

Topological-to-topological transition induced by on-site nonlinearity in a one-dimensional topological insulator

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(Received 2 February 2025; accepted 19 December 2025; published 24 February 2026)

Recent studies have extended the notion of band topology to nonlinear systems by defining nonlinear counterparts of eigenvalue problems. They have found the nonlinearity-induced topological transition regarding edge-localized eigenvectors, while it has required complicated nonlinearity such as off-diagonal one. Thus, the existence of nonlinearity-induced edge modes has been unclear under homogeneous on-site nonlinearity, which is ubiquitously found in nature. We here reveal that such on-site nonlinearity can induce transitions of topological modes, where topological modes converging to zero begin to converge to nonzero values. Since such nonlinearity-induced transition keeps the bulk band topology unchanged, we can regard it as a transition from a conventional topological mode to one unique to nonlinear systems. We analyze a nonlinear eigenvalue problem by rewriting it to a dynamical system in the spatial direction and clarify that the nonlinearity-induced transition is a result of the bifurcation in the spatial dynamics. We also propose a possible setup to observe the nonlinearity-induced transition that uses a gradual amplification of nonlinear waves. These results provide a general designing principle of topological insulators controlled by nonlinearity.

DOI: [10.1103/PhysRevResearch.8.L012045](https://doi.org/10.1103/PhysRevResearch.8.L012045)

Nonlinear effects appear in various physical systems, including photonics [1], fluids [2], and interacting bosons [3]. In particular, on-site nonlinearity lies at the center of studies of nonlinear physics, because it is obtained from the mean-field approximation of short-range many-body interactions. The leading order of such on-site nonlinearity is known as the Kerr nonlinearity [4] in photonics and also appears in the Gross-Pitaevskii equation [5,6] of cold atoms. The on-site nonlinearity induces unique wave phenomena such as solitons [7–9].

Recent studies have also investigated the effect of nonlinearity on band topology. Materials with nontrivial band topology, such as topological insulators [10,11], exhibit topological gapless modes localized at the edge of the sample, which is known as the bulk-boundary correspondence [12–16]. While the notion of band topology can be extended to dispersion relations in various classical and quantum systems [17–27], the nonlinear effect in such systems can alter the bulk-boundary correspondence [28–51]. By extending the eigenvalue problem to nonlinear systems, previous studies [52–56] have defined the nonlinear topological invariants and discussed their bulk-boundary correspondence. Some previous studies [50,53–55] have also revealed the amplitude dependence of edge-localized eigenvectors, which is termed the nonlinearity-induced topological transitions

and leads to unique behaviors such as remaining amplitude of topological modes in the bulk. However, while the effects of on-site nonlinearity on topological solitons and gapless band structures were discussed [30,52,57], its effects on edge-localized eigenvectors were unelucidated, and thus the existence of nonlinearity-induced edge eigenstates was unclear. From the theoretical viewpoint, since the on-site nonlinearity breaks the sublattice symmetry assumed in previous research [55,56], one needs to extend the analytical technique on nonlinear topological modes under the absence of such symmetries.

In this Letter, we reveal the existence of the transition of topological localized modes induced by uniform on-site nonlinearity. In this nonlinearity-induced transition, the long-range behavior of topological modes is changed; after the transition, they converge to nonzero even in the thermodynamic limit. Such localized modes with nonzero remaining amplitudes are similar to those observed in the nonlinearity-induced topological transition in previous studies. However, the nonlinearity-induced transition discussed here is not a topological transition; i.e., it accompanies no gap closing and thus the band topology remains unchanged. We analyze such nonlinearity-induced transitions by using a dynamical system in the spatial direction, which is a nonlinear extension of a transfer matrix. We find the bifurcation of fixed points in the spatial dynamics and that those bifurcations correspond to the nonlinearity-induced transitions of topological localized modes. We also propose the observation protocol of the nonlinearity-induced transition by the gradual amplification of nonlinear waves. These results clarify the universal correspondence between the transition of topological modes and the bifurcation in dynamical systems. In addition, because of the feasibility of on-site nonlinearity, our proposal is directly

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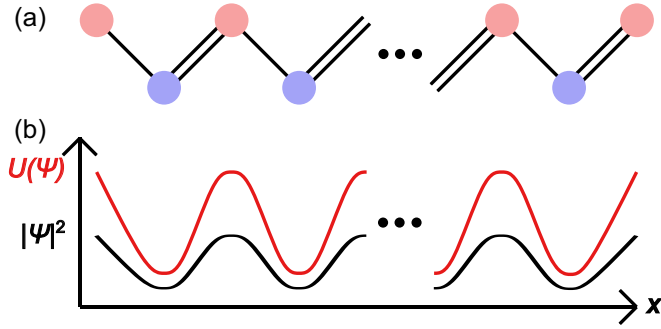


FIG. 1. Schematic of the nonlinear SSH model. (a) The nonlinear SSH model and the boundary condition. The model has two sublattices represented by the red and blue circles. The strengths of linear hoppings are alternately modulated. The double lines represent stronger hopping than the single lines. Sublattice A (red circles) lies at both open boundaries. Under this boundary condition, we obtain a localized mode at either end. (b) Effective potential induced by the on-site nonlinearity. The red curve shows the effective on-site potential, which is proportional to the square of the nonlinear wave (the black curve).

relevant to the experimental realization of nonlinearity-induced transitions of topological modes.

Setup. We consider the nonlinear Su-Schrieffer-Heeger (SSH) model [58] with on-site Kerr nonlinearity [29,49,50,53,59–65] shown in Fig. 1, which is a minimal setup of the nonlinearity-induced transition discussed in this Letter. Its time evolution is described as

$$i\partial_t \Psi_A(x) = a\Psi_B(x) + b\Psi_B(x-1) + U(\Psi_A(x))\Psi_A(x), \quad (1)$$

$$i\partial_t \Psi_B(x) = a\Psi_A(x) + b\Psi_A(x+1) + U(\Psi_B(x))\Psi_B(x), \quad (2)$$

$$U(\Psi_{A,B}(x)) = \kappa |\Psi_{A,B}(x)|^2, \quad (3)$$

where $U(\Psi_{A,B}(x))$ corresponds to the self-induced potential and determines the strength of the Kerr nonlinearity at each site. By assuming a periodic solution $\Psi_{A,B}(x; t) = e^{-iEt} \Psi_{A,B}(x)$, we obtain the nonlinear eigenvalue problem

$$E\Psi_A(x) = a\Psi_B(x) + b\Psi_B(x-1) + U(\Psi_A(x))\Psi_A(x), \quad (4)$$

$$E\Psi_B(x) = a\Psi_A(x) + b\Psi_A(x+1) + U(\Psi_B(x))\Psi_B(x). \quad (5)$$

We further assume spatially periodic solutions $\Psi_{A,B}(x) = e^{ikx} \psi_{A,B}(k)$ termed the Bloch ansatz [54,55] and finally obtain

$$\begin{pmatrix} \kappa |\psi_A|^2 & a + be^{-ik} \\ a + be^{ik} & \kappa |\psi_B|^2 \end{pmatrix} \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = E \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}, \quad (6)$$

which is a wave-number-space description of the nonlinear eigenvalue problem and corresponds to the Bloch Hamiltonian in linear systems.

Due to the absence of the superposition law, nonlinear eigenvectors depend on their norm. We note that scaling of the wavefunction $\psi \rightarrow \lambda\psi$ induces the change of the strength of nonlinearity $\kappa \rightarrow \lambda^2\kappa$. In the following, we fix κ to be $\kappa = 1$ and instead modify the strength of nonlinearity by considering eigenvectors with different norms. Specifically, in the calculation of bulk solutions, we focus on special solutions whose

amplitudes are constant independently of the wavenumber, $|\psi_A(k)|^2 + |\psi_B(k)|^2 = w$ [54,55].

By analytically solving Eq. (6), we obtain two types of bulk eigenvectors: ones obtained in the linear limit

$$(\psi_A(k), \psi_B(k)) = \sqrt{w/2}(1, \pm e^{i\theta}) \quad (7)$$

with the eigenvalues being $E = |a + be^{ik}| + \kappa w/2$ and θ representing $\theta = \arg(a + be^{ik})$, and ones without linear counterparts

$$(\psi_A(k), \psi_B(k)) = (r_{\pm}, e^{i\theta}|a + be^{ik}|/(\kappa r_{\pm})) \quad (8)$$

with the eigenvalues being $E = \kappa w$ and $r_{\pm}$ and θ representing $r_{\pm} = \sqrt{[\kappa w \pm \sqrt{(\kappa w)^2 - 4|a + be^{ik}|^2}]/(2\kappa)}$ and $\theta = \arg(a + be^{ik})$. The latter type of bulk eigenvectors appears only at large amplitude $w > 2|a + be^{ik}|/\kappa$ due to the spontaneous inversion-symmetry breaking by nonlinear effects. We note that there are no loop structures in the band structure that are reported in a previous paper [52], the reason for which is the absence of linear on-site terms. By using these bulk nonlinear eigenvectors, we can define the nonlinear Berry phase [53,56] as

$$\nu(w) = \frac{-i}{w} \int_0^{2\pi} dk [\psi_A(k)^* \partial_k \psi_A(k) + \psi_B(k)^* \partial_k \psi_B(k)], \quad (9)$$

where higher-order correction terms are absent due to the $U(1)$ symmetry. The nonlinear SSH model has the time-reversal and spatial inversion symmetries (see the Supplemental Material [66]), and the eigenvectors (7) also preserve them. The nonlinear Berry phases for such eigenvectors are quantized, $\nu(w) = 0, \pi \pmod{2\pi}$. However, the spatial inversion symmetry is spontaneously broken in the eigenvectors (8). In this case, since these eigenvectors are degenerate, we can use the sum of their Berry phases as a quantized topological invariant [66].

One can formulate the bulk-boundary correspondence of the nonlinear system as follows [54,55]: One considers the open boundary condition with a fixed edge amplitude, $|\Psi_A(1)|^2 + |\Psi_B(1)|^2 = w$ and $\Psi_B(0) = 0$, with w being that used to calculate the nonlinear Berry phase. Then, the nonzero (zero) Berry phase corresponds to the existence (absence) of localized gapless modes. While this bulk-boundary correspondence looks similar to the linear one, nonlinearity can produce antilocalized gapless modes in trivial systems with $\nu(w) = 0 \pmod{2\pi}$. However, the existence of antilocalized modes is consistent with the above bulk-boundary correspondence because it focuses on the existence or absence of localized modes. It is also noteworthy that nonlinear topological modes can converge to nonzero values in the limit of $x \rightarrow \infty$, which we cannot normalize and thus is not regarded as conventional eigenvectors in linear cases. In the nonlinear SSH model (6), we can numerically confirm this bulk-boundary correspondence in a wide range of the strength of nonlinearity, while we find that the on-site nonlinearity induces the transition of topological modes from decaying modes to extended modes.

Nonlinearity-induced transition of edge modes. By numerically solving the nonlinear eigenvalue problem [Eqs. (4) and (5)], we find the transition in the converging behavior of the topological edge modes at a critical amplitude. To solve

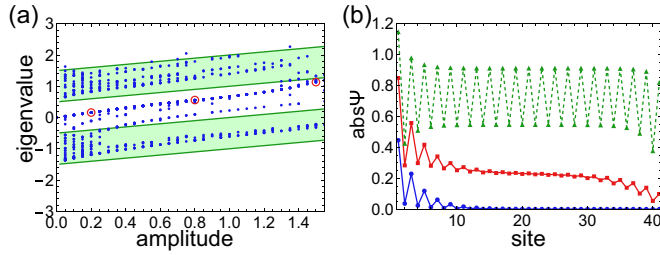


FIG. 2. Eigenvalues and eigenvectors at each amplitude. (a) Amplitude dependence of the nonlinear eigenvalue. Blue dots represent the nonlinear eigenvalues at each edge amplitude $|\Psi_A(1)|^2 + |\Psi_B(1)|^2 = w$ numerically obtained under the open boundary condition. The green bands show the analytically obtained bulk bands. (b) Nonlinear topological modes and their nonlinearity-induced transition. The blue, red, and green dashed polylines show the spatial distribution of absolute values of the topological modes, whose edge amplitudes are $w = 0.2$, $w = 0.8$, and $w = 1.5$, respectively. Their eigenvalues are presented in panel (a) by the red circles. If the edge amplitude is small (blue polyline), the topological mode converges to zero in the bulk, while above a critical amplitude, nonzero amplitudes remain in the bulk. If we consider larger amplitudes, the dimerization is broken, and two sublattices have different amplitudes as shown by the green dashed polyline.

the nonlinear eigenvalue problem, we use the quasi-Newton method [54,67]. We consider the open boundary condition in Fig. 1(a), where we cut the intercell (intracell) bonding at the left (right) edge. Under such a boundary condition, $\nu(w) = \pi \pmod{2\pi}$ corresponds to the appearance of edge modes at the left end, while even in the case of $\nu(w) = 0 \pmod{2\pi}$, we obtain edge modes at the right end [23]. The advantage of assuming this boundary condition is that we obtain exact solutions of edge modes without considering superpositions of left- and right-localized modes and thus can suppress the effect of the boundary condition at the edge opposite to the direction of the localization. In the following, we

fix the parameters $a = -0.5$ and $b = 1$ [68], where the nonlinear Berry phase of the eigenvector (7) becomes $\nu(w) = \pi \pmod{2\pi}$ at any w , and thus left-localized modes appear.

Figure 2(a) shows the amplitude dependence of the nonlinear eigenvalues. Since the effective on-site potential $U(\Psi)$ is proportional to the amplitude w , the nonlinear eigenvalues are shifted following the change of the amplitude, which is consistent with the analytical bulk solutions. We also confirm the existence of gapless modes. Such gapless modes are localized at the left edge, i.e., $|\Psi_A(x_{\text{bulk}})|^2 + |\Psi_B(x_{\text{bulk}})|^2 < |\Psi_A(1)|^2 + |\Psi_B(1)|^2$ at the bulk site $x_{\text{bulk}} \gg 1$. However, we also find that the edge modes have nonzero amplitudes at the bulk sites if the edge amplitude $|\Psi_A(1)|^2 + |\Psi_B(1)|^2$ is larger than a critical value $w_c \sim 0.6$ [cf. Fig. 2(b)], where the eigenvalue of the edge mode is also larger than that of bulk modes in the linear limit, $E_{\text{edge}} > a + b$. This amplitude dependence implies the nonlinearity-induced transition of topological modes from decaying localized modes to nonvanishing ones in the semi-infinite system. While nonvanishing localized modes are not normalizable, they can be observed as periodically oscillating states in a finite system since the nonlinear eigenvectors in Fig. 2 correspond to temporally periodic solutions. We note that remaining amplitudes in the limit of $x \rightarrow \infty$ can also be seen in edge modes emerging by nonlinearity-induced topological phase transitions discussed in previous papers [50,53–55]. Unlike them, the transition observed in Eqs. (4) and (5) accompanies no gap closing, and thus the bulk topology is unchanged, while it is still induced by the change of the strength of the nonlinearity $U(\Psi_{A,B}(x))$.

Dynamical system in the spatial direction. To clarify the existence of the nonlinearity-induced transition of topological in-gap states under the semi-infinite boundary condition, we analyze the spatial distribution of topological modes by rewriting the nonlinear eigenvalue problem into a discrete dynamical system in the spatial direction. By assuming the nonlinear eigenvalue E as a parameter, we obtain the following nonlinear dynamical system from Eqs. (4) and (5):

$$\Psi_A(x+1) = \frac{-a\Psi_A(x) - \kappa|\Psi_B(x)|^2\Psi_B(x) + E\Psi_B(x)}{b}, \quad (10)$$

$$\Psi_B(x+1) = \frac{-b\Psi_B(x) - \kappa|\Psi_A(x+1)|^2\Psi_A(x+1) + E\Psi_A(x+1)}{a}, \quad (11)$$

which corresponds to a nonlinear counterpart of the transfer-matrix representation of an eigenequation. Under the open boundary condition at the left edge $\Psi_B(0) = 0$, we can obtain the spatial distribution of a nonlinear eigenvector in the right semi-infinite system by iteratively calculating the above equations. We note that due to the time-reversal symmetry of the system, all the parameters in Eqs. (10) and (11) are real, and we can assume that topological modes also have real eigenvectors.

The dynamical system [Eqs. (10) and (11)] has some fixed points and periodic solutions. Except for the trivial fixed point $(\Psi_A(x), \Psi_B(x)) = (0, 0)$, those fixed points and periodic solutions correspond to bulk eigenvectors [Eqs. (7) and (8)] obtained from the Bloch ansatz, because periodicity in

Eqs. (10) and (11) indicates the spatial periodicity of the nonlinear eigenvector.

From the viewpoint of the dynamical system, we can regard the converging solutions from far points to fixed points or periodic solutions as localized modes because the corresponding nonlinear eigenvectors have larger amplitudes at the edge, $x = 1$, than in the bulk, $x \gg 1$. As demonstrated in Fig. 3(a), orbits of such converging solutions lie on the stable manifold of a fixed point (A_0, B_0) , which is defined as a point set of (Ψ_A, Ψ_B) such that $(\lim_{x \rightarrow \infty} \Psi_A(x), \lim_{x \rightarrow \infty} \Psi_B(x)) = (A_0, B_0)$ when the initial condition is $(\Psi_A(0), \Psi_B(0)) = (\Psi_A, \Psi_B)$. Figure 3(a) also shows the amplitude dependence of the stable manifold of the fixed point at $(0, 0)$. We can confirm that the stable manifold crosses the $\Psi_B = 0$ axis at

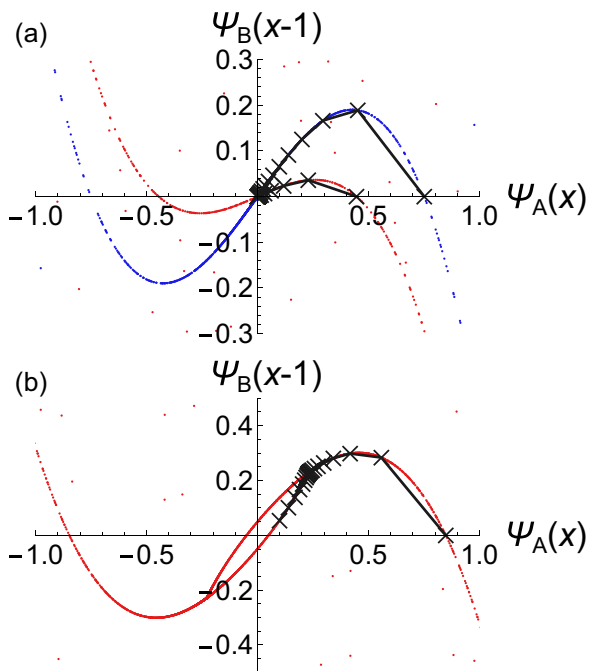


FIG. 3. Stable manifolds and bifurcation of the dynamical system in the spatial direction. (a) Stable manifolds at different eigenvalues. The red and blue dots show the stable manifolds corresponding to edge modes whose edge amplitudes are $w = 0.2$ and 0.5 , respectively. Due to the limitation of the numerical technique, there appear dots far from the curves, which are irrelevant to the stable manifold. The black crosses represent $(\Psi_A(x), \Psi_B(x-1))$ of the edge modes at each x . We can confirm that each component of the edge mode lies along the stable manifolds, and thus expect that the stable manifold determines the behavior of edge modes. (b) Stable manifolds after the transition. The red dots show the stable manifold corresponding to an edge mode whose edge amplitude is $w = 0.8$. The black crosses represent $(\Psi_A(x), \Psi_B(x-1))$ of the edge mode at each x . Nonzero fixed points $\Psi_A(x) = \Psi_B(x) \simeq \pm 0.23$ appear in this parameter regime. As in panel (a), the components of the edge mode lie along the stable manifolds.

a larger point as the nonlinear eigenvalue E becomes larger. Thus, the topological localized mode that satisfies the boundary condition $\Psi_B(0) = 0$ has a larger eigenvalue at a larger amplitude as is seen in Fig. 2(a). It is noteworthy that the dynamical system [Eqs. (10) and (11)] is discrete, while we still obtain curve-shaped stable manifolds at least in the range where we can numerically estimate their shapes [66].

The converging solutions exhibit a transition at $E = a + b$ due to the bifurcation of the fixed point at $(0, 0)$. Corresponding to the emergence of bulk modes at that eigenvalue, the stability of the fixed point $(0, 0)$ is changed; it is a saddle point at $E < a + b$ and becomes a center (i.e., linear stability is marginal) at $E > a + b$ [66]. Instead, we obtain steady solutions $(\Psi_A(x), \Psi_B(x)) = (\pm\sqrt{w/2}, \pm\sqrt{w/2})$, which have a one-dimensional stable manifold shown in Fig. 3(b). Therefore, the solutions corresponding to topological localized modes converge to the nonzero steady solutions $(\Psi_A(x), \Psi_B(x)) = (\pm\sqrt{w/2}, \pm\sqrt{w/2})$ if the nonlinear eigenvalue is larger than $a + b$. Since the eigenvalue becomes larger as the amplitude does, such a transition must occur at a certain

amplitude, which indicates the existence of the nonlinearity-induced transition of topological localized modes.

We note that the nonlinearity-induced transition without gap closing in Figs. 2 and 3 is unique to systems without the sublattice symmetry because under the sublattice symmetry, the eigenvalue of edge modes must be zero, and thus the amplitude change does not alter the spatial dynamics like Eqs. (10) and (11). Instead, nonlinearity-induced topological transitions occur, which are transitions between trivial antilocalized modes and topological localized modes. At the transition point, there emerge spatially homogeneous modes, which correspond to gapless bulk modes; band gaps must be closed at the nonlinearity-induced transition points under the sublattice symmetry. We also note that the bifurcations discussed in the sublattice-symmetric model in a previous paper [55] are different from those in the present work because they are induced by the change of an external parameter and do not solely induce the nonlinearity-induced transitions.

Further transition at a larger amplitude. Corresponding to the emergence of the additional bulk modes (8), the steady solutions $(\Psi_A(x), \Psi_B(x)) = (\pm\sqrt{w/2}, \pm\sqrt{w/2})$ of the dynamical system [Eqs. (10) and (11)] also exhibit a bifurcation. Since such a bifurcation changes the linear stability of the periodic solution and convergent values of the nonlinear dynamics, we can observe a further nonlinearity-induced transition of topological localized modes by considering larger edge amplitudes than $w'_c \sim 1.4$, at which the nonlinear eigenvalue of the localized mode becomes $E = 2(a + b)$. In fact, the localized mode represented by the green dashed poly-line in Fig. 2(b) shows different absolute values between A and B sublattices, unlike that obtained at smaller edge amplitudes and corresponding smaller eigenvalues, $a + b < E < 2(a + b)$. As in the nonlinearity-induced transition at $E = a + b$, we can analytically confirm the change of the linear stability of the periodic solution $(\Psi_A(x), \Psi_B(x)) = (\pm\sqrt{w/2}, \pm\sqrt{w/2})$ (see the Supplemental Material [66]). Thus, this further nonlinearity-induced transition is also understood as the change of the convergent value associated with the bifurcation of the periodic solution.

Observation protocol of the nonlinearity-induced transition by gradual amplification. We here propose a possible observation protocol of the nonlinearity-induced transition by the continuum deformation of nonlinearity. As we have discussed above, the change of the wave amplitude w also alters the nonlinear self-induced potential $U(\Psi_{A,B}(x)) = \kappa |\Psi_{A,B}(x)|^2$. Thus, we can continuously tune the strength of nonlinearity by gradually amplifying the nonlinear wave, which leads to possible observation of the nonlinearity-induced transition. The gradual amplification of the nonlinear wave is realized by simply introducing a uniform gain. We note that since the gain and the on-site Kerr nonlinearity are ubiquitous in, e.g., photonics, one can realize the nonlinear SSH model and the observation protocol in existing setups of topological photonics [60,63,69–71].

We numerically simulate the nonlinear dynamics [Eqs. (1) and (2)] under the existence of a temporally controlled gain (cf. Fig. 4). We consider the system size $L = 1000$. As an initial condition, we consider the excitation only at the edge site $\Psi_A(1) = \sqrt{0.1}$, $\Psi_A(x) = 0$ ($x \geq 2$), $\Psi_B(x) = 0$ and wait until $t = t_0 = 200$, when the bulk-propagating wave goes far from

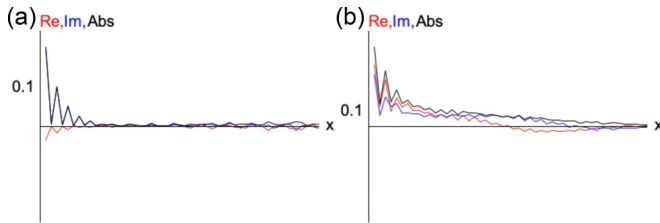


FIG. 4. Dynamical demonstration of the nonlinearity-induced transition. (a) Localized waves observed before introducing the gain. The red, blue, and black polylines show the real, imaginary parts, and absolute values of the nonlinear wave, respectively. The localized wave decays to zero in the bulk. (b) Localized waves after the amplification. We can confirm the nonzero amplitudes and dimerization in the bulk, which are consistent with the nonlinear eigenvector obtained in Fig. 2.

the left edge. Since the initial wave amplitude is small compared to the linear band gap, nonlinear effects are negligible, and thus the initial edge state is well approximated by linear edge modes. Then, we introduce a homogeneous gain whose strength depends on the time as $c(t) = [2(t - t_0) + 1/c_0]^{-1}$. This time dependence of $c(t)$ suppresses the exponential divergence of the strength of nonlinearity and instead amplifies the nonlinear wave almost linearly in time. We note that its modification by, e.g., nonlinear loss inherent in photonics is allowed as long as the suppression of divergence is realized. Finally, we switch off the gain at $t = 300$. In the numerical result, we can confirm that the localized wave penetrates into the bulk at a sufficiently large amplitude. Since the remaining amplitude is characteristic of topological modes after the nonlinearity-induced transition, this penetration implies the existence of the nonlinearity-induced transition and that the nonlinearity-induced edge modes are physical regardless of their nonnormalizability under the semi-infinite boundary condition.

Discussion. We revealed the existence of the nonlinearity-induced transition in the nonlinear SSH model with on-site nonlinearity. Before the transition, topological localized

modes decay to zero like those in linear cases, while after the transition, they converge to nonzero values in the limit of $x \rightarrow \infty$. To analyze the topological modes in the nonlinear SSH model, we rewrite the nonlinear eigenvalue problem into a dynamical system in the spatial direction, which is a nonlinear generalization of a transfer matrix. Then, the nonlinearity-induced transition is understood as a bifurcation of a fixed point at the origin.

From an experimental viewpoint, one can realize the nonlinearity-induced transition by gradual amplification of a nonlinear localized mode, which introduces the continuum deformation of nonlinearity. We note that nonlinear SSH models have been discussed in, e.g., photonics [60,63]. In realistic setups, however, higher-order nonlinearity than the third-order one considered in this work can affect the nonlinear dynamics. Thus, investigating the effect of such higher-order terms remains a crucial issue to experimentally realize the nonlinear topological phenomena.

While a previous study [55] has analyzed the bulk-boundary correspondence in nonlinear systems by using a dynamical system in the spatial direction, it has assumed the sublattice symmetry. Thus, its extension to two-dimensional cases is not straightforward because nontrivial band topology in two dimensions requires no symmetries. In contrast, the present work has extended such a technique to the system without the sublattice symmetry, which can also be extended to higher-dimensional systems. We also note that one can obtain the nonlinear SSH model from the dimensional reduction of a two-dimensional nonlinear Chern insulator [55,66,72], and thus the nonlinearity-induced transition seen in the nonlinear SSH model can also appear in higher-dimensional cases. Such extension to higher-dimensional cases should lead to a comprehensive understanding of the bulk-boundary correspondence in general nonlinear systems.

Acknowledgments. We thank Natsuko Ishida, Satoshi Iwamoto, and Tomonari Mizoguchi for valuable discussions. K.S. is supported by JSPS KAKENHI Grant No. JP24K22848. Y.H. is supported by JSPS KAKENHI Grant No. JP23K25788 and JST CREST, Grant No. JPMJCR19T1.

Data availability. The data supporting this study's findings are available within the article.

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