

Erratum: Improved local models and new Bell inequalities via Frank-Wolfe algorithms **[Phys. Rev. Research 5, 043059 (2023)]**

Sébastien Designolle , Gabriele Iommazzo , Mathieu Besançon ,
 Sebastian Knebel , Patrick Gelß , and Sebastian Pokutta 



(Received 5 October 2026; published 27 October 2026)

DOI: [10.1103/knpj-2k73](https://doi.org/10.1103/knpj-2k73)

Two errors in the treatment of marginal correlators affect the multipartite local models in the original paper. First, Lemma 1 of the Supplemental Material [1] holds for the Euclidean norm of full correlators, but not when marginal correlators are included. Its unit-ball formulation remains valid with marginals if the Euclidean norm is replaced by the sum of the Euclidean norms of the separate party-subset tensors. Second, the extension from finitely many measurements to all projective measurements requires an additional rescaling of the marginal correlators in the finite target. This rescaling was missing in the original computations.

Repeating the multipartite computations with these corrections gives

$$v_c^{\text{GHZ}_3} \geq 0.463679, \quad v_c^{\text{W}_3} \geq 0.491051, \quad (1)$$

in place of 0.468 838 and 0.491 675, respectively. The entries 0.4688 and 0.4917 in the “This work” lower-bound rows of Table II should thus read 0.4637 and 0.4911. The upper bounds are unaffected by these corrections. Since the corrected W-state lower bound is below the GHZ-state upper bound of approximately 0.4916, our bounds no longer prove $v_c^{\text{GHZ}_3} < v_c^{\text{W}_3}$. The claims of a proven strict ordering in the last paragraph of the Introduction and in the caption of Table II are therefore withdrawn. The corrected bounds leave this ordering open. The bipartite Werner-state bounds and their consequences for $K_G(3)$ are unaffected, since that construction uses only full correlators and has vanishing marginals.

We give the corrected definitions, lemma, and multipartite lower-bound construction below. They replace the affected parts of Secs. IV and V of the published Supplemental Material [1].

Marginal correlators and the norm. For N parties with m binary measurements each, let $\mathbf{r}_S$ denote the tensor of correlators for a nonempty subset $S \subseteq [N]$, where $[N] = \{1, \dots, N\}$. The corrected version of Eq. (19) of the Supplemental Material [1] is

$$(\mathbf{r}_S)_{x_S} = \sum_{a_S \in \{-1, +1\}^{|S|}} \left(\prod_{i \in S} a_i \right) p(a_S | x_S). \quad (2)$$

Here, $p(a_S | x_S)$ is the normalized marginal distribution of a nonsignaling behavior. No factor $m^{-(N-|S|)}$ is needed. The fixed normalization coordinate $r_{0,\dots,0} = 1$ is omitted from all correlation vectors and their norms. In particular, white noise is represented by $\mathbf{0}$. Define

$$\|\mathbf{r}\|_{1,2} = \sum_{\emptyset \neq S \subseteq [N]} \|\mathbf{r}_S\|_2 \quad \text{and} \quad \|\mathbf{r}\|_2^2 = \sum_{\emptyset \neq S \subseteq [N]} \|\mathbf{r}_S\|_2^2.$$

The first expression is a norm on the full space of nonempty correlators. If only the full-correlation block is retained, it reduces to the usual Euclidean norm.

Corrected Lemma 1. In a Bell scenario with two outputs per party, the closed unit ball for the norm $\|\cdot\|_{1,2}$ is contained in the local polytope including marginals. For full correlators alone, the closed Euclidean unit ball is contained in the corresponding local polytope.

The sign-absorption argument in the original proof cannot be applied to all blocks simultaneously: Flipping one party’s outputs negates only the correlators containing that party.

Corrected proof. First retain only full correlators. Write $\mathbf{d}(a)_{x_1 \dots x_N} = \prod_{i=1}^N a_{x_i}^{(i)}$, where each deterministic response $a_x^{(i)}$ is ± 1 . Let $\sum_a$ run over all 2^{Nm} assignments of these signs. For $w(a) = \langle \mathbf{r}, \mathbf{d}(a) \rangle$, averaging uniformly over all assignments gives

$$\frac{1}{2^{Nm}} \sum_a w(a) \mathbf{d}(a) = \mathbf{r} \quad \text{and} \quad \frac{1}{2^{Nm}} \sum_a w(a)^2 = \|\mathbf{r}\|_2^2. \quad (3)$$

Both $\mathbf{d}(a)$ and $-\mathbf{d}(a)$ are deterministic full-correlation tensors, so the first equality in Eq. (3) is a non-negative combination of such tensors with total weight bounded by the Cauchy-Schwarz inequality:

$$\frac{1}{2^{Nm}} \sum_a |w(a)| \leq \sqrt{\frac{1}{2^{Nm}} \sum_a w(a)^2} = \|\mathbf{r}\|_2.$$

If $\|\mathbf{r}\|_2 \leq 1$, complete this combination with the local point $\mathbf{0}$ to obtain a convex decomposition.

Now fix a nonempty subset S and a tensor $\mathbf{r}_S$ with $\|\mathbf{r}_S\|_2 \leq 1$. The preceding argument provides a local full-correlation model on S . Multiply all outputs of party $i \in S$ by a shared random sign s_i , uniformly distributed subject to $\prod_{i \in S} s_i = 1$. This preserves the S correlators and makes every nonempty proper marginal on S vanish, since every proper product of the signs has zero average. For a singleton S , the sign is simply fixed to 1. Give each party outside S an independent unbiased output. The resulting local model has $\mathbf{r}_S$ as its only nonzero block.

For arbitrary $\mathbf{r}$ with $\|\mathbf{r}\|_{1,2} \leq 1$, apply this construction to each nonzero block $\mathbf{r}_S/\|\mathbf{r}_S\|_2$ and mix the models with weights $\|\mathbf{r}_S\|_2$. Their sum is at most one; any remaining weight is assigned to $\mathbf{0}$. This proves the corrected lemma.

In Euclidean coordinates, the lemma implies

$$\|\mathbf{r}\|_2 \leq \frac{1}{\sqrt{2^N - 1}} \implies \mathbf{r} \text{ is local,} \quad (4)$$

because $\|\mathbf{r}\|_{1,2} \leq \sqrt{2^N - 1} \|\mathbf{r}\|_2$. This Euclidean radius is optimal: Set all $2^N - 1$ nonempty correlators at the input tuple $(1, \dots, 1)$ to $-t/(2^N - 1)$ and all other coordinates to zero. The norm is $t/\sqrt{2^N - 1}$, whereas $p(+1, \dots, +1|1, \dots, 1) = (1 - t)/2^N < 0$ for $t > 1$.

Residual correction and measurement shrinking. Let $\rho_v = v\rho + (1 - v)\mathbb{I}/2^N$, and let $\mathbf{p}_S$ be the correlator tensors of ρ for the chosen finite sets of projective measurements. Global white noise multiplies every nonempty block by the same visibility v . In contrast, simulating arbitrary measurements with a local Bloch-vector shrinking factor η multiplies a block involving $|S|$ parties by $\eta^{|S|}$. Applied to a local model for $v\rho$, this would give $v\eta^{|S|}\mathbf{p}_S$, rather than the correlators $v\eta^N\mathbf{p}_S$ of $\rho_{v\eta^N}$. Thus, the original use of Eq. (31) of the Supplemental Material [1] was not justified when proper marginals were nonzero.

To recover the common factor η^N , the finite membership computation must instead target $v_0\mathbf{q}$, where

$$\mathbf{q}_S = \eta^{N-|S|}\mathbf{p}_S \quad \text{for } \emptyset \neq S \subseteq [N]. \quad (5)$$

The full correlators are unchanged, while the lower-order ones are shrunk before optimization.

Given an exactly local iterate $\mathbf{x}_T$ and residual $\mathbf{e} = v_0\mathbf{q} - \mathbf{x}_T$, Eq. (10) of the Supplemental Material [1] keeps its form with the corrected norm: $v_2 = 1/(1 + \|\mathbf{e}\|_{1,2})$, $\mathbf{y} = \mathbf{e}/\|\mathbf{e}\|_{1,2}$, and $v_2v_0\mathbf{q} = v_2\mathbf{x}_T + (1 - v_2)\mathbf{y}$. For $\mathbf{e} \neq \mathbf{0}$, the lemma makes $\mathbf{y}$ local. If $\mathbf{e} = \mathbf{0}$, take $v_2 = 1$ and no residual correction is needed. A valid, possibly smaller, alternative using only the Euclidean residual is $v_2 = 1/(1 + \sqrt{2^N - 1}\|\mathbf{e}\|_2)$.

For completeness, η is chosen so that every unit Bloch vector $\hat{u}$ has a convex representation

$$\eta\hat{u} = \sum_{x=1}^m \sum_{s=\pm 1} \lambda_{x,s}^{\hat{u}} s\mathbf{a}_x$$

for $\lambda_{x,s}^{\hat{u}} \geq 0$ and $\sum_{x,s} \lambda_{x,s}^{\hat{u}} = 1$. Here, the sum runs over the finite measurement directions x and both signs s ; the latter are implemented by output relabeling. Applying this local simulation at each party to the model for $v_2v_0\mathbf{q}$ gives, for every nonempty S and every choice of projective measurements,

$$v_2v_0\eta^{|S|}\eta^{N-|S|}\mathbf{p}_S = v_2v_0\eta^N\mathbf{p}_S. \quad (6)$$

The construction therefore establishes $v_c \geq \eta^N v_2 v_0$, with the corrected finite target and residual factor. Changing the norm alone is insufficient: The finite local models must also be recomputed for Eq. (5).

Corrected tripartite computations. For both states, we use the $m = 61$ measurement polyhedron listed in Table I of the Supplemental Material [1], with $\eta \approx 0.9792$. The reruns give the following replacements for the numerical details in Sec. V C 1:

	GHZ ₃	W ₃
Initial v_0	0.494	0.524
Number of deterministic strategies	54486	33529
Analyticity factor v_2 (approx.)	0.9997	0.9981
Lower bound $\eta^3 v_2 v_0$	0.4637	0.4911

The upper-bound calculations in Sec. V C 2 do not use the marginal correction. For the equatorial GHZ measurements, all proper marginals vanish, so the planar shrinking construction is likewise unaffected by the errors corrected here.

There is also a separate error in the W-state “Entanglement threshold” row of Table II. The value 0.2096 attributed to Szalay [2] is the positive-partial-transpose boundary $3/(3 + 8\sqrt{2})$, not the full-separability threshold. This row should instead give 0.1780, attributed to Chen and Jiang [3], with the year 2020.

Acknowledgment. GPT 6 Astra assisted in identifying and correcting the marginal-correlator error in the lemma.

-
- [1] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/PhysRevResearch.5.043059> for original paper, Secs. IV and V.
[2] S. Szalay, Separability criteria for mixed three-qubit states, *Phys. Rev. A* **83**, 062337 (2011).
[3] X.-Y. Chen and L.-Z. Jiang, Noise tolerance of Dicke states, *Phys. Rev. A* **101**, 012308 (2020).