

Intermittency and nonuniversality of pair dispersion in isothermal compressible turbulence

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(Received 23 July 2025; accepted 19 December 2025; published 11 February 2026)

We study statistical properties of the pair dispersion of Lagrangian particles in transonic to supersonic compressible turbulence of an isothermal ideal gas in two dimensions, driven by large-scale solenoidal and irrotational stirring forces, via direct numerical simulations. We find that the scaling exponents of the order- p negative moments of the doubling and halving times of pair separations are nonlinear functions of p . Furthermore, the doubling- and halving-time statistics are different. The halving-time exponents are universal—they satisfy their multifractal model-based prediction, irrespective of the nature of the stirring. The doubling-time exponents are not. In the solenoidally stirred flows, the doubling-time exponents can be expressed solely in terms of the multifractal scaling exponents obtained from the structure functions of the solenoidal component of the velocity. Moreover, they depend strongly on the Mach number, Ma . In contrast, in the irrotationally stirred flows, the doubling-time exponents do not satisfy any known multifractal model-based relation, and are independent of Ma . Our work is a generalization of Richardson's law of pair dispersion to compressible turbulence.

DOI: 10.1103/kr1z-rf6b

One of the pillars of our understanding of turbulence is *Richardson's law* [1]: $\langle R^2(t) \rangle \sim t^3$, where R is the separation between a pair of Lagrangian particles or tracers, t is time, and $\langle \cdot \rangle$ denotes averaging over the statistically stationary turbulent state. This law holds for incompressible flows if R lies within the inertial range of length scales, across which the nonlinear interactions dominate over viscous dissipation. Here, we seek a generalization of Richardson's law to compressible turbulent flows, which are prevalent in many astrophysical systems [2–4].

We begin by summarizing the present status of Richardson's law in incompressible turbulence. Theoretically, it can be understood as a modeling of the evolution of $R(t)$ by a diffusive process [1,5] with a diffusion constant, $K(R) \sim R^{4/3}$, that stems [5] from Kolmogorov's 1941 (K41) theory [6]. Stated differently, Richardson's law implies that there exists a dynamic scaling exponent, $z = 2/3$, such that typical time scales, $\tau(r)$, of turbulent eddies of size r scale as $\tau(r) \sim r^z$. It is now well established, from experiments and direct numerical simulations (DNSs), that K41 must be generalized to include intermittency; i.e., the p th moment of the Eulerian velocity difference, δu , across a length scale r (the order- p velocity structure function), scales as $\langle \delta u^p(r) \rangle \sim r^{\zeta_p}$, where ζ_p

is a nonlinear concave function of p [7]. This also dictates that turbulence must, in fact, have an infinite number of dynamic exponents [8–13], suggesting the presence of intermittency corrections to Richardson's law, which itself enjoys feeble support from experiments [14–18] and DNS [16,19–23]—the observed scaling range of $\langle R^2 \rangle$ is at most a decade. A better approach to investigate the intermittency of pair dispersion in turbulence is to use *exit times*, defined as the time, $\tau_\sigma(R)$, it takes for R to cross σR for the first time, where σ is fixed. The choice $\sigma > 1$ corresponds to the *doubling times* [24], τ_D , for which [25–27]

$$\langle \tau_D^{-p}(R) \rangle \sim R^{\chi_p^D}, \quad \text{where} \quad \chi_p^D = p - \zeta_p. \quad (1)$$

This bridge relation between the dynamic exponents, χ_p^D , and ζ_p has been obtained by using the multifractal model of turbulence.

In compressible flows, in addition to doubling times, we need to define *halving times* $\tau_H(R)$, for which $\sigma < 1$ and the corresponding scaling exponents are χ_p^H . As the multifractal model does not distinguish between the doubling and halving times, χ_p^H are also expected naively to satisfy the same bridge relation [Eq. (1)]. Although there have been many studies of the statistics of velocity increments in compressible turbulence [28–33] and tracer dispersion in compressible surface flows are known not to obey Richardson's law [34], the intermittency of tracer pair dispersion has not been studied. We explore this via simulations of forced compressible turbulence of an isothermal ideal gas, governed by the Navier-Stokes equations in two dimensions.

In two-dimensional (2D) compressible turbulence forced at an intermediate scale, Refs. [35,36] had obtained two cascades of the kinetic energy: An inverse cascade from the energy-injection scale to large length scales and a forward

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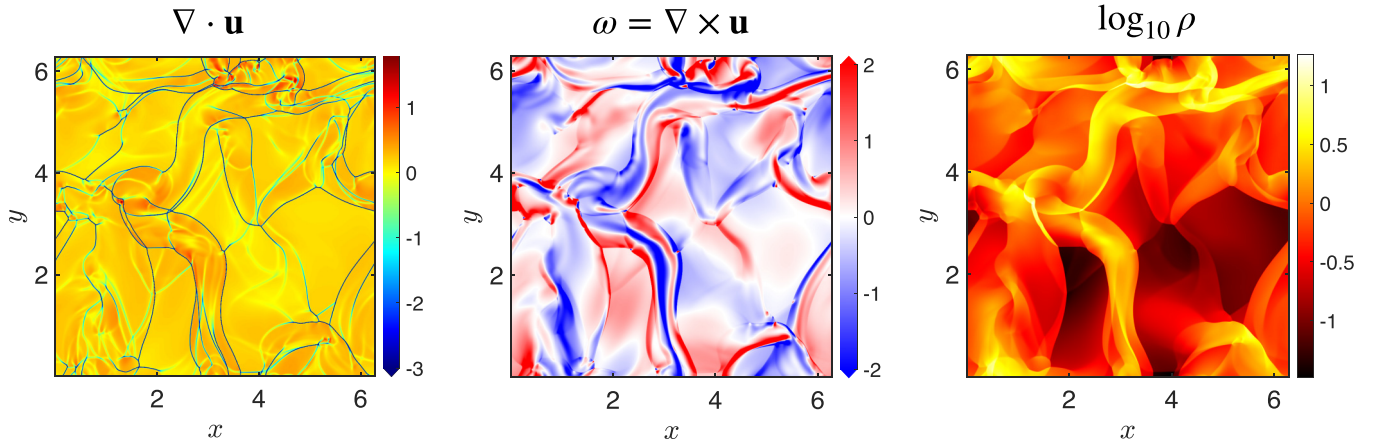


FIG. 1. Typical snapshots of $\nabla \cdot \mathbf{u}$, the vorticity $\omega = \nabla \times \mathbf{u}$, and $\log_{10} \rho$, for solenoidally driven supersonic turbulence, run S2 ($\nabla \cdot \mathbf{u}$ and ω are normalized by their respective root-mean-square values). Thin filaments in the $\nabla \cdot \mathbf{u}$ profile are the shocks. The vorticity, ω , is large in the vicinity of curved shocks. There are strong density gradients across shocks.

cascade to small length scales. In what follows, we choose the energy-injection scales to be close to the linear size of our simulation domain, and we focus on the forward-cascade regime, which we show to be intermittent. We unveil a power-law growth of $\langle R^2(t) \rangle$, but the scaling regime is too small to make robust quantitative measurements. Thus, we resort to the exit-time approach to calculate the dynamic exponents χ_p^H and χ_p^D . We find that χ_p^H obey the multifractal model-based bridge relation and are universal, i.e., they do not depend on how the turbulence is forced. However, contrary to the prediction of the multifractal model, $\chi_p^D \neq \chi_p^H$. Moreover, χ_p^D is not universal. If the external force is solenoidal, then (1) $\chi_p^D = p - \zeta_p^s$, where ζ_p^s are the scaling exponents of structure functions of the solenoidal component of the velocity field, and (2) χ_p^D depend on the Mach number (Ma). If the force is purely irrotational, χ_p^D neither follows any known bridge relation nor depends on Ma. We further identify, through the multifractal model, the dimensions of the structures that play the dominant role in pair dispersion.

We perform pseudospectral DNS of the equations of compressible, isothermal gas with a random white-in-time external force (at the Fourier modes with $|\mathbf{k}| = k_f = 3$) on a 2D doubly periodic square domain of side $L = 2\pi$ with $N^2 = 4096^2$ collocation points. The assumption of isothermality holds when the gas is optically thin (transparent to radiation), in which case the heat generated by the dissipation at shocks is rapidly radiated away. This is generally valid for molecular clouds that have low density and solar metallicity [2]. In addition to shear viscosity, we use a bulk viscosity, which is zero everywhere except near the shocks [37]. Shocks are generally resolved across 4–10 grid points. After the flow reaches a statistically stationary state, we seed it uniformly with $n = N^2$ Lagrangian particles and track them as the flow evolves. They cluster at the shocks and their number density is proportional to $\rho(\mathbf{x}, t)$.

Using the Helmholtz decomposition, we write $\mathbf{u} = \mathbf{u}^s + \mathbf{u}^c$, where $\mathbf{u}^s$ and $\mathbf{u}^c$ are, respectively, the solenoidal ($\nabla \cdot \mathbf{u}^s = 0$) and irrotational ($\nabla \times \mathbf{u}^c = 0$) components of the flow. The relative importance of $\mathbf{u}^s$ and $\mathbf{u}^c$ in the total kinetic energy is measured by the solenoidal ratio, $\Psi \equiv \langle \rho |\mathbf{u}^s|^2 \rangle / \langle \rho |\mathbf{u}|^2 \rangle$.

See the Supplemental Material [38] for further details on the numerical method.

We perform four sets of DNSs, S1 and S2 for solenoidal external force and C1 and C2 for irrotational external force (see the Supplemental Material [39]). Typically, subsonic and supersonic compressible flow can be qualitatively different. Hence, we perform a set of simulations that are transonic, runs S1 and C1, $\text{Ma} \simeq 1$, and another set, S2 and C2, that are supersonic, $\text{Ma} > 1$. In Fig. 1, we show representative pseudocolor plots of $\nabla \cdot \mathbf{u}$, $\omega = \nabla \times \mathbf{u}$, and $\log_{10} \rho$ for the run S2 (see the Supplemental Material [40]). The shocks are clearly visible as filamentary structures with large negative values of $\nabla \cdot \mathbf{u}$. Naively, we may expect that the presence of the shocks would not show up in pseudocolor plot of ω , but this is not the case because although a shock front that is a straight line does not contribute to vorticity, *curved* shock fronts do. In our simulations, (1) the density gradients are larger for the supersonic cases compared to the transonic ones and (2) the vorticity is more uniformly distributed in the transonic cases than the supersonic ones.

We can define the p th order longitudinal *equal-time* structure functions of $\mathbf{u}$ as well as its components, $\mathbf{u}^s$ and $\mathbf{u}^c$:

$$\delta u_{\parallel}(r) \equiv [\mathbf{u}(\mathbf{x} + \mathbf{r}) - \mathbf{u}(\mathbf{x})] \cdot \frac{\mathbf{r}}{r}, \quad (2a)$$

$$\text{and } \delta u_{\parallel}^{s,c}(r) \equiv [\mathbf{u}^{s,c}(\mathbf{x} + \mathbf{r}) - \mathbf{u}^{s,c}(\mathbf{x})] \cdot \frac{\mathbf{r}}{r}, \quad (2b)$$

$$\text{with } S_p(r) = \langle |\delta u(r)|^p \rangle, \quad S_p^{s,c}(r) = \langle |\delta u(r)^{s,c}|^p \rangle, \quad (2c)$$

$$\text{and } S_p(r) \sim r^{\zeta_p}, \quad S_p^s(r) \sim r^{\zeta_p^s}, \quad S_p^c(r) \sim r^{\zeta_p^c}. \quad (2d)$$

The scaling relations hold for $\eta \ll r \ll L_1$, where η is the energy dissipation scale. All the *equal-time* exponents, ζ_p , ζ_p^s , and ζ_p^c are nonlinear monotone increasing functions of p [29,30,32]. The spectra of turbulent fluctuations and the equal-time structure functions are shown in Ref. [41].

We calculate $\langle R^2(t) \rangle$ by averaging, at every instant, over the $2n$ separations where $n = N^2$ is the number of tracers. See Ref. [42] for plots of $\langle R^2(t) \rangle$ versus time for all the runs. At early times, $\langle R^2(t) \rangle$ grows exponentially for all the runs. This range of exponential growth is very small for C1 and C2. Similar results [43] have been obtained for incompressible 2D

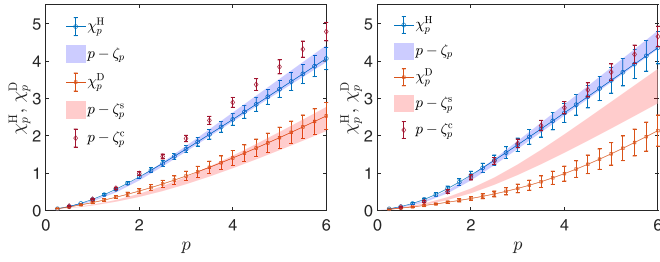


FIG. 2. Dynamic exponents for doubling and halving times (see text) χ_p^D and χ_p^H , respectively, as functions of the order p for the two supersonic runs [S2 (left panel) and C2 (right panel)]. The blue shaded region corresponds to the bridge relation, $p - \zeta_p$, derived from the multifractal model. The pink shaded region corresponds to the expression $p - \zeta_p^s$. In both S2 and C2, the multifractal prediction for χ_p^H is correct within errorbars, but not for χ_p^D . For S2, $\chi_p^D = p - \zeta_p^s$ is correct within error bars. For C2, χ_p^D does not satisfy any known bridge relation.

turbulence. Exponential growth up to a length scale typically implies that the flow across is smooth up to that scale. This holds for our simulations because at such small time scales most of the pair separations lie away from shocks. At late times, the behavior of $\langle R^2 \rangle$ is not universal, it depends on precisely how the DNS is forced.

If $R(t)$ lies in the inertial range, then $R^2 \sim t^{2/(1-h)}$, where h is the local Hölder exponent of the velocity fluctuations in space. If we use Kolmogorov's simple scaling hypothesis, $\zeta_2 = 2h$, which yields $\langle R^2(t) \rangle \sim t^{2z}$, where $z = 1/(1 - \zeta_2/2)$. In incompressible flows $\zeta_2 = 2/3$ and $z = 3/2$ that corresponds to Richardson's law. We calculate ζ_2 from the scaling of the second-order structure functions, $S_2(r)$. We find that there is a range of scales over which $\langle R^2 \rangle \sim t^{2z}$ in S1, S2, and C2, although not in C1. Crucially, in every case, the scaling regime is too small to make robust quantitative measurements of the values of z from the $\langle R^2(t) \rangle$ versus t plots. Hence, we turn to exit times.

In the stationary state, we choose $2n$ tracer pairs whose relative distance R lies in the inertial range and then calculate $\tau_H(R)$ and $\tau_D(R)$ as the first instant at which R goes below $3R/4$ or above $3R/2$, respectively [44]. The statistics of exit times are expected to show scaling (or multiscaling) if the distances R , $3R/2$, and $3R/4$ lie within the inertial range [45]. Specifically, the inertial range is defined as the range of scales over which the equal-time structure functions $S_p(r)$, $S_p^s(r)$, and $S_p^c(r)$ [see Eq. (2)] show scaling.

We find that the negative moments of the exit-time distributions scale as

$$\langle \tau_D^{-p}(R) \rangle \sim R^{-\chi_p^D} \quad \text{and} \quad \langle \tau_H^{-p}(R) \rangle \sim R^{-\chi_p^H}, \quad (3)$$

over almost a decade in R . We extract the exponents, χ_p^D , χ_p^H , ζ_p , ζ_p^c , and ζ_p^s , by linear least-square fitting of the log-log plots of $\langle \tau_D^{-p}(R) \rangle$, $\langle \tau_H^{-p}(R) \rangle$, $S_p(r)$, $S_p^s(r)$, and $S_p^c(r)$, respectively. In Fig. 2, we plot the dynamic exponents, χ_p^D and χ_p^H , for the runs S2 and C2, up to $p = 6$. Both are nonlinear functions of p , indicating *dynamic multiscaling*. The spreads of the shaded regions in Fig. 2 show the error bars of the ζ_p 's and ζ_p^s 's. The moments of exit times and the dynamic exponents

calculated from them for all the runs are shown in Ref. [46]. See Ref. [47] for our method of estimating error.

The multifractal model [7,48] does not distinguish between doubling and halving times, and hence, for both of them it predicts the same bridge relation, $\chi_p^{H,D} = p - \zeta_p$. This bridge relation agrees well with our data, for the halving times (see Fig. 2), but not for the doubling times. We find unambiguously that $\chi_p^D \neq \chi_p^H$, thereby violating the simple expectation based on the multifractal model.

For the two runs with a solenoidal external force, S1 and S2, we find $\chi_p^D = p - \zeta_p^s$; i.e., the bridge relation from the multifractal model holds for the doubling-time exponents, but with ζ_p replaced by ζ_p^s . Together, this demonstrates that, for the runs S1 and S2, (1) the solenoidal part of the velocity field is primarily responsible for the increase of R and (2) both the solenoidal and irrotational parts affect the decrease of R . Note that the bridge relations hold for the two values of Ma we have used, although the precise values of the exponents themselves depend on Ma .

For the two irrotationally forced runs, C1 and C2, besides $\chi_p^H = p - \zeta_p$, we find that the relation $\chi_p^H = p - \zeta_p^c$ also holds because the scaling exponents, ζ_p^c , of the irrotational component of the velocity are quite close to ζ_p . Crucially, $\chi_p^D \neq \chi_p^H$ again. Furthermore, unlike the runs S1 and S2, $\chi_p^D \neq p - \zeta_p^s$ (see Fig. 2 and Ref. [49]). This indicates that the growth of R is not determined solely by the solenoidal part of the velocity field, but is affected significantly by the regions of the flows with $\nabla \cdot \mathbf{u} > 0$. A quantitative understanding of these exponents remains an open question. Finally, given the accuracy of our data, we do not detect any dependence of these exponents on Ma .

In the multifractal model, the velocity field is constructed from an infinity of interwoven sets each characterized by a scaling exponent h and corresponding fractal dimension $D(h)$. It is straightforward to show [50] that the dynamic exponents χ_p and $D(h)$ are Legendre transforms of each other:

$$\chi_p = -\inf_h [ph + d - p - D(h)] \quad (4a)$$

$$\text{and} \quad D(h) = \inf_p (ph + d - p + \chi_p). \quad (4b)$$

Thus, from χ_p we determine $D(h)$ —the dimensionality of structures that contribute to dynamic scaling [51]. In addition, we must define $D_H(h)$ and $D_D(h)$, which are calculated from χ_p^H and χ_p^D , respectively, by using Eq. (4b). This reveals that the halving and doubling times are influenced by structures of different dimensionality. All the multifractal spectra are shown in Ref. [52].

We find that external force and Ma have very little effect on $D_H(h)$ —in all our runs, $D_H(h) \rightarrow 1$ as $h \rightarrow 0$. Thus, the halving times are controlled by shocks, which are one-dimensional structures characterized by $h \rightarrow 0$. As expected, $D_H(h)$ overlaps with the multifractal spectrum evaluated from equal time exponents ζ_p .

In contrast, the function $D_D(h)$ is not universal. In S1 ($Ma \simeq 1$), $D_D(h)$ does not go down to 1, implying that the structures that influence the scaling of $R(t)$ are more space filling than shocks. In S2 ($Ma \simeq 2.5$), we find $D_D(h) \simeq 1$ for $h \simeq 0.4 > 0$ we infer that these structures are not shocks but thin elongated patches of high vorticity, which appear in the

vicinity of the shocks, clearly visible in Fig. 1. Such patches do exist for S1, but they are relatively weaker and broader. Finally, we note that the function $D_D(h)$ takes values close to unity in the runs, C1 and C2, as well.

Note that we have previously undertaken similar studies in simpler models of compressible turbulence—the randomly forced Burgers equation in one [53] and two dimensions [54]. They have a few qualitative similarities with the results here—evidence of dynamic multiscaling and $\chi_p^D \neq \chi_p^H$. There are few key differences between the Burgers equation and the present, more realistic model: (1) zero pressure (tracers remain trapped on shocks forever), (2) zero vorticity at all times, and (3) bifractal velocity fluctuations. This additional complexity also precludes the formulation of building a theoretical framework for the exit times, which was possible in De *et al.* [54].

In summary, in trying to generalize Richardson’s law of pair dispersion to compressible flows, we learn three important lessons: (1) Compressible turbulence in two dimensions show dynamic multiscaling. (2) The dynamic multiscaling obtained here is qualitatively different from its incompressible counterpart; we need to consider both doubling and halving times and the two have different dynamic multiscaling exponents. The exponents χ_p^H satisfy a multifractal model-based bridge relation and are universal but χ_p^D are not. They depend on both the external force and the turbulent Mach number, Ma . This nonuniversality of the inertial-range pair-dispersion statistics is remarkable, given that the inertial-range fluctuations are supposed to be independent of the external force. (3) This dynamic multiscaling *cannot* be understood within the framework of the present multifractal model of turbulence. Consequently, a clear understanding of statistics of pair dispersion in compressible turbulence remains an open problem, whose solution may require that we go beyond the multifractal model.

Furthermore, note that in incompressible turbulence, the Richardson dispersion can be understood as a scale-dependent effective viscosity, and this serves as a measure of mixing

by turbulent flows. We find that in compressible turbulence, this description is fundamentally flawed because we have both doubling and halving times, which have different scaling properties. As turbulence in most astrophysical contexts is compressible, mixing in large-scale astrophysical systems, e.g., molecular clouds, must be revisited in the light of our results.

We speculate that the key qualitative content of our results would be valid in three dimensions (3D) as well. In particular, the necessity of the two kinds of exit times, dynamic multiscaling, and intermittency of pair dispersion should carry over to 3D.

Highly supersonic turbulent flows govern various astrophysical processes, e.g., star formation in molecular clouds, and the transport and mixing of gases along with their chemical kinetics in the interstellar media [2,4]. Our work sets out an effective paradigm that can potentially provide in-depth insights into these aspects.

Acknowledgments. D.M. and R.P. thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the program “Anti-diffusive dynamics: from subcellular to astrophysical scales,” where partial work on this paper was undertaken. This work was supported by EPSRC Grant No. EP/R014604/1. D.M. acknowledges the support of the Swedish Research Council Grant No. 638-2013-9243. Nordita is partially supported by NordForsk. S.D. thanks the PMRF (India) for support. R.P. and S.D. thank the Science and Engineering Research Board (SERB), Anusandhan National Research Foundation (ANRF), and the National Supercomputing Mission (NSM), India, for support, and the Supercomputer Education and Research Centre (IISc) for computational resources.

Data availability. The data that support the findings of this Letter are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

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- [38] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/kr1z-rf6b> for details on numerical algorithms, their implementation and additional figures and tables, which includes Refs. [20–23,55–60].
- [39] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/kr1z-rf6b> for all the parameters of the runs.
- [40] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/kr1z-rf6b> for representative snapshots of all the runs.
- [41] Section II.2 and II.3 of the Supplemental Material, respectively.
- [42] Section II.4 of the Supplemental Material.
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- [45] As long as this condition is satisfied, we expect our results to be independent of the particular choice of the thresholds, $3R/2$ or $3R/4$.
- [46] Section II.5 of the Supplemental Material.
- [47] Section II.7 of the Supplemental Material, which includes Ref. [61].
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