

(Dis)continuous buckling transition in elastic shell mediated by contact

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Snap buckling is a rapid shape transition in slender structures, appearing as a fundamental switching mechanism of natural and man-made systems. Boundary conditions of structures are crucial to predict and control their snap buckling behavior. However, the general framework that relates boundary conditions, geometry, and performance of structures is still absent to date. Here, we study the snap buckling of hemispherical shells in contact with rigid cylinders of different diameters to uncover the roles of boundary conditions in dynamic performance of shells. Specifically, we analyze the jumping dynamics of the pneumatically inverted shells placed on the rigid cylinder by combining experiments and analytical theory. We find the characteristic diameter of the cylinder determined from the shell geometry that can switch between continuous and discontinuous buckling associated with the jumping behavior. The analytical predictions for jumping performance of the shell supplemented with the characteristic diameter are in excellent agreement with our experimental results. Our study clarifies that contact geometry is crucial in predicting the pathway of snap buckling, indicating that dynamic performance of soft robots would be optimized by tuning their surface geometry.

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I. INTRODUCTION

Snap buckling is one of the bifurcation phenomena in a slender structure that transits from one stable state to the other [1–3]. The stored elastic energy of rods, plates, or shells is converted into kinetic energy, where rapid and large deformation, together with impulsive force or snapping sounds, follows the change of the control parameters. The snap buckling phenomena, which are found in nature and engineering applications, result from a complex interplay of elasticity and geometry of structures. For example, the geometry of structures [4] and shells [5–7], imperfections [8], contact friction [9], symmetry of boundary conditions [10–12], twist-bend coupling [13], elastomagnetic coupling [14], viscoelasticity [15–17], constraints [18], and boundary conditions [19] influence their buckling conditions and postbuckling behaviors. The tunable and predictive switching mechanism of snap buckling is found in nature, engineering designs, and chil-

dren's toys, e.g., nastic motion in plants [20], insects [21], sensors [22], soft actuators [23], and popping toys [24,25].

The jumping or eye popper toy is a children's toy made of an elastomeric hemispherical shell. The inverted shell flips and jumps, followed by snap buckling instability [4,24,25]. The inverted shell is usually placed on a flat substrate, and

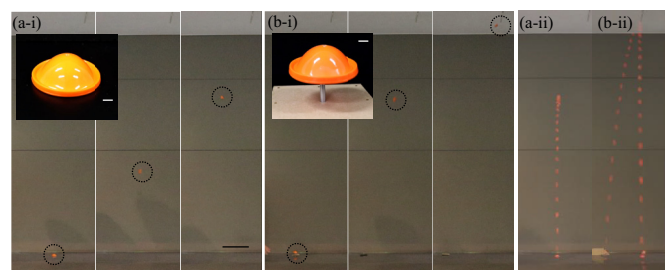


FIG. 1. The demonstration of the jumping experiment of popper toys on (a) the flat substrate (floor) and (b) rigid protrusion (6 mm screw). (b-i) The popping on the protrusion reaches the ceiling, not (a-i) on the flat substrate (scale bar 30 cm). The insets represent the corresponding initial (buckled) configurations (scale bar 10 mm). The multiple exposures of the popping phenomena (available as Movie S1 in the Supplemental Material [38]) on (a-ii) the flat substrate and (b-ii) protrusion. Note that all the experimental data analyzed in this paper are obtained in the jumping experimental system in Fig. 2.

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it pops out with a snapping sound. When we place it on a rigid protrusion, the shell reaches a much higher height than the flat substrate (see Fig. 1). Although this remarkable difference in jumping performance of the popper highlights the fact that the boundary condition (contact between shell and substrate) is crucial in forecasting snap buckling instability, the general framework for the jumping behavior of the popper has not been investigated thus far [24,25]. In our previous work, we have proposed an analytical framework against jumping performance of pneumatically controlled poppers on a flat substrate, which is consistent with experiments and simulations [26]. Our experimental system enables us to systematically study the roles of shell geometry, material stiffness, and net weight. Nevertheless, the role of the substrate geometry, i.e., why does the popper jump higher on the protrusion, is not clarified to date.

The solutions to the seemingly simple question about the jumping popper presented in this paper would be applied to extend the activity of soft robots to date. Various designs of soft jumpers have been proposed over the last few decades, including those utilizing the buckling instability of structures [27,28]. Given the material properties and design of soft jumpers, their jumping performance (e.g., speed and height) has been evaluated experimentally and numerically under ideal conditions (e.g., on a flat plate). One of the recent challenges is to predict their performance quantitatively under more realistic conditions. For example, given that many soft jumpers require less power amplifiers (compared with rigid robots), they could be lightweight and potentially useful in unknown environments such as lunar or space exploration [29–31]. The fundamental difficulty in their quantitative prediction is multibody dynamics and the methodology to consider the interaction with the compliant circumstances [32,33]. The framework established under the ideal condition is not necessarily valid; rather, it must be adapted to environments. Indeed, we show that jumping performance on the protrusion differs from that on the flat plate, combining experiments and theory. In other words, our model system study provides insights into expanding the playground of current soft robot designs.

The fundamental aspect of snap-through dynamics, the power source of soft jumpers, has been studied within the context of the mechanics of materials recently [15,16]. For example, Gomez *et al.* formulate the viscoelastic snap-through dynamics of arches, employing the analysis of the von Mises truss [16]. They predict the timescale for the snap-through dynamics, highlighting the effect of the viscoelasticity of materials. Although several features of snap-through dynamics have been applied to engineering problems, e.g., microelectromechanical systems [34], shock protections [35], and soft robots [36], the prediction of their performances is challenging due to inherent coupling between material properties and nonlinear geometry.

Here, we systematically study the roles of substrate geometry in jumping performance of elastomeric shells. We experimentally analyze the jumping process of the shell on a rigid cylinder of small diameter d . We propose a predictive framework for jumping performance on a sufficiently narrow (pointlike) substrate by applying the analytical formula for the point-indentation test of the shell. The analytical formula is

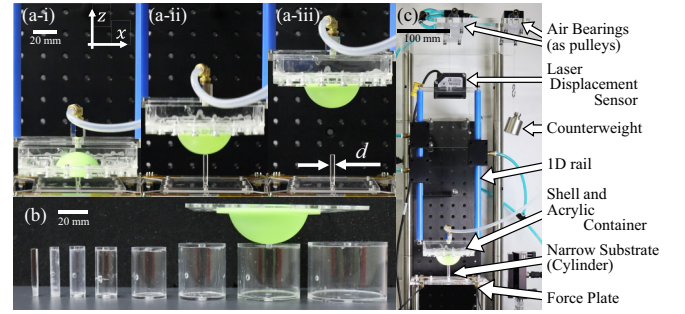


FIG. 2. The pneumatically controlled elastic shell jumps on a rigid cylinder. (a-i)–(a-iii) The front view of the typical jumping process of our elastic spherical shell on a cylinder $[(R, h) = (25, 2.0) \text{ mm}, d = 4 \text{ mm}]$. (b) Photographs of rigid (acrylic) cylinders with different diameters $d \in [4, 50] \text{ mm}$. (c) Photograph of the full experimental apparatus. The set of the container and shell (popper) rises vertically along one-dimensional (1D) rails. The popper is connected to the counterweight via air bearings. In each jumping test, the vertical position, pressure of the container, and normal reaction force of the substrate are measured simultaneously by combining the laser displacement sensor, the pressure sensor, and the custom-made force sensor plate connected with the cylinder. Jumping experiments are performed on different substrates by replacing the cylinder on the force plate.

derived and compared with the experimental results, showing in excellent agreement with experimental data without requiring any fitting parameters. We find that when the diameter is sufficiently larger than the threshold, the maximum jumping height converges to that on the flat substrate reported in the previous study [26], whereas, in the other limit, the jumping height becomes large and is predicted by analytical formula proposed in this paper. Our findings would be applicable in predicting performance of the jumping robot in complex or irregular (bumpy) environments.

II. EXPERIMENTAL METHOD

We fabricate elastomeric shells made of silicone elastomer [Elite Double (ED) 22 and ED32, Zhermack, Italy] with homogeneous thickness h , the radius of curvature R , and the polar angle φ . The viscoelasticity of this elastomer is sufficiently small, such that the mechanical performances of our samples could be modeled as incompressible hyperelastic materials of Young's modulus E (Poisson's ratio $\nu = 0.5$) [37]. Young's moduli for ED22 and ED32 are, respectively, given as $E = 0.47$ and 1.2 MPa . The base and catalyst are mixed in the centrifugal mixer (AR-100, Thinky Corporation, Japan) and degassed in the vacuum chamber to reduce possible defects by removing air bubbles. The mixed liquid elastomer is injected into the three-dimensional printed mold and is cured at room temperature. After curing ($\sim 30 \text{ min}$), the shells are demolded and glued to the acrylic container [see Figs. 2(a) and 2(b)]. We systematically fabricate shells of different sizes and aspect ratios, fixing the polar angle φ as $\varphi = 80^\circ$ throughout.

The set of shell and container, which we call *popper* here, is fixed with two rigid shafts acting as 1D rails. The rigid shafts can slide along air bearings (13 mm ID Air Bushing, Newway, USA) smoothly under ideally low friction. The rigid

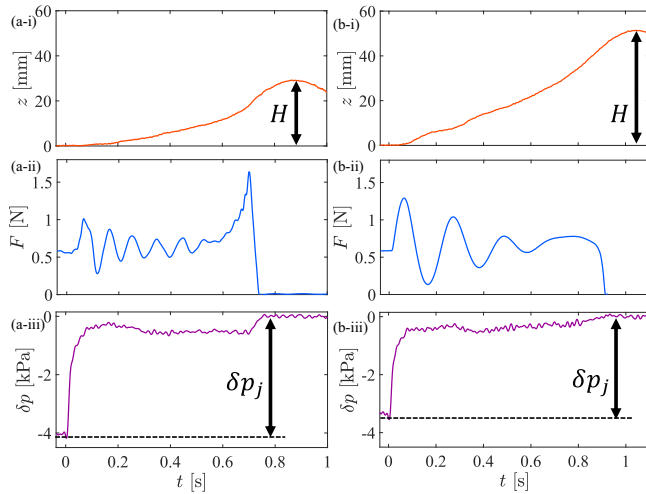


FIG. 3. Comparison of the jumping phenomena on (a) the flat substrate and (b) narrow substrate $[(R, h) = (30, 2.0) \text{ mm}, mg = 0.6 \text{ N}]$. Typical time series data for the jumping process on the flat substrate [panels (a-i)–(a-iii)] and on the narrow substrate [panels (b-i)–(b-iii)]: (i) the vertical position of the container, z , (ii) the reaction force of the substrate, F , and (iii) the pressure difference between inner and atmospheric pressures, δp .

shafts are connected with the counterweight via a Nitinol wire and air bearings (acting as pulleys) to precisely control the net weight of the popper, mg [Fig. 2(c)]. The vertical location of the popper, z , the pressure difference inside the container from the atmospheric pressure, δp , and the reaction force F are measured by the laser displacement sensor (IL-300, KEYENCE, Japan, 3 kHz sampling rate), pressure sensor (AP-C30, KEYENCE, Japan, 400 Hz sampling rate), and the custom-made force plate attached to the strain gauge, respectively. The time series data for z , δp , and F are time synchronized to correlate their dynamics. The rigid cylinder of diameter d and length 30 mm fixed to the force plate is replaceable [Fig. 2(b)]. We refer to the substrate with the cylinder as the narrow substrate.

The experimental protocol is identical to our previous work [26] and is detailed briefly as follows. The container is depressurized as $\delta p < 0$ by a syringe. When the pressure difference exceeds the critical value, the shell is inverted (shell buckling). The inverted shell and container under the pressure difference δp_j is lowered along the rail and is grounded with the substrate, where the top of the cylinder and the shell apex are in contact. The central axes of the cylinder and shell are set to be aligned coaxially. After grounding the popper, we open the valve connected to the container at time $t = 0$ so that the internal pressure reaches nearly atmospheric pressure. The shell recovers its original shape, and the popper is launched from the initial height at $z = 0$.

III. COMPARISON BETWEEN FLAT AND POINTLIKE SUBSTRATE

We show the typical time series data for the popper height z , internal pressure, δp and the reaction force F in Fig. 3. We compare the results on the flat substrate and those on

the cylinder of diameter $d = 6 \text{ mm}$ for $(R, h) = (30, 2.0) \text{ mm}$ and $mg = 0.6 \text{ N}$ in Figs. 3(a) and 3(b), respectively. When the valve is opened at $t = 0$, the air flows into the container instantaneously, and the shell starts to recover its original shape. The longitudinal wave of the wire connecting the popper and the counterweight results in the damped oscillation right after the valve opens, while pressure saturates to a nearly constant value immediately. The popper is then launched and reaches the maximum height H . The remarkable difference between flat and narrow substrates is the force right before the jump [Figs. 3(a) and 3(b-ii)]. The characteristic impulsive force exists when the popper jumps on the flat substrate, while it does not when the popper launches from the narrow substrate. It should be noted that even if the impulsive force does not exist for the narrow substrate, the popper can jump higher than on the flat substrate [Figs. 3(a) and 3(b-i)], consistent with our demonstration (Fig. 1).

The characteristic impulsive force on a flat substrate originates from the fact that the apex of a shell “kicks” the substrate. The buckled shell, whose rim is in contact with the substrate, abruptly snaps and transitions to the flat areal contact configuration when the contact radius reaches the critical value as $r \simeq r^*$ [detailed later in Eq. (8)] [26,38]. In contrast, as the jumping of the shell on the pointlike substrate progresses, the shell recovers to its natural hemispherical shape from the buckled configuration, while the apex stays in contact with a cylinder. The buckled radius r decreases continuously to zero during the jumping process, and then the popper leaves the substrate when r reaches zero, $r \simeq 0$. In other words, the buckling transition occurs reversely and *continuously* upon jumping. Based on this observation, we expect that jumping phenomena on pointlike substrates could be predicted by applying the mechanics of point indentation of shells.

IV. POINT-INDENTATION TEST FOR PRESSURIZED SHELL

We now consider the point-indentation test for an elastic shell based on previous literatures [2,5,6]. Suppose that the shell is point indented at the apex with the displacement e . For simplicity, the pressure difference between inside and outside of the shell is set to zero, the effects of which will be considered later. The shell buckles with the rim radius r as $r \sim \sqrt{eR}$ by geometry. The rim forms a ridge whose width δ decreases with the shell thickness h as $\delta \sim \sqrt{hR}$ [2,5,6]. The ridge width δ is determined by the balance of the stretching and bending energy as $\varepsilon_{st} \sim \varepsilon_b$. The stretching, ε_{st} , and bending energy per unit area, ε_b , are, respectively, given by $\varepsilon_{st} \sim E h e \delta^2 / R^3$ and $\varepsilon_b \sim E h^3 e / (R \delta^2)$, from which we obtain $\delta \sim \sqrt{hR}$ [38]. Assuming that the area of the ridge region is estimated as $\sim 2\pi r \delta \sim 2\pi R \sqrt{eh}$ and that the dominant contribution of elastic energy comes from the ridge, we obtain the elastic energy for the point indentation [2]:

$$\mathcal{E}_{\text{point}} \sim E \frac{h^{5/2} e^{3/2}}{R}. \quad (1)$$

The scaling relation for the indentation force F_{point} is now obtained by differentiating Eq. (1) with respect to e , $F_{\text{point}} \sim \partial \mathcal{E}_{\text{point}} / \partial e$, i.e., $F_{\text{point}} \sim (E h^{5/2} / R) e^{1/2}$. The rigorous

expressions for the elastic energy and the indenting force are, respectively, derived in the previous literature [7] as

$$\mathcal{E}_{\text{point}} = \frac{2\beta}{3} \frac{Eh^{5/2}}{\{12(1-\nu^2)\}^{3/4}R} e^{3/2}, \quad (2)$$

$$F_{\text{point}} = \beta \frac{Eh^{5/2}}{\{12(1-\nu^2)\}^{3/4}R} e^{1/2}, \quad (3)$$

with the dimensionless prefactor $\beta = 10.48$. In Ref. [7], the energy landscape for the indentation is further derived when the shell is pressurized from inside by pressure p . The contribution from the pressure difference is linearly added to Eqs. (2) and (3) by taking into account the volume change, $\sim \pi e^2 R/2$. The necessary work to indent the shell by the displacement e under the compressive pressure, p (the difference of outer pressure from inner pressure), $\mathcal{E}_{\text{point}}(e, p)$, is represented as

$$\mathcal{E}_{\text{point}}(e, p) = \frac{2\beta}{3} \frac{Eh^{5/2}}{\{12(1-\nu^2)\}^{3/4}R} e^{3/2} - \frac{\pi}{2} e^2 R p. \quad (4)$$

We use the above analytical formula, Eq. (4), to predict the jumping behavior of the popper.

V. ANALYTICAL PREDICTION FOR THE POINTLIKE SUBSTRATE

We expect that the maximum jumping height is derived by assuming that the necessary work to indent the shell (stored total energy of the system), $\mathcal{E}_{\text{point}}(e, p)$, is fully converted into the gravitational potential energy, mgH_{point} . In addition to the shell elastic energy, we would need to consider the work done by the internal air because the shell deforms (e decreases) against the pressure difference. The compressive pressure of the shell for a given indentation e is assumed to be determined by minimizing $\mathcal{E}_{\text{point}}(e, p)$ together with the gravitational potential, i.e., $\partial \mathcal{E}_{\text{point}} / \partial e = mg$, which yields the condition to relate p and e as

$$F_{\text{point}}(e) - \pi e R p = mg. \quad (5)$$

By substituting Eq. (5) into the energy conservation law as $\mathcal{E}_{\text{point}}(e, p(e)) = mgH_{\text{point}}$, we obtain the analytical prediction for the maximum jumping height as

$$H_{\text{point}} = \frac{\beta}{6\{12(1-\nu^2)\}^{3/4}} \frac{Eh^{5/2}e^{3/2}}{mgR} + \frac{e}{2}. \quad (6)$$

Equation (6) provides the maximum jumping height of the popper indented by e .

We measure the maximum jumping height of the popper, H , for various sets of shell geometry, (h, R) , Young's modulus E , net weight mg , and the initial displacement of the apex e , while we set the diameter of the cylinder as $d = 6$ mm throughout. The experimental parameters are coordinated by our analytical prediction, Eq. (6), in Fig. 4(a). We find that Eq. (6) (dashed line) is in excellent agreement with experimental data without any fitting parameters. The possible origin of their difference is the relation between the compressive pressure $p(e)$ and the apex displacement e [Eq. (5)], where the dynamics of the air is neglected. Although more sophisticated (computational) modeling by taking into account the fluid-structure interactions will improve the predictions, we leave the extension as a future study. The excellent agreement between the experimental results for $d = 6$ mm and

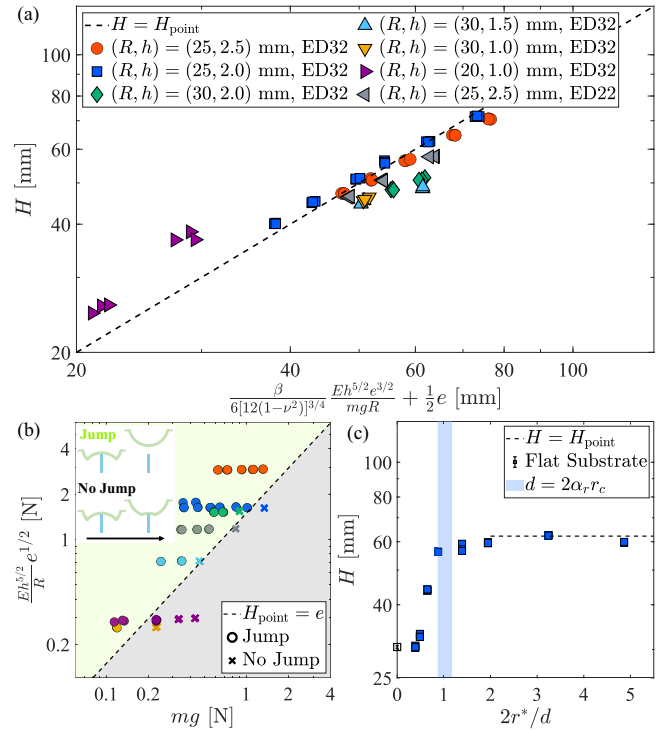


FIG. 4. Jumping performance of elastic shells on the narrow substrates. (a) The jumping height of the shell, H , on the pointlike substrate ($d = 6$ mm) plotted against the analytical formula (dashed line) [Eq. (6)]. (b) The phase diagram for the jump on the pointlike substrate ($d = 6$ mm). The dashed line represents the analytical prediction [Eq. (6)]. (c) The jumping height as a function of the rescaled substrate diameter, d/r_c , for the same loading conditions $[(R, h) = (25, 2.0)$ mm, $e \simeq 38$ mm, $mg = 0.45$ N]. The blue shaded region is the critical diameter for the contact geometry transition, $d = 2r^* = 2\alpha_r \sqrt{hR}$ [see Eq. (8)]. The dashed line represents the analytical formula for the jumping height on the narrow substrate ($d/2r^* \ll 1$). We plot the corresponding experimental data (empty square) for the flat substrate, $2r^*/d \rightarrow 0$.

theory based on the point-indentation model indicates that contact geometry of the point-indented pressurized shell is crucial in predicting the dynamics of the popper.

VI. JUMPING PHASE DIAGRAM

We have shown that Eq. (6) provides sufficient prediction for the maximum jumping height of the popper. We further apply Eq. (6) for another experiment, where we uncover whether the popper can jump or not. According to Eq. (6), H_{point} increases monotonically with e ; i.e., the popper can jump higher as we indent the apex largely. In other words, when the apex displacement is small, the stored elastic energy would be insufficient to launch the popper. The condition that the popper can jump or not corresponds to that H_{point} is larger than the initial displacement e or not; $H_{\text{point}} = e$. The critical condition for the pros and cons of the jump is

$$\frac{\beta}{3\{12(1-\nu^2)\}^{3/4}} \frac{Eh^{5/2}e^{1/2}}{R} = mg. \quad (7)$$

We have performed the systematic experiment by varying the net weight mg to identify the critical condition. We classify the experimental outcomes where the popper can/cannot leave the substrate as *jump/no jump*, respectively, in Fig. 4(b). We find that Eq. (7) is in excellent agreement with the experimental data, indicating again that the contact mechanics is essential to predict jumping performance of the popper.

VII. CRITICAL DIAMETER FOR THE SUBSTRATE

The jumping phenomena of the popper on the pointlike substrate ($d \rightarrow 0$) are now entirely predictable by applying the contact mechanics of the shell. It is demonstrated that the stored total energy of the system is fully converted into the mechanical energy when the diameter of the cylinder is sufficiently small. Here, we discuss the effect of cylinder diameter d on the maximum jumping height H and examine the relation with the prediction on the flat substrate validated in Ref. [26]. We perform the jumping experiment to measure H for various values of d , while we set $(h, R) = (2.0, 25)$ mm, $mg = 0.45$ N, and $e \simeq 38$ mm [Fig. 4(c)]. We find that the maximum height of the popper increases as we decrease the diameter of the substrate ($d \rightarrow 0$); i.e., the popper jumps higher on the pointlike substrate than on the flat one, which is consistent with our daily observations. Interestingly, the maximum height significantly increases once the diameter falls behind the critical length scale $2r^*$, the critical diameter for the contact geometry transition.

When the buckled shell is placed on the flat substrate, it jumps as the shell recovers its original shape, with a snapping sound. The shell snaps when the contact radius reaches critical one, r^* (derived below), where the ringlike contact shape (via the rim of the buckled shell) transits to the areal disklike contact [26]. The critical radius at the transition, r^* , is obtained by balancing the stretching and bending energy [38] as

$$r^* \sim \sqrt{hR}. \quad (8)$$

When the diameter of the cylinder is sufficiently large, the contact geometry is similar to the flat substrate. In other words, the apex of the shell cannot come into contact with the substrate prior to the contact geometry transition. When the contact radius r reaches the critical r^* , the shell snaps and pops out. However, when the diameter is sufficiently smaller than the critical radius, $d \ll 2r^*$, the cylinder is always in contact with the apex (the buckled rim is free), and the radius of the rim, r , decreases to zero without passing the contact geometry transition. Hence, the cylinder prohibits the contact geometry transition. Introducing the prefactor α_r ($1.2 \leq \alpha_r \leq 1.6$) into the scaling relation (8) as $r^* = \alpha_r \sqrt{hR}$ [26,38], we find that the critical radius provides a threshold for the substrate as shown in Fig. 4(c). When the diameter is on the order of the critical radius, $d \sim 2r^*$, one may need to take into account the geometry of the top surface of the cylinder. Indeed, it is reported that the contact geometry is crucial in the snap buckling of the shell via axisymmetric numerical simulation [19]. Although we speculate that the maximum height H follows a logistic curvelike function of $2r^*/d$, we leave the prediction for the intermediate regime $d \sim 2r^*$ as an intriguing future theoretical study, which is beyond the scope of this paper.

VIII. SUMMARY

In this paper, we have studied the jumping dynamics of elastomeric shells on a rigid cylinder. We experimentally analyze jumping performance on the rigid cylinder of diameter d . In contrast to the jumping dynamics of the popper on the flat substrate [26], the popper does not generate impulsive force upon jumping. Despite the absence of the impulsive force, the popper can offer sufficient jumping power. We have derived the analytical formula for the maximum jumping height on the pointlike substrate ($d \rightarrow 0$). The maximum jumping height results from the complex interplay of elasticity, geometry, and contact mechanics. Our prediction is in excellent agreement with the experimental data, not only for the maximum height but also for the jumping ability of the popper. Finally, we systematically vary the diameter of the cylinder to clarify the critical condition of the substrate being pointlike or flat. We identify that threshold is close to the critical diameter for the contact geometry transition of the shell in contact with the flat substrate [39].

Our model experimental results successfully validate the proposed analytical framework. We now examine whether our model provides useful prediction for the popping toy (e.g., Fig. 1) beyond the idealized experimental system. Given that our popping toys are sufficiently light, it would be allowed to take the limit of $mg \rightarrow 0$ and keep the leading order terms only. The maximum height on the flat substrate, H_{flat} , has been derived by taking into account elasticity and geometry for the contact geometry transitions [26]. The leading term for $mg \rightarrow 0$ is obtained by setting $e \sim h$ in Eq. (1) as $\mathcal{E} \sim Eh^4/R$. Assuming that the elastic energy at transition is fully converted into the gravitational potential energy, we get $H_{\text{flat}} \simeq \mathcal{E}/mg \sim Eh^4/mgR$. The maximum height on the pointlike substrate scales as $H_{\text{point}} \sim Eh^{5/2}e^{3/2}/mgR$. When the popper is fully inverted before jump, the apex displacement is approximately written as $e \approx 2R$. The ratio of the maximum jumping height between flat and pointlike substrate obeys the following scaling relation:

$$\frac{H_{\text{point}}}{H_{\text{flat}}} \sim \left(\frac{R}{h}\right)^{3/2}, \quad (9)$$

implying that the thicker shells would jump higher on the narrow substrate than on the flat one. In our demonstration, we have used the popper toys of $(R, h) \approx (20, 2)$ mm. After computing all the dimensionless prefactors [40], we get the ratio between H_{point} and H_{flat} as $H_{\text{point}}/H_{\text{flat}} \approx 1.4$, consistent with our demonstration. Although we need to be careful about the validity of the scaling relation, Eq. (9) [41], we believe that our predictive framework for the jumping behavior could be extended to more complex popping robots.

We note the difference between fundamental mechanisms behind the jumping ability on the flat and narrow substrates from the bifurcation point of view. We additionally perform two different loading/unloading tests using either a plate or a narrow cylinder in Fig. 5, where the force F and displacement Δ are normalized by the characteristic scales as $\tilde{F} = F/(Eh^3/(12(1-\nu^2)R))$ and $\tilde{\Delta} = \sqrt{12(1-\nu^2)}\Delta/h$. Upon plate indentation, the shell contacts the plate via disk (areal) contact, and the contact geometry transits to the ring, which appears as the upper branch in Fig. 5(a). For the

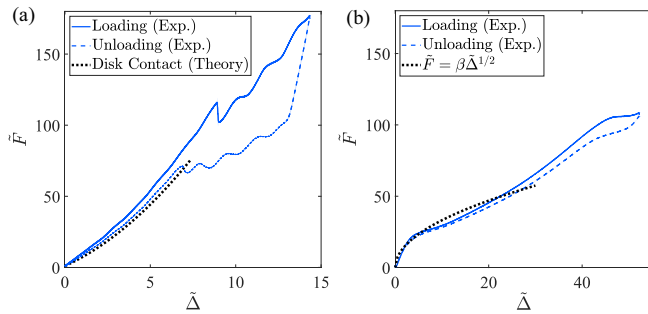


FIG. 5. The rescaled force-displacement curves of the shell $[(R, h) = (25, 2.0) \text{ mm}]$ for the indentation test using (a) the flat plate and (b) narrow cylinder ($d = 6 \text{ mm}$). The solid and dashed blue lines correspond to the experimental loading and unloading processes, respectively. The black dashed line in panel (a) represents the analytical theory assuming the disk (areal) contact [2], while that in panel (b) is the analytical prediction for the point-indentation test [7]. The theory has no adjustable parameters.

unloading protocol, the force follows a different path together with the change in contact geometry from ring to disk (lower branch). This contact geometry transition plays a crucial role in jumping on the flat substrate, which is characterized by the hysteretic response in the indentation test [Fig. 5(a)]. The force-displacement curve exhibits a subcritical nature, where the loading and unloading processes track different paths. The shell is indented to store the elastic energy, which is released upon jumping. The indenting and jumping processes correspond to the loading and unloading protocols, respectively. The difference in the loading/unloading curves, i.e., the area enclosed in the cyclic test, represents the energy dissipation in the jumping process on the flat substrate (highlighted by the snapping sound). However, when the shell snaps on the narrow substrate, the contact geometry transition does not exist. In contrast to the flat-indentation test, the force-displacement curve by the point-indentation exhibits almost no hysteresis [Fig. 5(b)], where there is no subcritical bifurcation. Given that the loading and unloading process follows nearly

identical paths, the energy dissipation is less pronounced upon jumping. In other words, the jumping process on the pointlike substrate utilizes the stored total energy of the system efficiently. Hence, the jumping height on the narrow substrate observed in our experiment is higher than that on the flat one.

Throughout this paper, we utilize the idealized setup, where experimental dissipation is minimized as much as possible. Our analytical framework in the limit of $d \rightarrow 0$ stems from the energy conservation law. It is consistent with the experiments for finite but small d , which is identified to be violated above the critical diameter as $d \gg 2r^*$. When the popping model demonstrated in this study is applied in more realistic engineering applications, one would need to take into account dissipation in the system. For example, the effects of the friction on the rail and the viscoelasticity of the material need to be carefully examined in the future.

We have revealed that jumping on the pointlike substrate is more efficient than that on the flat substrate, at least with respect to the maximum height. Although jumping on the narrow substrate is energetically efficient, the impulsive force triggered by snap buckling does not exist (see Fig. 3). The jumping process on the narrow substrate is similar to the springlike actuation [42]. The switching mechanism between snap buckling and springlike actuation is mediated by contact geometry between the shell and substrate. We expect that the role of contact geometry uncovered in this study would be applied in optimizing the actuation mechanism of soft and flexible robots in the near future. For example, by optimizing the surface geometry of the shell, one may be able to control jumping performance by design [8].

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DATA AVAILABILITY

The data that support the findings of this article are openly available [39].

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