

Stability analysis of relativistic electron dynamics in Lagrange waveguides

Majid G. Nazarlou¹, Haokun Luo¹, Yunxuan Wei¹, Georgios G. Pyrialakos¹, Mercedeh Khajavikhan^{1,2,*}, and Demetrios N. Christodoulides^{1,2,†}

¹Ming Hsieh Department of Electrical and Computer Engineering, *University of Southern California*, Los Angeles, California 90089, USA

²Department of Physics and Astronomy, *University of Southern California*, Los Angeles, California 90089, USA



(Received 16 June 2025; accepted 8 November 2025; published 22 December 2025)

Recently, the prospect of guiding charged particles by means of Lagrange points has been put forward. What makes this methodology possible is the action of Coriolis forces that dynamically provide stability in static helicoidal potential landscapes. In such an environment, particle guidance can occur indefinitely with minimal losses, without the need for magnetic fields, in a manner analogous to photon transport in optical fibers. Here, we provide a rigorous stability analysis of the relativistic charged-particle dynamics when launched into a Lagrange waveguide—an aspect crucial in establishing the necessary criteria for stable trapping. By considering all the force components involved, our results delineate the stability boundaries under both attractive and repulsive electrostatic conditions. A first-order perturbation approach yields both the spectral eigenvalues and their corresponding eigenvectors associated with the stability of particle dynamics in a six-dimensional phase space. The Larmor radiation is also investigated in terms of pertinent parameters, along with the radiated energy spectral density. Our results enable the design and realization of this new class of Lagrange systems capable of guiding charged particles across a wide range of energy scales, with potential utility in a broad spectrum of applications.

DOI: [10.1103/kst6-rz68](https://doi.org/10.1103/kst6-rz68)

I. INTRODUCTION

Over the years, guiding and controlling the transport behavior of charged particles has had a profound impact on many diverse areas, ranging from electron microscopies [1–4] and lithographies [5–7] to particle accelerators [8–12] and quantum information platforms [13–20]. In accelerator facilities, for example, charged particles are commonly steered and focused by means of powerful magnetic lenses that employ the so-called strong focusing principle, a concept first proposed by Christofilos [21] and independently by Courant *et al.* [22]. This same principle is at work in FODO lattices [23], where a periodic sequence of alternating focusing and defocusing quadrupole magnets ensures beam stabilization. In general, beam stability is of paramount importance as it directly impacts the performance of such systems—an issue that needs to be systematically investigated and addressed for a particular setting under consideration. This aspect also applies to confining setups, as in Paul [24–26] and Penning traps [27,28], as well as in situations where beam particle transport is essential [29,30]. Indeed, charged-particle stability has been at the core of a multitude of efforts aimed at developing relativistic accelerators and free-electron lasers [31,32], to mention a few.

Recently, a new methodology has been proposed for guiding charged particles with minimal energy loss [33], in a manner analogous to photon transport in optical fibers, which have revolutionized modern communications [34–41]. This approach leverages the dynamic behavior of Lagrange points—equilibrium positions encountered in celestial mechanics where gravitational attraction is balanced by centrifugal forces [42,43]. Counterintuitively, at these locations, dynamical stability is introduced via Coriolis forces that, in a Sisyphian manner, tend to capture a body from a distance, even though the potential landscape is locally unstable (a maximum) when viewed within the corotating frame. On this same principle, a Lagrange waveguide can indefinitely trap and transport charged particles [33]; however, unlike celestial mechanics, it relies solely on a single helicoidal electrostatic source to stably sustain guidance, as schematically shown in Figs. 1(a) and 1(b). As opposed to traditional methods that primarily rely on complex arrangements of magnetic or electrostatic lenses to steer particles, here confinement over small cross-sectional areas is achieved in a continuous fashion and with almost zero expenditure of energy, using a simple and versatile arrangement. In general, an electron Lagrange waveguide can be deployed using either attractive or repulsive potentials and can be operated over a wide range of energies. Furthermore, this same system is capable of supporting particle transport in the quantum domain where the guided eigenfunctions are formally analogous to optical modes in dielectric waveguides—an aspect successfully demonstrated in a recent experimental study where light guidance was observed in a twisted defocusing refractive index environment [44]. As indicated in Ref. [33], in principle, one can always identify an equilibrium position in a Lagrange waveguide

[†]Contact author: demetri@usc.edu

^{*}Contact author: khajavik@usc.edu

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI.

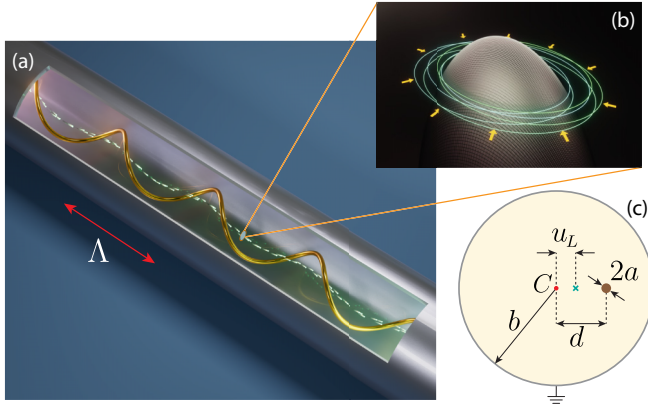


FIG. 1. (a) Schematic of charged-particle beam trajectory in a Lagrange waveguide with helical pitch Λ : The electrons are trapped at a stable Lagrange point and follow a helical trajectory. (b) The effective potential around the Lagrange point, when viewed within the corotating frame, and the restorative effect of the Coriolis force counteracting deviations from equilibrium. (c) The cross section of the Lagrange waveguide under study, showing structural parameters: a grounded outer tube with radius b and center C , encompassing a helical wire of radius a at a distance d from C . The location of the Lagrange point is at a distance u_L from the center.

where, even under relativistic conditions, the electrostatic attraction/repulsion is exactly balanced by the centrifugal force. Nevertheless, at this point, it is by no means clear if indeed dynamic stability is guaranteed, given that acceleration along different axes might influence the various forces involved, especially for high-energy particles. In this respect, a series of important questions arise: Will, for example, any parametric instabilities arise as an electron traverses this periodic beamguide in the relativistic regime, where nonlinearities proliferate? If not, what would determine the dynamical spectrum of an electron beam when injected in the vicinity of a Lagrange equilibrium point?

In this study, we formally investigate the stability properties of relativistic charged particles traversing a Lagrange waveguide. Using rigorous analysis and first-order perturbation methods, we account for all the forces governing electron dynamics within the corresponding six-dimensional phase space. For a given Lagrange structure, we delineate the regions where an electron is stably captured in a confined manner as a function of electron energy and helix voltage. Moreover, we identify the eigenfrequencies and eigenvectors of the linearized dynamical system, which accurately describe the intricate particle dynamics. Notably, our study also provides insight into the radiation spectrum emitted by relativistic electrons trapped along these helical trajectories.

II. THEORETICAL FORMULATION

The cross section of the structure used throughout this study, capable of inducing a Lagrange point, is illustrated in Fig. 1(c). Essentially, this arrangement is comprised of a grounded metallic hollow cylindrical tube of radius b that encompasses a periodically twisted (helix) metallic wire of radius a , biased at a potential V_0 . The helix has a pitch Λ and is positioned at a distance d from the center C of the grounded tube. The location of the induced Lagrange point u_L (to be derived) falling on the line connecting C and the wire is also illustrated in the same figure. Assuming vacuum conditions, we begin our analysis by considering the relativistic equation of motion [45] within the stationary or laboratory frame $m_0 \gamma d\vec{v}/dt = \vec{F} - (\vec{F} \cdot \vec{v}/c^2)\vec{v}$, where m_0 is the rest electron mass, c is the vacuum speed of light, $\vec{v} = (\dot{x}, \dot{y}, \dot{z})$ is the velocity vector, and $\vec{F} = -\nabla G$, where $G(x, y, z)$ represents a general electric potential function. Here, γ is the Lorentz factor $\gamma = 1/\sqrt{1 - \beta^2}$ with $\beta = v/c$. While in principle this problem can be investigated in the actual laboratory coordinates (x, y, z) , it proves more convenient to analyze it within a corotating frame that revolves at an angular spatial frequency Ω , coinciding with that of the helical electrostatic source ($\Omega = 2\pi/\Lambda$). In this corotating frame, the coordinates are defined according to $u = x \cos(\Omega z) + y \sin(\Omega z)$, $v = -x \sin(\Omega z) + y \cos(\Omega z)$, and $\xi = z$. What simplifies matters in this respect is the fact that in (u, v, ξ) , the periodically swirling potential now becomes dependent solely on the transverse coordinates (u, v) . Moreover, analyzing the dynamics in the corotating frame not only provides intuition about different forces in play but is also instrumental in readily identifying the locations of possible Lagrange points.

In the (u, v, ξ) coordinates, the perceived acceleration $\vec{a}_\Omega$ involves three distinct inertial forces in addition to that acting in the laboratory frame $\vec{a}_s$, given that $\vec{a}_s = \vec{a}_\Omega + 2\Omega\dot{\xi}(\dot{u}\hat{v} - \dot{v}\hat{u}) + \Omega^2\dot{\xi}(u\hat{v} - v\hat{u}) - \Omega^2\dot{\xi}^2(u\hat{u} + v\hat{v})$, where the second, third, and fourth terms on the right-hand side represent the Coriolis, Euler, and centrifugal forces, respectively [46]. The electrostatic potential associated with the configuration depicted in Fig. 1(c) is here well approximated by the logarithmic function $G(u, v) = eV_0 \ln(\sqrt{(u-d)^2 + v^2}/b) / \ln(a/b)$, where e denotes the electron charge. This approximation is valid when $a, d \ll b$ and $d \ll \Lambda$, as indicated by COMSOL simulations (see Supplemental Material 2 [46]). Moreover, in the corotating frame, the ∇ operator acting on the electrostatic potential transforms into $\nabla = \hat{u}\partial_u + \hat{v}\partial_v + \hat{\xi}(\Omega(v\partial_u - u\partial_v) + \partial_\xi)$, where the partial derivative ∂_ξ vanishes since the helicoidal source in the (u, v, ξ) system is independent of ξ . Based on this, we arrive at the following nonlinear system of equations governing the particle dynamics in the Lagrange waveguide under study (see Supplemental Material 3 [46]):

$$\begin{cases} \ddot{u} = +2\Omega\dot{v}\dot{\xi} + u\Omega^2\dot{\xi}^2 + \frac{eV_0}{m_0\gamma \ln(a/b)} \frac{1}{(u-d)^2 + v^2} [(\dot{u}^2/c^2 - 1)(u-d) + v\dot{u}\dot{v}/c^2 + \Omega^2 v^2 d], \\ \ddot{v} = -2\Omega\dot{u}\dot{\xi} + v\Omega^2\dot{\xi}^2 + \frac{eV_0}{m_0\gamma \ln(a/b)} \frac{1}{(u-d)^2 + v^2} [(u-d)\dot{u}\dot{v}/c^2 + (\dot{v}^2/c^2 - 1)v - \Omega^2 u v d], \\ \ddot{\xi} = \frac{eV_0}{m_0\gamma \ln(a/b)} \frac{1}{(u-d)^2 + v^2} [(u-d)\dot{u}\dot{\xi}/c^2 + v\dot{v}\dot{\xi}/c^2 + \Omega v d]. \end{cases} \quad (1)$$

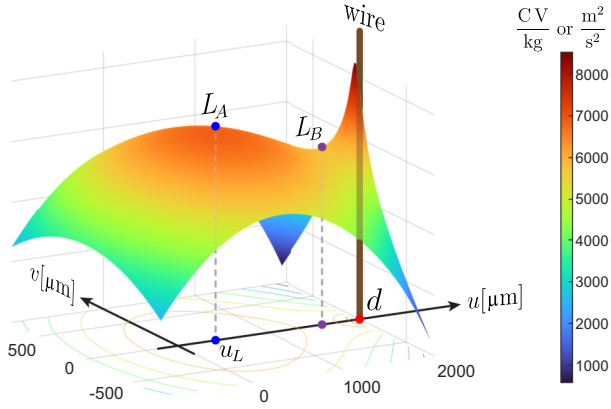


FIG. 2. Effective potential landscape of the Lagrange waveguide in the corotating frame (u, v, ξ) , resulting from the superposition of electrostatic and centrifugal potentials. Two Lagrange points are present: L_A , which is potentially stable, and L_B , a saddle (unstable) point. The brown line indicates the wire center in this frame, while the blue point marks the stable Lagrange point of interest. Parameters are $d = 2$ mm, $V_0 = -10$ kV, $a = 0.5$ mm, $b = 3$ cm, and $\Lambda = 30$ cm.

This set of dynamical equations can, in turn, be used to determine the position (u_L, v_L) of a Lagrange point. Along these lines, (u_L, v_L) can be obtained by assuming that the charged particle moves on its helical path at a constant velocity v_z along the longitudinal direction ($\dot{\xi} = v_z$, $\ddot{\xi} = 0$), while remaining fixed in the transverse plane ($\dot{u} = \dot{v} = 0$). Under these conditions, the location of an induced Lagrange point is given by

$$(u_L, v_L) = \left(\frac{d \pm \sqrt{d^2 + \frac{4eV_0}{m_0\gamma\Omega^2 v_z^2 \ln(a/b)}}}{2}, 0 \right), \quad (2)$$

where, since the particle moves on a helical path of radius u_L , $\gamma = 1/\sqrt{1 - v_z^2(1 + \Omega^2 u_L^2)/c^2}$, and the resulting equation is to be solved for u_L . We note that, irrespective of V_0 , this particular configuration admits two Lagrange points, and here, we focus on the potentially stable one, L_A , designated by the negative sign in Eq. (2). The other one, L_B , being a saddle point of the effective potential, as shown in Fig. 2, is always unstable [46]. We emphasize that if a charged particle is positioned exactly at a stable Lagrange point (u_L, v_L) while injected at an angle that matches its helicoidal path of pitch Λ (ideally, $\dot{x} = 0$ and $\dot{y} = u_L \Omega v_z$ in the laboratory frame), it will remain indefinitely stationary within the corotating frame, thus exhibiting no oscillations or other deviations.

III. STABILITY ANALYSIS

In order to study the stability of these Lagrange orbits, we now perturb Eq. (1) to the first order around their equilibrium values, i.e., $u(t) = u_L + \delta u(t)$, $v(t) = v_L + \delta v(t)$, and $\xi(t) = v_z t + \delta \xi(t)$. After substituting these expressions into Eq. (1) and by neglecting higher-order nonlinear terms, we obtain the following linearized system of equations for the perturbations

(see Supplemental Material 4 [46]):

$$\begin{cases} \delta \ddot{u} = \Omega^2 v_z^2 \left(1 + \frac{u_L}{u_L - d}\right) \delta u + 2\Omega v_z \delta \dot{v} + 2\Omega^2 v_z u_L \delta \dot{\xi}, \\ \delta \ddot{v} = -2\Omega v_z \delta \dot{u} + \Omega^2 v_z^2 \left(1 + \Omega^2 u_L^2\right) \left(1 - \frac{u_L}{u_L - d}\right) \delta v, \\ \delta \ddot{\xi} = \frac{\Omega^2 v_z^3 u_L}{c^2} \delta \dot{u} + \frac{\Omega^2 v_z^3 u_L d}{u_L - d} \delta \dot{v}. \end{cases} \quad (3)$$

The eigenvalues λ of this system can be directly obtained by assuming the form $\delta u(t) = u_0 e^{\lambda t}$, $\delta v(t) = v_0 e^{\lambda t}$, and $\delta \xi(t) = \xi_0 e^{\lambda t}$ [47]. The resulting characteristic polynomial involves terms of orders λ^6 , λ^4 , and λ^2 , displaying two degenerate zero eigenvalues. The remaining nonzero eigenvalues can be subsequently determined from

$$\begin{aligned} \mu^2 - \Omega^2 v_z^2 \left(\frac{\Omega^2 u_L^2 d}{d - u_L} + \frac{2\Omega^2 u_L^2 v_z^2}{c^2} - 2 \right) \mu \\ + \Omega^4 v_z^4 \left(\left(1 + \Omega^2 u_L^2\right) \left(1 - \frac{u_L^2}{(u_L - d)^2} + \frac{2\Omega^2 u_L^2 d}{d - u_L} \frac{v_z^2}{c^2} \right) \right. \\ \left. + \frac{4\Omega^2 u_L^2 d}{u_L - d} \right) = 0, \end{aligned} \quad (4)$$

where $\mu = \lambda^2$. Evidently, the dynamical system will be stable provided that the roots of the corresponding quadratic equation $\mu^2 - S\mu + P = 0$ [Eq. (4)] are both real and negative, in which case $\lambda = \pm\sqrt{\mu}$ is purely imaginary; otherwise, the solutions will grow unboundedly. This stability condition is always guaranteed if $S < 0$, $P > 0$, and $S^2 - 4P \geq 0$. Interestingly, under nonrelativistic conditions, these criteria reduce to a single expression, as detailed in Supplemental Material 6 [46].

In what follows, we characterize the stability properties of a Lagrange waveguide system with fixed structural parameters. We do so by varying the electrostatic voltage of the helix and the velocity/energy of the electron and examining the aforementioned stability criteria pointwise by means of a $\beta_z - V_0$ diagram, where $\beta_z = v_z/c$ represents the predominant relativistic ratio along the z coordinate. The $\beta_z - V_0$ diagram displayed in Fig. 3(a) corresponds to a configuration with parameters $\Lambda = 30$ cm, $a = 0.5$ mm, $b = 3$ cm, and $d = 2$ mm. The boundaries between stable and unstable regions [as obtained from Eq. (4)] are depicted, indicating that stable guidance is possible under both positive and negative bias V_0 conditions. In this respect, although for $V_0 < 0$ the stable region is significantly shrunk, it still presents certain advantages since the required bias voltage can be significantly lower compared to $V_0 > 0$ under relativistic conditions while simultaneously maintaining guidance. As one would expect, for a given β_z , as Λ decreases, the bias voltage should increase to overcome centrifugal forces.

From our simulations, we find that the electron, once perturbed from the ideal launching conditions, remains more confined around the Lagrange point when the system operates close to the stability boundaries. This behavior is, in a sense, linked to the difference between the two nonzero eigenvalues, $\Delta\mu = \lambda_1^2 - \lambda_2^2 = \sqrt{S^2 - 4P}$, of Eq. (4). As illustrated in Fig. 3(b), $\Delta\mu$ reaches its maximum in the vicinity of the stability boundary. Notably, this transition is very sharp for negative voltages, with the maximum difference occurring near the boundary, while no such clear

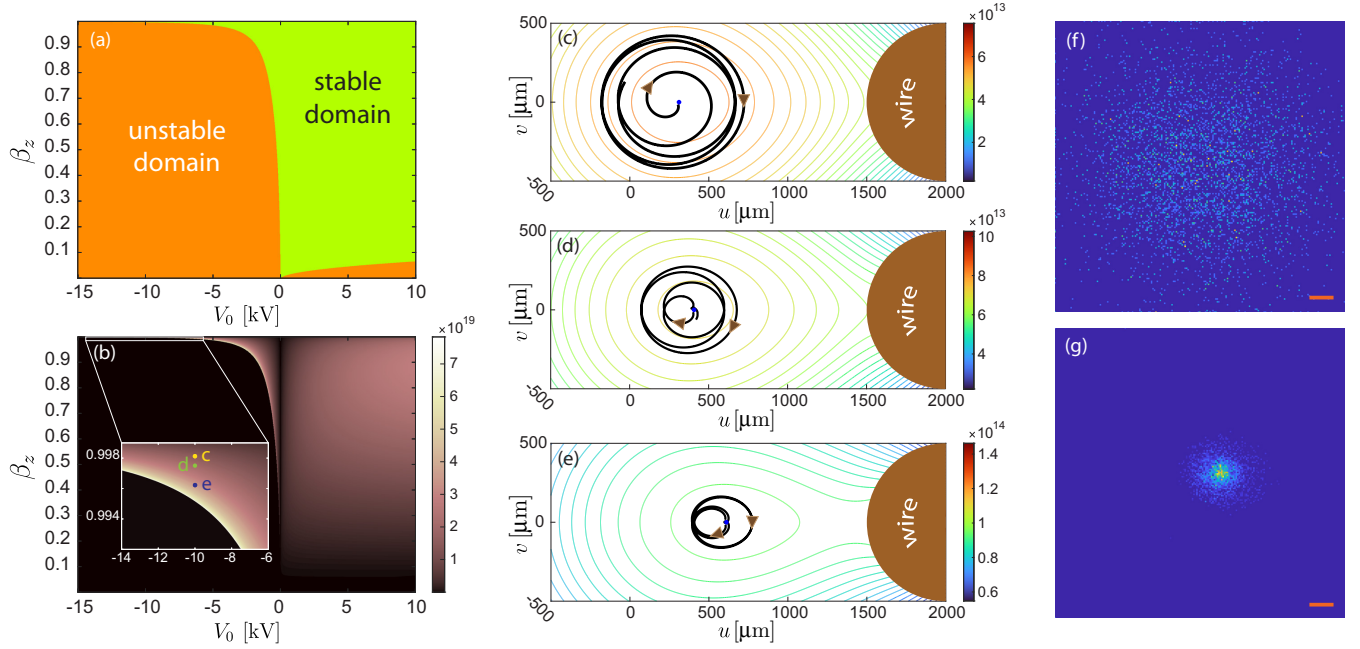


FIG. 3. (a) Stability and instability regions for a structure with parameters $\Lambda = 30$ cm, $a = 0.5$ mm, $b = 3$ cm, and $d = 2$ mm, illustrated for both repulsive and attractive potentials. (b) The inset represents an expanded view of the stability diagram in panel (a), which provides, in addition, information as to the difference between the two eigenvalues μ_1 and μ_2 for each point (V_0, β_z) ; brighter colors in the plot correspond to larger differences between the eigenvalues that happen near the stability boundary and provide more confinement. (c)–(e) Electron trajectories over a propagation distance of 1.4 m with energies of 10, 8, and 6 MeV, corresponding to points c, d, and e in panel (b), respectively, under identical initial conditions and a wire voltage of -10 kV. The isocontour map represents the effective potential in the corotating frame, which is the sum of the electrostatic potential of the helical wire and the centrifugal potential. The brown semicircle denotes the helical wire. (f) The electron beam spreads when the waveguide is turned off ($V_0 = 0$). (g) The electron beam is captured and guided by the waveguide when it is activated with $V_0 = -10$ kV. In panels (f) and (g), the orange-colored bar represents a transverse length scale of 500 μm .

transition appears for positive voltages. To investigate this effect, we selected three sample operational points near the stability boundary. The corresponding electron dynamics for these points with $V_0 = -10$ kV and different values of energies, 10, 8, and 6 MeV, respectively, are shown in Figs. 3(c)–3(e), when observed in the corotating frame. For comparison purposes, in all three cases, we apply the same perturbation in phase space around the ideal Lagrange trapping conditions. Clearly, the electron trajectories become more confined when the arrangement is operated close to the stability boundary.

In the absence of an induced Lagrange waveguide ($V_0 = 0$), a relativistic 8 MeV beam with normalized emittance of 0.15 mm mrad and an initial diameter of 20 μm expands to a spot size of ~ 1.84 mm after 1.4 m of propagation [Fig. 3(f)]. Meanwhile, under the same conditions but with $V_0 = -10$ kV, the beam is stably confined to a spot size of ~ 275 μm [Fig. 3(g)]. The emittance acceptance of this Lagrange waveguide was computed as the normalized rms emittance of electrons that successfully propagate through the waveguide, relative to the stable Lagrange point, yielding values of $\epsilon_{nx} = 3.37$ mm mrad and $\epsilon_{ny} = 3.21$ mm mrad. Since our analysis focuses on single-electron dynamics, the simulation treats electrons individually in a statistical, one-at-a-time manner. We note that in our configuration, space-charge effects can be neglected for electron currents below mA levels [33,48]. In this regard, collective effects such as wakefields are also negligible here due to the low-current beam

and minuscule radiation (see Sec. V), though they may become significant in high-current or bunched-beam scenarios [30,49].

IV. EIGENVALUES' AND EIGENVECTORS' VALIDATION

Next, we proceed to validate the perturbative approach by comparing the eigenvalues of the perturbed system of Eq. (3) with the spectrum obtained numerically from the dynamics of Eq. (1). The frequency spectrum along the u direction for an 8 MeV electron and $V_0 = -10$ kV is depicted in Fig. 4(a). The structural parameters used here are the same as those in Fig. 3. In the corotating frame, two distinct frequency peaks appear at $f_1 = 0.878$ GHz and $f_2 = 1.105$ GHz, along with a trivial peak at $f_0 = 0$ GHz. The zero-frequency component corresponds to a constant in time displacement from the Lagrange point along the u direction. This zero frequency is absent from the spectral analysis of $v(t)$, as $G(u, v) = G(u, -v)$. For clarification, Fig. 4(b) presents these same results on a logarithmic scale. The frequency lines of Figs. 4(a) and 4(b) are in excellent agreement with the frequency eigenvalues of 0, 0.86, and 1.118 GHz anticipated from theory ($\pm \text{Im}\{\sqrt{\mu}\}/2\pi$), i.e., Eq. (4). The smaller additional lines appearing in these spectra are due to nonlinear dynamics that are not accounted for by Eq. (3). Transitioning to the laboratory frame, the particle dynamics are expected to exhibit different frequency spectra given that f_0 , f_1 , and f_2 will generate sidebands around the

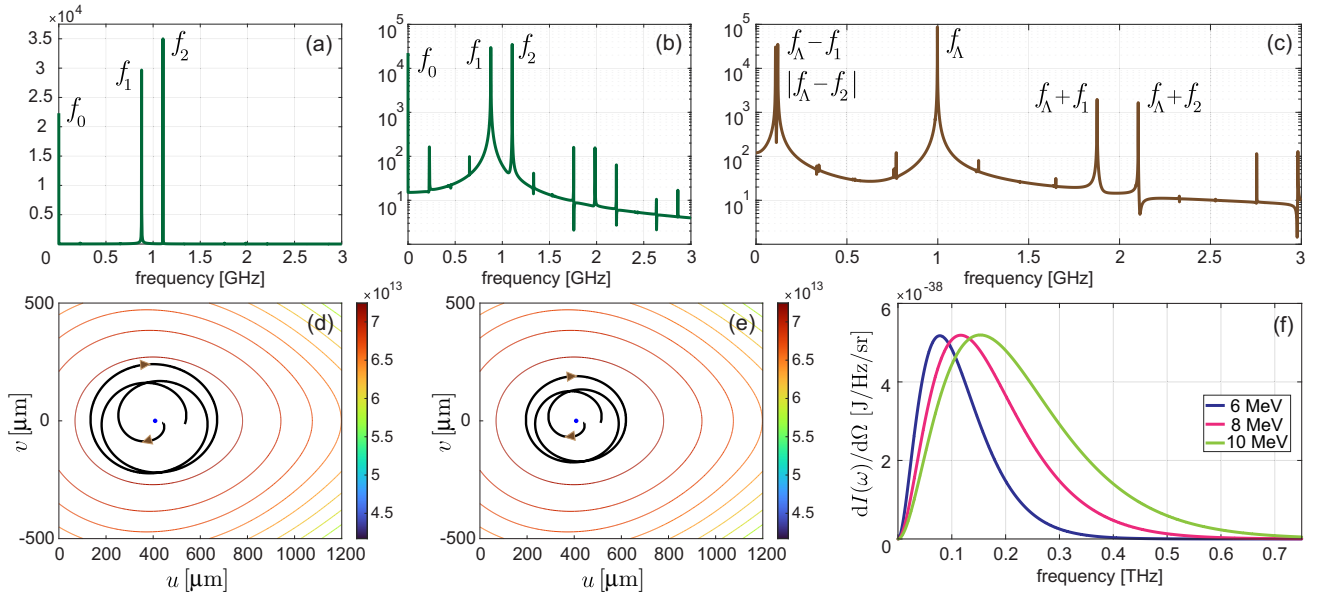


FIG. 4. Frequency spectra associated with Lagrange dynamics: (a) Fourier power spectrum of the particle's dynamics in the u coordinate (corotating frame). (b) Same as in panel (a) but on a logarithmic scale. (c) Fourier power spectrum associated with the particle's $x(t)$ trajectory (laboratory frame), represented in a logarithmic scale. (d) Electron trajectory as obtained from Eq. (1) and (e) the corresponding trajectory predicted by the first-order perturbation analysis. (f) Frequency spectrum of the electromagnetic radiation emitted along the central axis of the tube for 6, 8, and 10 MeV electrons captured in the Lagrange waveguide.

helix's own frequency $f_\Lambda = v_z/\Lambda$ (see Supplemental Material 5 [46]), as shown in a logarithmic scale in Fig. 4(c).

The perturbed dynamics discussed before can also be described by means of the eigenvectors of Eq. (3). The six phase-space eigenvectors can be directly obtained from $d\vec{\delta}_i/dt = \vec{A}\vec{\delta}_i = \lambda_i\vec{\delta}_i$, where $\vec{\delta}_i$ is the i th eigenvector having a form of $\vec{\delta} = (\delta\dot{u}, \delta\dot{v}, \delta\dot{\xi}, \delta u, \delta v, \delta\xi)$ and $\vec{A}$ is the associated coefficient matrix. Given that $\lambda_i = \pm\sqrt{\mu_i} = \pm i\omega_i$, any perturbation from the ideal Lagrange conditions will evolve according to $\vec{\delta}(t) = c_0\vec{\delta}_0 + c_{0^*}\vec{\delta}_{0^*} + \sum_n c_n\vec{\delta}_n e^{i\omega_n t}$, where $n \in \{-2, -1, 1, 2\}$ and c_i are coefficients to be determined from initial conditions. In addition, $\omega_{-i} = -\omega_i$, and $\vec{\delta}_{0^*}$ is a generalized eigenvector associated with the degenerate zero eigenvalue. Figures 4(d) and 4(e) present a comparison between the actual perturbation dynamics and those analytically predicted for the same set of initial conditions. These results indicate that the perturbative eigenvectors essentially capture the features of the original dynamics in a self-similar manner, despite the fact that nonlinear processes are also in action when considering Eq. (1). It is also worth considering the nonharmonic part of the vectorial perturbation (the eigenvectors associated with the zero eigenvalues), as derived in Supplemental Material 5 [46]:

$$\vec{\delta}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{\delta}_{0^*} = t \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ -\frac{u_L}{v_z} \frac{u_L - d}{u_L - d/2} \\ 0 \\ 0 \end{bmatrix}. \quad (5)$$

Weighted by their respective coefficients, they yield

$$c_0\vec{\delta}_0 + c_{0^*}\vec{\delta}_{0^*} = \begin{bmatrix} 0 \\ 0 \\ c_{0^*} \\ -\frac{u_L}{v_z} \frac{u_L - d}{u_L - d/2} c_{0^*} \\ 0 \\ c_0 + c_{0^*}t \end{bmatrix}. \quad (6)$$

This expression reveals that there may be a constant displacement c_0 in the ξ direction, as well as a constant velocity deviation c_{0^*} from v_z . Importantly, in the transverse plane, no constant displacement occurs in the v direction, whereas a constant displacement arises in the u direction. These findings are consistent with the statement regarding the zero-frequency eigenvalue in the spectrum of particle dynamics in the u direction.

V. ADDITIONAL REMARKS

Finally, we estimate the energy loss rate associated with a relativistic charged particle trapped in a Lagrange waveguide. Since the particle follows a helical trajectory, it is expected to radiate energy because of transverse and longitudinal accelerations. In this case, the relativistic radiation loss rate can be calculated using Liénard's generalized Larmor formula [46,50]. For an 8 MeV electron traveling in a Lagrange waveguide with the same parameters used in Fig. 3, the radiated power is estimated to be 850 eV/s, which translates to a negligible loss of 1.38×10^{-12} dB/km. Furthermore, the frequency spectrum of radiation as observed along the axis of the helix for a single electron moving along the helical path of this Lagrange waveguide is shown in Fig. 4(f) for different

energies of 6, 8, and 10 MeV (see Supplemental Material 8 [46]).

VI. CONCLUSION

We have formally analyzed the stability of relativistic electrons when captured in a Lagrange waveguide. We provided the stability criteria and domains where such a guiding arrangement can operate. This work lays the groundwork for the development and design of Lagrange waveguide systems and sets the stage for further analytical and experimental studies. Our findings could be useful across several fields of science and technology, such as electron microscopies, quantum transport/communications, and particle accelerators.

ACKNOWLEDGMENTS

This research was partially supported by the Air Force Office of Scientific Research (AFOSR) Multidisciplinary University Research Initiative (MURI) award on Novel light-matter interactions in topologically non-trivial Weyl semimetal structures and systems (Award No. FA9550-20-1-0322) (M.G.N., H.L., Y.W., G.G.P., M.K.,

and D.N.C.), the U.S. Department of Energy, Office of Basic Energy Sciences, Division of Materials Sciences and Engineering under Award No. DE-SC0025224 (Developing the theory) (M.G.N., H.L., Y.W., G.G.P., M.K., and D.N.C.), Army Research Office (ARO) (W911NF-23-1-0312), ONR MURI award on the classical entanglement of light (Award No. N00014-20-1-2789) (M.G.N., H.L., Y.W., G.G.P., M.K., and D.N.C.), AFOSR MURI on “Programmable systems with non-Hermitian quantum dynamics” (Award No. FA9550-21-1-0202) (M.G.N., H.L., Y.W., G.G.P., M.K., and D.N.C.), and Department of Energy Grant No. DESC0022282 (M.G.N., H.L., G.G.P., and D.N.C.), W.M. Keck Foundation (M.G.N., H.L., G.G.P., and D.N.C.), MPS Simons collaboration (Simons Grant No. 733682) (M.G.N., H.L., G.G.P., and D.N.C.), U.S. Air Force Research Laboratory (FA86511820019) (M.G.N., H.L., G.G.P., and D.N.C.), and an Annenberg fellowship from the University of Southern California (M.G.N.).

DATA AVAILABILITY

The data supporting this study’s findings are available within the article.

-
- [1] X. Yang, L. Wang, J. Maxson, A. C. Bartnik, M. Kaemingk, W. Wan, L. Cultrera, L. Wu, V. Smaluk, T. Shaftan, S. McSweeney, C. Jing, R. Kostin, and Y. Zhu, Towards construction of a novel nanometer-resolution MeV-STEM for imaging thick frozen biological samples, *Photonics* **11**, 252 (2024).
 - [2] L. Reimer, *Transmission Electron Microscopy: Physics of Image Formation and Microanalysis*, 1984th ed. (Springer, New York, 2013).
 - [3] C. J. Peddie, C. Genoud, A. Kreshuk, K. Meechan, K. D. Micheva, K. Narayan, C. Pape, R. G. Parton, N. L. Schieber, Y. Schwab, B. Titze, P. Verkade, A. Aubrey, and L. M. Collinson, Volume electron microscopy, *Nat. Rev. Methods Primers* **2**, 51 (2022).
 - [4] R. Dahan, A. Gorlach, U. Haeusler, A. Karnieli, O. Eyal, P. Yousefi, M. Segev, A. Arie, G. Eisenstein, P. Hommelhoff, and I. Kaminer, Imprinting the quantum statistics of photons on free electrons, *Science* **373**, eabj7128 (2021).
 - [5] Y. Chen, Nanofabrication by electron beam lithography and its applications: A review, *Microelectron. Eng.* **135**, 57 (2015).
 - [6] C. Zhu, H. Ekinici, A. Pan, B. Cui, and X. Zhu, Electron beam lithography on nonplanar and irregular surfaces, *Microsyst. Nanoeng.* **10**, 52 (2024).
 - [7] C. Vieu, F. Carcenac, A. Pépin, Y. Chen, M. Mejias, A. Lebib, L. Manin-Ferlazzo, L. Couraud, and H. Launois, Electron beam lithography: Resolution limits and applications, *Appl. Surf. Sci.* **164**, 111 (2000).
 - [8] L. J. Wong, K.-H. Hong, S. Carbajo, A. Fallahi, P. Piot, M. Soljačić, J. D. Joannopoulos, F. X. Kärtner, and I. Kaminer, Laser-induced linear-field particle acceleration in free space, *Sci. Rep.* **7**, 11159 (2017).
 - [9] S. M. Hooker, Developments in laser-driven plasma accelerators, *Nat. Photon.* **7**, 775 (2013).
 - [10] E. Esarey, C. B. Schroeder, and W. P. Leemans, Physics of laser-driven plasma-based electron accelerators, *Rev. Mod. Phys.* **81**, 1229 (2009).
 - [11] E. A. Peralta, K. Soong, R. J. England, E. R. Colby, Z. Wu, B. Montazeri, C. McGuinness, J. McNeur, K. J. Leedle, D. Walz, E. B. Sozer, B. Cowan, B. Schwartz, G. Travish, and R. L. Byer, Demonstration of electron acceleration in a laser-driven dielectric microstructure, *Nature (London)* **503**, 91 (2013).
 - [12] T. Chlouba, R. Shiloh, S. Kraus, L. Brückner, J. Litzel, and P. Hommelhoff, Coherent nanophotonic electron accelerator, *Nature (London)* **622**, 476 (2023).
 - [13] C. D. Bruzewicz, J. Chiaverini, R. McConnell, and J. M. Sage, Trapped-ion quantum computing: Progress and challenges, *Appl. Phys. Rev.* **6**, 021314 (2019).
 - [14] D. Cozzolino, B. Da Lio, D. Bacco, and L. K. Oxenløwe, High-dimensional quantum communication: Benefits, progress, and future challenges, *Adv. Quantum Technol.* **2**, 1900038 (2019).
 - [15] T. K. Paraíso, T. Roger, D. G. Marangon, I. De Marco, M. Sanzaro, R. I. Woodward, J. F. Dynes, Z. Yuan, and A. J. Shields, A photonic integrated quantum secure communication system, *Nat. Photon.* **15**, 850 (2021).
 - [16] D. Kielpinski, C. Monroe, and D. J. Wineland, Architecture for a large-scale ion-trap quantum computer, *Nature (London)* **417**, 709 (2002).
 - [17] V. Krutyanskiy, M. Galli, V. Krcmarsky, S. Baier, D. A. Fioretto, Y. Pu, A. Mazloom, P. Sekatski, M. Canteri, M. Teller, J. Schupp, J. Bate, M. Meraner, N. Sangouard, B. P. Lanyon, and T. E. Northup, Entanglement of trapped-ion qubits separated by 230 meters, *Phys. Rev. Lett.* **130**, 050803 (2023).
 - [18] A. Walther, F. Ziesel, T. Ruster, S. T. Dawkins, K. Ott, M. Hettrich, K. Singer, F. Schmidt-Kaler, and U. Poschinger, Controlling fast transport of cold trapped ions, *Phys. Rev. Lett.* **109**, 080501 (2012).

- [19] C. Monroe and J. Kim, Scaling the ion trap quantum processor, *Science* **339**, 1164 (2013).
- [20] K. R. Brown, J. Kim, and C. Monroe, Co-designing a scalable quantum computer with trapped atomic ions, *npj Quantum Inf.* **2**, 16034 (2016).
- [21] N. Christofilos, Focussing system for ions and electrons, US Patent 2,736,799 (1956), <https://patents.google.com/patent/US2736799A/en>.
- [22] E. D. Courant, M. S. Livingston, and H. S. Snyder, The strong-focusing synchrotron—A new high energy accelerator, *Phys. Rev.* **88**, 1190 (1952).
- [23] H. Wiedemann, *Particle Accelerator Physics* (Springer, Cham, 2015).
- [24] W. Paul and H. Steinwedel, Notizen: Ein neues massenspektrometer ohne magnetfeld, *Z. Naturforsch. A* **8**, 448 (1953).
- [25] W. Paul, Electromagnetic traps for charged and neutral particles, *Rev. Mod. Phys.* **62**, 531 (1990).
- [26] G. Harari, Y. Ben-Aryeh, and A. Mann, Propagator for the general time-dependent harmonic oscillator with application to an ion trap, *Phys. Rev. A* **84**, 062104 (2011).
- [27] L. S. Brown and G. Gabrielse, Geonium theory: Physics of a single electron or ion in a Penning trap, *Rev. Mod. Phys.* **58**, 233 (1986).
- [28] S. Sasidharan, O. Bezrodnova, S. Rau, W. Quint, S. Sturm, and K. Blaum, Penning-trap mass measurement of helium-4, *Phys. Rev. Lett.* **131**, 093201 (2023).
- [29] K. Y. Ng, *Physics of Intensity Dependent Beam Instabilities* (World Scientific Publishing Company, Singapore, 2010).
- [30] A. Chao, *Physics of Collective Beam Instabilities in High Energy Accelerators* (John Wiley & Sons, Nashville, TN, 1993).
- [31] D. A. G. Deacon, L. R. Elias, J. M. J. Madey, G. J. Ramian, H. A. Schwettman, and T. I. Smith, First operation of a free-electron laser, *Phys. Rev. Lett.* **38**, 892 (1977).
- [32] P. G. O'Shea and H. P. Freund, Free-electron lasers. Status and applications, *Science* **292**, 1853 (2001).
- [33] H. Luo, Y. Wei, G. G. Pyrialakos, M. Khajavikhan, and D. N. Christodoulides, Guiding charged particles in vacuum via Lagrange points, *Nat. Commun.* **15**, 6882 (2024).
- [34] D. Marcuse, Propagation of light rays through a lens-waveguide with curved axis, *Bell Syst. Tech. J.* **43**, 741 (1964).
- [35] S. E. Miller, Communication by laser, *Sci. Am.* **214**, 19 (1966).
- [36] A. W. Snyder and J. D. Love, *Optical Waveguide Theory*, 1st ed. (Springer, New York, NY, 1983).
- [37] J. C. Knight, Photonic crystal fibres, *Nature (London)* **424**, 847 (2003).
- [38] T. A. Birks, J. C. Knight, and P. S. Russell, Endlessly single-mode photonic crystal fiber, *Opt. Lett.* **22**, 961 (1997).
- [39] M. Ibanescu, Y. Fink, S. Fan, E. L. Thomas, and J. D. Joannopoulos, An all-dielectric coaxial waveguide, *Science* **289**, 415 (2000).
- [40] Y. Plotnik, M. C. Rechtsman, D. Song, M. Heinrich, J. M. Zeuner, S. Nolte, Y. Lumer, N. Malkova, J. Xu, A. Szameit, Z. Chen, and M. Segev, Observation of unconventional edge states in “photonic graphene”, *Nat. Mater.* **13**, 57 (2014).
- [41] C. W. Hsu, B. Zhen, A. D. Stone, J. D. Joannopoulos, and M. Soljačić, Bound states in the continuum, *Nat. Rev. Mater.* **1**, 16048 (2016).
- [42] E. Bannikova and M. Capaccioli, *Foundations of Celestial Mechanics*, 1st ed. (Springer International Publishing, Cham, Switzerland, 2023).
- [43] A. Pérez-Villegas, M. Portail, C. Wegg, and O. Gerhard, Revisiting the tale of Hercules: How stars orbiting the Lagrange points visit the Sun, *Astrophys. J. Lett.* **840**, L2 (2017).
- [44] H. Luo, Y. Wei, F. O. Wu, G. G. Pyrialakos, D. N. Christodoulides, and M. Khajavikhan, Guiding Trojan light beams via Lagrange points, *Nat. Phys.* **20**, 95 (2024).
- [45] C. Moller, *Theory of Relativity*, 2nd ed. (Oxford University Press, London, UK, 1972).
- [46] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/kst6-rz68> for derivations, additional details, and supporting information.
- [47] M. W. Hirsch, S. Smale, and R. L. Devaney, *Differential Equations, Dynamical Systems, and an Introduction to Chaos*, 3rd ed. (Academic Press, San Diego, CA, 2017).
- [48] S. E. Tsimring, *Electron Beams and Microwave Vacuum Electronics* (John Wiley & Sons, Inc., Hoboken, NJ, 2010).
- [49] M. Reiser, *Theory and Design of Charged Particle Beams*, 2nd ed. (Wiley-VCH Verlag, Weinheim, 2008).
- [50] A. Zangwill, *Modern Electrodynamics* (Cambridge University Press (Virtual Publishing), Cambridge, UK, 2019).