

Acoustically controlled periodic γ -optical signals described by semiclassical theoryVlastimil Vrba , Michal Hausner , Aleš Stejskal , and Vít Procházka ^{*}*Department of Experimental Physics, Faculty of Science, Palacký University Olomouc,
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This study provides a detailed theoretical description of periodic γ -ray intensity signals. These signals are generated by the tabletop time coincidence experimental setups which utilize nuclear resonant interaction of γ -photons from a radioactive source with an absorber exhibiting ultrasonic vibrations. Transmission of the γ -radiation field through the absorber is described by semiclassical theory considering arbitrary periodic motion of the absorber. A key aspect of the presented work is the application of the Fourier analysis. A fundamental relation between the Fourier coefficients of the γ -ray intensity signal and the Fourier coefficients of a complex exponential of the absorber motion is derived. Application of this relation is shown on the example of the trapezoid motion, which demonstrates its usability for systematic inspection of the discussed signals. The results of the analysis expand the knowledge of the coherent manipulation of γ -photons and may find usage in a broad range of research fields utilizing γ -ray sources.

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I. INTRODUCTION

Radioactive sources provide a basis for the generation of highly controllable γ -photon intensity waveforms [1]. Such optical signals include, but are not limited to, ultrashort single or multiple γ -photon pulses with frequencies above 1 MHz [2]. In recent years, tabletop experimental setups have been designed to generate and study these intensity signals [1,3] with the possible application fields covering quantum information [2] and quantum memories [4], high-resolution measurements [5–8], or cryptography using single γ -photons [9].

One of the approaches to controlling the γ -radiation properties is based on the nuclear resonant interaction of γ -photons with absorbers containing suitable nuclear ensembles [10,11]. This method allows for a very fine tuning of the radiation spectral and temporal properties by utilizing the recoilless absorption and emission of the γ -photons (Mössbauer effect) and Doppler modulation of the γ -ray energy. Furthermore, application of ultrasonic vibrations to the absorber significantly expands the possibilities of controlling the transmission of the recoilless γ -photons by modifying the parameters of the absorber motion [1,2,11–15].

Generation of the γ -radiation intensity signals is based on the time coincidence detections of single γ -photons, which utilize the ^{57}Co radioactive source and nuclear transitions of ^{57}Fe [1,3,16]. A simplified scheme of the technique and

the physical principle for the considered system is shown in Fig. 1. The nuclear decay of ^{57}Co into a ^{57}Fe nuclide may result in a subsequent emission of two photons corresponding to the transitions of ^{57}Fe from its second excited state $|e_2\rangle$ to the first excited state $|e_1\rangle$ and then from $|e_1\rangle$ to the ground state $|g\rangle$. The first photon references the formation time of ^{57}Fe in $|e_1\rangle$. The second emitted photon can then be transmitted through the absorber (containing ^{57}Fe nuclide) and detected, which gives rise to the measured γ -optical signal. The control of the ^{57}Fe absorber motion by a piezoelement capable of MHz frequency modulation enables the modification of the transmitted γ -photon field. The intensity signals are measured as the histograms of the γ -photon detections at different time delays with respect to a starting event. The starting signal can be produced either by the detections of the starting photons from the $|e_2\rangle \rightarrow |e_1\rangle$ transition or by the arbitrary function generator driving the piezoelement. For the latter case, the radiation waveform can be transformed into a periodic signal featuring repeating modifications of the transmission of γ -photons in time.

The strong connection between the radiation waveform and the absorber motion can be exploited to precisely control and optimize the transmission of γ -photons in time for specific applications. Along with the development of the experimental equipment and the detection systems, progress in the theoretical understanding of the acoustic modulation of γ -rays is important for further expanding its applicability. The modulated γ -photon waveforms have been theoretically described using the semiclassical formalism, where the solution to the wave equation is found in the reference frame of the vibrating absorber [1,12,17]. The formation of so-called spectral combs and its connection to the temporal properties of the transmitted γ -ray field have been extensively studied [1,2,5,18–21]. The focus has been on the analysis and optimization of the γ -ray pulses for the case of the harmonic vibrations and a single

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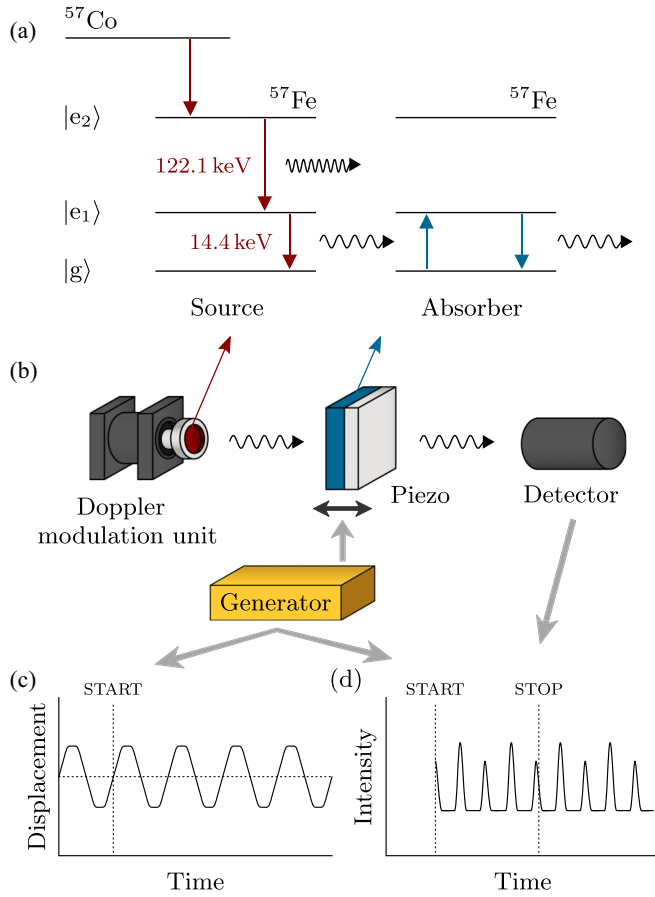


FIG. 1. (a) Nuclear level transitions corresponding to the ^{57}Co radiation source (red arrows) and absorber containing ^{57}Fe nuclei (blue arrows). The energies of γ -photons for the $|e_2\rangle \rightarrow |e_1\rangle$ and $|e_1\rangle \rightarrow |g\rangle$ transitions are 122.1 keV and 14.4 keV, respectively. (b) Experimental arrangement utilizing the transitions for the time coincidence detections. Bottom panels: Illustrations of the synchronized signals describing (c) the absorber motion and (d) the γ -radiation intensity.

nuclear transition being utilized in the absorber. The effect of multifrequency absorber vibrations has also been studied to derive optimal conditions for inducing short and intense γ -ray pulses [22,23]. Additionally, the possibility of transforming the exponentially decreasing intensity signals into pulses of controllable shapes using the rapid absorber displacements with suitable time profiles has been investigated [24]. The controlled absorber motion can also be used to modify the spectral and temporal properties of x-ray photons or to study nuclear dynamics in experiments utilizing synchrotron radiation sources [13,25–27].

In this work we focus on the γ -ray intensity signals in the form of the periodic time-delay histograms. The acoustically induced coherent control of the γ -photon waveforms is described semiclassically by introducing the coherent scattering amplitude and applying the index of refraction model [28,29] in the absorber reference frame. The description is given in a general form enabling the inclusion of γ -radiation polarization effects and absorbers exhibiting arbitrary nuclear hyperfine interactions into the theoretical models. The

periodic nature of the resulting intensity signals is further exploited by applying Fourier analysis in the description. The main focus is concentrated on analyzing the connection between the periodic absorber motion and properties of the generated γ -optical signals. General motion profiles consisting of an arbitrary number of higher harmonics are considered. These profiles can be described directly by their fundamental parameters, which may simplify the optimization problems and physical interpretation of the results.

The fundamental connection between the generated time-delay histogram and the absorber motion is derived by decomposing the intensity signal into its harmonic components. The corresponding Fourier coefficients are directly influenced by the overlap between the emission and absorption spectral lines and by motion-dependent coefficients. These two contributions are separated so the parameters of the spectral overlapping and the parameters of the absorber motion can be evaluated independently on each other. Moreover, the motion-dependent coefficients are found to be equal to the Fourier coefficients of a complex exponential of the motion profile. These findings are particularly interesting for general nonharmonic vibrations, where the computing complexity in various optimization problems can be significantly reduced.

A special issue within the discussed topics is the solving of the inverse problem in which for a given intensity signal a particular motion profile that would generate this signal is being searched. The potential to easily set any time dependence of the γ -ray intensity can be very interesting for γ -ray pump-and-probe experiments, investigations of single γ -photon quantum entanglement, or applications in metrology. In the latter case, the vibrating system would be considered as a studied sample whose motion could be determined within extremely high precision as the displacement time evolution below the level of the γ -photon wavelength can be achieved. The results of this work might present a useful tool for analyzing such a problem directly in the frequency domain instead of the time domain. Therefore, the Fourier coefficients of the absorber motion, the complex motion exponential, and time-delay histograms would be evaluated to determine and optimize the former.

The paper is divided into four main sections. A general theoretical framework is introduced in Sec. II, leading to equations describing the γ -ray intensity signals under the considered physical conditions. The application of the Fourier series is then described in Sec. III, where the general form of the Fourier coefficients is derived. The calculations are based on the generalization of the conventionally derived forms of the intensity signals as shown in Sec. II. Agreement between the introduced description and existing theory is shown for a simplified case in Sec. IV. The application of the Fourier series description is shown in the case example of the trapezoid motion profile in Sec. V. A detailed analysis of specific intensity signals and the corresponding Fourier coefficients is presented, which demonstrates the applicability of the introduced description.

II. THEORETICAL FRAMEWORK

The presented description is based on the semiclassical theory of nuclear resonant scattering [10,28]. We consider

the coherent processes and the forward scattering direction with respect to the direction of the γ -photons incident on the absorber (i.e., the transmission geometry in Mössbauer experiments). The formalism is described directly in the reference frame of the moving absorber as the observable quantity (intensity of the transmitted radiation) is not altered by the transform back to the laboratory reference frame [1].

A. Semiclassical theory

The electric field of the incident γ -radiation can be described in its vector form using the linear polarization basis $\{\hat{\mathbf{e}}_\sigma, \hat{\mathbf{e}}_\pi\}$ as

$$\mathbf{E}_0(t) = E_0(t)\mathbf{A}, \quad (1)$$

where $\mathbf{A} = A_\sigma \hat{\mathbf{e}}_\sigma + A_\pi \hat{\mathbf{e}}_\pi$ is a complex vector. For the radioactive sources of resonant photons utilizing single nuclear transition, the time dependence of the incident field in the absorber reference frame is given as

$$E_0(t) = e^{-i\omega_s \tau(t)} e^{-\frac{\Gamma}{2\hbar} \tau(t)} \Theta[\tau(t)], \quad (2)$$

where $\hbar\omega_s$ is the mean transition energy of the source, Γ is the natural linewidth, and $\tau(t) = t - t_0 - l(t)/c$. The reference time t_0 marks the formation of the excited nuclear state and $l(t)$ defines the absorber motion profile.

In this work we consider an arbitrary periodic motion with the angular frequency $\Omega = 2\pi/T$, where T is the motion period. The function $l(t)$ describes the absorber's displacement from its equilibrium position with respect to the γ -photon source. The source-absorber and the absorber-detector mean distances can be omitted from the description [1].

Transmission of the resonant γ -photons through the absorber can be solved in the frequency (energy) domain by calculating the transmission of the field's elementary spectral contributions [30]. The decomposition of $\mathbf{E}_0(t)$ into its elementary components with the angular frequency ω and the amplitude $\mathbf{E}_0(\omega) = E_0(\omega)\mathbf{A}$ is given by Fourier transform as follows:

$$\mathbf{E}_0(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{E}_0(\omega) e^{-i\omega t} d\omega. \quad (3)$$

The transmission of $\mathbf{E}_0(\omega)$ through the absorber can then be solved using the semiclassical formalism. The absorber is taken as a continuum described by a complex index of refraction, which is related to the coherent scattering amplitude $f(\omega)$ of the system [28]. The complex amplitude of the transmitted field is then given as [28,31]

$$\mathbf{E}_{\text{tr}}(\omega) = e^{ikd} e^{i\frac{2\pi\eta}{k} f(\omega)d} \mathbf{E}_0(\omega), \quad (4)$$

where k is the field wave number, d is the absorber thickness, and η is the number density of Mössbauer nuclei.

The transmitted γ -photon field $\mathbf{E}_{\text{tr}}(t)$ consists of the individual transmitted elementary waves $\mathbf{E}_{\text{tr}}(\omega) e^{-i\omega t}$. The Fourier transform of the complex amplitude $\mathbf{E}_{\text{tr}}(\omega)$ can therefore be utilized again to obtain the transmitted field in the form

$$\mathbf{E}_{\text{tr}}(t) = X(t)\mathbf{A}, \quad (5)$$

where $f(\omega)$ and $X(t)$ are 2×2 complex matrices related by the following equation:

$$X(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikd} e^{i\frac{2\pi\eta}{k} f(\omega)d} E_0(\omega) e^{-i\omega t} d\omega. \quad (6)$$

The outlined description of the transmitted γ -radiation field can be applied for Mössbauer absorbers exhibiting arbitrary hyperfine structure and for arbitrary periodic absorber motion described by $l(t)$. The hyperfine interactions are described in the coherent scattering amplitude, which may also include combined electric and magnetic interactions or simultaneous contributions from multiple nuclear surroundings [10,31]. The index of refraction model can be connected to other standard approaches for describing nuclear resonant scattering in the forward direction, including the multiple scattering in Mössbauer transmission and the nuclear forward scattering of synchrotron radiation [29–31].

For the incident γ -photon field in a pure polarization state, the intensity $I_{\text{tr}}(t)$ corresponding to the measured time-delay histograms could be calculated by taking $|\mathbf{E}_{\text{tr}}(t)|^2$. For generalizing the description to also include the mixed polarization states, the density matrix formalism can be applied [28,32]. The transmitted radiation intensity is then given as

$$I_{\text{tr}}(t) = |\mathbf{A}|^2 \text{Tr}[X(t)\rho X^\dagger(t)], \quad (7)$$

where $X^\dagger(t)$ is the Hermitian conjugate of $X(t)$ and ρ is the density matrix of the radiation coming from the source. For the special case of the unpolarized source radiation, the corresponding density matrix ρ is equal to $\mathbb{I}/2$, where $\mathbb{I}$ is 2×2 identity matrix. For a general case, the density matrix can be used to describe the polarization effects in nuclear resonant scattering induced, e.g., by magnetically split Mössbauer absorbers using the external magnetic field. Such conditions can be interesting for the experiments, where the vibrational control of the transmitted radiation intensity can be combined with the magnetic control of the γ -ray polarization.

The intensity of the transmitted field in Mössbauer time coincidence measurements is obtained by registering the time-delayed photons passing through the absorber into the detector and measuring the time-delay interval. When the time delay is taken with respect to a driving signal of the absorber motion, the intensity $I_{\text{tr}}(t)$ must be averaged over all possible excited state formation times (e.g. the first excited state of ^{57}Fe recorded by detection of the 122.1-keV photon at time t_0). This leads to the resulting form of the detected time-delay histograms:

$$\bar{I}_{\text{tr}}(t) = C \int_{-\infty}^{\infty} I_{\text{tr}}(t) dt_0, \quad (8)$$

where C is a constant. In real time coincidence detections, these histograms can also be affected by nonresonant background, additional exponential count rate decrease, and finite time resolution of the experimental setup [3]. These effects will not be considered in this work as they have no direct influence on the discussed results.

B. Harmonic expansion of absorber motion

In the literature the time-delay histograms have been described by expanding the periodic motion as a sum of

harmonic contributions with integer multiples of the fundamental angular frequency Ω [22,23] as follows:

$$l(t) = \sum_{k=1}^K \alpha_k \sin(k\Omega t + \vartheta_k). \quad (9)$$

The integer K can, in principle, be infinitely high and Eq. (9) defines the Fourier decomposition of $l(t)$ with amplitudes α_k and phases ϑ_k . The incident field $\mathbf{E}_0(t)$ can be expanded by substituting Eq. (9) into Eq. (2) and applying the Jacobi-Anger

expansion of the complex exponential terms [12]:

$$e^{i\frac{\omega_s}{c}\alpha_k \sin(k\Omega t + \vartheta_k)} = \sum_{n_k=-\infty}^{\infty} J_{n_k}(p_k) e^{in_k(k\Omega t + \vartheta_k)}, \quad (10)$$

where $J_{n_k}(p_k)$ is the Bessel function of the first kind and order n_k . The argument p_k of the Bessel function is the modulation index defined as $p_k = \frac{\omega_s \alpha_k}{c}$. The incident field expansion can then be rewritten analogously to previous descriptions [22,23]:

$$\mathbf{E}_0(t) \approx \sum_{n_1} \sum_{n_2} \cdots \sum_{n_K} \mathbf{A} e^{i\omega_s t_0} e^{-\frac{\Gamma}{2\hbar}(t-t_0)} \Theta(t-t_0) J_{n_1}(p_1) e^{in_1 \vartheta_1} J_{n_2}(p_2) e^{in_2 \vartheta_2} \cdots J_{n_K}(p_K) e^{in_K \vartheta_K} e^{-i[\omega_s - (n_1 + 2n_2 + \cdots + Kn_K)\Omega]t}, \quad (11)$$

where the sums run from $-\infty$ to ∞ .

Following the steps in Sec. II A using Eq. (11), the time-delay histogram can be described by the formula

$$\begin{aligned} \bar{I}_{\text{tr}}(t) &= \frac{C|\mathbf{A}|^2}{2\pi} \sum_{n_1} \sum_{n_2} \cdots \sum_{n_K} \sum_{m_1} \sum_{m_2} \cdots \sum_{m_K} J_{n_1}(p_1) J_{n_2}(p_2) \cdots J_{n_K}(p_K) J_{m_1}(p_1) J_{m_2}(p_2) \cdots J_{m_K}(p_K) \\ &\times e^{i[(n_1 - m_1)\vartheta_1 + (n_2 - m_2)\vartheta_2 + \cdots + (n_K - m_K)\vartheta_K]} e^{i[(n_1 - m_1) + 2(n_2 - m_2) + \cdots + K(n_K - m_K)]\Omega t} Q_{n_1, n_2, \dots, n_K}^{m_1, m_2, \dots, m_K} \end{aligned} \quad (12)$$

consisting of $2K$ sums, where

$$Q_{n_1, n_2, \dots, n_K}^{m_1, m_2, \dots, m_K} = \int_{-\infty}^{\infty} \frac{\text{Tr}\{e^{i\frac{2\pi\eta}{k}f[\omega - (n_1 + 2n_2 + \cdots + Kn_K)\Omega]d} \rho e^{-i\frac{2\pi\eta}{k}f^\dagger[\omega - (m_1 + 2m_2 + \cdots + Km_K)\Omega]d}\}}{(\omega - \omega_s)^2 + \left(\frac{\Gamma}{2\hbar}\right)^2} d\omega. \quad (13)$$

The intensity $\bar{I}_{\text{tr}}(t)$ depends on the motion parameters p_k and ϑ_k and on the spectral overlaps of the source emission line with the absorber transition lines shifted by multiples of Ω . These spectral lines create a so-called spectral comb, which is the result of the γ -photon transmission in the reference frame of the vibrating absorber [1,5].

The complex terms in Eqs. (12) and (13) could be further simplified using their harmonic expansion [1]. Difficulties of this calculation for a general periodic absorber motion come mainly from the summation over a large number of indexes n_k and m_k , $k = 1, 2, \dots, K$.

III. FOURIER SERIES DESCRIPTION

The time-delay histograms $\bar{I}_{\text{tr}}(t)$ described by Eq. (12) exhibit a periodicity with the fundamental angular frequency Ω and as such they can be decomposed using the Fourier series expansion. We use the complex form of the expansion,

$$\bar{I}_{\text{tr}}(t) = \sum_{M=-\infty}^{\infty} G_M e^{iM\Omega t}, \quad (14)$$

where this form can be directly derived from Eq. (12) without computing the integrals for the coefficients. This can be achieved by separating the terms satisfying the condition $(n_1 - m_1) + 2(n_2 - m_2) + \cdots + K(n_K - m_K) = M$. Let us assume for convenience that $\frac{C|\mathbf{A}|^2}{2\pi} = 1$, so the complex Fourier coefficients take the form

$$\begin{aligned} G_M &= \sum_{n_1} \sum_{n_2} \cdots \sum_{n_K} \sum_{m_1} \sum_{m_2} \cdots \sum_{m_K} J_{n_1}(p_1) e^{in_1 \vartheta_1} J_{n_2}(p_2) e^{in_2 \vartheta_2} \cdots J_{n_K}(p_K) e^{in_K \vartheta_K} J_{m_1}(p_1) e^{-im_1 \vartheta_1} \\ &\times J_{m_2}(p_2) e^{-im_2 \vartheta_2} \cdots J_{m_K}(p_K) e^{-im_K \vartheta_K} Q_{n_1 + 2n_2 + \cdots + Kn_K, m_1 + 2m_2 + \cdots + Km_K}^{m_1 + 2m_2 + \cdots + Km_K} \delta_{n_1 + 2n_2 + \cdots + Kn_K, m_1 + 2m_2 + \cdots + Km_K + M}. \end{aligned} \quad (15)$$

By rewriting the summation indexes, the coefficients can be written in a simplified form, in which the motion-dependent and energy-dependent terms can be separated, leading to the equation

$$G_M = e^{iM\vartheta_1} \sum_{N=-\infty}^{\infty} P_{N+M} P_N^* Q_{N+M}^N, \quad (16)$$

where $*$ denotes the complex conjugate,

$$P_N = \sum_{n_2} \sum_{n_3} \cdots \sum_{n_K} J_{n_2}(p_2) e^{in_2(\vartheta_2 - 2\vartheta_1)} J_{n_3}(p_3) e^{in_3(\vartheta_3 - 3\vartheta_1)} \cdots J_{n_K}(p_K) e^{in_K(\vartheta_K - K\vartheta_1)} J_{N - (2n_2 + 3n_3 + \cdots + Kn_K)}(p_1), \quad (17)$$

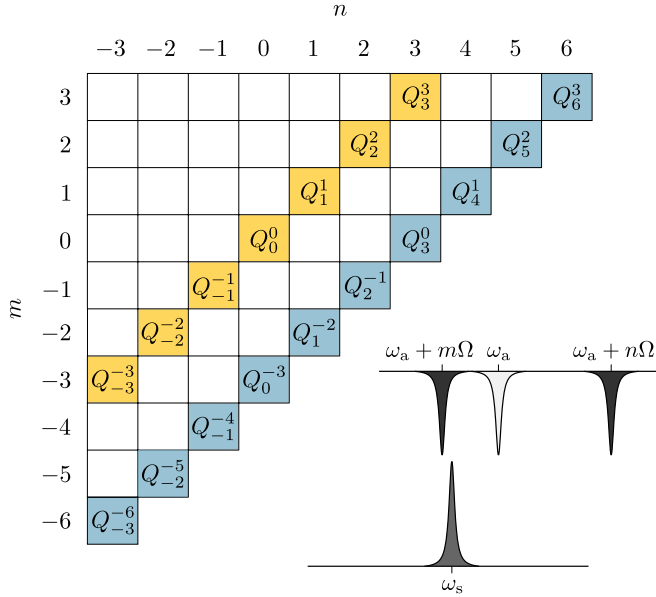


FIG. 2. Top left: Schematic representation of the Q_n^m matrix. Yellow squares show the diagonal matrix elements Q_n^N . Blue squares show the elements Q_{N+M}^N shifted from the diagonal by $M = 3$. These are the elements used for calculating the G_3 Fourier coefficient of the γ -ray intensity signal. Bottom right: Schematic illustration of the overlap between the emission spectral line with the mean angular frequency ω_s and the absorption spectral lines with mean angular frequencies ω_a , $\omega_a + n\Omega$, and $\omega_a + m\Omega$. The overlap with the shifted absorption resonances by $n\Omega$ and $m\Omega$ influences the corresponding Q_n^m matrix element value according to Eq. (18).

and the spectral contribution Q_{N+M}^N now depends only on two indexes, N and M , according to

$$Q_n^m = \int_{-\infty}^{\infty} \frac{\text{Tr}\{e^{i\frac{2\pi\eta}{k}f(\omega-n\Omega)d} \rho e^{-i\frac{2\pi\eta}{k}f^*(\omega-m\Omega)d}\}}{(\omega - \omega_s)^2 + (\frac{\Gamma}{2\hbar})^2} d\omega. \quad (18)$$

Equations (16) and (18) show that Fourier coefficients G_M depend on all the M th off-diagonal elements of the Q_n^m matrix, which describes spectral overlapping of the source line with n th and m th shifted nuclear resonances. This is graphically represented in Fig. 2.

Further simplification of the problem can be achieved by applying the integral form of Bessel functions:

$$J_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(ny - x \sin y)} dy. \quad (19)$$

By substituting Eq. (19) into Eq. (17), the G_M coefficients in Eq. (16) can be rewritten as

$$G_M = \sum_{N=-\infty}^{\infty} R_{N+M} R_N^* Q_{N+M}^N, \quad (20)$$

where the motion-dependent terms take a simple form:

$$R_N = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i[Ny - \sum_{k=1}^K p_k \sin(ky - \vartheta_k)]} dy. \quad (21)$$

Furthermore, by substituting $y = -\Omega t$ in Eq. (21), we get the time integral form of the R_N coefficient:

$$R_N = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{i[\sum_{k=1}^K p_k \sin(\Omega t + \vartheta_k) - N\Omega t]} dt, \quad (22)$$

where T is the motion period. By comparing the sum in the exponent of the integrand with Eq. (9) defining the absorber motion, we can see that they differ only in the factor of ω_s/c . Consequently, Eq. (22) can be expressed as

$$R_N = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{i\frac{\omega_s}{c}l(t)} e^{-iN\Omega t} dt. \quad (23)$$

Equation (23) shows that the R_N coefficients, which are used in the Fourier series description of the time-delay histograms (14) using complex coefficients G_M in Eq. (20), are the Fourier coefficients of $e^{i\frac{\omega_s}{c}l(t)}$, i.e.,

$$e^{i\frac{\omega_s}{c}l(t)} = \sum_{N=-\infty}^{\infty} R_N e^{iN\Omega t}. \quad (24)$$

Equations (20) and (24) introduce an elegant way of finding the relation between the Fourier series of the time-delay histograms and the absorber periodic motion. This can bring a deeper insight into the problem of the transmission of γ -photons through an acoustically modified absorber. Additionally, in comparison to results in Sec. II B and analogous descriptions of the modulated γ -ray intensity in the literature, the presented results (considering the periodic intensity waveforms) allow one to reduce the number of summation indices and decrease the computation time. At the same time, Eq. (20) shows that the parameters of the spectral conditions and the parameters of the periodic motion are separated in the corresponding Q_{N+M}^N and $R_{N+M} R_N^*$ terms. This can be an advantage over numerical calculations [14] for specific purposes, e.g., where only one set of parameters is being evaluated and the other is kept unchanged.

IV. SIMPLE HARMONIC MOTION OF SINGLET ABSORBER

In this section we show the equivalency of the Fourier series approach to describing time-delay histograms with the description presented in the literature for a special case [1]. Namely, let us consider a singlet absorber, where ^{57}Fe does not exhibit additional splitting of the nuclear energy levels [11] and only a single transition with the mean energy $\hbar\omega_a$ can be induced. The matrix element Q_n^m of the time-delay histogram spectral contribution then simplifies to

$$Q_n^m = \int_{-\infty}^{\infty} \frac{e^{-i\frac{d_{\text{eff}}}{2} \left(\frac{\frac{\Gamma}{2\hbar}}{\omega - n\Omega - \omega_a + i\frac{\Gamma}{2\hbar}} - \frac{\frac{\Gamma}{2\hbar}}{\omega - m\Omega - \omega_a - i\frac{\Gamma}{2\hbar}} \right)}}{(\omega - \omega_s)^2 + (\frac{\Gamma}{2\hbar})^2} d\omega, \quad (25)$$

where d_{eff} is the effective thickness of the absorber [11]. Next, we assume a simple harmonic absorber motion in the form

$$l(t) = L \sin(\Omega t + \varphi), \quad (26)$$

where L is the harmonic vibration amplitude and φ is the initial phase. Since according to the Jacobi-Anger expansion,

the complex motion exponential $e^{i\frac{\omega_s}{c}l(t)}$ can be written as

$$e^{i\frac{\omega_s}{c}L \sin(\Omega t + \varphi)} = \sum_{N=-\infty}^{\infty} J_N(p) e^{iN(\Omega t + \varphi)}, \quad (27)$$

where $p = \omega_s L/c$, the general coefficients R_N in Eq. (24) simplify to

$$R_N = J_N(p) e^{iN\varphi}, \quad (28)$$

and

$$G_M = \sum_{N=-\infty}^{\infty} J_{N+M}(p) e^{iM\varphi} J_N(p) Q_{N+M}^N, \quad (29)$$

where the spectral contribution terms are given by Eq. (25). Substituting Eq. (14) gives the resulting time-delay histogram described for the considered conditions:

$$\bar{I}_{tr}(t) = \sum_{M,N=-\infty}^{\infty} J_{N+M}(p) J_N(p) Q_{N+M}^N e^{iM(\Omega t + \varphi)}. \quad (30)$$

By substituting the indexes $N + M = n$ and $N = m$ and taking the real part of Eq. (30), the time dependence $\bar{I}_{tr}(t)$, which is a real-valued function, can be simplified into the formula of Vagizov *et al.* [1].

V. TRAPEZOID ABSORBER MOTION

The full utilization of the Fourier series description is demonstrated on the application of the trapezoid motion profile. The trapezoid motion has been used in γ -optical experiments to generate the sequences of γ -pulses and to induce short-term absorption [14]. Such intensity signals may find applications in metrology, quantum information, or γ -optical communication [2–5,9].

The trapezoid motion profile can be defined as

$$l(t) = \begin{cases} -\frac{2L}{\xi}t - \frac{LT}{\xi}, & t \in \left(-\frac{T}{2}, -\frac{T-\xi}{2}\right), \\ -L, & t \in \left(-\frac{T-\xi}{2}, -\frac{\xi}{2}\right), \\ \frac{2L}{\xi}t, & t \in \left(-\frac{\xi}{2}, \frac{\xi}{2}\right), \\ L, & t \in \left(\frac{\xi}{2}, \frac{T-\xi}{2}\right), \\ -\frac{2L}{\xi}t + \frac{LT}{\xi}, & t \in \left(\frac{T-\xi}{2}, \frac{T}{2}\right), \end{cases} \quad (31)$$

where $l(t)$ follows the periodicity relation $l(t + T) = l(t)$. Figure 3 shows the plot of the trapezoid time dependence on the interval $\langle -\frac{T}{2}, \frac{T}{2} \rangle$. The parameters describing $l(t)$ are the motion amplitude L and the parameter ξ describing the time interval of the linear translation with constant velocity $v = \frac{2L}{\xi}$, where $0 < \xi < \frac{T}{2}$. Special limit cases of the trapezoid motion are the rectangular motion ($\xi \rightarrow 0$) and the triangular motion ($\xi \rightarrow \frac{T}{2}$).

The resulting R_N coefficients can be derived directly by substituting Eq. (31) into Eq. (23). For the index values $N \neq -\frac{pT}{\pi\xi}$, 0, and $\frac{pT}{\pi\xi}$, the coefficients take a simplified form:

$$R_N = \left(\frac{\frac{\xi}{T}}{p - \pi N \frac{\xi}{T}} + \frac{1}{\pi N} \right) \sin \left(p - \pi N \frac{\xi}{T} \right) + \left(\frac{\frac{\xi}{T}}{p + \pi N \frac{\xi}{T}} - \frac{1}{\pi N} \right) (-1)^N \sin \left(p + \pi N \frac{\xi}{T} \right), \quad (32)$$

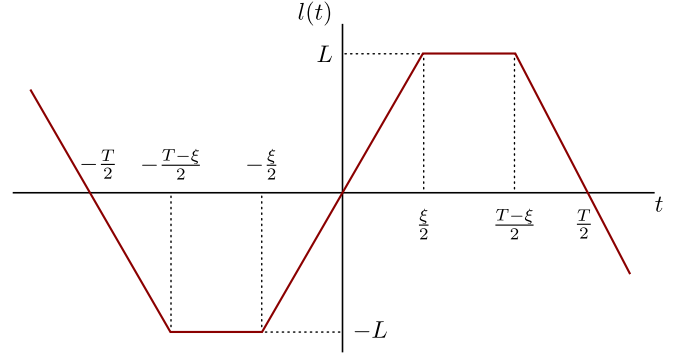


FIG. 3. Trapezoid time dependence of the absorber motion given by the displacement $l(t)$. The motion profile is shown on the selected interval showing one period of the absorber vibration.

where $p = \omega_s L/c$. For $N = 0$ and $N = \pm \frac{pT}{\pi\xi}$ (if such N exist), the remaining coefficients are

$$R_0 = \cos(p) - 2 \frac{\xi}{T} \left[\cos(p) - \frac{\sin(p)}{p} \right], \quad (33)$$

$$R_{N=-\frac{pT}{\pi\xi}} = \frac{\xi}{T} \left[(-1)^N + \frac{\sin(2p)}{2p} \right], \quad (34)$$

$$R_{N=\frac{pT}{\pi\xi}} = \frac{\xi}{T} \left[1 - (-1)^N \frac{\sin(2p)}{2p} \right]. \quad (35)$$

We can see that all R_N coefficients are real. Additionally, the symmetry property $R_{-N} = R_N$ can be derived for even N values and $R_{-N} = -R_N$ for odd N values.

The dependence of the time-delay histograms (14) with the Fourier series coefficients (20) on the trapezoid motion parameters can now be analyzed for the selected spectral conditions. For simplicity, let us again consider the singlet absorber as described in Sec. IV. The motion period $T = 100$ ns corresponding to the fundamental frequency $\Omega/2\pi = 10$ MHz is further used for the presented simulations.

Figure 4 shows the dependence of the calculated time-delay histograms on the absorber displacement amplitude L . The simulations were performed for the constant value of the ξ/T ratio set to 0.25. This corresponds to the case where the

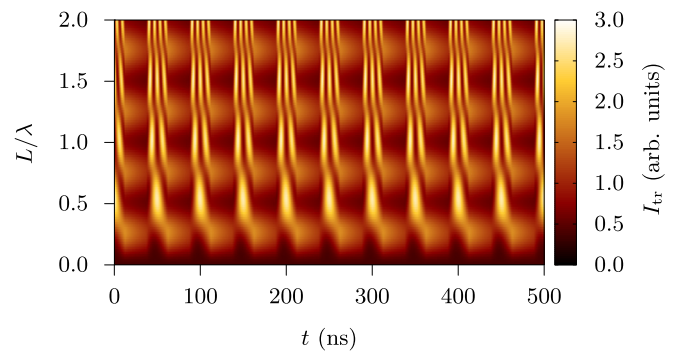


FIG. 4. Intensity signals in the form of the γ -ray time-delay histograms. The horizontal axis shows time delay of the γ -photons with respect to the zero point of the trapezoid motion time dependence. The vertical axis shows the displacement amplitude in units of the radiation wavelength. Intensity values are given in the color scale.

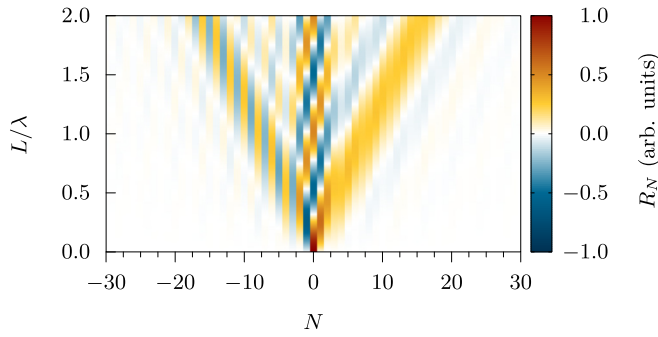


FIG. 5. Fourier coefficients of the complex motion exponential $e^{i \frac{\Omega}{c} l(t)}$ for the trapezoid displacement $l(t)$. The horizontal axis shows the index N of the corresponding Fourier coefficient. The vertical axis shows the displacement amplitude in units of the radiation wavelength. The R_N values are given in the color scale.

absorber spends one half of the period in rest and the other half performing a linear motion towards or from the γ -photon source (see Fig. 3). The amplitude L is shown in the multiples of the γ -photon wavelength $\lambda = \frac{2\pi c}{\omega_s}$ (which is approximately 86 pm for 14.4-keV photons corresponding to the nuclear transition of ^{57}Fe). The resonance condition $\omega_s = \omega_a$ was set for the calculations.

The intensity waveforms exhibit significant qualitative changes for L values near specific levels, given as either even or odd multiples of the quarters of the radiation wavelength. These conditions can be expressed by the following equations:

$$L \approx j \frac{\lambda}{2}, \quad (36)$$

$$L \approx (2j + 1) \frac{\lambda}{4}, \quad (37)$$

where j is a positive integer. Namely, for the amplitude values satisfying condition (36), a set of j well-resolved pulses is being formed in the same time intervals in which the absorber exhibits its linear motion. The total duration of the set of j pulses is therefore equal to ξ and the pulse width decreases with increasing j . In time intervals where the absorber velocity is zero, a monotonous slowly decreasing profile can be seen. For the amplitudes corresponding to condition (37), the γ -ray pulses are suppressed and the monotonous part is enhanced. Moreover, an abrupt shift of the pulses and the formation of an additional pulse can be seen when the amplitude L exceeds the corresponding $(2j + 1) \frac{\lambda}{4}$ region.

The time-delay histograms can be further inspected using the Fourier series description. The dependence of the R_N coefficients on the displacement amplitude L is shown in Fig. 5. The corresponding Fourier coefficients of the time-delay histograms G_M are shown in Fig. 6. Regions where the time-delay histograms change their overall shape can be clearly seen in the frequency domain. These are the regions in which the real or imaginary parts of selected Fourier coefficients approach zero levels when transitioning between their positive and negative values (see the white horizontal zones in Fig. 6).

Figure 5 exhibits a clear V shape, which shows that the highest significant N th harmonic contributions to $\bar{I}_{tr}(t)$

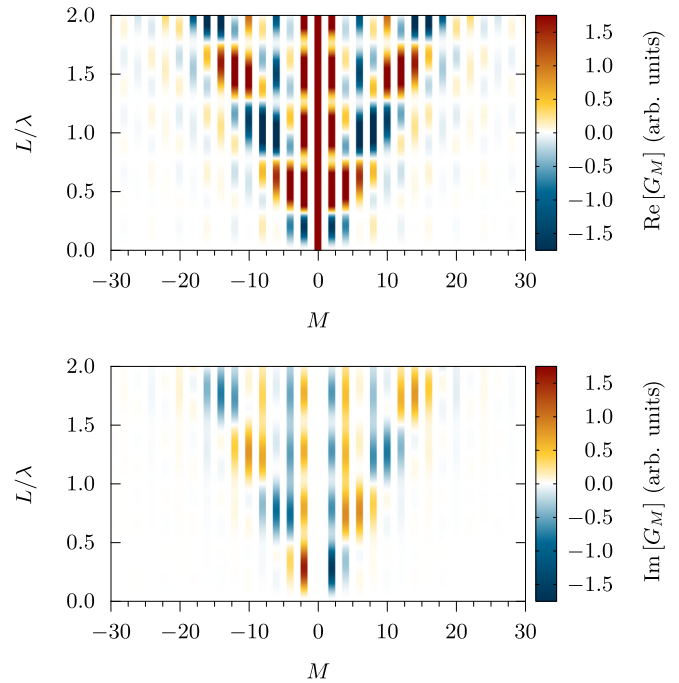


FIG. 6. Real and imaginary parts of the γ -radiation intensity Fourier coefficients. Horizontal axes show the index M of the corresponding Fourier coefficient. The vertical axis shows the displacement amplitude in units of the radiation wavelength. The $\text{Re}[G_M]$ and $\text{Im}[G_M]$ values are given in the color scale.

linearly increase with increasing L . This is also reflected in the shape of the G_M graph in Fig. 6, showing the close connection between the Fourier decomposition of $\bar{I}_{tr}(t)$ and of $e^{i \frac{\Omega}{c} l(t)}$. Additionally, Fig. 6 shows that the resonance condition $\omega_s = \omega_a$ results in $G_M = 0$ for all odd values of M . Consequently, the time dependence of the radiation intensity exhibits a periodicity with double the fundamental frequency in comparison to the motion frequency $\Omega/2\pi$. This can be seen in Fig. 4 where the period of the formed γ -photon pulses is 50 ns, which is half the motion period.

Figure 7 displays time-delay histograms together with the corresponding R_N and G_M coefficients for selected amplitudes of the absorber displacement, namely, $L = j \frac{\lambda}{2}$ with $j = 1, 2, 3$, and 4. The R_N coefficients can be calculated by substituting the respective $p = j\pi$ values in Eqs. (32)–(35). For $N \neq -\frac{pT}{\pi\xi}$, 0, and $\frac{pT}{\pi\xi}$, the coefficients take the form

$$R_N = (-1)^{j+1} \frac{2j^2 \sin(\pi N \frac{\xi}{T})}{\pi N (j - N \frac{\xi}{T})(j + N \frac{\xi}{T})} \quad (38)$$

for even N values and

$$R_N = (-1)^{j+1} \frac{2j \frac{\xi}{T} \sin(\pi N \frac{\xi}{T})}{\pi (j - N \frac{\xi}{T})(j + N \frac{\xi}{T})} \quad (39)$$

for odd N values. Equations (38) and (39) describe the oscillating sinc-like modulation of R_N coefficients with increasing N (see the top-right plots with the blue bars in Fig. 7). Analogous modulation can also be observed for the corresponding G_M coefficients (bottom-right plots with green and red bars in

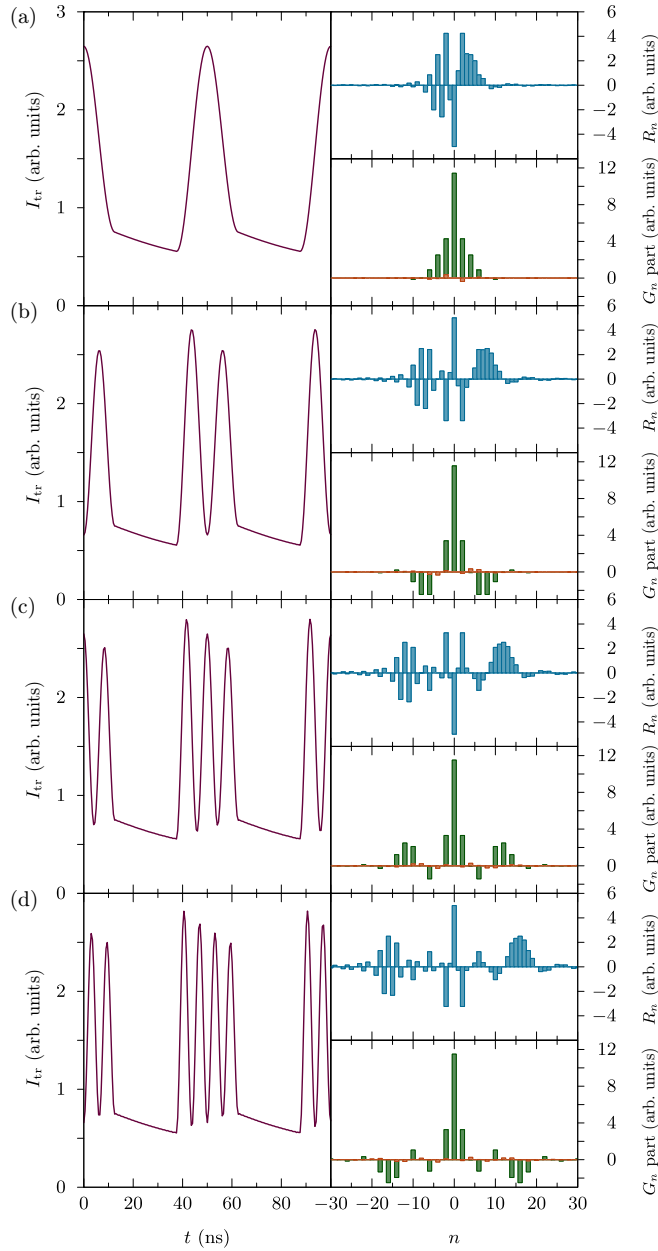


FIG. 7. Selected γ -optical intensity signals (left, purple lines), corresponding Fourier coefficients (bottom-right, $\text{Re}[G_n]$ with green rectangles, $\text{Im}[G_n]$ with red rectangles), and the Fourier coefficients of the complex motion exponential (top-right, blue rectangles). Simulations correspond to the vibration amplitude values (a) $L = \frac{\lambda}{2}$, (b) $L = \lambda$, (c) $L = \frac{3\lambda}{2}$, and (d) $L = 2\lambda$.

Fig. 7). This results in the increase of the number of pulses within the $\frac{T}{4} = 25$ ns time interval with increasing L .

VI. CONCLUSIONS

The theoretical description of γ -radiation intensity signals realized by the time coincidence detections of recoilless γ -photons was presented. Periodic waveforms of γ -rays were described semiclassically by applying the index of refraction model in the absorber reference frame together with the Fourier analysis of the periodic time profiles. The γ -optical signals can be decomposed into their harmonic contributions with their amplitudes tunable by the absorber motion parameters.

A general relation between the Fourier coefficients of the intensity signal and the complex exponential of the motion profile was derived. The results can be applied for an arbitrary periodic motion and can find usage in the analysis and optimization problems within experiments utilizing γ -photon sources. These include optimization of arbitrarily selected γ -optical signal shapes, analysis of experiments utilizing general nuclear environments and photon polarization states, or extracting information on a studied motion by inspecting the frequency domain of the measured time-delay histograms. Additionally, the potential of the presented formalism for nuclear resonant scattering experiments utilizing synchrotron radiation as a source of resonant x-ray photons could be further investigated. However, such a study would require incorporating the synchronized nature and the temporal and spectral properties of the synchrotron radiation pulses into the theoretical framework, which is beyond the scope of the present work.

The Fourier series description was applied for analyzing particular intensity signals generated using the trapezoid absorber motion. The selected simulations demonstrate the connection between the γ -radiation waveforms and the corresponding motion profiles. Namely, the systems of γ -ray multipulse signals with increasing number of pulses within one period and decreasing pulse widths are described in the frequency domain. The modulation of the Fourier coefficients and its evolution with increasing the absorber displacement amplitude are described.

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