

Phase-space topology and spectral flow in screened magnetized plasmas

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(Received 12 February 2026; accepted 10 June 2026; published 24 June 2026)

Topological wave phenomena in continuous media are fundamentally challenged by unbounded spectra and the absence of a compact Brillouin zone, which obstruct conventional bulk-interface formulations. We develop a unified phase-space framework for screened magnetized plasma based on a pseudo-Hermitian formulation with a positive-definite metric, enabling a generalized Schrödinger description and a Weyl-symbol analysis of the bulk generator. We show that the bulk Weyl symbol hosts isolated band degeneracies acting as Berry-Chern monopoles, including a higher-order spin-1 degeneracy with topological charge +2 that generically splits into two spin- $\frac{1}{2}$ Weyl points for finite k_z . To characterize topology in this noncompact setting, we introduce a strip-gap Chern number associated with finite real-frequency strips of the spectrum of the phase-space symbol, extending band Chern topology to continuum systems. This invariant governs the spectral flow of interface modes induced by spatial variations of the magnetic field and establishes a bulk-interface correspondence at the level of phase-space symbols. By solving the interface eigenvalue problem, we demonstrate that the net spectral flow across the strip gap is determined by the enclosed monopole charge. We further show that this correspondence persists under weak collisional damping as long as the complex spectrum retains a finite real-part strip gap. Our results provide a systematic phase-space framework for topological wave transport in continuous media beyond compact-band and idealized Hermitian settings.

DOI: [10.1103/kfhx-hp37](https://doi.org/10.1103/kfhx-hp37)

I. INTRODUCTION

Topological phases of waves have emerged as a unifying concept across condensed matter physics [1,2], photonics [3,4], acoustics [5,6], fluid dynamics [7,8], and plasma physics [9–12] revealing that unidirectional transport protected by band topology can arise from global features of the underlying spectrum rather than microscopic details. While early studies focused primarily on lattice systems with discrete Bloch bands [13,14], it has become increasingly clear that similar topological phenomena can also occur in continuous media, including fluids, plasmas, and geophysical flows [7,8,15,16]. Prominent examples include equatorial waves in rotating fluids, magnetized plasma waves, and electromagnetic continua, where topology manifests itself through spectral flow and protected interface modes rather than conventional edge states [10,17,18].

Topological waves in magnetized plasmas have been investigated in several important settings. Previous works on cold magnetized plasmas have established bulk Chern-number characterizations of plasma-wave bands and demonstrated bulk-edge correspondence for plasma-vacuum and plasma-plasma interfaces [9]. These studies showed that

gyrotropic plasma dynamics can support chiral surface or interface modes whose existence is controlled by the topology of the homogeneous bulk spectrum [9,10]. Related developments have further connected plasma-wave topology to electromagnetism continua, non-Hermitian wave physics, and more general fluid and magnetohydrodynamic models [19,20]. These results provide the essential band-topological background for the present work.

The focus of this paper is complementary. Rather than focusing primarily on the homogeneous cold-plasma band problem in momentum space, we study a screened magnetized plasma in an inhomogeneous continuous setting. Screening modifies the plasma-wave spectrum and provides a controlled setting in which the relevant band degeneracies can be analyzed. Moreover, the interface problem naturally depends on both position and wave vector, so that the corresponding topological description is formulated in phase space rather than solely in momentum space [21,22]. This motivates a strip-gap formulation of topology and spectral flow for noncompact continuous spectra.

Despite this progress, establishing a systematic bulk-interface correspondence for continuous media remains challenging [8,10,15]. Unlike lattice systems, continuous media generally lack a compact Brillouin zone and exhibit unbounded spectra, rendering standard band-topological invariants inapplicable [15,17]. In particular, the notion of a band gap must be generalized to accommodate spectral continua, and the definition of topological invariants requires careful treatment in phase space [15,23]. These difficulties are further exacerbated in plasma systems, where the dynamical variables are subject to constraints such as Gauss's law and screening, leading to noncanonical Hamiltonian structures [24].

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In this work, we address these challenges by formulating the dynamics of a screened magnetized plasma as a pseudo-Hermitian system with a positive-definite metric. This structure allows the linearized equations to be cast into a generalized Schrödinger form with a real spectrum in the absence of dissipation [25]. By employing the Weyl-Wigner formalism, we analyze the Weyl symbol of the effective generator and identify isolated band degeneracies in phase space [26,27]. To characterize topology in this continuous setting, we introduce a strip-gap Chern number, defined for real-frequency intervals free of spectrum on a sufficiently large phase-space surface. This invariant generalizes the usual band-gap topology to continuous media and connects naturally to spectral-flow formulations of bulk-interface correspondence [10,15,28].

It is useful to distinguish two closely related spaces in our formulation. For the homogeneous bulk problem, the reduced magnetic field μ is treated as an external parameter, and the Weyl symbol is analyzed in the extended parameter space (μ, k_x, k_y) , where isolated degeneracies act as Berry-Chern monopoles. For the interface problem, μ becomes a spatial profile $\mu(x)$, and the corresponding symbol is obtained by pulling back the bulk symbol to the interface phase space (x, k_x, k_y) . The strip-gap Chern number governing spectral flow is defined in this interface phase space, while the bulk parameter-space monopoles provide a convenient way to identify which degeneracies are swept by a given $\mu(x)$ profile.

Within this framework, the screened magnetized plasma considered here realizes a pseudo-Hermitian continuum system hosting a spin-1 band degeneracy at $k_z = 0$. We show that this higher-order degeneracy carries a topological charge +2 and generically splits into two spin- $\frac{1}{2}$ Weyl points under symmetry breaking, $k_z = 0 \rightarrow k_z \neq 0$. When the background magnetic field varies spatially, the associated strip-gap Chern number governs the spectral flow of interface modes. By solving the corresponding one-dimensional eigenvalue problem, we confirm that the net number and chirality of gap-crossing modes are dictated by the total monopole charge swept by the profile $\mu(x)$. Finally, we show that this spectral flow persists under weak collisional damping as long as the complex spectrum retains a finite real-part strip gap [28,29]. When this real-part spectral separation is destroyed, the strip-gap spectral flow is no longer well defined. Exceptional points may provide one possible mechanism for such a breakdown, but their explicit identification is not assumed in our criterion. These results provide a phase-space interpretation of chiral interface waves in screened magnetized plasmas and clarify the regime in which the bulk-interface correspondence extends beyond ideal Hermitian models.

II. WAVE TOPOLOGY OF SCREENED MAGNETIZED PLASMA

In this section, we develop the bulk topological characterization of a screened magnetized plasma at the level of its phase-space symbol. Our goal is to identify the relevant band degeneracies of the Weyl symbol, define the associated strip-gap Chern numbers, and establish the bulk topological invariants that underlie the bulk-interface correspondence discussed in the Introduction. We first cast the linearized plasma

dynamics into a pseudo-Hermitian Schrödinger-type form, and then analyze the topology of the resulting Weyl symbol in phase space.

With a uniform background magnetic field $\mathbf{B}_0$ and vanishing background flow, the linearized dynamics of a magnetized cold plasma is governed by

$$\begin{aligned} m_e \frac{\partial \mathbf{v}_1}{\partial t} &= e \nabla \phi_1 + e \mathbf{v}_1 \times \mathbf{B}_0, \\ \frac{\partial n_1}{\partial t} &= -n_0 \nabla \cdot \mathbf{v}_1, \\ (\nabla^2 - \lambda^2) \phi_1 &= 4\pi e n_1. \end{aligned} \quad (1)$$

Here, the displacement current is neglected, and the electric field is assumed to arise solely from the perturbed electron density n_1 . $\mathbf{v}_1$ denotes the perturbed electron velocity, λ^{-1} is the screening length of the plasma, and ϕ_1 is the electrostatic potential. The electron charge and mass are denoted by e and m_e , respectively, and n_0 is the uniform background density.

We normalize time by the inverse plasma frequency ω_{pe}^{-1} , with $\omega_{pe} = \sqrt{4\pi n_0 e^2 / m_e}$, and length by the screening length λ^{-1} . The electrostatic potential is normalized by $4\pi n_0 |e| / \lambda^2$, and the velocity by ω_{pe} / λ . Introducing the state vector

$$\Xi = (\mathbf{v}_1, \phi_1)^T,$$

and defining the dimensionless cyclotron frequency $\mu = (|e| \mathbf{B}_0 / m_e) / \omega_{pe}$, the system (1) can be cast into a generalized Schrödinger-type form [25,30]

$$i \hat{\eta} \partial_t \Xi = \hat{H} \Xi. \quad (2)$$

Here, $\hat{H}$ is a Hermitian operator, while $\hat{\eta}$ is a positive-definite Hermitian metric operator,

$$\hat{H} = \begin{pmatrix} i\mu \times & -i\nabla \\ -i\nabla \cdot & 0 \end{pmatrix}, \quad \hat{\eta} = \begin{pmatrix} I & 0 \\ 0 & -\nabla^2 + 1 \end{pmatrix}. \quad (3)$$

Equation (2) therefore defines a pseudo-Hermitian dynamical system, in which the generator $\hat{\eta}^{-1} \hat{H}$ is similar to a Hermitian operator and possesses a purely real spectrum as long as $\hat{\eta}$ remains positive definite [25].

We consider inhomogeneous magnetic configurations of the form $\mu = \mu(x) \mathbf{e}_z$, varying only along the x direction. Under this assumption, a Fourier transform can be performed along the homogeneous directions (y, z, t) , such that the transverse wave numbers k_y and k_z are conserved parameters. The resulting eigenvalue problem along x thus depends parametrically on (k_y, k_z) .

For continuous media, the spectral and topological properties are encoded in the phase-space structure of the associated Weyl symbol [21,26]. Applying the Wigner transform along the x direction, we obtain the exact Weyl symbol of the effective generator $\hat{\eta}^{-1} \hat{H}$,

$$\text{sym}(\hat{\eta}^{-1} \hat{H}) = \begin{pmatrix} 0 & -i\mu(x) & 0 & k_x \\ i\mu(x) & 0 & 0 & k_y \\ 0 & 0 & 0 & k_z \\ \frac{k_x}{k^2+1} & \frac{k_y}{k^2+1} & \frac{k_z}{k^2+1} & 0 \end{pmatrix}, \quad (4)$$

where $k^2 = k_x^2 + k_y^2 + k_z^2$. The symbol satisfies the pseudo-Hermitian symmetry relation

$$\text{sym}(\hat{\eta}) \text{sym}(\hat{\eta}^{-1} \hat{H}) \text{sym}(\hat{\eta}^{-1}) = \text{sym}(\hat{\eta}^{-1} \hat{H})^\dagger,$$

and the positivity of the screening-induced metric $\hat{\eta}$ guarantees that this symmetry is unbroken. Let the symbol of $\hat{\eta}$ be denoted by η , and the symbol of $\hat{H}$ by H . Then, it is evident that

$$\text{sym}(\hat{\eta}^{-1}\hat{H}) = \eta^{-1}H. \quad (5)$$

Strictly speaking, $\text{sym}(\hat{\eta}^{-1}\hat{H}) = \text{sym}(\hat{\eta}^{-1}) \star H$, where $\star$ denotes the Moyal product. However, $\hat{\eta}^{-1}$ acts only on the fourth row of the operator $\hat{H}$, and it is independent of the parameter x . The x -dependent terms appear only in μ , which enters the first and second rows of $\hat{H}$. Therefore, the equality $\text{sym}(\hat{\eta}^{-1}\hat{H}) = \eta^{-1}H$ holds exactly.

The symbol $\text{sym}(\hat{\eta}^{-1}\hat{H})$ possesses four real eigenvalues, which can be written compactly as

$$(\pm)_1 \frac{\sqrt{k^2 + (1+k^2)\mu^2 + (\pm)_2 \sqrt{(k^2 + (1+k^2)\mu^2)^2 - 4(1+k^2)k_z^2\mu^2}}}{\sqrt{2}\sqrt{1+k^2}}.$$

The four branches are ordered as $\omega_1 \geq \omega_2 \geq \omega_3 \geq \omega_4$, with $\omega_1 = -\omega_4$ and $\omega_2 = -\omega_3$. Before proceeding to the interface problem, we clarify the spaces in which the topological invariants are defined. For a homogeneous bulk plasma, the reduced magnetic field μ is treated as an external parameter. For fixed k_z , the bulk Weyl symbol

$$h_b(\mu, k_x, k_y; k_z) = \eta^{-1}H(\mu, k_x, k_y; k_z)$$

is defined over the extended parameter space (μ, k_x, k_y) . Isolated degeneracies of this symbol act as Berry-Chern monopoles. The charge of an isolated monopole p_a is computed on a small closed surface $S_a \subset (\mu, k_x, k_y)$ enclosing it:

$$q_a = \frac{i}{2\pi} \int_{S_a} \text{Tr}(P_b dP_b \wedge dP_b),$$

where

$$dP_b = \partial_\mu P_b d\mu + \partial_{k_x} P_b dk_x + \partial_{k_y} P_b dk_y.$$

Here, P_b is the spectral projector onto the spectral subspace separated from the considered strip gap on its lower-real-frequency side. This projector is defined purely by the spectral separation of the linear wave operator. It is not an occupation projector and does not involve a Fermi level, a Fermi-Dirac distribution, or any thermodynamic occupation.

For the interface problem, μ is no longer an independent parameter but a smooth spatial profile $\mu = \mu(x)$. The relevant Weyl symbol is the pullback

$$h_{\text{int}}(x, k_x, k_y; k_z) = h_b(\mu(x), k_x, k_y; k_z),$$

defined over the interface phase space (x, k_x, k_y) . Throughout this work, a strip gap refers to a real-frequency interval $I_\omega \subset \mathbb{R}$ such that, outside a sufficiently large compact region in the phase-space variables (x, k_x, k_y) , the spectrum of h_{int} does not intersect I_ω . This condition is the phase-space analog of a band gap for continuous media with unbounded spectra.

Since the phase space is noncompact, the gap condition has to be checked at phase-space infinity. For simplicity, we assume that $\mu(x)$ is smooth, bounded, and approaches constant asymptotic values,

$$\mu(x) \rightarrow \mu_1 \quad (x \rightarrow -\infty), \quad \mu(x) \rightarrow \mu_2 \quad (x \rightarrow +\infty). \quad (6)$$

This assumption is compatible with the interface problem considered in the following numerical sections. It is imposed to ensure that the asymptotic behavior of the interface operator

h_{int} remains controllable for sufficiently large $|x|$. We then show that, under suitable conditions, h_{int} does not encounter any bulk degeneracy points of h_b in the asymptotic region of phase space.

For the gap between the two positive-frequency branches ω_1 and ω_2 , this condition can be verified explicitly. As shown in Appendix A, the large- K asymptotics gives

$$\{\omega_1, \omega_2, \omega_3, \omega_4\} \rightarrow \{\sqrt{1 + \mu(x)^2}, 0, 0, -\sqrt{1 + \mu(x)^2}\},$$

uniformly for bounded $\mu(x)$. Hence, the gap between ω_1 and ω_2 remains uniformly open in the large-momentum region.

For fixed $k_z \neq 0$, let

$$\mu_0 = \sqrt{\frac{k_z^2}{1 + k_z^2}}.$$

The asymptotic bulk gaps at $x \rightarrow \pm\infty$ are given by

$$(\min(\mu_0, |\mu_{1,2}|), \max(\mu_0, |\mu_{1,2}|)).$$

Therefore, if

$$(|\mu_2| - \mu_0)(|\mu_1| - \mu_0) > 0,$$

the two asymptotic gaps have a nonempty common intersection. In this case, one can choose

$$I_\omega = \bigcap_{i=1,2} (\min(\mu_0, |\mu_i|), \max(\mu_0, |\mu_i|)) \cap (\epsilon, 1),$$

with a suitable $0 < \epsilon < \mu_0$. This interval is avoided by the spectrum of h_{int} outside a sufficiently large compact region in (x, k_x, k_y) , and hence defines a uniform strip gap at phase-space infinity.

In particular, for $k_z = 0$, one has $\mu_0 = 0$. Provided that $|\mu_{1,2}| > 0$, the strip gap can be chosen as

$$I_\omega = (\epsilon, \min(|\mu_1|, |\mu_2|, 1)),$$

with $0 < \epsilon < \min(|\mu_1|, |\mu_2|, 1)$.

Under these conditions, the gap does not close at phase-space infinity. Any gap closing relevant to the interface topology can therefore occur only at finite phase-space points $(x_a, 0, 0)$, which are pulled back from the bulk degeneracies $(\mu_a, 0, 0)$ through the relation $\mu(x_a) = \mu_a$.

The interface strip-gap Chern number is then defined on a closed surface $\Sigma \subset (x, k_x, k_y)$ enclosing the relevant finite

phase-space degeneracies:

$$C_{\text{gap}}^{\text{int}} = \frac{i}{2\pi} \int_{\Sigma} \text{Tr}(P_{\text{int}} dP_{\text{int}} \wedge dP_{\text{int}}), \quad (7)$$

where P_{int} is the spectral projector of h_{int} onto the spectral subspace separated from the considered strip gap on its lower-real-frequency side, and

$$dP_{\text{int}} = \partial_x P_{\text{int}} dx + \partial_{k_x} P_{\text{int}} dk_x + \partial_{k_y} P_{\text{int}} dk_y.$$

Since P_{int} is smooth on any surface avoiding the degeneracies, $C_{\text{gap}}^{\text{int}}$ is invariant under smooth deformations of Σ that do not cross a degeneracy. A change of the Chern number occurs only when Σ crosses a Berry-Chern monopole.

Under the pullback map

$$\pi : (x, k_x, k_y) \mapsto (\mu(x), k_x, k_y),$$

a bulk monopole at $(\mu_a, 0, 0)$ appears as an interface phase-space degeneracy at $(x_a, 0, 0)$, where $\mu(x_a) = \mu_a$. For a monotonic profile crossing the degeneracy values transversely, the interface Chern number equals the sum of the swept monopole charges, with the orientation determined by the profile

$$C_{\text{gap}}^{\text{int}} = \text{sgn}(\mu_2 - \mu_1) \sum_{a: \mu_a \in (\min(\mu_1, \mu_2), \max(\mu_1, \mu_2))} q_a.$$

We now evaluate the same Chern form for the homogeneous bulk symbol in the extended parameter space (μ, k_x, k_y) , in order to determine the monopole charges entering $C_{\text{gap}}^{\text{int}}$. Let the right and left eigenvectors of $\text{sym}(\hat{\eta}^{-1}\hat{H})$ corresponding to the eigenvalue ω_i be denoted as $|\Psi_i^R\rangle$ and $\langle\Psi_i^L|$, respectively. Then,

$$\eta^{-1}H |\Psi_i^R\rangle = \omega_i |\Psi_i^R\rangle, \quad (8)$$

$$\langle\Psi_i^L| \eta^{-1}H = \omega_i \langle\Psi_i^L|. \quad (9)$$

Due to the presence of pseudo-Hermitian symmetry with respect to $\hat{\eta}$, the eigenvectors can be chosen to satisfy the biorthogonality condition [25]:

$$\langle\Psi_i^L|\Psi_j^R\rangle = \delta_{ij} \quad (10)$$

and

$$|\Psi_i^L\rangle = \eta |\Psi_i^R\rangle. \quad (11)$$

For the strip gap between ω_1 and ω_2 , and with the ordering $\omega_1 \geq \omega_2 \geq \omega_3 \geq \omega_4$, the spectral projector onto the lower-real-frequency side of the gap is

$$P = \frac{1}{2\pi i} \oint_{\Gamma} (z - h_b)^{-1} dz = \sum_{i=2}^4 |\Psi_i^R\rangle \langle\Psi_i^L|.$$

Here, Γ is a closed contour in the complex-frequency plane enclosing the spectral cluster $\{\omega_i\}_{i=2}^4$ and no other eigenvalues. This projector is defined purely by the spectral separation of the linear wave operator. It should not be interpreted as an occupation projector, and no Fermi level, Fermi-Dirac distribution, or thermodynamic occupation is involved in the present plasma-wave problem.

Using the eigenvalue equations satisfied by $|\Psi_i^{R/L}\rangle$, namely,

$$\omega_i |\Psi_i^R\rangle = \eta^{-1}H |\Psi_i^R\rangle, \quad (12)$$

$$\omega_i |\Psi_i^L\rangle = (\eta^{-1}H)^\dagger |\Psi_i^L\rangle, \quad (13)$$

one can derive the expression for the gap Chern number. For the bulk-monopole calculation below, we use the bulk version of the same Chern form in the extended parameter space (μ, k_x, k_y) . In this case, the integration surface is a closed surface $S_b \subset (\mu, k_x, k_y)$ enclosing the relevant bulk degeneracy. The two branches ω_1 and ω_2 remain separated on S_b , while their degeneracy is located inside the enclosed region. Here, ∇ denotes the gradient with respect to these bulk parameters:

$$\nabla = (\partial_\mu, \partial_{k_x}, \partial_{k_y}).$$

The corresponding bulk gap Chern number can be written as

$$C_{\text{gap}}^{\text{b}} = -\frac{i}{2\pi} \sum_{i=2}^4 \int_{S_b} \text{Im} \left[\sum_{j \neq i} \frac{\mathbf{A}_{ij} \times \mathbf{A}_{ji}}{(\omega_i - \omega_j)^2} \right] \cdot d\mathbf{S} \\ =: \sum_{i=2}^4 C_i, \quad (14)$$

where $\mathbf{A}_{ij} = \langle\Psi_i^L|\nabla h_b|\Psi_j^R\rangle$. Here, S_b may be chosen as a small surface enclosing a single bulk monopole, in which case $C_{\text{gap}}^{\text{b}}$ gives the monopole charge, or as a larger surface enclosing several bulk monopoles, in which case it gives the sum of their charges.

For the interface problem, the same Chern form is applied after the pullback $\mu \mapsto \mu(x)$ satisfying Eq. (6). The corresponding surface is then $\Sigma \subset (x, k_x, k_y)$, enclosing all finite phase-space degeneracies between bands 1 and 2 associated with the considered strip gap. The exterior derivative is taken with respect to the interface phase-space variables, as specified in Eq. (7):

$$dP_{\text{int}} = \partial_x P_{\text{int}} dx + \partial_{k_x} P_{\text{int}} dk_x + \partial_{k_y} P_{\text{int}} dk_y.$$

The interface invariant $C_{\text{gap}}^{\text{int}}$, rather than $C_{\text{gap}}^{\text{b}}$ itself, is the quantity that enters the spectral-flow formula. The role of the bulk calculation is to determine the Berry-Chern monopole charges that are swept by a given profile $\mu(x)$. We denote C_i the Chern number of band i corresponding to eigenvalue ω_i . Similarly, the gap Chern number for gap between band $j-1$ and band j is $\sum_{i=j}^4 C_i$. The expressions of C_i s correspond to the Berry-curvature integral for bands of a pseudo-Hermitian system. One can see that, when η continuously approaches the identity matrix I , $\sum_{i=j}^4 C_i$ smoothly reduces to the gap Chern number in the Hermitian case. Therefore, the topology associated with a pseudo-Hermitian symbol is homotopic to that of its Hermitian counterpart. Because the metric operator η is positive definite and varies smoothly in the phase space (x, k_x, k_y) , the associated pseudo-Hermitian vector bundle is homotopic to a Hermitian vector bundle. As a result, the Berry curvature and Chern numbers can be computed using the biorthogonal eigenvectors without introducing additional metric-connection corrections.

It is important to clarify the scope of this statement. In the lossless problem considered in Secs. II and III, the metric symbol η is positive definite and smooth over the relevant phase-space region. Therefore, the pseudo-Hermitian symbol

$$h = \eta^{-1}H$$

is similar to the Hermitian symbol

$$h_H = \eta^{-1/2}H\eta^{-1/2}.$$

This similarity transformation preserves the real strip gap and does not generate exceptional points. Hence, for the line-gap or strip-gap topology relevant to the present spectral-flow problem, the biorthogonal Chern number computed from the spectral projector is homotopic to the Chern number of the Hermitian representative.

This statement should not be interpreted as claiming that all possible pseudo-Hermitian or non-Hermitian topological structures are exhausted by the strip-gap Chern number. Additional invariants may arise in systems with indefinite or singular metrics, point gaps in the complex-frequency plane, spectral winding, nonreciprocal skin effects, or exceptional degeneracies. These cases are outside the scope of the lossless pseudo-Hermitian problem considered here. In the present model, the metric is positive definite, the lossless spectrum is real, and the topology relevant to the observed interface spectral flow is the strip-gap topology characterized by $C_{\text{gap}}^{\text{int}}$.

Numerical integration shows that, for $k_z = 0$, the degenerate point contributes a topological charge of +2. When $k_z \neq 0$, the two degenerate points each contribute a topological charge of +1. In other words, the total topological charge between band 1 and band 2 is conserved. The above conservation law of the gap Chern number, which is independent of k_z , holds for any gap between band $j - 1$ and band j .

Near $k = 0$, where $\hat{\eta} \simeq I$, the local dynamics reduces to that of the shallow-water equations describing equatorial geostrophic waves, and the degeneracy carries a topological charge $C = +2$, originating from two copropagating chiral modes [7].

III. BULK-INTERFACE CORRESPONDENCE FOR SCREENED MAGETIZED PLASMA

The dynamical relevance of $C_{\text{gap}}^{\text{int}}$ is its equality to the spectral flow of the one-dimensional interface operator. For fixed k_z , the interface problem is obtained by replacing $k_x \rightarrow -i\partial_x$, while k_y serves as the spectral-flow parameter. As k_y varies from $-\infty$ to $+\infty$, discrete interface branches may detach from the continuum associated with ω_2 , traverse the strip gap, and reconnect with the continuum associated with ω_1 . The net number of such oriented gap-crossing branches satisfies

$$\text{SF} = C_{\text{gap}}^{\text{int}}.$$

Thus, the Chern number is a wave-spectral invariant that predicts the number and chirality of interface plasma modes, rather than an occupation-based invariant.

We now consider a configuration in which the reduced magnetic field $\mu(x)$ varies along the spatial x direction, while the transverse wave numbers k_y and k_z are held fixed. We assume that the profile $\mu(x)$ is such that the symbol $\eta^{-1}H$ admits

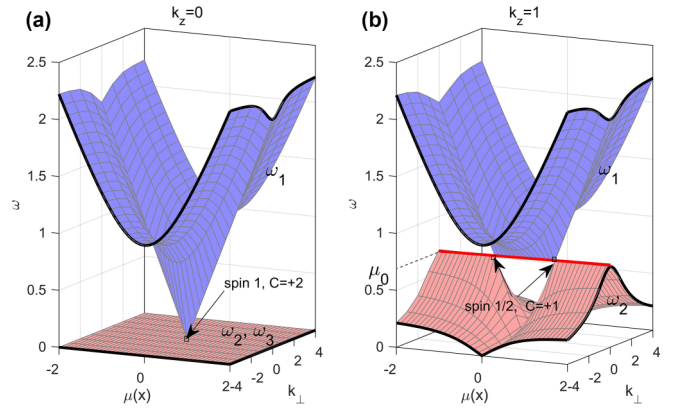


FIG. 1. Schematic illustration of the bulk band structure of the Weyl symbol $\eta^{-1}H$ in the parameter space (μ, k_x, k_y) for fixed k_z . Isolated band degeneracies appear as Weyl points (marked by arrows), each carrying a topological charge. (a) For $k_z = 0$, the spectrum exhibits a single higher-order Weyl point with topological charge +2. (b) For $k_z \neq 0$, this degeneracy splits into two Weyl points, each carrying a unit topological charge +1.

a common strip gap for sufficiently large $\|(x, k_x, k_y)\|$. Under this condition, the problem reduces to a one-dimensional eigenvalue problem parametrized by k_y , whose spectral flow we analyze.

Importantly, the behavior differs qualitatively between the cases $k_z = 0$ and $k_z \neq 0$, as well as for different asymptotic values $\mu(\pm\infty)$. This distinction stems from the structure of the bulk degeneracies: When $k_z \neq 0$, the single degenerate point carrying a topological charge of +2 splits into two isolated degeneracies, each carrying unit charge +1.

To demonstrate this behavior, we solve the associated eigenvalue problem on a finite interval $[-L, L]$ with Dirichlet boundary conditions. We adopt the smooth profile

$$\mu(x) = \frac{\mu_1 + \mu_2}{2} + \frac{\mu_2 - \mu_1}{2} \tanh\left(\frac{x}{\delta}\right), \quad (15)$$

which interpolates between the asymptotic values μ_1 and μ_2 .

For $k_z = 0$, the parameters μ_1 and μ_2 can be chosen as arbitrary nonzero values such that the common strip-gap condition discussed above is satisfied. In contrast, when $k_z \neq 0$, a common strip gap exists only if

$$\mu_1, \mu_2 \in (-\mu_0, +\mu_0) \quad \text{or} \quad \mu_1, \mu_2 \notin [-\mu_0, +\mu_0]. \quad (16)$$

This restriction can be understood from the bulk structure shown in Fig. 1.

When the bulk-interface correspondence, equivalently the spectral-flow-monopole correspondence, holds in the present model, the following behavior is expected.

For $k_z = 0$, the spectral flow within the strip gap equals 2 $\text{sign}(\mu_2 - \mu_1)$ when

$$\mu_1\mu_2 < 0, \quad (17)$$

and vanishes when

$$\mu_1\mu_2 > 0. \quad (18)$$

For $k_z \neq 0$, the spectral flow equals $2 \operatorname{sign}(\mu_2 - \mu_1)$ when one of $\mu_{1,2}$ satisfies

$$\mu_{1/2} < -\mu_0, \quad (19)$$

while the other satisfies

$$\mu_{2/1} > \mu_0. \quad (20)$$

When

$$-\mu_0 < \mu_{1/2} < \mu_0, \quad (21)$$

the spectral flow vanishes.

In contrast, when one of $\mu_{1,2}$ lies in the interval $(-\mu_0, \mu_0)$ while the other lies outside $[-\mu_0, \mu_0]$, the strip-gap condition is violated at $(k_x, k_y) = (0, 0)$ for arbitrary values of $\mu(x)$. Although the system crosses a spin- $\frac{1}{2}$ degeneracy carrying unit topological charge under the continuous deformation of $\mu(x)$ from μ_1 to μ_2 , no well-defined spectral flow can be assigned within the strip gap. In this situation, the strip gap is violated only at $(k_x, k_y) = (0, 0)$ for sufficiently large $\|(x, k_x, k_y)\|$. Consequently, for $k_y \neq 0$ the strip gap remains open, and nontrivial topology may still persist.

A. Numerical results and spectral flow

We now present the numerical results for the interface eigenvalue problem and demonstrate the spectral flow predicted by the gap Chern number.

For fixed transverse wave numbers k_y and k_z , we solve the generalized eigenvalue problem,

$$\hat{H}(k_y, k_z; \mu(x)) \Psi(x) = \omega \hat{\eta}(k_y, k_z) \Psi(x), \quad (22)$$

on a finite interval $x \in [-L, L]$, subject to Dirichlet boundary conditions. The Dirichlet boundary condition is imposed solely for numerical convenience. We have verified that the existence and chirality of the interface modes are insensitive to the specific choice of boundary conditions, provided that the system size L is sufficiently large. Here, $\hat{H}(k_y, k_z; \mu(x))$ denotes the one-dimensional differential operator obtained by replacing $k_x \rightarrow -i\partial_x$ in the bulk symbol H , and $\hat{\eta}$ is the corresponding metric operator. The spatial profile $\mu(x)$ is chosen as in Eq. (15), varying smoothly between the asymptotic values μ_1 and μ_2 . The system size L is taken to be much larger than the interface width δ , so that the boundary does not affect the localized interface modes. We have verified that the spectral features reported below are insensitive to further increases of L .

Figure 2 shows the eigenvalue spectrum as a function of k_y for a representative interface configuration. Within the strip gap, two discrete branches emerge from the bulk continuum, traverse the gap monotonically, and merge into the opposite bulk band as k_y varies from $-\infty$ to $+\infty$. For $k_z = 0$, the bulk parameter space contains a charge-+2 degeneracy at $(\mu, k_x, k_y) = (0, 0, 0)$. When the profile $\mu(x)$ changes sign, this monopole is pulled back to the interface phase space at the point $(x_0, 0, 0)$, where $\mu(x_0) = 0$. The closed surface $\Sigma \subset (x, k_x, k_y)$ enclosing this phase-space degeneracy therefore carries $C_{\text{gap}}^{\text{int}} = +2$, leading to two chiral branches in Fig. 2(a). For $k_z \neq 0$, the charge-+2 degeneracy splits into two charge-+1 Weyl points at $\mu = \pm\mu_0$ in the bulk parameter space. If the interpolation $\mu(x)$ crosses both values, the interface phase

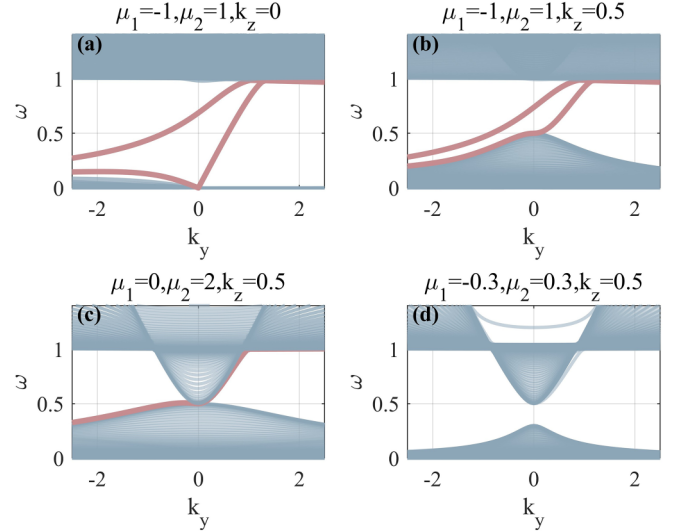


FIG. 2. Eigenvalue spectra of the interface problem as functions of k_y for different magnetic configurations $\mu(x)$. Discrete eigenvalues inside the strip gap are marked by plus symbols. (a), (b) Cases with a common strip gap. For $k_z = 0$, the variation of $\mu(x)$ crosses a single higher-order degeneracy carrying topological charge +2; for $k_z \neq 0$, it crosses two Weyl points each carrying unit topological charge +1. In both cases, two chiral branches traverse the strip gap, yielding a net spectral flow of +2, consistent with the gap Chern number. (c) Case without a strip gap at $k_y = 0$. Although the gap is locally closed, the variation crosses a single Weyl point with unit topological charge, resulting in a single chiral branch connecting the two continuous spectra. (d) Case with a strip gap but no band degeneracy crossed during the variation, for which no discrete interface modes appear within the gap.

space contains two corresponding degeneracies, and the total interface Chern number remains +2, consistent with the two gap-crossing branches in Fig. 2(b).

Figure 2(c) illustrates a relatively “anomalous” phenomenon. In this case, the system does not possess a strip gap at $k_y = 0$; nevertheless, the numerical simulation still reveals a chiral branch that connects the two continuous spectra. Notably, as $\mu(x)$ varies from μ_1 to μ_2 , the system crosses a single degenerate point carrying a topological charge of +1. Therefore, this chiral spectrum can also be regarded as consistent with the bulk-interface correspondence. From a topological perspective, this behavior does not contradict the bulk-interface correspondence. Although the global strip-gap condition is not satisfied in the strict sense, the numerical spectrum still exhibits a chiral branch associated with the crossing of a single Weyl point. We therefore regard this case as a useful illustration of the monopole mechanism beyond the strictly gapped setting, rather than as an example where the strip-gap Chern number is rigorously defined.

If the profile $\mu(x)$ does not sweep any ω_1 - ω_2 monopole, the corresponding closed surface in (x, k_x, k_y) encloses no degeneracy and the interface Chern number vanishes. This explains the absence of discrete gap modes in Fig. 2(d).

IV. SPECTRAL FLOW UNDER COLLISIONAL DAMPING

In realistic plasma systems, collisional effects and dissipation inevitably introduce damping into the electron dynamics.

Such effects render the governing equations non-Hermitian and break the pseudo-Hermitian symmetry underlying the idealized model discussed above. It is therefore natural to ask whether the spectral flow predicted by the gap Chern number remains robust in the presence of damping. In this section, we address this question by incorporating a simple collisional damping term and analyzing its impact on the interface spectrum.

A. Damped model and non-Hermitian perturbation

We introduce a phenomenological damping rate $\nu > 0$ into the electron momentum equation,

$$m_e \frac{\partial \mathbf{v}_1}{\partial t} = e \nabla \phi_1 + e \mathbf{v}_1 \times \mathbf{B}_0 - m_e \nu \mathbf{v}_1, \quad (23)$$

while keeping the continuity equation and the screened Poisson equation unchanged. After normalization using the same conventions as in Sec. II, the resulting generalized Schrödinger-type equation takes the form

$$i \hat{\eta} \partial_t \Xi = (\hat{H} + i \nu \hat{D}) \Xi, \quad (24)$$

where $\hat{H}$ and $\hat{\eta}$ are the Hermitian operators defined previously, and

$$\hat{D} = \begin{pmatrix} -I & 0 \\ 0 & 0 \end{pmatrix}$$

acts only on the velocity components. The additional term $i \nu \hat{D}$ explicitly breaks the pseudo-Hermitian symmetry and renders the spectrum generally complex.

B. Persistence and breakdown of spectral flow

In the absence of damping ($\nu = 0$), the interface spectral flow is protected by the gap Chern number defined in the phase space (x, k_x, k_y) . The lossless symbol is pseudo-Hermitian with respect to a positive-definite metric and therefore has a real spectrum.

When ν is finite, the Weyl symbol becomes non-Hermitian and the eigenfrequencies are generally complex. Since the lossless problem has a uniform real-frequency strip gap outside a sufficiently large compact region in phase space, this spectral separation is stable under sufficiently small non-Hermitian perturbations. Hence, for weak damping, the real-frequency strip gap is continuously continued to a real-part strip gap of the complex spectrum. The relevant condition is that the complex spectrum can still be separated into two spectral clusters whose real parts are separated by a common interval.

As long as this real-part spectral separation remains open, the spectral projector associated with the separated spectral cluster can be continuously deformed from the lossless one. Consequently, the spectral flow of the real parts of the eigenvalues remains well defined. In particular, the net number of eigenvalue branches whose real parts traverse the strip gap as k_y varies from $-\infty$ to $+\infty$ is unchanged from the lossless case and continues to coincide with the gap Chern number. The non-Hermitian continuation of the strip-gap condition is discussed in Appendix B.

When the damping becomes sufficiently strong, the real-part strip gap may close. In this situation, the spectral

separation required to define the strip-gap spectral flow is lost, and a well-defined spectral flow within the gap can no longer be assigned. Exceptional points provide one possible mechanism for such a non-Hermitian gap closing, since they involve the coalescence of both eigenvalues and eigenvectors. However, the closure of the real-part strip gap alone does not by itself constitute a complete identification of exceptional points. Such an identification would require additional eigenvector diagnostics, such as checking eigenvector defectiveness or the vanishing of the corresponding biorthogonal norm. Therefore, in the present work, we use the closing of the real-part strip gap as the criterion for the breakdown of the topological spectral-flow protection, without claiming a full classification of the exceptional-point structure.

C. Numerical illustration

To substantiate the above analysis, we perform numerical calculations for finite damping rates $\nu = 0.5, 1.5$, and 2.5 . Throughout this subsection, we examine how weak to moderate damping affects the interface spectrum and the associated spectral flow.

For $\nu = 0.5$, complex eigenfrequencies already appear, as shown in Fig. 3(a). Nevertheless, the real parts of the spectrum still exhibit a well-defined strip gap. Correspondingly, in the complex-frequency plane [Fig. 3(d)], the spectral cloud clearly reveals two discrete branches traversing between the two continuous spectra as k_y varies. These branches represent a spectral flow that remains identical to the lossless case, despite the presence of finite imaginary parts.

When the damping is increased to $\nu = 1.5$, the strip gap in the real part of the spectrum becomes noticeably narrower, as shown in Fig. 3(b). The corresponding spectral cloud in Fig. 3(e) indicates that the separation between the two continuous spectra is reduced. Nevertheless, two gap-crossing branches can still be identified, indicating that the spectral flow persists as long as a finite real-part strip gap remains open.

For a larger damping rate $\nu = 2.5$, the real-part strip gap closes completely. As shown in Fig. 3(c), the two continuous spectra merge in the real-frequency projection. In this regime, the spectral cloud [Fig. 3(f)] consists of two families of discrete eigenvalues parametrized by k_y , which can be viewed as the continuation of the gap-crossing branches observed at smaller damping. However, since no real-part strip gap exists, a well-defined spectral flow within a gap can no longer be assigned.

These numerical results illustrate that the spectral flow predicted by the gap Chern number remains robust under finite damping as long as the real parts of the complex spectrum retain a strip gap. Once this real-part spectral separation is destroyed, the strip-gap condition breaks down and the topological protection of the spectral flow is lost. We emphasize that this criterion is based on the persistence or closure of the real-part strip gap. Although exceptional points may provide a possible mechanism for such a non-Hermitian gap closing, their explicit identification requires additional eigenvector analysis and is not assumed in the present numerical criterion.

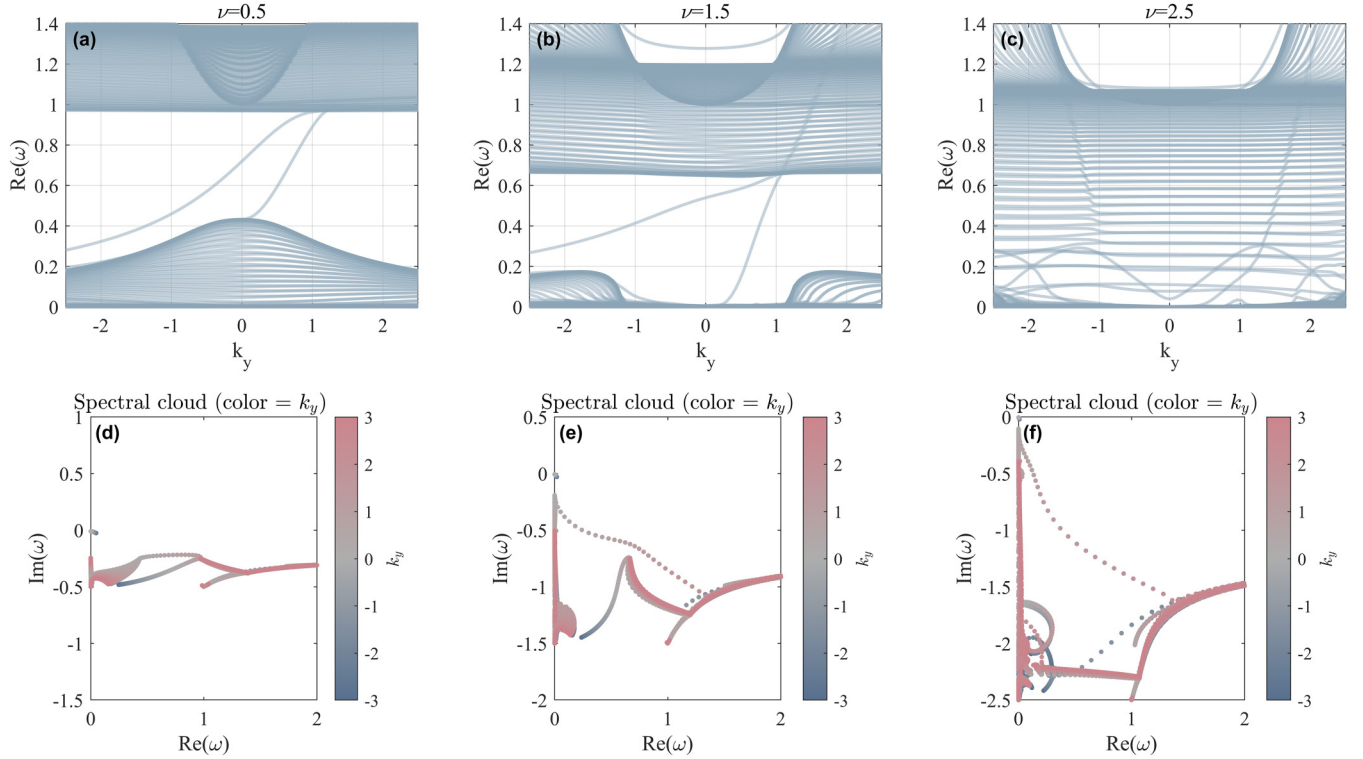


FIG. 3. Interface eigenvalue spectra in the presence of finite damping for $\nu = 0.5, 1.5$, and 2.5 . Panels (a)–(c) show the real parts of the eigenfrequencies as functions of k_y , while panels (d)–(f) display the corresponding spectral clouds in the complex-frequency plane. (a), (d) For $\nu = 0.5$, complex eigenfrequencies appear, but the real parts of the spectrum retain a finite real-part strip gap, within which two discrete branches traverse between the two continuous spectra. (b), (e) For $\nu = 1.5$, the real-part strip gap is reduced, and the separation between the two continuous spectra in the spectral cloud becomes narrower, while the gap-crossing branches remain identifiable. (c), (f) For $\nu = 2.5$, the real-part strip gap closes completely and the two continuous spectra merge in the real-frequency projection; the spectral cloud consists of two families of discrete eigenvalues parametrized by k_y , corresponding to the continuation of the gap-crossing branches observed at smaller damping.

V. CONCLUSION

We have established a unified phase-space topological framework for wave dynamics in screened magnetized plasmas based on a pseudo-Hermitian formulation with a positive-definite metric. By recasting the linearized plasma equations into a generalized Schrödinger form, we identified a mathematically controlled setting in which bulk topology can be defined despite the absence of a compact Brillouin zone and conventional band gaps.

At the level of the Weyl-Wigner symbol, we showed that the bulk spectrum hosts isolated band degeneracies in phase space that act as Berry-Chern monopoles carrying quantized topological charges. In particular, a higher-order spin-1 degeneracy with charge $+2$ generically splits into two spin- $\frac{1}{2}$ Weyl points under symmetry breaking, while preserving the total strip-gap topological charge. These monopoles provide the microscopic origin of the spectral flow observed in the interface problem.

To characterize topology away from such degeneracies in a continuous medium with unbounded phase space, we introduced the strip-gap Chern number. This invariant extends the notion of band Chern number to continuous media by assigning topology to finite real-frequency strips rather than isolated

bands. It governs the spectral flow of eigenvalues across the strip gap and provides a robust topological characterization of continuum wave systems.

By solving the interface eigenvalue problem generated by a smooth spatial variation of the magnetic field, we demonstrated that in the strip-gapped regime, the net number of interface eigenvalue branches traversing the strip gap is determined by the total enclosed topological charge. Our numerical results confirm this bulk-interface correspondence across a broad parameter range, including situations in which the strip gap is locally violated at isolated points in phase space. This gives the strip-gap Chern number a direct physical meaning: It predicts not only the existence of gap-crossing eigenvalue branches, but also the net chirality of dynamically propagating interface plasma waves. In this sense, the bulk Berry-Chern monopoles of the screened plasma symbol are converted into observable spectral flow along an inhomogeneous plasma interface.

We further examined the robustness of this correspondence in the presence of non-Hermitian effects arising from collisional damping. Although dissipation renders the spectrum complex and breaks pseudo-Hermitian symmetry, the spectral flow of the real parts of the eigenvalues remains quantized as long as the complex spectrum retains a finite

real-part strip gap. Once the strip gap closes, eigenvalues can no longer be continuously tracked, and the notion of spectral flow within the gap ceases to be well defined. This sharply delineates the regime in which a strip-gap-based topological description remains meaningful in dissipative continua.

More broadly, while topological wave phenomena in continuous media have been explored in various contexts, our results provide a concrete phase-space framework in which pseudo-Hermiticity, strip-gap topology, and bulk-interface correspondence are treated on equal footing. The formulation developed here can in principle be extended to fluid, plasma, and electromagnetic systems with constraints and long-range interactions, and may serve as a foundation for understanding robust chiral wave transport beyond idealized lattice models.

$$(\pm)_1 \frac{\sqrt{k^2 + (1 + k^2)\mu^2 + (\pm)_2 \sqrt{(k^2 + (1 + k^2)\mu^2)^2 - 4(1 + k^2)k_z^2 \mu^2}}}{\sqrt{2}\sqrt{1 + k^2}}.$$

Let $K = \sqrt{k_x^2 + k_y^2}$, with $k^2 = K^2 + k_z^2$. For bounded $\mu(x)$, as $K \rightarrow \infty$,

$$\{\omega_1, \omega_2, \omega_3, \omega_4\} \rightarrow \{\sqrt{1 + \mu(x)^2}, 0^+, 0^-, -\sqrt{1 + \mu(x)^2}\}.$$

For bounded $\mu(x)$, this convergence is uniform over the range of the profile. Because $\sqrt{1 + \mu(x)^2} \geq 1$, for any fixed $0 < \epsilon < 1$, there exists $K_\epsilon > 0$ such that the interval $(\epsilon, 1)$ lies between ω_2 and ω_1 for all $K > K_\epsilon$, uniformly for bounded $\mu(x)$.

For sufficient large $|x|$, we first assume that $\mu(x)$ is smooth, bounded, and approaches constant asymptotic values,

$$\mu(x) \rightarrow \mu_1 \quad (x \rightarrow -\infty), \quad \mu(x) \rightarrow \mu_2 \quad (x \rightarrow +\infty).$$

The following result holds. For any fixed μ and k_z , ω_1 is a monotonically increasing function of $\sqrt{k_x^2 + k_y^2}$, whereas ω_2 is a monotonically decreasing function of $\sqrt{k_x^2 + k_y^2}$. The proof is as follows:

We introduce

$$q = k^2 = K^2 + k_z^2, \quad a = \mu^2, \quad b = k_z^2.$$

Consider the two positive-frequency branches

$$f(K) := \omega_1 \quad \text{and} \quad g(K) := \omega_2.$$

For compactness, define $A(q) = a + (1 + a)q$ and $D(q) = A(q)^2 - 4(1 + q)ab$. Then, the squares of the two branches can be written as $f^2(K) = \lambda_+(q)$, $g^2(K) = \lambda_-(q)$, where

$$\lambda_{\pm}(q) = \frac{A(q) \pm \sqrt{D(q)}}{2(1 + q)}.$$

Equivalently, $\lambda_{\pm}$ are the two roots of the quadratic equation

$$(1 + q)\lambda^2 - A(q)\lambda + ab = 0. \quad (\text{A1})$$

ACKNOWLEDGMENT

This work was supported by the National Natural Science Foundation of China (Grant No. 11775220).

DATA AVAILABILITY

The code and data supporting the findings of this work are available in Ref. [31].

APPENDIX A: STRIP-GAP CONDITION OF ω_1 AND ω_2 AT PHASE-SPACE INFINITY

The gap condition must therefore be checked at phase-space infinity.

The large-momentum behavior can be checked explicitly. The symbol $\text{sym}(\hat{\eta}^{-1}\hat{H})$ has four real eigenvalues,

We now determine the monotonicity of $\lambda_{\pm}$ with respect to q . Differentiating Eq. (A1) with respect to q , we obtain

$$\lambda^2 + 2(1 + q)\lambda \frac{d\lambda}{dq} - (1 + a)\lambda - [a + (1 + a)q] \frac{d\lambda}{dq} = 0.$$

Therefore,

$$\frac{d\lambda}{dq} = \frac{\lambda(1 + a - \lambda)}{2(1 + q)\lambda - A(q)}.$$

Since

$$\lambda_{\pm}(q) = \frac{A(q) \pm \sqrt{D(q)}}{2(1 + q)},$$

we have

$$2(1 + q)\lambda_{\pm} - A(q) = \pm\sqrt{D(q)}.$$

Thus,

$$\frac{d\lambda_{\pm}}{dq} = \frac{\lambda_{\pm}(1 + a - \lambda_{\pm})}{\sqrt{D(q)}},$$

whereas

$$\frac{d\lambda_{\pm}}{dq} = -\frac{\lambda_{\pm}(1 + a - \lambda_{\pm})}{\sqrt{D(q)}} = \frac{\lambda_{\pm}(\lambda_{\pm} - 1 - a)}{\sqrt{D(q)}}.$$

Moreover, by Vieta's formula, we have

$$\lambda_{+} + \lambda_{-} = \frac{a + (1 + a)q}{1 + q} = 1 + a - \frac{1}{1 + q},$$

and for the present branches one has $\lambda_{+} \geq \lambda_{-} \geq 0$. Hence,

$$0 \leq \lambda_{-} \leq \lambda_{+} < 1 + a = 1 + \mu^2.$$

Therefore, in the nondegenerate region where $D(q) > 0$, one has

$$\frac{d\lambda_{+}}{dq} > 0 \quad \text{and} \quad \frac{d\lambda_{-}}{dq} < 0.$$

On the other hand, $q = K^2 + k_z^2$, so that $dq/dK = 2K > 0$.

Hence, for $f(K) = \sqrt{\lambda_+(q)}$ and $g(K) = \sqrt{\lambda_-(q)}$, we have

$$\frac{df(K)}{dK} = \frac{1}{2\sqrt{\lambda_+}} \frac{d\lambda_+}{dq} \frac{dq}{dK} > 0$$

and

$$\frac{dg(K)}{dK} = \frac{1}{2\sqrt{\lambda_-}} \frac{d\lambda_-}{dq} \frac{dq}{dK} < 0.$$

Therefore, for $K > 0$ and $D(q) > 0$, $f(K)$ increases monotonically with K , whereas $g(K)$ decreases monotonically with K . Moreover, for $K > 0$, $\mu \neq 0$, $k_z \neq 0$, and away from degeneracies, these monotonicities are strict.

Therefore, for finite K , in the asymptotic limits $x \rightarrow \pm\infty$, $\omega_1(K, \mu)$ and $\omega_2(K, \mu)$ are separated by a spectral gap given by

$$\begin{aligned} &(\omega_2(0, \mu_{1,2}), \omega_1(0, \mu_{1,2})) \\ &= (\min(\mu_0, |\mu_{1,2}|), \max(\mu_0, |\mu_{1,2}|)), \end{aligned} \quad (\text{A2})$$

with

$$\mu_0 = \sqrt{\frac{k_z^2}{1 + k_z^2}}. \quad (\text{A3})$$

Consequently, for fixed $k_z \neq 0$, one has

$$0 < \mu_0 = \sqrt{\frac{k_z^2}{1 + k_z^2}} < 1.$$

We may therefore choose $\epsilon > 0$ such that $\epsilon < \mu_0$. Provided that

$$(|\mu_2| - \mu_0)(|\mu_1| - \mu_0) > 0, \quad (\text{A4})$$

the two asymptotic values $|\mu_2|$ and $|\mu_1|$ lie on the same side of the critical value μ_0 . Hence, the intersection

$$\mathcal{G} = \bigcap_{i=1,2} (\min(\mu_0, |\mu_i|), \max(\mu_0, |\mu_i|)) \cap (\epsilon, 1) \quad (\text{A5})$$

is nonempty. In this case, we obtain a strip gap

$$I_\omega := \mathcal{G}. \quad (\text{A6})$$

Together with the large- K estimate above, this implies that there exists $R > 0$ such that, whenever

$$\sqrt{x^2 + k_x^2 + k_y^2} > R,$$

the spectrum of the Weyl symbol avoids I_ω . Therefore, I_ω is a uniform strip gap at phase-space infinity in the variables (x, k_x, k_y) , independently of the direction in which infinity is approached.

In particular, for the case $k_z = 0$, one has $\mu_0 = 0$. If $|\mu_{1,2}| > 0$, the strip gap exists and can be chosen as

$$I_\omega = (\epsilon, \min(|\mu_1|, |\mu_2|, 1)), \quad (\text{A7})$$

with $0 < \epsilon < \min(|\mu_1|, |\mu_2|, 1)$.

APPENDIX B: REAL-PART STRIP GAP UNDER WEAK DAMPING

In the lossless problem, the relevant spectral separation is a real-frequency strip gap. When weak collisional damping is introduced, the Weyl symbol becomes non-Hermitian and the eigenfrequencies are generally complex. In this case, we use the real-part continuation of the strip gap. More precisely, we say that the damped symbol $h_\nu : \eta^{-1}(H + i\nu D)$ satisfies a real-part strip-gap condition if its spectrum can be separated into two disjoint spectral clusters,

$$\sigma(h_\nu) = \sigma_-(h_\nu) \cup \sigma_+(h_\nu), \quad \sigma_-(h_\nu) \cap \sigma_+(h_\nu) = \emptyset,$$

such that, outside a sufficiently large compact region in phase space, there exists a real interval (ω_-, ω_+) satisfying

$$\text{Re } \sigma_-(h_\nu) < \omega_- < \omega_+ < \text{Re } \sigma_+(h_\nu).$$

Here, ν denotes the damping strength, and the interval (ω_-, ω_+) is continuously connected to the real-frequency strip gap of the lossless system.

As long as this spectral separation is preserved, the spectral projector onto one of the two spectral clusters is well defined by the Riesz projection

$$P_\nu = \frac{1}{2\pi i} \oint_\Gamma (z - h_\nu)^{-1} dz,$$

where Γ is a closed contour in the complex-frequency plane enclosing $\sigma_-(h_\nu)$ and excluding $\sigma_+(h_\nu)$. Since the contour does not intersect the spectrum, the resolvent $(z - h_\nu)^{-1}$ is well defined on Γ . Moreover, if the real-part strip gap remains open as ν is increased from zero, the contour Γ and the projector P_ν can be chosen continuously with ν . Therefore, P_ν is homotopic to the lossless projector P_0 , and the corresponding Chern number is unchanged under weak damping.

This provides the sense in which the interface spectral flow persists in the weakly damped problem. The relevant stability condition is the preservation of the real-part strip gap, or equivalently the absence of a non-Hermitian spectral degeneracy that destroys the separation between the two spectral clusters. When the damping strength becomes sufficiently large and the real-part strip gap closes, the spectral projector can no longer be continuously connected to the lossless one, and the topological protection of the spectral flow breaks down.

Exceptional points provide one possible mechanism for such a breakdown, because both eigenvalues and eigenvectors coalesce at an exceptional point. However, the closure of the real-part strip gap alone does not by itself constitute a full identification of an exceptional point. A direct identification of exceptional points would require additional eigenvector diagnostics, such as checking the defectiveness of the eigenvectors or the vanishing of the corresponding biorthogonal norm. In the present work, we therefore use the closing of the real-part strip gap as the criterion for the breakdown of the topological spectral-flow protection, without claiming a complete classification of the exceptional-point structure.

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