

Unifying framework for amplification mechanisms: Spectral criticality, resonance, and nonnormality

Virgile Troude  and Didier Sornette 

Institute of Risk Analysis, Prediction and Management (Risks-X), Academy for Advanced Interdisciplinary Sciences, Southern University of Science and Technology, Shenzhen, China



(Received 19 May 2025; accepted 20 August 2025; published 1 December 2025)

We bring together three key amplification mechanisms in linear dynamical systems: spectral criticality, resonance, and nonnormality. We present a unified linear framework that both distinguishes and quantitatively links these effects through two fundamental parameters: (1) the spectral distance to a conventional bifurcation or to a resonance and (2) a nonnormal index K (or condition number κ) that measures the obliqueness of the eigenvectors. Closed-form expressions for the system's response in the form of the variance v_∞ of the observable responding to both Gaussian noise and periodic forcing reveal a general amplification law $v_\infty = v_0(1 + \mathcal{G}(K))$ with nonnormal gain $\mathcal{G}(K) \propto K^2$ represented in universal phase diagrams. By reanalyzing a model of remote earthquake triggering based on breaking of Hamiltonian symmetry, we illustrate how our two-parameter framework significantly expands both the range of conditions under which amplification can occur and the magnitude of the resulting response, revealing a broad pseudocritical regime associated with large κ that previous single-parameter approaches overlooked. Similarly, in the non-Hermitian extensions of quantum optics provided by forward four-wave mixing experiments, we show the presence of a counterintuitive gain-from-loss effect that directly manifests nonnormal amplification in a propagating-wave setting. This predicts the possibility to engineer transient optical energy amplification without the need for true lasing or exact $\mathcal{PT}$ -symmetry breaking. Our framework applies to many other physical, natural, and social systems and offers diagnostic tools to distinguish true critical behavior from transient amplification driven by nonnormality.

DOI: [10.1103/kdgv-shxf](https://doi.org/10.1103/kdgv-shxf)

Many natural and engineered systems exhibit sudden, disproportionate responses to small disturbances or subtle shifts in control parameters—well before classical indicators like spectral instability or resonance emerge. Earthquake ruptures, hydrodynamic bursts, and financial crashes are striking examples where minor perturbations can trigger massive departures from equilibrium. Such abrupt, switchlike transitions are pervasive across complex systems: epileptic seizures in the brain [1,2], ecosystem collapses [3,4], market crashes [5], and rapid epigenetic reprogramming [6] all involve transient amplifications that vastly exceed background dynamics. These phenomena highlight the need for a broader framework to understand how small inputs can drive large-scale responses.

Classical bifurcation theory seeks to explain such events by the slow drift of a control parameter until an eigenvalue of the linearized Jacobian $\mathbf{J}$ crosses the imaginary axis, generating well-known early warning signals such as critical slowing down and rising variance [7]. Two enduring puzzles, however, limit this picture: (1) critical points are measure-zero targets

in all dimensional parameter spaces and (2) criticallylike signatures are frequently observed far from any spectral instability [8].

When $\mathbf{J}$ is nonnormal ($[\mathbf{J}, \mathbf{J}^\dagger] \neq 0$), the ε pseudospectrum $\sigma_\varepsilon(\mathbf{J}) = \{\lambda \in \mathbb{C} \mid \|(\mathbf{J} - \lambda I)^{-1}\| \geq \varepsilon^{-1}\}$ can inflate dramatically [9]. The eigenbasis condition number κ (defined as the largest singular value divided by the smallest singular value of the eigenbasis transformation matrix) quantifies this inflation; for $\kappa \gg 1$, a finite perturbation is transiently amplified proportionally to κ^2 [8]. These pseudocritical transients mimic all the hallmarks of true criticality, even though the real parts of the eigenvalues remain negative. In essence, nonnormality unfolds a continuum of criticallylike states surrounding the bifurcation manifold. Pseudocritical bursts have been reported in balanced neural networks [10], hierarchical food webs [11], turbulent shear flows [12], and minute-scale DNA-methylation surges [13]. The ubiquity of nonnormal amplification calls for a comprehensive theoretical approach.

Here, we develop such a unified analytical framework that integrates three distinct amplification mechanisms—(1) critical bifurcations, (2) resonant amplification in underdamped systems, and (3) pseudocritical transients driven by nonnormality—into a single coherent formalism with compatible concepts and notations. By combining pseudospectral analysis with condition-number geometry, we (1) derive closed-form criteria that clearly distinguish these regimes, (2) quantify their joint effects on system responses, and (3)

offer practical diagnostic tools—such as pseudospectral growth bounds and *in situ* estimates of κ —to detect pseudocritical amplification in both laboratory and field settings.

This framework charts the full landscape of dynamical amplification, revealing how nonnormality opens a continuum of criticallylike states even when eigenvalues remain stable and providing a comprehensive guide for interpreting large fluctuations across disciplines.

Generalized dynamical system. The following general setting captures these three amplification routes. Consider an n -dimensional state vector $\mathbf{x}(t) \in \mathbb{R}^n$ governed by the generalized Langevin equation

$$\ddot{\mathbf{x}} + \gamma \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(t), \quad \gamma > 0, \quad (1)$$

where γ is an isotropic damping coefficient and $\mathbf{g}(t)$ is an external forcing to be specified, e.g., Gaussian white noise or sinusoidal forcing. In full generality, the force field $\mathbf{f}(\mathbf{x})$ can be nonvariational. Employing the Helmholtz decomposition [14], it can be expressed as

$$\mathbf{f}(\mathbf{x}) = -\nabla U(\mathbf{x}) + (\nabla^\dagger \mathbf{A}(\mathbf{x}))^\dagger, \quad \mathbf{A}^\dagger(\mathbf{x}) = -\mathbf{A}(\mathbf{x}), \quad (2)$$

where $U(\mathbf{x})$ captures the conservative contributions while the solenoidal term arising from the anti-Hermitian tensor $\mathbf{A}(\mathbf{x})$ generates circulation in phase space and breaks detailed balance.

Expanding Eq. (1) around a stable equilibrium $\mathbf{x}^* = 0$ yields the linear system

$$\ddot{\mathbf{x}} + \gamma \dot{\mathbf{x}} = \mathbf{J}\mathbf{x} + \mathbf{g}(t), \quad \mathbf{J} := [\nabla \mathbf{f}(\mathbf{x})]_{\mathbf{x}=0}, \quad (3)$$

where stability implies that all eigenvalues of $\mathbf{J}$ have negative real part. The component $(\nabla^\dagger \mathbf{A}(\mathbf{x}))^\dagger$ of the force in Eq. (2) is responsible for nonnormality of $\mathbf{J}$, which appears naturally as a large subclass of the nonvariational family. nonnormality implies that $\mathbf{J}$ cannot be diagonalized by a unitary transformation.

The intuition behind this model is to view it as a system of coupled, damped oscillators, where each oscillator interacts in a nonsymmetric and hierarchical manner. We consider stable and nonnormal systems, meaning $\mathbf{J}$ is nonnormal and has eigenvalues with negative real part. The degree of nonnormality is quantified by the condition number κ of the eigenbasis transformation matrix $\mathbf{P}$ defined through the eigendecomposition of $\mathbf{J}$ as $\mathbf{J} = \mathbf{P}\mathbf{\Lambda}\mathbf{P}^{-1}$, where $\mathbf{\Lambda} = \text{Diag}(\lambda \mid \lambda \in \sigma(\mathbf{J}))$, with $\sigma(\mathbf{J})$ the spectrum of $\mathbf{J}$. The condition number κ is defined as the ratio between the largest and smallest singular values of $\mathbf{P}$. A condition number $\kappa = 1$ means that $\mathbf{J}$ is normal, while $\kappa > 1$ characterizes a nonnormal system. Strongly nonnormal systems ($\kappa \gg 1$) can exhibit dynamical behaviors that closely resemble those near true bifurcations, despite being far from any actual bifurcation point, a phenomenon that has been referred to as a “pseudobifurcation” [8].

A simplified scenario that nevertheless captures the essence of nonnormality is provided by a matrix $\mathbf{P}$ whose singular values are all equal to 1, except for the smallest one, which is $1/\kappa$. The smallness of this singular value directly determines the magnitude of the condition number κ , which quantifies the degree of nonnormality in the system. In this simplified case, the matrix $\mathbf{P}$ is approximately unitary in all directions except along one dimension characterized by a specific nonnormal mode $\hat{\mathbf{n}}$. It has been shown [8] that, up to a unitary

transformation, one can write $\mathbf{P} = \mathbf{I} + (\kappa^{-1} - 1)\hat{\mathbf{n}}\hat{\mathbf{n}}^\dagger$, and correspondingly, $\mathbf{P}^{-1} = \mathbf{I} + (\kappa - 1)\hat{\mathbf{n}}\hat{\mathbf{n}}^\dagger$, with $\mathbf{I}$ being the identity matrix. This specific structure reveals that excitations along the nonnormal mode $\hat{\mathbf{n}}$ induced by external forcing will experience amplification by a factor κ , a hallmark of pseudocritical behavior.

For later reference, we assume the system is damped, so that the homogeneous solution decays to zero and only the particular solution remains (see the Supplemental Material [15]). The resulting solution is given by

$$\mathbf{x}(t) = \int_0^t G(\mathbf{J}, t-s) \mathbf{g}(s) ds. \quad (4)$$

When system (1) is driven by an additive Gaussian white noise

$$\mathbf{g}(t) = \sqrt{2\delta} \boldsymbol{\eta}(t), \quad \langle \eta_i(t) \eta_j(s) \rangle = \delta_{ij} \delta(t-s), \quad (5)$$

with noise variance δ , since the deterministic contribution from the initial condition decays exponentially, the stationary statistics are governed by the particular solution $\mathbf{x}$ (4). The long-term mean-squared deviation (MSD) is then given by (see the Supplemental Material [15])

$$v_\infty := \langle \|\mathbf{x}(t)\|^2 \rangle = 2\delta \int_0^\infty \|G(\mathbf{J}, \tau)\|_F^2 d\tau, \quad (6)$$

where $\|G(\mathbf{J}, t)\|_F$ is the Frobenius norm of $G(\mathbf{J}, t)$.

Using the spectral decomposition of $\mathbf{J}$ and $\mathbf{P} = \mathbf{I} + (\kappa^{-1} - 1)\hat{\mathbf{n}}\hat{\mathbf{n}}^\dagger$, $\hat{\mathbf{n}} \sim (1, 1)/\sqrt{2}$, in the two-dimensional case with a unique nonnormal mode, $\|G(\mathbf{J}, \tau)\|_F^2$ can be written as see the Supplemental Material [15])

$$\begin{aligned} \|G(\mathbf{J}, t)\|_F^2 &= |G(\lambda_+, t)|^2 + |G(\lambda_-, t)|^2 \\ &\quad + K^2 |G(\lambda_+, t) - G(\lambda_-, t)|^2, \end{aligned} \quad (7)$$

where the magnitude of nonnormality is defined as

$$K := \frac{1}{2} |\kappa - \kappa^{-1}|. \quad (8)$$

The third term in Eq. (7), which vanishes for $K = 0$ ($\kappa = 1$), encodes transient energy bursts that enhance the MSD by a factor proportional to K^2 .

Let us define $\theta(\lambda_i) = \sqrt{(\frac{\gamma}{2})^2 + \lambda_i}$, with $\lambda_i \in \sigma(\mathbf{J})$. Starting from the spectral decomposition $\mathbf{J} = \mathbf{P}\mathbf{\Lambda}\mathbf{P}^{-1}$ of $\mathbf{J}$, then $G(\mathbf{J}, t) = \mathbf{P}G(\mathbf{\Lambda}, t)\mathbf{P}^{-1}$ is the spectral decomposition of $G(\mathbf{J}, t)$, where $G(\mathbf{\Lambda}, t)$ is the diagonal matrix with elements

$$G(\lambda_i, t) = e^{-\gamma t/2} \frac{\sinh[\theta(\lambda_i)t]}{\theta(\lambda_i)}, \quad \lambda_i \in \sigma(\mathbf{J}). \quad (9)$$

When $\mathbf{J}$ is nonnormal ($[\mathbf{J}, \mathbf{J}^\dagger] \neq 0$), the kernels exhibit direction-dependent transient growth with amplitude quantified by a magnitude of nonnormality K [Eq. (8)], dependent on the condition number κ of the eigenbasis transformation. For a normal matrix ($\kappa = 1$), the magnitude of nonnormality is $K = 0$. The Frobenius norm $\|G(\mathbf{J}, t)\|_F$ of $G(\mathbf{J}, t)$ (where $\|\mathbf{J}\|_F = \sqrt{\text{Tr}(\mathbf{J}^\dagger \mathbf{J})}$ is the Frobenius norm of matrix $\mathbf{J}$) may therefore increase as the result of a competition between exponentials that can create a finite-time maximum before ultimately decaying, a hallmark of nonnormal amplification.

Reduced nonnormal forms. Our aim in this section is to simplify the parametrization of a two-dimensional system

with a real Jacobian matrix so that its stability, degeneracies, and nonnormality can be analyzed in the most transparent way.

A real 2×2 Jacobian has eigenvalues that either are both real or form a complex-conjugate pair. We write them in the unified form

$$\lambda_{\pm} = \lambda \pm \Delta \chi, \quad (10)$$

where $\lambda, \Delta \in \mathbb{R}$ and $\chi \in \{1, i\}$ distinguishes the purely real ($\chi = 1$) from the complex-conjugate ($\chi = i$) case. In both cases, the spectrum depends on three control parameters $(\lambda, \Delta, \kappa)$, with κ measuring the degree of nonnormality.

However, treating all three parameters simultaneously can obscure the essential control mechanisms. To disentangle them, we introduce two alternative time rescalings, each leaving only one independent parameter in the spectrum:

$$\alpha = -\frac{\lambda}{\Delta}, \quad t \rightarrow t \Delta \quad (\alpha \text{ form}), \quad (11a)$$

$$\beta = -\frac{\Delta}{\lambda}, \quad t \rightarrow t |\lambda| \quad (\beta \text{ form}). \quad (11b)$$

We assume stability, $\lambda < 0$, and, without loss of generality, $\Delta > 0$. The corresponding eigenvalues are then

$$\lambda_{\pm} = \begin{cases} -\alpha \pm \chi & (\alpha \text{ form}), \\ -1 \pm \beta \chi & (\beta \text{ form}). \end{cases} \quad (12)$$

Up to a unitary transformation and the above time rescaling in Eq. (11), the system's Jacobian $\mathbf{J}$ in Eq. (3) reduces to (see the Supplemental Material)

$$\Gamma_{\chi}^{\alpha} = \begin{pmatrix} -\alpha & \chi \kappa \\ \chi \kappa^{-1} & -\alpha \end{pmatrix}, \quad \Gamma_{\chi}^{\beta} = \begin{pmatrix} -1 & \chi \beta \kappa \\ \chi \beta \kappa^{-1} & -1 \end{pmatrix}. \quad (13)$$

The α form normalizes time to the spectral gap Δ between the two eigenvalues. The dynamics is measured in units of the inverse splitting between eigenvalues. This makes α a distance-to-criticality parameter, which is best suited for tracking how nonnormality interacts with critical slowing down near a bifurcation. The β form normalizes time to the mean decay rate $|\lambda|$ of the modes. The dynamics is measured in units of the inverse average damping time. This makes β a degeneracy parameter, best suited for describing smooth spectral transitions and mode collisions, rather than approaching an instability threshold.

(1) Overdamped dynamics:

$$\dot{\mathbf{x}} = \Gamma_{\chi}^{\alpha/\beta} \mathbf{x}. \quad (14)$$

In this case, criticality occurs when $\Re(\lambda_{\pm}) \rightarrow 0^-$. In the α form, this corresponds to $\alpha \rightarrow \alpha_c^+$, with $\alpha_c = 1$ for $\chi = 1$ and $\alpha_c = 0$ for $\chi = i$. Thus, α directly measures proximity to criticality, while κ modulates the impact of nonnormality. Changing χ switches between oscillatory and purely exponential damping without altering stability.

(2) Underdamped dynamics:

$$\ddot{\mathbf{x}} = \Gamma_{\chi}^{\alpha/\beta} \mathbf{x}. \quad (15)$$

The four eigenvalues are $\pm\sqrt{\lambda_{\pm}}$. In the α form, this gives $\pm i\sqrt{\alpha \pm \chi}$. For $\chi = 1$, a bifurcation occurs at $\alpha \rightarrow 1^+$, similar to the overdamped case. For $\chi = i$, the system is always unstable: switching from $\chi = 1$ to $\chi = i$ then induces a discontinuous bifurcation. Thus, the α form is less suited to describing smooth transitions from real to complex-conjugate eigenvalues. For such smooth transitions, the β form is more appropriate: eigenvalues collide when $\beta = 0$, corresponding to a degeneracy point of the system.

The two reduced forms serve complementary purposes:

- (1) The α form parametrizes the distance from criticality.
- (2) The β form captures degeneracy and smooth transitions between real and complex-conjugate eigenvalues.

Together, they form a minimal yet complete framework for analyzing how nonnormality combines with criticality or resonance to produce amplification phenomena.

Overdamped system driven by a Gaussian white noise. We begin with this simple case for pedagogical clarity, as it provides intuition for the more general scenarios discussed later. For this, we quantify how nonnormality and criticality influence the response to external forcing when system (1), reduced to system (14) for its deterministic part, is driven by additive Gaussian white noise (5). The long-term MSD then takes the form given in Eq. (6).

The overdamped limit of Eq. (1), i.e., $\gamma \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(t)$, is recovered for $\gamma \gg \lambda_{\pm}$. In this regime, the propagator simplifies to $G(\mathbf{J}, t) \rightarrow e^{\mathbf{J}t/\gamma}$, maintaining a direct link to analytically tractable overdamped models studied in Refs. [8,13].

Transient growth in the response—defined by a nonmonotonically decaying kernel $\|G(\mathbf{J}, t)\|_F^2$ —occurs for $K > K_c \geq 0$. For the α and β forms, thresholds are given by

$$(\alpha \text{ form}) \quad K_c = \begin{cases} \sqrt{\frac{\alpha^2 - 1}{\alpha - \sqrt{\alpha^2 - 1}}}, & \alpha > 1, \quad \chi = 1, \\ \sqrt{\frac{\alpha}{\alpha^2 + 1 - \alpha}}, & \alpha > 0, \quad \chi = i, \end{cases} \quad (16a)$$

$$(\beta \text{ form}) \quad K_c = \begin{cases} \sqrt{\frac{\sqrt{1 - \beta^2}}{1 - \sqrt{1 - \beta^2}}}, & 1 > \beta \geq 0, \quad \chi = 1, \\ \sqrt{\frac{1}{\beta^2 + 1 - 1}}, & \beta > 0, \quad \chi = i. \end{cases} \quad (16b)$$

(1) For $K > K_c$ (with K_c depending on α or β and χ), the system exhibits amplified fluctuations, qualifying as pseudocritical, since it temporarily mimics the behavior of a truly critical system.

(2) For $K < K_c$, no amplification occurs, and Gaussian perturbations decay exponentially as in a normal system, indicating resilience to external forcing.

We can summarize the main observations as follows:

(1) α form: Near criticality, $K_c \rightarrow 0$, the system displays pseudocritical behavior before becoming truly critical. Far from criticality, a useful heuristic is $K > \sqrt{2}\alpha$ ($\alpha \gg 1$) for transient amplification to occur.

(2) β form: As $\beta \rightarrow 0$ (degeneracy), $K_c \approx \sqrt{2}/\beta \rightarrow \infty$; here, $\mathbf{J}$ approaches a scalar multiple of the identity, making it invariant under basis changes and suppressing nonnormal effects. Away from degeneracy, $K_c \rightarrow 0$, and nonnormal amplification is generally present.

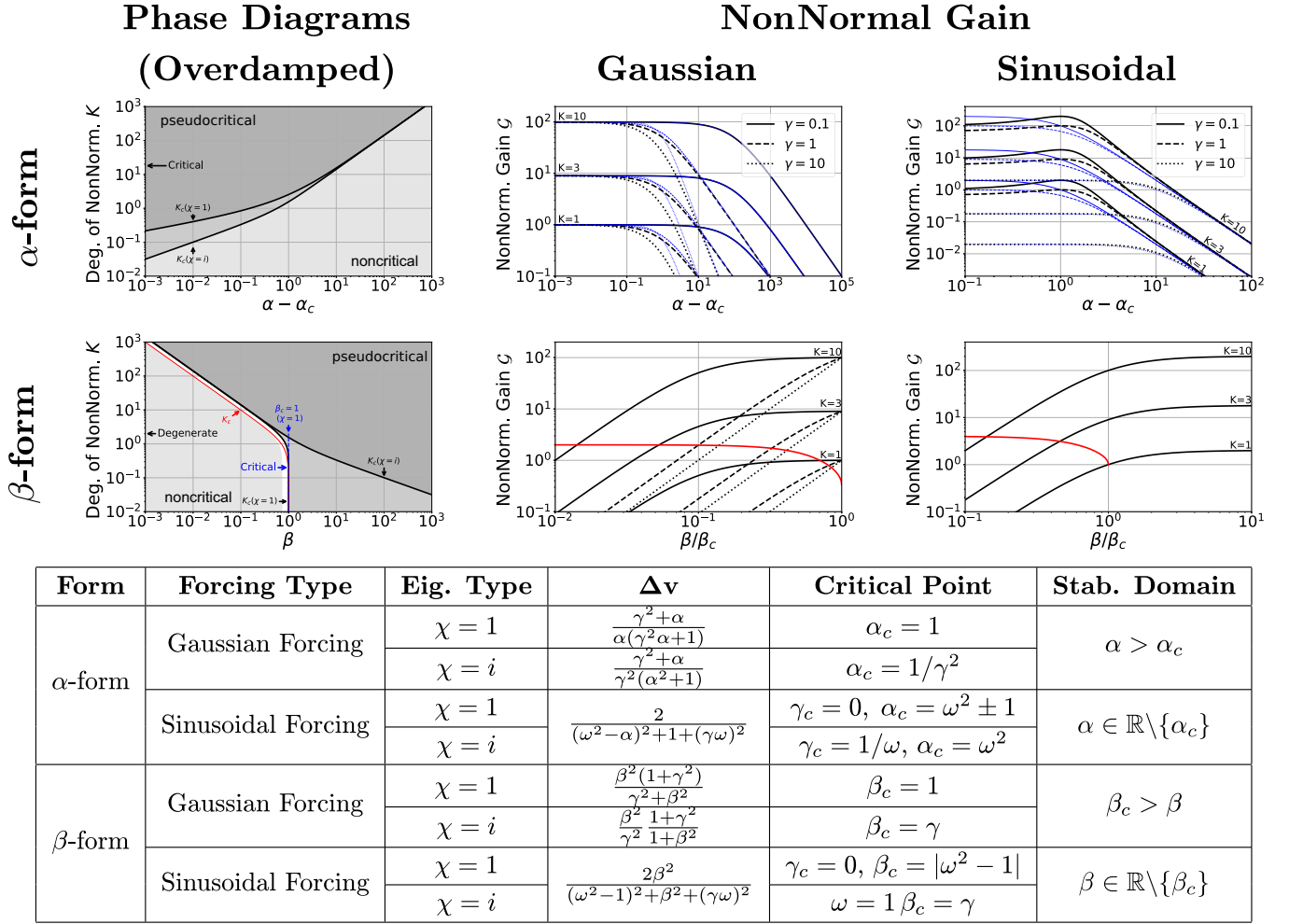


FIG. 1. Phase diagrams with boundaries between noncritical and pseudocritical regimes given by Eq. (16) and unified amplification gain (24) for the mechanisms of spectral criticality, nonnormality, and resonance. Each row corresponds to one formulation (top: α form; bottom: β form). (Left column) Pseudocritical phase diagrams for an overdamped system with Gaussian forcing. When $K > K_c$, the system is in a pseudocritical phase where each noise perturbation is transiently amplified before decaying. When $K < K_c$, perturbations decay exponentially. The threshold K_c is given in Eq. (16). For the α form, the critical point is $\alpha_c = 1$ for $\chi = 1$ and $\alpha_c = 0$ for $\chi = i$. For the β form, the critical point is $\beta_c = 1$ for $\chi = 1$ (shown in blue), while no critical point exists for $\chi = i$. In both forms, the corresponding K_c values are indicated for $\chi = 1$ and $\chi = i$. Dark gray regions mark the pseudocritical phase; light gray regions indicate the noncritical phase, including degenerate points ($\beta = 0$) and critical points ($\alpha = \alpha_c, \beta = \beta_c$). (Middle and right columns) nonnormal gain for two types of forcing (middle: Gaussian; right: sinusoidal). Results are shown for three magnitudes of nonnormality, $K = 1, 3$, and 10 . For all α -form cases, and for the β form with Gaussian forcing, we use three damping values: $\gamma = 0.1$ (solid), $\gamma = 1$ (dashed), and $\gamma = 10$ (dotted). In the α form, black lines correspond to $\chi = 1$ and thin blue lines to $\chi = i$. For the β form with Gaussian forcing, only $\chi = 1$ is plotted, since the nonnormal gain as a function of β/β_c is invariant under $\gamma \rightarrow 1/\gamma$. For the β form with sinusoidal forcing, both $\chi = 1$ and $\chi = i$ produce identical curves as a function of β/β_c when the system pulsation and damping are set to their critical values. For the β form, the parametrization $[\beta(\epsilon), K(\epsilon)]$ (28) [16] is shown in red, and with $\gamma = 1$ for the Gaussian noise nonnormal gain figure. (Bottom) Table listing the coefficient Δv [entering in the definition of the nonnormal gain (24)], as well as the critical point and stability domain for each form and forcing type. The overdamped limit can be recovered as the leading order term in power of $\gamma \gg 1$.

In summary, within the Gaussian overdamped framework, the closer the system is to criticality, the easier it is to observe pseudocritical transient amplification. Moreover, near eigenvalue degeneracy, the system becomes insensitive to nonnormal effects. Figure 1 (left column) illustrates these pseudocritical regions in (K, α) and (K, β) diagrams, which we refer to as phase diagrams since the structural response changes qualitatively as K crosses K_c , depending on the approach to either criticality or degeneracy.

General system driven by a Gaussian white noise. We now quantify how nonnormality and criticality influence the long-time variance when system (1) is driven by an external additive Gaussian white noise (5).

In the two-dimensional case, where $\mathbf{J}$ is a real matrix, the eigenvalues $\lambda_{\pm}$ for the α and β forms are given by Eq. (12). The kernel is expressed in Eq. (9), and the variance can be computed from Eqs. (6) and (7). From these relations, we obtain the following expressions for the MSD

(see the Supplemental Material [15]):

$$(\alpha \text{ form}) \quad v_\infty = \begin{cases} \frac{2\delta}{\gamma} \frac{\alpha}{\alpha^2-1} \left[1 + K^2 \frac{\gamma^2+\alpha}{\alpha(\gamma^2\alpha+1)} \right], & \alpha > 1, \quad \chi = 1, \\ \frac{2\delta}{\gamma} \frac{1}{\alpha-\gamma^{-2}} \left[1 + K^2 \frac{\gamma^2+\alpha}{\gamma^2(\alpha^2+1)} \right], & \alpha > \gamma^{-2}, \quad \chi = i, \end{cases} \quad (17a)$$

$$(\beta \text{ form}) \quad v_\infty = \begin{cases} \frac{2\delta}{\gamma} \frac{1}{1-\beta^2} \left[1 + K^2 \beta^2 \frac{1+\gamma^2}{\gamma^2+\beta^2} \right], & 1 > \beta > 0, \quad \chi = 1, \\ \frac{2\delta}{\gamma} \frac{1}{1-(\beta/\gamma)^2} \left[1 + K^2 \left(\frac{\beta}{\gamma} \right)^2 \frac{1+\gamma^2}{1+\beta^2} \right], & \gamma > \beta > 0, \quad \chi = i. \end{cases} \quad (17b)$$

In all cases, the MSD exhibits the general form

$$v_\infty = v_0(1 + \mathcal{G}(K)), \quad \mathcal{G}(K) = K^2 \Delta v, \quad (18)$$

where v_0 is the MSD of the corresponding normal system (i.e., $K = 0$). We refer to $\mathcal{G}(K)$ as the nonnormal gain, which grows quadratically with K , while Δv is a coefficient depending only on the system's spectral structure.

The critical point corresponds to the special case where v_0 diverges. The critical parameters are

$$\alpha_c = \begin{cases} 1, & \chi = 1, \\ \gamma^{-2}, & \chi = i, \end{cases} \quad \beta_c = \begin{cases} 1, & \chi = 1, \\ \gamma, & \chi = i. \end{cases} \quad (19)$$

At criticality, $\Delta v = 1$ is maximal. Far from criticality, the scaling reduces to

$$\Delta v \sim \frac{1}{\gamma^2 \alpha} \quad (\alpha \text{ form}), \quad \Delta v \sim (1 + \gamma^{-2}) \beta^2 \quad (\beta \text{ form}). \quad (20)$$

Therefore, the results derived in the overdamped case from the analysis of the critical K_c [Eq. (16)] remain valid in the general case. Achieving a nontrivial nonnormal gain of order unity requires $K \gtrsim 1$ at criticality, or K larger than the distance to criticality when the system is far from it.

Deterministic periodic forcing. We now consider the case of periodic forcing, which further confirms that the response kernel $G(\mathbf{J}, t)$ forms the fundamental basis for all amplification mechanisms. Specifically, let the external forcing in Eq. (1) be $\mathbf{g}(t) = \sqrt{2\delta} \hat{\mathbf{n}} \sin(\omega t)$, $\omega > 0$, where $\mathbf{g}$ is decomposed along the unique nonnormal mode $\mathbf{n}$, which dominates the leading nonnormal dynamics (terms scaling as K^2 , with all remaining terms being $\mathcal{O}(1)$ or smaller). The parameter δ sets the forcing amplitude.

The appropriate measure of potential amplification is the time-averaged mean-square deviation (MSD) (see the Supplemental Material [15])

$$\begin{aligned} v_\infty &:= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \|\mathbf{x}(t)\|_2^2 dt \\ &= 2\delta \pi \|\hat{G}(\mathbf{J}, \omega) \hat{\mathbf{n}}\|_2^2, \end{aligned} \quad (21)$$

where $\hat{G}(\mathbf{J}, \omega) := \frac{1}{\sqrt{2\pi}} \int_0^\infty G(\mathbf{J}, t) e^{-i\omega t} dt$

denotes the Fourier transform of the kernel evaluated at frequency ω (see the Supplemental Material [15]).

As with Gaussian forcing, in the two-dimensional case, the MSD takes the form (18) with

$$\Delta v = \begin{cases} \frac{2}{(\omega^2 - \alpha)^2 + 1 + (\gamma\omega)^2} & (\alpha \text{ form}), \\ \frac{2\beta^2}{(\omega^2 - 1)^2 + \beta^2 + (\gamma\omega)^2} & (\beta \text{ form}), \end{cases} \quad K = \sqrt{\frac{\kappa^2 - 1}{2}}, \quad (22)$$

where the value $\chi = 1$ or i does not affect Δv .

The normal MSD (i.e., when $K = 0$) diverges near resonance or at critical points given by

$$(\alpha \text{ form}) \quad \begin{cases} \alpha_c = \omega^2 \pm 1, & \gamma_c = 0, & \chi = 1, \\ \alpha_c = \omega^2, & \gamma_c = \omega^{-1}, & \chi = i, \end{cases} \quad (23a)$$

$$(\beta \text{ form}) \quad \begin{cases} \beta_c = |\omega^2 - 1|, & \gamma_c = 0, & \chi = 1, \\ \beta_c = \gamma_c, & \omega = 1, & \chi = i. \end{cases} \quad (23b)$$

Similarly to the Gaussian forcing case, $\Delta v = 1$ at resonance, but this value is not maximal. The maximal value becomes $\Delta v = 1$ when $\alpha = \omega^2$ and $\gamma = 0$ in the α form, and for $\omega = 1$, $\beta = 1$, and $\gamma = 0$ in the β form.

Furthermore, as the system moves away from resonance, Δv decreases quadratically with the distance to resonance. Hence, achieving a nonnormal gain $\mathcal{G}(K) = K^2 \Delta v$ greater than unity requires the magnitude of nonnormality K to exceed the distance to resonance. This is consistent with the conclusion already reached from the overdamped Gaussian analysis and the critical value K_c given in Eq. (16).

Unified amplification formalism. All previous results for the MSD of the response of system (3) can be expressed in the unified form (18), encompassing various types of external forcing—Gaussian noise (17) and sinusoidal forcing (18) with Eq. (22)—as well as different parameter regimes.

In this framework, the nonnormal (amplification) gain is defined as

$$\mathcal{G}(K) := \frac{v_\infty(K)}{v_\infty(K=0)} - 1 = K^2 \Delta v, \quad (24)$$

where K quantifies the magnitude of nonnormality and Δv depends solely on the spectrum, decaying quadratically with the distance to criticality or resonance.

Figure 1 summarizes the unified formalism. For each forcing type (Gaussian or sinusoidal), the magnitude of nonnormality is determined by the eigenbasis condition number κ , but with distinct functional forms:

$$K = \begin{cases} \frac{1}{2} |\kappa - \kappa^{-1}| & (\text{Gaussian}), \\ \sqrt{\frac{\kappa^2 - 1}{2}} & (\text{sinusoidal}). \end{cases} \quad (25)$$

The phase diagrams in Fig. 1 plot $\mathcal{G}(K)$ [Eq. (24)] as a function of the distance $\alpha - \alpha_c$ to criticality for the α form, and β/β_c for the β form. These diagrams reveal the values of K needed to achieve various gain levels $\mathcal{G}(K)$.

Two general trends emerge:

(1) *Far from criticality.* As the system moves away from the critical line, larger K values are required to sustain large nonnormal gain.

(2) *Close to criticality.* Δv approaches its maximum, so amplification is primarily governed by K . Conversely, near eigenvalue degeneracy ($\beta = 0$), the nonnormal term is suppressed and the gain remains close to its maximal value away from criticality.

By unifying criticality, nonnormality, and resonance into a single response map, the phase diagrams in Fig. 1 act as “risk maps.” They identify regions of parameter space where even weak stochastic or periodic perturbations can produce disproportionately large responses. With experimentally estimated pair (β, K) or (α, K) , one can perform the following tasks:

(1) *Predict variance inflation.* Use $\mathcal{G}(K)$ to quantify the MSD increase relative to the normal ($K = 0$) case.

(2) *Diagnose the dominant amplification route.* Points near criticality or resonance indicate classical mechanisms, whereas large K at moderate distances from criticality signal a pseudocritical regime.

Application to remote earthquake triggering. Dynamical triggering refers to the induction of earthquakes by weak, low-frequency seismic waves originating from distant large events. These waves can provoke ruptures thousands of kilometers away, well beyond typical aftershock zones [17–19]. Despite extensive study, the phenomenon remains incompletely understood, as its causal mechanisms have yet to be definitively identified, with several competing hypotheses still unvalidated [20].

Recently, Refs. [16,21] proposed a mechanism based on the breaking of Hamiltonian symmetry caused by the interplay of rotational effects and friction within fault gouges. This mechanism predicts that stressed systems develop extreme sensitivity to small perturbations of any frequency, without requiring resonance. Here, we reinterpret the formulation of Refs. [16,21] within our framework, revealing a significantly broader domain of applicability.

Starting from the general linear dynamical equation (3), consider the reduced “normal form” proposed in [21],

$$\mathbf{J} = \begin{pmatrix} -1 + \delta & -\eta \\ \eta & -1 - \delta \end{pmatrix}. \quad (26)$$

A straightforward spectral decomposition (see the Supplemental Material [15]) yields

$$\lambda_{\pm} = -1 \pm \beta, \quad \beta = \sqrt{\delta^2 - \eta^2}, \quad \kappa = \sqrt{\frac{\delta + \eta}{\delta - \eta}}. \quad (27)$$

The critical point is reached as $\eta \rightarrow \delta$, where the two eigenvalues coalesce at -1 , signaling a bifurcation in the underdamped system.

The condition number $\kappa > 1$ diverges as $\eta \rightarrow \delta$. Reference [16] introduced the parameter $\epsilon := 1 - \frac{\eta}{\delta} > 0$, which measures the distance to criticality. For $\epsilon \ll 1$, $\beta \sim \sqrt{\epsilon}$.

However, this parametrization also implies $\kappa \sim \frac{1}{\sqrt{\epsilon}}$, collapsing the Hopf degeneracy ($\beta \rightarrow 0$) and nonnormality ($\kappa \rightarrow \infty$) into a single control parameter ϵ . This obscures which mechanism dominates amplification.

To disentangle these effects and clarify how criticality, Hopf degeneracy, and nonnormality counteract one another along the path defined by this parametrization, we extract (β, K) for each case:

$$\beta = \delta \sqrt{2\epsilon - \epsilon^2}, \quad K = \begin{cases} \frac{2(1-\epsilon)}{\sqrt{2\epsilon - \epsilon^2}} & \text{(Gaussian),} \\ \sqrt{\frac{1-\epsilon}{\epsilon}} & \text{(sinusoidal),} \end{cases} \quad (28)$$

where we choose $\delta > \eta$ to ensure operation in the β framework with real eigenvalues, i.e., $\chi = 1$. For $\epsilon = 0$, the system is degenerate. For $\epsilon = 1$, the system is critical ($\beta = \delta$). We use the parametrization [Eq. (28)] to move from degeneracy to criticality by setting $\delta = \beta_c$. Depending on the regime, the correspondence can be read from the table in Fig. 1 (bottom). This parametrization, shown in red in Fig. 1, satisfies $K\beta \sim \beta_c$, so the divergence of K is exactly compensated by the vanishing of β as $\epsilon \rightarrow 0$. The resulting phase diagram and nonnormal gain are plotted in Fig. 1. Despite the divergence of K , the system never reaches pseudocriticality, and no non-normal amplification occurs.

For this application, our main contribution is to reveal that, while the parametrization $\epsilon = 1 - \eta/\delta$ naturally monitors the distance to criticality, it fails to detect the existence of a “highly nonnormal” regime ($K > K_c$). To address this, we propose re-parameterizing $\mathbf{J}$ as

$$\begin{cases} \delta = \frac{\beta}{2}(\kappa + \kappa^{-1}), \\ \eta = \frac{\beta}{2}(\kappa - \kappa^{-1}), \end{cases} \quad \beta \geq 0, \quad \text{for } \delta > \eta. \quad (29)$$

For $\eta > \delta$, the two expressions are permuted. This parametrization allows independent exploration of spectral degeneracy through β and nonnormality through κ . It also shows that $\delta, \eta > 1$ is permissible as long as $\beta = \sqrt{\delta^2 - \eta^2}$ [Eq. (27)] ensures asymptotic stability.

Varying κ at fixed β corresponds to moving vertically in the phase diagram of Fig. 1, thereby exploring the full pseudocritical sector that was previously excluded by the single-parameter ϵ approach of Refs. [16,21].

Our analysis demonstrates that nonnormality provides an independent amplification mechanism, capable of producing large responses to weak periodic perturbations—even when the system is spectrally far from resonance or criticality. Recognizing nonnormality as a separate axis of fault instability significantly expands the hazard landscape: faults considered “subcritical” by eigenvalue analysis may nonetheless reside within the pseudocritical region of Fig. 1, where small disturbances can be strongly amplified.

Incorporating the nonnormality parameter K alongside the spectral parameter β thus elevates the risk assessment of remote triggering and suggests diagnostic metrics—such as transient growth bounds or local condition number estimates—that could be derived from seismic data.

Non-hermitian quantum optics: pseudo-critical regime. Non-Hermitian quantum physics has rapidly become a focal point of modern science, not only for its elegant description of

open systems with gain and loss, but also for its far-reaching implications in quantum technologies. By extending quantum theory beyond the Hermitian paradigm, it offers insights into the measurement problem, decoherence, and control of quantum states—key challenges for quantum computing and sensing. Central to this framework are exceptional points (EPs), spectral singularities where eigenvalues and eigenvectors merge, acting as critical organizing centers for wave dynamics and enabling phenomena such as mode switching, unidirectional transport, and extreme sensitivity [22,23]. Near an EP, small perturbations can induce disproportionately large responses, paralleling the spectral criticality analyzed in this work. Here, we demonstrate that, for appropriate parameter regimes, non-Hermitian quantum optical systems can be recast into our α -form framework and exhibit pseudocritical amplification.

Non-Hermitian extensions of quantum optics, such as the forward four-wave mixing (FFWM) experiment [24], can be expressed in the general linear form

$$\frac{\partial}{\partial z}\mathbf{x} = \mathbf{J}\mathbf{x} + \mathbf{g}, \quad \mathbf{J} = \begin{pmatrix} -a + i\frac{\Delta k}{2} & ic \\ -ic & l - i\frac{\Delta k}{2} \end{pmatrix}, \quad (30)$$

where $a > 0$ models linear absorption due to imperfect electromagnetically induced transparency, l denotes the small Raman gain from the pump, c is the nonlinear coupling coefficient, and Δk represents the phase mismatch. The forcing term $\mathbf{g}$ accounts for injected probe and conjugate fields. Although the wave propagates along the spatial coordinate z , mathematically this is analogous to temporal wave propagation.

In the FFWM model (30), the exceptional point—where eigenvalues and eigenvectors coalesce—naturally emerges within the β framework (see the Supplemental Material [15]). At this point, the eigenvalue collision suppresses nonnormal amplification, mirroring behavior observed in frictional amorphous solids. By imposing an appropriate hierarchy among the hyperparameters ($a, l, c, \Delta k$), the system can be mapped onto the α framework, yielding

$$\alpha \approx \frac{a - l}{\sqrt{\Delta k^2 - 4c^2}}, \quad \kappa \approx \sqrt{\left| \frac{\Delta k + 2c}{\Delta k - 2c} \right|}, \quad (31)$$

which identifies the parameter regime

$$\Delta k \approx 2c \gg |\Delta k - 2c| \gg a + l > a - l > 0, \quad (32)$$

under which pseudocritical amplification arises despite the presence of optical damping. Here, α quantifies the distance to spectral criticality, while the pair (α, κ) jointly governs the interplay between criticality and nonnormality.

Since $\mathbf{x}$ represents the slowly varying field amplitudes, its norm squared, $\|\mathbf{x}\|^2$, is directly proportional to the optical energy. In the pseudocritical regime [$K > K_c(\alpha)$, see Eq. (16)], transient growth can occur: the wave transiently extracts net energy from the medium despite frictionlike optical damping before ultimately decaying. This counterintuitive gain-from-loss effect is a direct manifestation of nonnormal

amplification in a propagating-wave context. Thus, by tuning FFWM parameters into the α -form pseudocritical regime, it is possible to engineer transient optical energy amplification without requiring true lasing or exact $\mathcal{PT}$ symmetry breaking.

Conclusion. We have presented a unified linear framework that both distinguishes and quantitatively links two key amplification mechanisms: conventional criticality/resonance (governed by the spectral gap) and pseudocriticality (driven by nonnormality, measured by the condition number κ or magnitude of nonnormality K). For both Gaussian noise and sinusoidal forcing, we have derived closed-form expressions for the MSD, showing that these mechanisms contribute together. This yields a universal master law visualized in phase diagrams (Fig. 1), where any linear system can be placed and its susceptibility to large deviations assessed.

Reanalyzing the “giant amplification” model of remote earthquake triggering from Refs. [16,21], we showed that its single-parameter setup constrains the system below the pseudocritical threshold $K = K_c$. Our two-parameter reformulation lifts this limit, allowing κ to grow independently of λ and thus accessing the full nonnormal regime that has a much broader domain of existence.

The same analysis applies to parity-time symmetry breaking in non-Hermitian physics. In particular, the four-wave mixing in cold atoms [24] corresponds to an overdamped Langevin framework, with a linear dynamics matrix matching (26) up to a $\pi/2$ phase shift. Thus, our conclusions extend to exceptional points in non-Hermitian systems, where an emphasis on bifurcation and critical phenomena often obscures the role of nonnormality.

The proposed framework is universal, applying to systems from seismology to non-Hermitian photonics and ecological networks. A nonlinear extension suggests the coupling between the two key amplification mechanisms persists beyond linearity. This opens various diagnostic paths: (1) in seismology, transient growth bounds and field estimates of κ could complement traditional eigenvalue-based hazard metrics; and (2) in laboratory or numerical studies of exceptional points, the phase diagrams developed here give a straightforward recipe for separating genuine critical point and degeneracy effects from nonnormal transients. More broadly, any discipline that diagnoses “criticallike” bursts should test whether they originate from criticality, large K , or the potential combination of both—a distinction now made precise by the framework laid out in this Letter.

Acknowledgments. D.S. was partially supported by the National Natural Science Foundation of China (Grant No. T2350710802 and No. U2039202), Shenzhen Science and Technology Innovation Commission Project (Grants No. GJHZ20210705141805017 and No. K23405006), and the Center for Computational Science and Engineering at Southern University of Science and Technology.

Data availability. No data were created or analyzed in this study.

- [1] M. I. Maturana, C. Meisel, K. Dell, P. J. Karoly, W. D'souza, D. B. Grayden, A. N. Burkitt, P. Jiruska, J. Kudlacek, J. Hlinka *et al.*, Critical slowing down as a biomarker for seizure susceptibility, *Nat. Commun.* **11**, 2172 (2020).
- [2] J. Royer, B. C. Bernhardt, S. Larivière, E. Gleichgerrcht, B. J. Vorderwülbecke, S. Vulliémoz, and L. Bonilha, Epilepsy and brain network hubs, *Epilepsia* **63**, 537 (2022).
- [3] S. Tang and S. Allesina, Reactivity and stability of large ecosystems, *Front. Ecol. Evol.* **2**, 21 (2014).
- [4] M. Hirota, M. Holmgren, E. H. Van Nes, and M. Scheffer, Global resilience of tropical forest and savanna to critical transitions, *Science* **334**, 232 (2011).
- [5] D. Sornette, *Why Stock Markets Crash: Critical Events in Complex Financial Systems* (Princeton University Press, Princeton, NJ, USA, 2017), Vol. 49, 2nd printing.
- [6] L. Busto-Moner, J. Morival, H. Ren, A. Fahim, Z. Reitz, T. L. Downing, and E. L. Read, Stochastic modeling reveals kinetic heterogeneity in post-replication DNA methylation, *PLoS Comput. Biol.* **16**, e1007195 (2020).
- [7] M. Scheffer, J. Bascompte, W. Brock, V. Brovkin, S. R. Carpenter, V. Dakos, H. Held, E. H. Van Nes, M. Rietkerk, and G. Sugihara, Early-warning signals for critical transition, *Nature (London)* **461**, 53 (2009).
- [8] V. Troude, S. Lera, K. Wu, and D. Sornette, Pseudo-bifurcations in stochastic nonnormal systems, *arXiv:2412.01833*.
- [9] M. Embree and L. N. Trefethen, *Spectra and Pseudospectra: The Behavior of Nonnormal Matrices and Operators* (Princeton University Press, Princeton, NJ, 2005).
- [10] B. K. Murphy and K. D. Miller, Balanced amplification: A new mechanism of selective amplification of neural activity patterns, *Neuron* **61**, 635 (2009).
- [11] J. D. O'Brien, K. A. Oliveira, J. P. Gleeson, and M. Asllani, Hierarchical route to the emergence of leader nodes in real-world networks, *Phys. Rev. Res.* **3**, 023117 (2021).
- [12] L. N. Trefethen, A. E. Trefethen, S. C. Reddy, and T. A. Driscoll, Hydrodynamic stability without eigenvalues, *Science* **261**, 578 (1993).
- [13] V. Troude and D. Sornette, Accelerated transition rates in generalized Kramers problems for non-variational, nonnormal system, *arXiv:2502.05251*.
- [14] E. Glötzl and O. Richters, Helmholtz decomposition and potential functions for n -dimensional analytic vector fields, *J. Math. Anal. Appl.* **525**, 127138 (2023).
- [15] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/kdgv-shxf> for derivations details and further explanation.
- [16] H. Charan, O. Gendelman, I. Procaccia, and Y. Sheffer, Giant amplification of small perturbations in frictional amorphous solids, *Phys. Rev. E* **101**, 062902 (2020).
- [17] F. F. Pollitz, R. S. Stein, V. Sevilgen, and R. Bürgmann, The 11 April 2012 East Indian Ocean earthquake triggered large aftershocks worldwide, *Nature (London)* **490**, 250 (2012).
- [18] M. M. Mendoza, A. Ghosh, and S. S. Rai, Dynamic triggering of small local earthquakes in the central Himalaya, *Geophys. Res. Lett.* **43**, 9581 (2016).
- [19] R. T. O'Malley, D. Mondal, C. Goldfinger, and M. J. Behrenfeld, Evidence of systematic triggering at teleseismic distances following large earthquakes, *Sci. Rep.* **8**, 11611 (2018).
- [20] C. Qi, M. Wang, G. Kocharyan, A. Kunitskikh, and Z. Wang, Dynamically triggered seismicity on a tectonic scale: A review, *Deep Underground Sci. Eng.* **3**, 1 (2024).
- [21] H. Charan, A. Pomyalov, and I. Procaccia, Generic mechanism for remote triggering of earthquakes, *Phys. Rev. E* **104**, 044903 (2021).
- [22] A. Krasnok, N. Nefedkin, and A. Alù, Parity-time symmetry and exceptional points, *IEEE Antennas Propag. Mag.* **63**, 110 (2021).
- [23] R. El-Ganainy, K. G. Makris, M. Khajavikhan, Z. H. Musslimani, S. Rotter, and D. N. Christodoulides, Non-Hermitian physics and PT symmetry, *Nat. Phys.* **14**, 11 (2018).
- [24] Y. Jiang, Y. Mei, Y. Zuo, Y. Zhai, J. Li, J. Wen, and S. Du, Anti-parity-time symmetric optical four-wave mixing in cold atoms, *Phys. Rev. Lett.* **123**, 193604 (2019).