuisance analysis.

Note that, in order to justify the invertibility of Q , one can also easily compute the determinant needed for its inverse as $\det Q = (1 - v^TA^{-1}u) \det A = \frac{1}{N} \det A > 0$ for all $N > 0$ and whenever all $\beta_j \neq 0$. Hence, Q is positive definite with a well-defined inverse. If any $\beta_j = 0$, $\det A = 0$ and thus Q becomes singular.

As stated in the main text, we focus on the scalar multiparameter quantum Cramér-Rao bound, which requires the trace of the inverse of the QFIM. To compute this, we simply recall that $\text{Tr}[\beta^{-1}(\beta^{-1})^T] = \sum_j \beta_j^{-2}$. Hence,

$$\text{Tr}[Q^{-1}] = \frac{N}{2} \sum_{j=2}^N \beta_j^{-2}. \quad (\text{C15})$$

The above expression reproduces the results for the trace of the inverse of the QFIM presented in the main text.

APPENDIX D: WEAK COMMUTATIVITY CONDITION: SATURATION OF ULTIMATE MULTIPARAMETER BOUND

Unlike the single-parameter case, in the multiparameter scenario the quantum Cramér-Rao bound cannot always be saturated. A sufficient condition for saturation is the strong commutativity of the SLDs, $[\mathcal{L}_k, \mathcal{L}_{k'}] = 0$, which guarantees the existence of a common eigenbasis, but this is not a necessary condition. A more general and less restrictive criterion is given within the framework of quantum local asymptotic normality from which a necessary and sufficient condition for the saturation of the multiparameter quantum Cramér-Rao

bound is the so-called weak commutativity condition,

$$\langle \psi | [\mathcal{L}_k, \mathcal{L}_{k'}] | \psi \rangle = 0, \quad (\text{D1})$$

which ensures the asymptotic attainability of the bound even when the SLDs do not commute strongly [26].

From Eq. (E3), it is straightforward to compute the product

$$\begin{aligned} \mathcal{L}_k \mathcal{L}_{k'} = & -\frac{4\beta_k \beta_{k'}}{N} \left[\frac{e^{i\beta_k \Phi_k}}{\sqrt{N}} |k\rangle \langle \psi| - e^{i(\beta_k \Phi_k - \beta_{k'} \Phi_{k'})} |k\rangle \langle k'| \right. \\ & \left. + \frac{e^{-i\beta_{k'} \Phi_{k'}}}{\sqrt{N}} |\psi\rangle \langle k'| \right], \end{aligned} \quad (\text{D2})$$

from which one can readily compute $\langle \psi | [\mathcal{L}_k, \mathcal{L}_{k'}] | \psi \rangle = 0$. This means that saturation of the multiparameter Cramér-Rao bound is, in principle, possible.

To illustrate that all phases Φ_j can be estimated simultaneously using an adaptive family of collective projective measurements, and that this measurement locally saturates the quantum Cramér-Rao bound around the operating point, we now explicitly analyze the first nontrivial case $N = 3$. As shown in the main text, using Eq. (49), we can compute the probabilities:

$$\begin{aligned} p_0 = & \frac{1}{9} (3 + 2 \cos[\beta_2(\Phi_2 - \vartheta_2)] + 2 \cos[\beta_3(\Phi_3 - \vartheta_3)] \\ & + 2 \cos[\beta_2 \Phi_2 - \beta_3 \Phi_3 - \beta_2 \vartheta_2 + \beta_3 \vartheta_3]), \end{aligned} \quad (\text{D3})$$

$$p_1 = \frac{2}{3} \sin^2 \left(\frac{1}{2} \beta_2 (\Phi_2 - \vartheta_2) \right), \quad (\text{D4})$$

$$\begin{aligned} p_2 = & \frac{1}{9} (3 + \cos[\beta_2(\Phi_2 - \vartheta_2)] - 2 \cos[\beta_3(\Phi_3 - \vartheta_3)] \\ & - 2 \cos[\beta_2 \Phi_2 - \beta_3 \Phi_3 - \beta_2 \vartheta_2 + \beta_3 \vartheta_3]), \end{aligned} \quad (\text{D5})$$

from which the entries of the classical Fisher information matrix can be built.

APPENDIX E: SINGLE-PARAMETER ESTIMATION: OPTIMAL MEASUREMENT BASIS VIA THE SYMMETRIC LOGARITHMIC DERIVATIVE EIGENBASIS

The SLD operator for pure states is given by

$$\mathcal{L}_k = 2(|\partial_k \psi\rangle \langle \psi| + |\psi\rangle \langle \partial_k \psi|). \quad (\text{E1})$$

We aim to determine the optimal measurement basis for a single-parameter estimation scenario. In this context, single parameter refers to the estimation process where only one phase Φ_k is estimated, while all other relative phases $\{\Phi_j\}_{j \neq k}$ are assumed to be known. For us to achieve the optimal measurement that saturates the single-parameter Cramér-Rao bound, that is, $Q = \mathcal{F}$, we need to compute the eigenbasis of the SLD operator. Recall the state described in Eq. (30) and its derivative with respect to Φ_k as

$$|\psi\rangle = \frac{1}{\sqrt{N}} \left(|1\rangle + \sum_{j=2}^N e^{i\beta_j \Phi_j} |j\rangle \right), \quad |\partial_k \psi\rangle = \frac{1}{\sqrt{N}} i\beta_k e^{i\beta_k \Phi_k} |k\rangle. \quad (\text{E2})$$

Therefore, the SLD operator can be calculated explicitly as follows:

$$\mathcal{L}_k = \frac{2i\beta_k}{N} \left[e^{i\beta_k\Phi_k} |k\rangle\langle 1| + \sum_{j=2}^N e^{i(\beta_k\Phi_k - \beta_j\Phi_j)} |k\rangle\langle j| - e^{-i\beta_k\Phi_k} |1\rangle\langle k| - \sum_{j=2}^N e^{i(\beta_j\Phi_j - \beta_k\Phi_k)} |j\rangle\langle k| \right]. \quad (\text{E3})$$

From Eq. (E1), it is straightforward to see that $\mathcal{L}_k$ has support only on $\text{span}\{|\psi\rangle, |k\rangle\}$. All remaining vectors that are orthogonal to both $|\psi\rangle$ and $|k\rangle$ have a zero eigenvalue, making them uninformative when constructing the corresponding probability distributions for classical Fisher information functions. Let us construct an orthonormal pair:

$$|e_1\rangle := |\psi\rangle, \quad (\text{E4})$$

$$|e_2\rangle := \sqrt{\frac{N}{N-1}} \left(e^{i\beta_k\Phi_k} |k\rangle - \frac{1}{\sqrt{N}} |\psi\rangle \right). \quad (\text{E5})$$

By construction, $\langle e_a | e_b \rangle = \delta_{ab}$ and $\langle \psi | e_2 \rangle = 0$. Using these definitions, the SLD operator $\mathcal{L}_k$ in its matrix form spanned in $\{|e_1\rangle, |e_2\rangle\}$ basis is

$$[\mathcal{L}_k]_{e_1, e_2} = \frac{2\beta_k\sqrt{N-1}}{N} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \quad (\text{E6})$$

From the above 2×2 SLD operator in matrix form, one can straightforwardly find its nonzero eigenvalues

$$\lambda_k^\pm = \pm \frac{2\beta_k\sqrt{N-1}}{N}, \quad (\text{E7})$$

with normalized eigenvectors

$$|v_k^\pm\rangle = \frac{1}{\sqrt{2}} (|e_1\rangle \pm i|e_2\rangle), \quad (\text{E8})$$

$$= \frac{1}{\sqrt{2}} \left[\left(1 \mp \frac{i}{\sqrt{N-1}} \right) |\psi\rangle \pm i \sqrt{\frac{N}{N-1}} e^{i\beta_k\Phi_k} |k\rangle \right]. \quad (\text{E9})$$

Finally, the optimal single-parameter measurement (for estimating Φ_k) is the optimal projective measurement

$$\Pi_\pm^{(k)} = |v_k^\pm\rangle\langle v_k^\pm|, \quad \text{with} \quad \Pi_0^{(k)} = \mathbb{I} - \Pi_+^{(k)} - \Pi_-^{(k)}. \quad (\text{E10})$$

Using the optimal measurement for the single-parameter scenario described in Eq. (E10), we can easily show that the classical Fisher information is equal to the quantum Fisher information. This means that the optimal measurement achieves the highest possible precision limit, as expected. To demonstrate this, we assume a fixed value of $\tilde{\Phi}_k$ and consider the optimal projective measurement. Under these conditions, the classical Fisher information can be expressed as follows:

$$\mathcal{F}(\tilde{\Phi}_k) = \left[\frac{1}{p_+(\Phi_k)[1-p_+(\Phi_k)]} \left(\frac{\partial p_+(\Phi_k)}{\partial \Phi_k} \right)^2 \right] \Bigg|_{\Phi_k=\tilde{\Phi}_k}, \quad (\text{E11})$$

where

$$p_+(\Phi_k) = \frac{1}{2} + \frac{\sqrt{N-1}}{N} \sin(\beta_k(\Phi_k - \tilde{\Phi}_k)). \quad (\text{E12})$$

Hence,

$$\begin{aligned} \mathcal{F}(\tilde{\Phi}_k) &= \frac{4\beta_k^2(N-1) \cos^2[\beta_k(\tilde{\Phi}_k - \Phi_k)]}{N^2 - 4(N-1) \sin^2[\beta_k(\tilde{\Phi}_k - \Phi_k)]} \Bigg|_{\Phi_k=\tilde{\Phi}_k} \\ &= \frac{4\beta_k^2(N-1)}{N^2}. \end{aligned} \quad (\text{E13})$$

The above expression exactly matches the quantum Fisher information for the single-parameter estimation scenario [see Eq. (66)].

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