

# Graded paraparticle algebra of Majorana fields for multidimensional quantum computing with structured light

Fabrizio Tamburini<sup>✉,\*</sup>, Nicoló Leone,<sup>†</sup> Matteo Sanna,<sup>‡</sup> and Roberto Siagri<sup>§</sup>  
*Rotonium—Quantum Computing, Le Village by CA, Piazza G. Zanellato, 23, 35131 Padova, Italy*



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We present a theoretical framework that integrates Majorana's infinite-component relativistic equation within the algebraic structure of paraparticles through the minimal nontrivial  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded Lie algebras and  $R$ -matrix quantization. By mapping angular momentum-dependent mass spectra of the Majorana Tower to graded sectors associated with generalized quantum statistics, we derive an equation describing Majorana quasiparticles of structured light carrying spin and orbital angular momentum. These quanta in the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded algebras and  $R$ -matrix formulations extend the previous results from superconducting qubits to photonic platforms and set up deterministic two-photon gates involving at least two qubits encoded in a single photon without nonlinear effects. This makes feasible the writing of general quantum computing pathways exploiting fractional statistics through Nelson's quantum mechanics and implementing a procedure for error correction in photonic platforms. Furthermore, this approach makes possible to set paraparticle-based quantum information processing, beyond fermions and bosons, using graded qudits.

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## I. INTRODUCTION

Paraparticles are hypothetical quantum particles that extend the standard classification of particles in quantum field theory beyond the well-known fermions and bosons [1]. Yet, after six decades of theoretical work, no experimental platform has demonstrated deterministic and scalable paraparticle statistics in the laboratory. These quantum entities emerge from parastatistics, a generalization of quantum statistics that allows for more complex symmetry properties under particle exchange [2,3]. While fermions like electrons obey the Pauli exclusion principle characterized by antisymmetric wavefunctions and bosons, like photons, follow symmetric wavefunctions and can occupy the same quantum state, paraparticles are theorized to obey intermediate symmetry rules characterized by a parameter,  $p$ , known as the parastatistics order.

For instance, a system of parafermions of order  $p$  allows at most  $p$  particles to occupy the same antisymmetric state, unlike ordinary fermions where only one is allowed. This framework arises naturally in mathematical formulations involving trilinear commutation relations rather than the standard bilinear ones, and it retains consistency with the spin-statistics theorem under certain conditions. Historically,

Wigner first asked whether the equations of motion fix the commutation rules and showed they do not, opening the door to generalized (para)statistics [4]. Paraparticles have surfaced in supersymmetry, quantum groups, and higher-dimensional field theories, exploring extensions of the Standard Model, but remain still experimentally elusive. They also sharpen our understanding of symmetry by extending to the full permutation group the Young-tableaux machinery [5] used for bosons and fermions. These diagrammatic tools are used in the representation theory of symmetric and general linear groups to describe the symmetry properties of bosonic systems or identical particles obeying parastatistics and in quantum computing and quantum information (for an introductory reference, see Ref. [6]).

A recent work [7] proposed a different second quantization theory of parastatistics based on a quadratic algebra with  $R$ -matrix commutation relations and also proposed their realizations in condensed matter physics such as quantum spin models. A quantum information application of  $R$  parastatistics was later proposed in Ref. [8]. The first theoretical detectability of  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded parafermions [9] and parabosons [10] was proposed by Toppan, proving that paraparticles beyond bosons/fermions and transforming under the permutation group are theoretically detectable.

Here, we propose a general mathematical formalism to be applicable to any type of qubit and focus our analysis to more general photonic applications, discussing the conceptual design of a direct photonic realization of paraparticle algebras using structured-light modes, providing also a method for continuous-variable and scalable platform to simulate also in this case exotic statistics beyond fermions and bosons.

This is achieved by using the fundamental properties of the Majorana Tower and Majorana photonic quasiparticles through the graded Lie superalgebras for paraparticles as proposed by Toppan [11] to build states of Majorana

\*Contact author: [fabrizio.tamburini@gmail.com](mailto:fabrizio.tamburini@gmail.com)

†Contact author: [nicolo.leone@rotonium.com](mailto:nicolo.leone@rotonium.com)

‡Contact author: [matteo.sannai@rotonium.com](mailto:matteo.sannai@rotonium.com)

§Contact author: [roberto.siagri@rotonium.com](mailto:roberto.siagri@rotonium.com)

paraparticles. In fact, braid-group statistics already allow for braided Majorana qubits [12,13], provide a minimal setting for braided quantum mechanics, and may offer a minimal setting to implement Kitaev's proposal for topological quantum computation [14]. Foundational field-theoretic/quantum-mechanical treatments of braid statistics and fractional spin appear in Refs. [15–17].

As an example, we describe the actual properties of an ideal platform with single-photon computation in a spin-orbit coupled waveguide that natively realizes the full  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded paraparticle algebra and describe the language of two-qubit photonic gates. This algebra, of course, can be extended to higher-dimensional qudits. After setting the necessary algebraic background, we derive a Jordan-Wigner map from graded paraboson/parafermion operators then applied to spin-orbital angular momentum photonic modes and, as an example, build a set of deterministic universal gates exploiting the properties of the photonic ququart.

Finally, we outline how the large- $p$  limit recovers Maxwell-Boltzmann behavior and apply Nelson's quantum mechanics and stochastic calculus to formulate a fractional Schrödinger-Majorana equation for diagnostic and error correction in photonic quantum circuits, establishing a bridge between paraparticle algebra, scalable photonic hardware, and the spectral geometry.

## II. STATISTICS OF PARABOSONS AND PARA-FERMIONS

Parabosons and parafermions that obey parastatistics of order  $p$  are associated with trilinear commutation or anticommutation relations and belong to representations of more general permutation groups related to Young diagrams rather than the permutation symmetric group  $S_N$  only used for bosons and fermions. Adopting the minimal nontrivial grading parabosons of order  $p$  are generalizations of bosons and satisfy the following trilinear commutation relations [18] for creation, different from standard bosonic commutators [19], with  $[a_i, a_j^\dagger] = \delta_{ij}$ ,

$$\begin{aligned} [a_i, [a_j^\dagger, a_k]] &= 2\delta_{ij}a_k, \quad [a_i, [a_j, a_k^\dagger]] = 0, \\ [a_i, [a_j^\dagger, a_k^\dagger]] &= 2\delta_{ij}a_k^\dagger - 2\delta_{ik}a_j^\dagger, \end{aligned} \quad (1)$$

and up to  $p$  identical parabosons can occupy the same quantum state. The states so constructed obey symmetrized Young tableaux, with at most  $p$  rows. When  $p = 1$ , the statistics reduce to ordinary bosons.

Parafermions, instead, obey the trilinear anticommutation relations:

$$\begin{aligned} \{f_i, \{f_j^\dagger, f_k\}\} &= 2\delta_{ij}f_k, \quad \{f_i, \{f_j, f_k^\dagger\}\} = 0, \\ \{f_i, \{f_j^\dagger, f_k^\dagger\}\} &= 2\delta_{ij}f_k + 2\delta_{ik}f_j. \end{aligned} \quad (2)$$

In both cases, the parameter  $p$  determines the degree of deviation from conventional statistics, with higher  $p$  values allowing more symmetry flexibility. Parastatistics find formal use in quantum field theory and group theory, but so far remain theoretical constructs without experimental evidence. Volkov developed a generalized quantization/ $S$ -matrix framework in which Green's trilinear parafield relations are consistently implemented [20,21]. For Lorentz-covariant

fields also according to Greenberg-Messiah construction [22], the commutator (paraboson) version of Eq. (1) must be used for integer spin, and the anticommutator (parafermion) version for half-integer spin [Eq. (2)]; otherwise, microcausality does not hold.

### Large- $p$ values

As the Green index  $p \rightarrow \infty$ , both parabosons and parafermions will change their characteristics. A paraboson system loses every trace of its generalized Bose exchange constraint and behaves, for all practical purposes, like a collection of distinguishable particles recovering the full Bose-Einstein condensation behavior and the exclusion constraint disappears making the occupation statistics (parastatistics) converge to Maxwell-Boltzmann instead of having a bosonic symmetrization. Hence, in the classical limit the quantum correlations and exchange effects that tell fermions from bosons are gradually erased. For parafermions, the operator algebra really collapses to canonical Fermi anticommutator  $\{f, f^\dagger\} = 1$ , while the occupancy statistics becomes Maxwell-Boltzmann.

Green trilinear relations for a single paraboson mode of order  $p$  of Eqs. (1) and (2) with  $p$ -deformed commutator reproduces Green's trilinear relations, implying the operator identity from the Green trilinear algebra:

$$[b, b^\dagger] = 1 + \frac{2}{p} b^\dagger b - \frac{2}{p} b b^\dagger. \quad (3)$$

Equation (3) is written explicitly in terms of commutators and anticommutators to describe a paraboson mode  $b, b^\dagger$  of order  $p$  through Greens trilinear identities,

$$[b, \{b^\dagger, b\}] = +2b, \quad (4a)$$

$$[b^\dagger, \{b^\dagger, b\}] = -2b^\dagger. \quad (4b)$$

with, as usual,  $\{A, B\} = AB + BA$  and  $[A, B] = AB - BA$ . The mixed use of square and curly brackets in Eqs. (4a) and (4b) is essential because a single paraboson mode of order  $p$  must satisfy a commutator acting on an anticommutator:  $[b, \{b^\dagger, b\}] = +2b$  and  $[b^\dagger, \{b^\dagger, b\}] = -2b^\dagger$ . The inner anticommutator  $\{b^\dagger, b\}$  provides the symmetric (even) combination of two operators, which is no longer a  $c$ -number for parabosons, whereas the outer commutator enforces the antisymmetric (odd) response required by the trilinear relations. Instead, the commutator  $[ , ]$  enforces the required antisymmetric (odd) response with respect to an additional creation or annihilation operator. This nested  $\{ \{ \} \}$  structure is the minimal algebraic closure that reproduces the Green trilinear identities and reduces to the ordinary bosonic relation when  $p = 1$ .

The number operator and  $p$ -deformed commutator are defined with the occupation operator  $\hat{N} := b^\dagger b$ , with  $\{b^\dagger, b\} = 2\hat{N} + 1$ . Here,  $p \in \mathbb{N} - \{0\}$  is a positive integer that represents the parastatistics order; taking  $p \rightarrow \infty$  is a purely formal limit that recovers ordinary bosons. Using Eq. (4), then one derives  $[\hat{N}, b^\dagger] = +b^\dagger$ ,  $[\hat{N}, b] = -b$  and the  $p$ -deformed single-mode commutator can be written also [23] making explicit the index number  $p$  in terms of the occupation operator  $\hat{N}$ :

$$[b, b^\dagger]_p = 1 + \frac{2}{p} (2\hat{N} + 1). \quad (5)$$

Equation (5) reduces to the canonical Bose commutator  $[b, b^\dagger] = 1$  when  $p \rightarrow \infty$ , while the full Green relations (4) remain intact for any finite  $p$ . This limit connects directly to the Wigner-Yang deformed-oscillator picture of the harmonic oscillator [24]. Related deformations with the reflection (Klein) operator underlie fractional-spin field equations and their planar realizations [25–28].

This is more evident with the  $p$ -deformed factorial and Fock ladder; the  $p$ -dependence sits only in a simple prefactor, making the limit  $p \rightarrow \infty$  literally a one-line observation. This is given by introducing

$$[n]!_p := \prod_{k=1}^n \left( k + \frac{k(k-1)}{p} \right), \quad (6)$$

and the normalized states

$$|n\rangle_p := \frac{(b^\dagger)^n}{\sqrt{[n]!_p}} |0\rangle, \quad b|0\rangle = 0. \quad (7)$$

The ladder matrix elements are

$$b^\dagger |n\rangle_p = \sqrt{n+1 + \frac{n(n+1)}{p}} |n+1\rangle_p, \quad (8a)$$

$$b |n\rangle_p = \sqrt{n + \frac{n(n-1)}{p}} |n-1\rangle_p. \quad (8b)$$

As  $p \rightarrow \infty$ , the square-root factors reduce to  $\sqrt{n+1}$  and  $\sqrt{n}$ , recovering the ordinary bosonic ladder exactly. For fixed  $n$ , then  $[n]!_p = n! [1 + \mathcal{O}(1/p)]$ , one obtains

$$|n\rangle_p \xrightarrow{p \rightarrow \infty} |n\rangle_{\text{boson}} = \frac{(b^\dagger)^n}{\sqrt{n!}} |0\rangle. \quad (9)$$

The probability weight for  $k$  identical parabosons occupying one mode contains the factor

$$\prod_{m=1}^{k-1} \left( 1 - \frac{m}{p} \right) = 1 - \mathcal{O}(k^2/p). \quad (10)$$

Thus, for  $p \gg k$ , the exclusion constraint disappears and the statistics converge to Maxwell-Boltzmann instead of Bose-Einstein. In this way, both the occupation-number normalization and all combinatorial weights visibly approach their Maxwell-Boltzmann (bosonic) forms term by term, making the convergence evident without additional algebraic work.

In other words, standard bosons and fermions ( $p = 1$ ) are therefore special, low-order points of the broader Green hierarchy; as  $p$  grows, exchange correlations fade and the system behaves as if the particles were distinguishable. Higher  $p$  just means that one is summing over  $p$  identical local fields. Green order  $p$  is an internal degeneracy, not a spin label. Historically, Greenberg used parastatistics in his paraquark model, an important conceptual step toward color [29]. The same limit for parafermions removes the Pauli cap and also yields Maxwell-Boltzmann populations. Similarly, for parafermions the Pauli cap is removed and the occupations approach Maxwell-Boltzmann statistics (the operator algebra remains fermionic).

For the sake of completeness, in this case, Green relations for single mode order  $p$  can be also written in a

mixed commutator/anticommutator notation,  $[f, \{f^\dagger, f\}]_+ = 2f$ , and  $[f^\dagger, \{f^\dagger, f\}]_+ = -2f^\dagger$  where “+” and “−” indicate integer spin and half-integer spin, respectively. The  $p$ -deformed anticommutator with  $\hat{N} = f^\dagger f$  is  $\{f, f^\dagger\}_{(p)} = 1 - \frac{2}{p}(2\hat{N} - 1)$ . In the large- $p$  limit,  $\lim_{p \rightarrow \infty} \{f, f^\dagger\}_{(p)} = 1$  and  $n_{\text{max}} = p \rightarrow \infty$ , so the exclusion cap disappears and the statistics approach Maxwell-Boltzmanns, mirroring the paraboson result  $[b, b^\dagger]_{(p)} \rightarrow 1$ .

$$\begin{aligned} \text{parabosons } p \rightarrow \infty &\rightarrow \begin{cases} \text{Bose algebra (operator level),} \\ \text{MB statistics (occupation level),} \end{cases} \\ \text{parafermions } p \rightarrow \infty &\rightarrow \begin{cases} \text{Fermi algebra (operator level),} \\ \text{MB statistics (occupation level).} \end{cases} \end{aligned} \quad (11)$$

Physically, large- $p$  systems can be used to model systems with extremely high internal symmetry or large hidden degrees of freedom, where particles behave as if they were distinguishable or noninteracting. These set of particles can be used to build quantum computational states. When parabosons are considered at very large order  $p$ , their Fock space becomes indistinguishable from the bosonic one; likewise, as  $p$  grows for parafermions, the occupation cap disappears and the resulting Fock states approach classical field modes. These asymptotic limits indicate that ordinary bosons and fermions are simply the lowest-order manifestations of a wider algebraic framework.

Consider the example of one-mode photon Fock ladder for large- $p$  parabosons used to describe photonic qudits with orbital angular momentum (OAM) and polarization. Let  $b^\dagger, b$  obey the Green paraboson algebra of order  $p$ , then  $[b, \{b^\dagger, b\}]_- = 2b$ ,  $[b^\dagger, \{b^\dagger, b\}]_- = -2b^\dagger$ , with the vacuum  $b|0\rangle = 0$ . From this, one defines the  $p$ -deformed occupation ladder as  $|n\rangle_p = \frac{1}{\sqrt{[n]!_p}} (b^\dagger)^n |0\rangle$  and  $[n]!_p := \prod_{k=1}^n (k + \frac{k(k-1)}{p})$ . The deformation enters only through the “parabosonic factorial”  $[n]!_p$ ; every coefficient or normalization factor that would ordinarily contain the usual factorial  $n!$  is simply replaced by its  $p$ -deformed counterpart, while the rest of the algebraic structure remains unchanged.

Matrix elements are defined as  $b^\dagger |n\rangle_p = \sqrt{n+1 + \frac{n(n+1)}{p}} |n+1\rangle_p$  and  $b |n\rangle_p = \sqrt{n + \frac{n(n-1)}{p}} |n-1\rangle_p$ . For large- $p$  limit at fixed photon number  $n$ ,  $[n]!_p \xrightarrow{p \rightarrow \infty} n!$  and the ladder coefficients reduce to  $\sqrt{n+1}$  and  $\sqrt{n}$ ; hence,  $|n\rangle_p \xrightarrow{p \rightarrow \infty} |n\rangle_{\text{boson}}$ . All  $p$ -dependent corrections are  $\mathcal{O}(1/p)$ ; suppressed by the factor  $1/p$ , the algebra approaches the canonical commutator  $[b, b^\dagger] = 1$ , and the Fock space is therefore indistinguishable from the photonic one.

In the two-mode extension, for modes  $b_1$  and  $b_2$ , the deformed factorial becomes  $[n_1]!_p [n_2]!_p$  and the basis states  $|n_1, n_2\rangle_p = (b_1^\dagger)^{n_1} (b_2^\dagger)^{n_2} |0\rangle / \sqrt{[n_1]!_p [n_2]!_p}$  again converge to  $|n_1, n_2\rangle_{\text{boson}}$  as  $p \rightarrow \infty$ . A photonic qudit encoded in OAM or polarization can therefore be viewed as a large- $p$  parabosonic register where the  $p$  corrections quantify how fast true Bose statistics are approximated.

For parafermions, the analogous ladder has an occupation cap  $N \leq p$ , but as  $p \rightarrow \infty$  the cap recedes and  $|n\rangle_p^{(\text{paraF})} \rightarrow |n\rangle_{\text{MB}}$ , i.e., Maxwell-Boltzmann populations with *no* exchange symmetry. Thus, the bosonic paraboson tower interpolates to Bose condensation, whereas the parafermionic Majorana tower interpolates to a classical, fully distinguishable gas of particles. This explicit Fock ladder, the ordered sequence of Fock number states, shows that photon states can be embedded in the broader paraboson algebra, with ordinary optics recovered when the order  $p$  is large, supporting the interpretation of bosons and fermions as low-order corners of a unified statistical hierarchy. In models of high-energy physics or quantum gravity, where exotic symmetries or hidden sectors dominate, the  $p \rightarrow \infty$  regime may serve as an effective approximation. In this way is also provided a formal connection between quantum and classical statistics, suggesting how indistinguishability and statistical exclusion emerge from deeper algebraic origins. Thus, paraparticles of high order tend to lose their distinct quantum statistical identities, merging toward a classical, bosonlike limit that dissolves the constraints of both Bose and Fermi exclusion principles.

### III. PHOTONIC OAM MAJORANA QUASIPARTICLES AND PARAPARTICLE STATISTICS

As we all know, boson and fermions follow different statistics. From Wess and Zumino to String theory [30–32], including other different formulations of high-energy physics and several theories of everything, there are different approaches to unify and extend these two statistics in a single multiplet. A first attempt was made in 1932 by Majorana [33], who introduced infinite-component relativistic wavefunctions for arbitrary spin values related to the energy-mass of the particle. Majorana proposed to cast both bosonic and fermionic solutions in a denumerably infinite spectrum of integer and half-integer values of the spin, the so-called Majorana Tower. In this way, Majorana investigated theoretical extensions to particle statistics and field theories that transcend the standard fermionic and bosonic dichotomy [34,35]. The Majorana Tower finds several applications in condensed matter theories in photonics and in high-energy physics related to scattering mechanisms and the Riemann zeta function [36,37].

Majorana's approach is rooted in relativistic wave equations for arbitrary spin particles and introduces the concept of a “mass-spin Tower” through the relationship where their Majorana masses follow the spin  $s$  and mass  $M$  defined in the following equation:

$$M = \frac{m}{s + \frac{1}{2}}, \quad (12)$$

which relates the effective mass  $M$  and the universal Majorana mass scale  $m$ , defined by the energy of these quanta in the laboratory rest frame. Equation (12) leads to a unification framework where bosons and fermions appear as different states of the same underlying entity. A clear example is found in photonics: the spiral spectrum of photons carrying OAM and spin angular momentum (SAM) forms photonic quasiparticle states equivalent to the spin spectrum of the Majorana Tower [38]. In 2+1 dimensions, the Majorana equation is explicitly tied to fractional-spin (anyon)

dynamics [39,40]. Spin blocks, Lorentz multiplets, and the fixed graded bracket are present in the Tower. Each tower component  $\Psi_s(x)$  carries definite spin  $s \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \dots\}$  and transforms in the Lorentz representation, with the two-spinor notation where the (proper) Lorentz group is locally  $\text{SL}(2, \mathbb{C}) \simeq \text{SU}(2)_L \times \text{SU}(2)_R$  as  $D^{(s,0)} \oplus D^{(0,s)}$  [irreducible for  $\text{SL}(2, \mathbb{C})$ ], so that the full field splits as  $\Psi(x) = \bigoplus_s \Psi_s(x)$ . The infinite-dimensional irreducible representations of the (proper) Lorentz group for the Majorana Tower are labeled by a pair of half-integers  $(j_L, j_R)$  that we write as  $D^{(j_L, j_R)}$  and they have dimension  $(2j_L + 1)(2j_R + 1)$ . In our case,  $D^{(s,0)}$  has completely symmetric rank- $2s$  undotted spinor [left handed,  $\text{SU}(2)_L$ ] carrying spin  $s$ . Instead,  $D^{(0,s)}$  is a completely symmetric rank- $2s$  dotted spinor [right handed,  $\text{SU}(2)_R$ ], the complex-conjugate chiral partner of  $D^{(s,0)}$ . Both have dimension  $2s + 1$  when we consider a finite subset of  $s$  spin states of the Tower.

For every block the correct (graded) exchange sign is

$$[\Psi_s(x), \Psi_s(y)]_{\pm} = 0 \quad (13)$$

when  $(x - y)^2 < 0$ . As before, we take the “+” sign (commutator) for integer spins  $s \in \mathbb{Z}$  and the sign “−” (anticommutator) for half-integer spin  $s \in \mathbb{Z} + \frac{1}{2}$  because a  $2\pi$  rotation multiplies the Wightman two-point function  $W_s(x - y) = \langle 0 | \Psi_s(x) \Psi_s(y) | 0 \rangle$  by  $(-1)^{2s}$ ; micro causality then forces the exchange bracket to carry the same phase, reproducing the spinstatistics pairing [41]. This fixes the sign before any  $\mathbb{Z}_2 \times \mathbb{Z}_2$  grading is introduced, ensuring that each spin block remains local and Lorentz covariant.

*Majorana fields and Majorana quasiparticles: Conceptual distinction and algebraic correspondence.* The original Majorana extended solution to Dirac's equation is characterized by an infinite spectrum of spin components. Differently, the Majorana quasiparticle has to include the paraparticle index  $p$ . In Majorana's relativistic, infinite-component construction, the field  $\Psi(x) = \bigoplus_{s \in \{0, \frac{1}{2}, 1, \dots\}} \Psi_s(x)$  carries, for each spin  $s$ , a component transforming in the Lorentz representation  $D(s, 0) \oplus D(0, s)$ . The tower exhibits the spin-mass relation  $M_s = m/(s + \frac{1}{2})$  [Eq. (12)] and obeys microcausality with the usual spin-statistics pairing (commutator for integer spin, anticommutator for half-integer spin) as in Eq. (13). These describe a relativistic elementary field and its Wightman two-point functions, and initially was not formulated to describe excitations of a medium.

By contrast, “Majorana quasiparticles” in condensed matter [e.g., Bogoliubov–de Gennes (BdG) zero modes] or in the photonic setting considered here are *emergent* low-energy degrees of freedom of quanta. They need not transform in  $(s, 0) \oplus (0, s)$  under the Lorentz group, nor obey relativistic microcausality. The label “Majorana” in this case refers to algebraic properties inherited from his 1937 work [42] (self-conjugacy  $\gamma^\dagger = \gamma$  and  $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$  in BdG systems) or, in our photonic case, to the fact that the mode content can be organized in blocks that mirror the Majorana tower pattern and exchange rules. In particular, our structured-light modes are bosonic; the “Majorana” terminology here is *algebraic* and organizational, not a claim of topological Majorana zero modes in optics.

*Algebraic correspondence (operator dictionary).* The bridge we use has two steps. First, we reorganize the infinite-component field into  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded sectors; second, we realize the same grading on photonic SAM/OAM modes.

(1) *From Majorana tower blocks to graded sectors.* We associate with each spin block  $\Psi_s$  a grading label  $\chi(s) = (a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_2$ . A convenient choice, used later for photonics, is the cyclic assignment

$$\begin{aligned} s = 0 &\mapsto (0, 0), & s = \frac{1}{2} &\mapsto (1, 0), \\ s = 1 &\mapsto (0, 1), & s = \frac{3}{2} &\mapsto (1, 1), \end{aligned}$$

extended periodically in  $s$  [see the mapping used around Eq. (36)]. With the projectors  $P_{(a,b)}$  onto each homogeneous sector, the graded gamma operators are  $\Gamma_{(a,b)}^\mu := P_{(a,b)} \otimes \gamma^\mu$ , so that each component obeys the blockwise Dirac/Majorana equation

$$(\Gamma_{(a,b)}^\mu \partial_\mu - M_{(a,b)}) \Psi_{(a,b)}(x) = 0, \quad M_{(a,b)} = \frac{m}{s_{(a,b)} + \frac{1}{2}} \quad (14)$$

and the graded Clifford relations

$$\Gamma_{(a,b)}^\mu \Gamma_{(c,d)}^\nu + (-1)^{(a,b) \cdot (c,d)} \Gamma_{(c,d)}^\nu \Gamma_{(a,b)}^\mu = 2g^{\mu\nu} \delta_{(a,b),(c,d)}.$$

This reproduces, sector by sector, the standard spin-statistics sign of Eq. (13).

(2) *From graded sectors to photonic modes.* On the optical side, we map physical modes labeled by OAM  $\ell \in \mathbb{Z}$  and SAM (helicity)  $\sigma = \pm 1$  (or, for  $\ell = 0$ , two orthogonal waveguide modes  $A, B$ ) to the same grading via

$$\varphi : (\ell, \sigma) \mapsto (a, b) = (\ell \bmod 2, (1 - \sigma)/2); \quad (15)$$

see Eq. (56) and Table III for the explicit ququart used in our device model. Creation/annihilation operators  $\psi_{(a,b)}^\dagger$  and  $\psi_{(a,b)}$  attached to those modes obey the graded bracket

$$\begin{aligned} [X, Y]_{\text{gr}} &:= XY - (-1)^{(a,b) \cdot (a',b')} YX, \\ X &\in \mathfrak{g}_{(a,b)}, \quad Y \in \mathfrak{g}_{(a',b')}, \end{aligned} \quad (16)$$

and, when trilinear Green/Volkov relations are required, the corresponding nested identities with the  $R$ -matrix deformation used elsewhere in the paper.

Equation (14) organizes the photonic mode algebra in the same  $\mathbb{Z}_2 \times \mathbb{Z}_2$  blocks as the relativistic tower, but it does not endow photons with a Lorentz-invariant rest mass;  $M_{(a,b)}$  is an *organizational label* inherited from the Majorana pattern [cf. Eq. (12)], and all dynamical, platform-dependent details (dispersion, group velocity reduction, mode coupling) remain in the photonic Hamiltonian. Likewise, we do not claim BdG-type self-conjugate fermionic zero modes in our optical setting.

*Minimal worked example (two low-spin levels).* Take the truncated set  $\{s = 0, \frac{1}{2}\}$ . With the assignment above,

$$s = 0 \leftrightarrow (0, 0), \quad s = \frac{1}{2} \leftrightarrow (1, 0).$$

Let  $\psi_{(0,0)}$  denote the bosonic-grade mode (e.g.,  $|\ell = 0, A\rangle$ ) and  $\psi_{(1,0)}$  the parafermionic-grade mode (e.g.,  $|\ell = +1, \sigma = +1\rangle$ ). Then, at equal time,

$$\begin{aligned} [\psi_{(0,0)}, \psi_{(0,0)}^\dagger] &= 1, & \{\psi_{(1,0)}, \psi_{(1,0)}^\dagger\} &= 1, \\ [\psi_{(0,0)}, \psi_{(1,0)}] &= 0, \end{aligned}$$

because  $(0, 0) \cdot (1, 0) = 0$ . This reproduces (1) commutator for the  $s = 0$  block, (2) anticommutator for the  $s = \frac{1}{2}$  block, and (3) mutual locality between integer/half-integer sectors, in direct correspondence with Eq. (13). Extending to  $\{s = 0, \frac{1}{2}, 1, \frac{3}{2}\}$  gives the full four-sector pattern used in Sec. III and in the photonic ququart in the following text [cf. Eqs. (21) and (39), and Table III].

Majorana's infinite-component field theory provides the *representation-theoretic* and *exchange-algebra* template (tower decomposition, spin-statistics sign) [Eqs. (12) and (13)]. Our  $\mathbb{Z}_2 \times \mathbb{Z}_2$  grading implements the same template algebraically through projectors, graded Clifford blocks, and graded (tri)linear brackets. Photonic SAM/OAM modes are then *assigned* to these graded sectors via the map (15), yielding an explicit dictionary between the relativistic Majorana tower, the graded paraparticle algebra, and the optical mode operators that we manipulate in the device-level proposals of Secs. IV and VI.

On the other hand, paraparticles are governed by their parastatistics through algebraic structures like the smallest noncyclic Abelian group  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded Lie superalgebras. In this way, paraparticle sectors are algebraically represented [11]. This algebraic structure offers four symmetry sectors, enabling encoding of multiple particle types.  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded Lie algebras can implement the formulation of trilinear commutation and anticommutation relations and systematically classify them with graded Jacobi identities. This approach recovers the structure of Green's parastatistics and connects it to modern Lie algebra representation theory enabling an extension of supersymmetric (SUSY) to para-SUSY-like theories [43,44]. Closely related constructions connect the deformed Heisenberg algebra to fractional-spin fields, supersymmetry without fermions, and hidden nonlinear supersymmetries in pure parabosonic systems [45–47].

To generalize the Green's trilinear algebra to the graded setting, we then postulate the graded trilinear relation  $[\psi_{(a,b)}^-, [\psi_{(a',b')}^+, \psi_{(a'',b'')}^-]] = \sum_{(c,d)} f_{(a,b),(a',b'),(a'',b'')}^{(c,d)} \psi_{(c,d)}^-$ , where the structure constants  $f_{(*,*)}^{(*,*)}$  encode the graded Lie triple system underlying the paraparticle algebra and are compatible with the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded Jacobi identity.

A bridge between the Majorana Tower and the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded Lie superalgebras used to describe paraparticles in terms of qubits and qudits (or even with continuous variable) can be set directly using the fundamental properties of the algebra itself. Each homogeneous subspace  $\mathfrak{g}_{(a,b)}$  of degree  $(a, b)$  can host a distinct type of field/operator/mode. In our four-sector decomposition the even-degree sectors  $(0, 0)$  and  $(1, 1)$  are bosonic or parabosonic-like, whereas the odd-degree sectors  $(1, 0)$  and  $(0, 1)$  are used for fermionic or parafermionic-like sectors.

The algebra is graded by the following Abelian group:

$$G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}, \quad (17)$$

with multiplication table reported in Table I.

Each element of the algebra  $X_{(a,b)} \in \mathfrak{g}_{(a,b)}$  is assigned a *degree*  $(a, b)$ . The full algebra is then a direct sum of the algebraic elements:

$$\mathfrak{g} = \bigoplus_{(a,b) \in \mathbb{Z}_2 \times \mathbb{Z}_2} \mathfrak{g}_{(a,b)}. \quad (18)$$

TABLE I. Multiplication table for  $\mathbb{Z}_2 \times \mathbb{Z}_2$ .

| $\oplus$ | (0, 0) | (0, 1) | (1, 0) | (1, 1) |
|----------|--------|--------|--------|--------|
| (0, 0)   | (0, 0) | (0, 1) | (1, 0) | (1, 1) |
| (0, 1)   | (0, 1) | (0, 0) | (1, 1) | (1, 0) |
| (1, 0)   | (1, 0) | (1, 1) | (0, 0) | (0, 1) |
| (1, 1)   | (1, 1) | (1, 0) | (0, 1) | (0, 0) |

Commutation relations are described by the graded bracket of two homogeneous elements  $X \in \mathfrak{g}_{(a,b)}$  and  $Y \in \mathfrak{g}_{(a',b')}$  defined as  $[X, Y] = XY - (-1)^{(a,b) \cdot (a',b')} YX$ , where the scalar product is defined as  $(a, b) \cdot (a', b') = aa' + bb' \pmod{2}$ . This ensures bosonic behavior when the scalar product is even, implying that the bracket is symmetric. The fermionic behavior is obtained when the scalar product is odd; then, the bracket is antisymmetric. To precisely illustrate the implications of the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  grading, we explicitly define the graded commutation relations. Given two graded elements  $x \in \mathcal{A}_{(a,b)}$  and  $y \in \mathcal{A}_{(a',b')}$ , their product respects the grading rule  $xy = (-1)^{aa'+bb'} yx$ . This relation ensures consistency with the algebraic structure, clearly dictating the statistics of paraparticle excitations within this formalism.  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded algebras arise naturally in superconformal mechanics and double-graded SUSY (see Refs. [48–50]).

Majorana's infinite-component wavefunction anticipates the unification of different spin states, indirectly supporting that higher-order statistics (e.g., two-bit parastatistics) yield physically distinguishable observables, thereby challenging the “conventionality” thesis of parastatistics. When paraparticles such as parabosons and parafermions are cast within the Majorana Tower as in Eq. (12), the structure introduces a profound connection between bosonic and fermionic sectors. This Majorana solution enforces that both bosons and fermions fall into mass-degenerate multiplets governed by a common ratio, indicating a form of supersymmetric or presupersymmetric symmetry that Majorana explored even before the full formalism of supersymmetry was developed. Majorana low-energy limits independently from each spin value recover Schrödinger dynamics. In such a scheme, the ordinary distinction between paraboson and parafermion statistics becomes intertwined due to the unified mass-spin structure.

Since particles of different spin share the same mass scale  $m$ , they are likely to be treated as components of a single irreducible representation of an extended symmetry group that includes both spin and internal parastatistics degrees of freedom in a unified representation space to which symmetry constraints are then imposed. The parastatistics of order  $p$  must remain consistent across the Majorana Tower to preserve this multiplet structure. Hence, if a paraboson of spin  $s = 0$  and a parafermion of spin  $s = 1/2$  belong to the same Tower level, their respective parastatistics orders (and algebraic commutation structure) are constrained to coexist without contradiction, implying a graded Lie algebra or a more exotic trilinear superalgebra unifying their statistics.

The trilinear commutation/anticommutation relations for parabosons and parafermions would be indexed not just by mode labels but also by Tower level characterized by a discrete quantum number  $n$  corresponding to the Majorana

mass  $m$  (or equivalently spin  $s$ ), introducing relations such as  $[a_i, [a_j^\dagger, a_k]] = 2\delta_{ij}a_k^n$ , where the Kronecker delta  $\delta_{ij}$  enforces algebraic consistency by constraining the operators within the same mode sector. The superscript  $n$  explicitly denotes the Tower level, differentiating clearly among states with distinct Majorana masses. Furthermore, these relations can be generalized to mixed commutation rules when interactions between parabosonic and parafermionic modes within the same Tower are considered, maintaining internal algebraic closure and coherence.

Physical implications of the Majorana Tower in this scenario are quite evident. This type of embedding implies an extended symmetry principle beyond supersymmetry, where paraparticles form a deeper layer of structure connecting mass and spin in a unified algebraic way. If such a Tower were physical, it could provide a framework for preonic models [51], unification theories, or nonstandard quantum fields, where symmetry unification at the level of mass and spin dictates the allowable statistics and interactions.

To consistently classify the infinite-spin Majorana Tower within a  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded framework, we associate each field component of spin  $s$  to a grading label  $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_2$ , where integer spin values ( $s \in \mathbb{Z}$ ) are mapped to bosoniclike sectors and half-integer spins ( $s \in \mathbb{Z} + 1/2$ ) to fermioniclike sectors. The graded Lie bracket for homogeneous elements  $X \in \mathfrak{g}_{(a,b)}$  and  $Y \in \mathfrak{g}_{(a',b')}$  ensures that components corresponding to integer spins commute symmetrically, while half-integer spin components anticommute, in agreement with the spin-statistics theorem. Moreover, the mass-spin relation intrinsic to the Majorana Tower remains invariant under the grading assignment, since the classification acts on symmetry properties without altering the spin or mass eigenvalues. The graded structure further satisfies closure under commutation and obeys the graded Jacobi identity, thus confirming that the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded assignment provides a coherent, physically faithful, and algebraically consistent classification of the Majorana infinite-spin Tower.

#### A. Embedding Majorana tower particles into the paraparticle framework

To express the Majorana Tower of particles in terms of paraparticles, specifically those arranged in the mass-spin Tower and characterized by the mass-spin relationship in Eq. (12), one must reinterpret their infinite-component field structure within the algebraic language of parastatistics, particularly that of  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded Lie superalgebras.

Majorana's framework naturally lends itself to paraparticle embedding because the wavefunction  $\Psi$  spans an infinite-dimensional Hilbert space accommodating arbitrary spin states, which is structurally similar to the Fock spaces of trilinear parafield algebras. The reinterpretation starts by associating the spin-dependent mass levels with graded sectors of a paraparticle algebra, such that each spin state  $s$  corresponds to a representation in a  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded module. Integer spin states are naturally associated with parabosons and could reside in the (0, 0) and (1, 1) sectors, and half-integer spin particles instead associated with parafermions and encoded in the (1, 0) and (0, 1) sectors.

These sectors obey trilinear (anti)commutation relations and respect exclusion or aggregation rules consistent with

the statistics of the particles. The infinite-component Majorana wavefunction then becomes a superposition over these sectors, and transitions between spin states correspond to algebraic operations involving braided tensor products and graded commutators. Crucially, the graded Hopf algebra framework used in paraparticle quantization provides a formal coproduct structure that mimics Majoranas relativistic treatment of multiparticle states. More details are in the Appendixes B and C.

Furthermore, observables in the  $(0, 0)$  sector (bosonic) of the graded algebra reproduce the effective dynamics and degeneracies predicted by Majorana's theory, particularly in the nonrelativistic limit. Therefore, Majorana particles can be embedded as graded paraparticles, with the Tower structure arising from graded symmetry assignments and algebraic constraints that generalize both their spin and statistical behavior, integrating relativistic covariance and quantum statistics in a unified framework.

The graded algebra multiplication is succinctly summarized by the following projection and exchange rules, ensuring internal algebraic consistency: First  $P_{(a,b)}P_{(a',b')} = \delta_{(a,b),(a',b')}P_{(a,b)}$  involving the projectors  $P_{(a,b)}$ , which bundles idempotency ( $P_{(a,b)}^2 = P_{(a,b)}$ ) when the labels coincide, and orthogonality ( $P_{(a,b)}P_{(a',b')} = 0$ ) when they differ. Second, the relationships ruling the exchange matrices  $X_{(a,b),(a',b')}$  given by  $X_{(a,b),(a',b')}X_{(a'',b''),(a''',b''')} = \delta_{(a',b'),(a'',b'')}X_{(a,b),(a''',b''')}$ . Here, the product is nonvanishing only if the target index pair of the first exchange operator matches the source index pair of the second. When this middle pair coincides, the two-step hop collapses to a single hop from  $(a, b)$  to  $(a''', b''')$ ; otherwise, the composition is null. Together, these rules guarantee a block-matrix structure for any algebra element

$$\begin{aligned} \mathcal{O} &= \sum_{(a,b),(a',b')} (P_{(a,b)} \mathcal{O} P_{(a',b')}) \\ &= \sum_{(a,b),(a',b')} c_{(a,b),(a',b')} X_{(a,b),(a',b')}, \end{aligned} \quad (19)$$

with coefficients  $c_{(a,b),(a',b')} = \langle (a, b) | \mathcal{O} | (a', b') \rangle$ . The projector relation keeps diagonal blocks independent, while the exchange-matrix relation controls off-diagonal transitions, ensuring that graded multiplication never mixes incompatible sectors.

As further step, Majorana's original wave equation for arbitrary spin particles can be written also in the following form:

$$[\gamma^\mu p_\mu - M]\Psi = 0, \quad (20)$$

with  $c = \hbar = 1$  and the mass  $M$  depends on the particle spin from Eq. (12). Positive-energy relativistic infinite-component equations in the Majorana spirit were developed by Dirac [52,53]. The wavefunction  $\Psi$  spans an infinite-dimensional representation space that can be decomposed into graded sectors reflecting parastatistics; specifically,  $(0, 0)$  and  $(1, 1)$  sectors represent parabosons with integer spins and  $(1, 0)$  and  $(0, 1)$  are parafermions, with half-integer spins.

In this case, to define the paraparticle algebra we use trilinear commutation relations for the operators  $a_i$ ,  $a_i^\dagger$  or  $f_i$ ,  $f_i^\dagger$  that indicate parabosons and parafermions, respectively, and obtain the relationships in Eqs. (1) and (2). We then rewrite Majorana's equation as an

operator-valued equation acting on  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded tensor products of state vectors  $\Psi_{(a,b)} \in H_{(a,b)}$ , with  $(a, b) \in \{(0, 0), (0, 1), (1, 0), (1, 1)\}$  and the infinite-component field becomes a graded vector,  $\Psi = \sum \Psi_{(a,b)}$  and  $\deg(\Psi_{(a,b)}) = (a, b)$  for which is satisfied a graded Dirac-like equation

$$[\Gamma_{(a,b)}^\mu \partial_\mu - M_{(a,b)}]\Psi_{(a,b)} = 0, \quad (21)$$

where  $\Gamma_{(a,b)}^\mu$  are graded gamma operators, the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded analogs of gamma matrices, acting inside the corresponding subsector  $\mathfrak{g}_{(a,b)}$ , following the graded commutation relations

$$\Gamma_{(a,b)}^\mu \Gamma_{(c,d)}^\nu + (-1)^{(a,b),(c,d)} \Gamma_{(c,d)}^\nu \Gamma_{(a,b)}^\mu = 2g^{\mu\nu} \delta_{(a,b),(c,d)}, \quad (22)$$

with a Majorana spin-mass term  $M_{(a,b)} = m/(s_{(a,b)} + 1/2)$  associated with each specific graded sector  $(a, b)$ . Instead,  $\delta_{(a,b),(c,d)}$  is the Kronecker delta on grading that is valued 1 if  $(a, b) = (c, d)$ , else 0. Equation (22) implies that the family  $\{\Gamma_{(a,b)}^\mu \mid (a, b) \in \mathcal{I}, \mu = 0, \dots, 3\}$  realizes a graded Clifford algebra.  $\mathcal{I}$  is the index set of all grade labels that appear in the current algebra. The index  $\mu = 0, 1, 2, 3$  refers to the time (0) and spatial (1, 2, 3) spacetime coordinates. Because the Clifford (Dirac) algebra is built from a set of one single fermionic generator,  $\gamma$  matrix, per spacetime dimension, exactly four of them in  $(3+1)\text{D}$  are needed. Extending the present model to  $d$  dimensions, one replaces the previous 4D range to  $\mu = 0, 1, \dots, d-1$  and uses the corresponding  $d \times d$  metric tensor  $g^{\mu\nu}$ .

For a fixed grade  $(a, b)$ , the relation reduces to the standard Dirac algebra, while for different grades the operators either commute or anticommute according to their combined parity. This graded structure is what allows one to describe generalized Majorana paraparticles without sacrificing Lorentz covariance or internal consistency.

Unlike circuit-based simulations that implement paraparticle behavior numerically, our framework introduces a  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded Lie superalgebra structure that can be encoded in a single photon with OAM and SAM states, embedding Majorana's infinite-component wavefunctions directly into a relativistic, algebraic field theory and in structured photonic systems.

For the sake of completeness, we show that  $\Gamma_{(a,b)}^\mu$  furnish a Lorentz-covariant representation [50,54] in the following lemma.

**Definition ( $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded Clifford formulation).** Let  $G = \mathbb{Z}_2 \times \mathbb{Z}_2$  and let  $\chi : G \times G \rightarrow \{\pm 1\}$  be the symmetric bicharacter  $\chi((a, b), (c, d)) = (-1)^{ac+bd}$ . On the  $G$ -graded spinor space  $\mathcal{S}_G = \bigoplus_{g \in G} \mathcal{S}_g$  with central projectors  $P_g \in \mathbb{C}[G]$ , set  $\Gamma_g^\mu := P_g \otimes \gamma^\mu$  and  $\Gamma^\mu := \sum_{g \in G} \Gamma_g^\mu$ . The  $G$ -graded Clifford algebra  $\text{Cl}_G(\eta)$  is the unital algebra generated by the homogeneous  $\{\Gamma_g^\mu\}$  subject to the single sectorwise relation

$$\Gamma_g^\mu \Gamma_h^\nu + \chi(g, h) \Gamma_h^\nu \Gamma_g^\mu = 2\eta^{\mu\nu} \delta_{g,h} P_g. \quad (23)$$

Equivalently, the block-sum satisfies the ordinary Clifford relation  $\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu} \mathbf{1}$ , while different sectors anticommute trivially:  $\{\Gamma_g^\mu, \Gamma_h^\nu\} = 0$  for  $g \neq h$ .

*Physical role.* The projectors  $P_g$  label the four parity sectors of the Majorana tower [e.g.,  $s = 0, \frac{1}{2}, 1, \frac{3}{2} \mapsto (0, 0), (1, 0), (0, 1), (1, 1)$  as used later], and the bicharacter  $\chi$  implements the exchange sign between operators whose actions lie in different sectors. Thus, Eq. (23) packages Lorentz-covariant dynamics (within each sector) together with the graded sign rules needed when composing operations across sectors and, in the photonic realization, across SAM/OAM parity subspaces.

*Lemma 1 (graded-Clifford closure).* Let  $\{\gamma^\mu\}_{\mu=0}^3$  be the Dirac matrices on a spinor space  $\mathcal{S}$ , satisfying  $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbf{1}_{\mathcal{S}}$  with  $\eta = \text{diag}(+, -, -, -)$ . Let  $\{P_{(a,b)}\}$  ( $a, b \in \{0, 1\}$ ) be the four *central idempotents* of the group algebra  $\mathbb{C}[\mathbb{Z}_2 \times \mathbb{Z}_2]$ :

$$P_{(a,b)}P_{(c,d)} = \delta_{ac}\delta_{bd}P_{(a,b)}, \quad \sum_{a,b} P_{(a,b)} = \mathbf{1}, \quad P_{(a,b)}^\dagger = P_{(a,b)}. \quad (24)$$

On the Hilbert space  $\mathcal{H} = (\bigoplus_{a,b} \mathbb{C}_{(a,b)}) \otimes \mathcal{S}$ , define the block-diagonal operators

$$\Gamma_{(a,b)}^\mu := P_{(a,b)} \otimes \gamma^\mu, \quad \Gamma^\mu := \sum_{a,b} \Gamma_{(a,b)}^\mu. \quad (25)$$

Then, set the function  $\{\Gamma^\mu\}$  that furnishes a Hermitian, Lorentz-covariant representation of the Clifford algebra  $\text{Cl}_{1,3}$ :

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}\mathbf{1}_{\mathcal{H}} \quad (26)$$

and, sectorwise,  $\{\Gamma_{(a,b)}^\mu, \Gamma_{(c,d)}^\nu\} = 2\eta^{\mu\nu}\delta_{ac}\delta_{bd}P_{(a,b)}$ .

*Proof.* The proof is articulated in five points for the sake of clarity:

(1) Intrasector anticommutator. For fixed  $(a, b)$ , we have

$$\{\Gamma_{(a,b)}^\mu, \Gamma_{(a,b)}^\nu\} = P_{(a,b)} \otimes \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}P_{(a,b)}. \quad (27)$$

(2) Intersector anticommutator. If  $(a, b) \neq (c, d)$ , then, by Eq. (24),  $P_{(a,b)}P_{(c,d)} = 0$ ; hence,

$$\{\Gamma_{(a,b)}^\mu, \Gamma_{(c,d)}^\nu\} = 0. \quad (28)$$

(3) Global closure. Summing Eq. (27) over all four sectors and using  $\sum_{a,b} P_{(a,b)} = \mathbf{1}$  yields Eq. (26).

(4) Hermiticity. Since each  $P_{(a,b)}$  is Hermitian and  $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$ , every  $\Gamma_{(a,b)}^\mu$  inherits the standard Hermiticity property; consequently, so does  $\Gamma^\mu$ .

(5) Lorentz covariance. Defining  $\Sigma^{\mu\nu} := \frac{i}{4}[\Gamma^\mu, \Gamma^\nu]$ , one finds  $\Sigma^{\mu\nu} = \sum_{a,b} P_{(a,b)} \otimes \sigma^{\mu\nu}$ , reproducing the usual Lorentz generators blockwise.

The Hilbert space, throughout this work, is the graded spinor Hilbert space

$$\mathcal{H} = \left( \bigoplus_{(a,b) \in \{0,1\}^2} \mathbb{C}_{(a,b)} \right) \otimes \mathbb{C}^4, \quad (29)$$

endowed with the inner product  $\langle v, w \rangle_{\mathcal{H}} = \sum_{a,b} \bar{v}_{(a,b)} w_{(a,b)} \psi_{(a,b)}^\dagger \phi_{(a,b)}$ . The first factor keeps track of the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  charge. The second one is the usual Dirac spinor space.

Let  $\mathcal{S} = \mathbb{C}^4$  be the usual Dirac spinor space with the inner product  $\langle \psi, \phi \rangle_{\mathcal{S}} = \psi^\dagger \phi$ . Then, let

$$\mathbb{C}[\mathbb{Z}_2 \times \mathbb{Z}_2] = \bigoplus_{(a,b) \in \{0,1\}^2} \mathbb{C}_{(a,b)} \quad (30)$$

be the group algebra of the Klein four-group. Each one-dimensional summand

$$\mathbb{C}_{(a,b)} = \text{span}\{P_{(a,b)}\} \quad (31)$$

is spanned by a central idempotent  $P_{(a,b)}$  and carries the inner product

$$\langle z_1, z_2 \rangle_{(a,b)} = \bar{z}_1 z_2. \quad (32)$$

Here, the operator “span” means the complex-linear span, the smallest complex vector space that contains the listed element(s). Because only one element is listed, that span is one dimensional with Eq. (31) giving  $\mathbb{C}_{(a,b)} = \{\lambda P_{(a,b)} | \lambda \in \mathbb{C}\}$ . The group algebra  $\mathbb{C}[\text{Klein}]$  of the Klein four-group  $V_4 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$  is abelian and its group algebra decomposes into a direct sum of four one-dimensional, central, pairwise-orthogonal idempotents:

$$\mathbb{C}[\text{Klein}] = \bigoplus_{(a,b) \in \{0,1\}^2} \mathbb{C}_{(a,b)},$$

for which every group-algebra element can be expressed uniquely as a sum  $\sum_{(a,b)} \lambda_{(a,b)} P_{(a,b)}$ . Equation (31) declares that each block in the decomposition is the one-dimensional complex vector space generated by the corresponding central idempotent  $P_{(a,b)}$ .

Then, the Hilbert space for the graded model is defined as the external tensor product:

$$\mathcal{H} = \left( \bigoplus_{a,b} \mathbb{C}_{(a,b)} \right) \otimes \mathcal{S} = \bigoplus_{a,b} (\mathbb{C}_{(a,b)} \otimes \mathbb{C}^4) \quad (33)$$

with inner product

$$\langle v, w \rangle_{\mathcal{H}} = \sum_{a,b} \langle v_{(a,b)}, w_{(a,b)} \rangle_{(a,b)} \langle \psi_{(a,b)}, \phi_{(a,b)} \rangle_{\mathcal{S}}. \quad (34)$$

Here,  $v_{(a,b)} \otimes \psi_{(a,b)}$  is the component of  $v$  in the  $(a, b)$  sector.

With this choice, one ensures from grading information tools that the four one-dimensional subspaces encode the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  charge of the state.  $\mathcal{S}$  carries the usual  $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$  Lorentz representation through spin degrees of freedom. For block-diagonal operators, they respect the grading—e.g.,  $\Gamma_{(a,b)}^\mu$  act diagonally in the first factor and as ordinary Dirac matrices in the second. This feature is precisely what enables the graded-Clifford proof to succeed.

## B. $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded sector decomposition

Each spinor component  $\Psi_s$  is assigned to a  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded sector, with the full field given by  $\Psi = \sum_{(a,b)} \Psi_{(a,b)}$ , where  $(a, b) \in \{(0, 0), (0, 1), (1, 0), (1, 1)\}$  with the usual index correspondence, integer spins (parabosons)  $\rightarrow$  sectors  $(0, 0), (1, 1)$ , and half-integer spins (parafermions)  $\rightarrow$  sectors  $(0, 1), (1, 0)$ . Each component satisfies the Dirac-like equation of Eq. (21).

Transition operators between sectors in this graded particle framework are essential to describe how states or

fields in one graded symmetry class evolve or transform into another. These transitions must respect both the graded algebraic structure and the underlying relativistic invariance of the theory. To achieve this, one constructs the operators using two key mathematical tools such as braided coproducts and graded projectors and exchange matrices ensuring relativistic covariance and  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded symmetry and in an explicit operator form.

Braided coproducts in Hopf algebra language define how single-particle operators extend to multiparticle systems. For paraparticles, this coproduct must be braided, meaning it includes a nontrivial exchange rule governed by a braiding matrix  $R$ . This structure guarantees that tensor products of fields or operators respect the parastatistical exchange rules and follow a deformed symmetry algebra. Explicitly, for an operator  $\hat{\psi}_{(a,b)}$ , the braided coproduct  $\Delta_B$  acts as follows:

$$\Delta_B(\hat{\psi}_{(a,b)}) = \hat{\psi}_{(a,b)} \otimes \mathbb{I} + \sum_{(c,d)} R_{(a,b)}^{(c,d)} (\mathbb{I} \otimes \hat{\psi}_{(c,d)}), \quad (35)$$

ensuring that exchange between particles follows the prescribed parastatistics.

The graded projectors and exchange matrices  $X$  are operators that project a field or state into a specific graded subspace or exchange components between sectors.

The exchange matrix acts on the internal indices (the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  indices) and encodes how sectors transform into one another. This can include sign factors or matrix rotations, depending on whether the sector obeys bosoniclike  $((0,0), (1,1))$  or fermioniclike  $((0,1), (1,0))$  symmetries. We define the exchange operation on tensor products of graded fields via  $\Psi_{(a,b)} \otimes \Psi_{(a',b')} \rightarrow (-1)^{(a,b) \cdot (a',b')} \Psi_{(a',b')} \otimes \Psi_{(a,b)}$ , which is implemented by the exchange matrix acting as a symmetry operator  $X_{(a,b) \cdot (a',b')} = (-1)^{(a,b) \cdot (a',b')}$ , ensuring graded (anti)symmetry in multifield operations.

An exchange operator  $E$  can act as  $E_{(a,b)}^{(c,d)} \Psi_{(c,d)}(x) = \sum_{(e,f)} X_{(a,b)}^{(e,f)} \Psi_{(e,f)}(x)$ , with  $X_{(a,b)}^{(e,f)} \in GL(n)$  the components of a sector-intertwining transformation, ensuring the full field  $\Psi$  remains covariant under both Lorentz transformations and graded algebra automorphisms. In this way, we define explicit operator-valued maps that realize graded symmetry transformations in both the algebra and field content of the theory. This structure is crucial for preserving locality, microcausality, and internal consistency in theories that unify Majorana's spin-mass spectrum with paraparticle statistics.

To define coproducts for the Majorana Tower embedded in a  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded paraparticle framework, we must structure the Tower's components—each labeled by a spinlike index (e.g.,  $s = 0, 1/2, 1, \dots$ ) as sectors in a graded Hopf algebra. Each component field  $\Psi_s$  (or its operator-valued mode  $\hat{\psi}_s$ ) then transforms under this algebra, and the coproduct determines how it acts in multiparticle Fock spaces or tensor product representations. In a braided Hopf algebra with  $\mathbb{Z}_2 \times \mathbb{Z}_2$  grading, the coproduct for an operator  $\hat{\psi}_{(a,b)}$  is modified by the braiding between graded sectors. The coproduct map  $\Delta$  defines how an operator acts on a two-particle space:

$$\Delta(\hat{\psi}_{(a,b)}) = \hat{\psi}_{(a,b)} \otimes \mathbb{I} + \sum_{(c,d)} R_{(a,b)}^{(c,d)} (\mathbb{I} \otimes \hat{\psi}_{(c,d)}). \quad (36)$$

TABLE II. Example of braiding coefficients with the grading maps in Eq. (39).

| $s$ | $s'$ | $(a,b)_s \cdot (a',b')_{s'}$ | $R_s^{s'}$                |
|-----|------|------------------------------|---------------------------|
| 0   | 0    | 0                            | $+1 + \theta_{(0,0)}$     |
| 0   | 1/2  | 0                            | $+1 + \theta_{(0,1/2)}$   |
| 1/2 | 1/2  | 1                            | $-1 + \theta_{(1/2,1/2)}$ |
| 1   | 0    | 0                            | $+1 + \theta_{(1,0)}$     |
| 1/2 | 1    | 1                            | $-1 + \theta_{(1/2,1)}$   |

Here,  $R_{(a,b)}^{(c,d)}$  are the coefficients from the  $R$  matrix satisfying the Yang-Baxter equation, encoding exchange statistics and graded symmetry.

Let  $\hat{\psi}_s$  denote the field operator associated with spin  $s$  and assume the mass/spin relation in Eq. (12). Each  $s$  labels a component in the Majorana Tower and is mapped to a grading  $(a,b) \in \mathbb{Z}_2 \times \mathbb{Z}_2$ . Recalling Eq. (35), the graded coproduct becomes

$$\Delta(\hat{\psi}_s) = \hat{\psi}_s \otimes \mathbb{I} + \sum_{s'} R_s^{s'} (\mathbb{I} \otimes \hat{\psi}_{s'}), \quad (37)$$

which ensures the correct behavior under particle exchange, consistent with parastatistics. The  $R$  coefficients  $R_s^{s'}$  can be constructed based on the parity of spin  $s$ , whether  $s$  maps to a parafermionic or parabosonic sector and the internal symmetry imposed by the graded structure.

To construct the braiding coefficients  $R_s^{s'}$  in a  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded paraparticle framework, we associate with each spin  $s$  a graded label  $(a,b)_s$ . The general form of the braiding coefficient is

$$R_s^{s'} = (-1)^{(a,b)_s \cdot (a',b')_{s'}} \delta_{ss'} + \theta_{ss'}, \quad (38)$$

where  $\theta_{ss'}$  is a deformation parameter and the dot product imposes the appropriate graded symmetry (see Appendix A).

As an example, we can build up a limited set of states using the grading map with the first three spin states of the Majorana tower  $s = \{0, 1/2, 1, 3/2\}$  or an equivalent set of photon OAM Majorana quasiparticles [38] with spin ( $\sigma = \pm 1$ ) and orbital angular momentum  $\ell = \pm 1$  entangled together  $s' = \{(\ell = -1) \otimes (\sigma = -1), (\ell = 0) \otimes (\sigma = -1), (\ell = 0) \otimes (\sigma = +1), \ell = -1 \otimes (\sigma = +1)\}$  that can be encoded in quantum information quantum bit structures from the following assignments as summarized in Table II:

$$\begin{aligned} s = 0 &\rightarrow (0,0) \text{ (paraboson)}, \\ s = 1/2 &\rightarrow (1,0) \text{ (parafermion)}, \\ s = 1 &\rightarrow (0,1) \text{ (parafermion)}, \\ s = 3/2 &\rightarrow (1,1) \text{ (paraboson)}. \end{aligned} \quad (39)$$

Then, we assign the grading labels cyclically, identifying  $s = 2$  with  $(0,0)$  to reflect the periodicity induced by the algebraic relations. This cyclic structure ensures that the grading is compatible across sectors when considering the whole tower modulo the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  structure. In this way, the correct exchange statistics and grading rules are preserved across all components in the Majorana Tower. This correspondence is summarized in Fig. 1. The deformation parameter  $\theta_{ss'}$  encodes deviations from conventional Bose or Fermi statistics in the

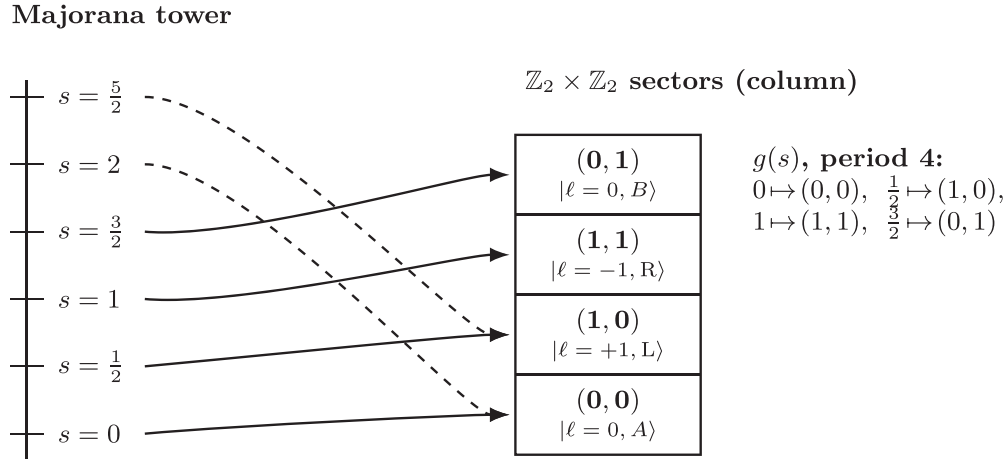


FIG. 1. Correspondence between the first Majorana-tower levels and the four  $\mathbb{Z}_2 \times \mathbb{Z}_2$  sectors using the explicit choice  $g(0)=(0, 0)$ ,  $g(\frac{1}{2}) = (1, 0)$ ,  $g(1) = (1, 1)$ , and  $g(\frac{3}{2}) = (0, 1)$  (then periodic in  $s$ ). The photonic SAM-OAM ququart mapping shown inside the cells matches Table III.

graded  $R$ -matrix formalism. It introduces braiding phases or statistical twist terms that differentiate paraparticle exchange from standard symmetrization rules.

It is important to emphasize that the coefficients  $\theta_{ss'}$  represent algebraic couplings and selection-rule weights, not physical masses, kinetic energies, or direct dynamical interactions. Their role is to encode which sector transitions are symmetry-allowed and how the graded algebra constrains composite operations in the photonic realization. A general expression is given by

$$\theta_{ss'} = \epsilon_{ss'}(q^{s+s'} - 1), \quad (40)$$

where  $\epsilon_{ss'} = (-1)^{2s \cdot 2s'}$  accounts for spin parity and  $q \in \mathbb{C}$  is a deformation parameter such that  $|q| = 1$ . This ensures that when  $q = 1$ , the deformation vanishes, recovering standard (anti)commutation relations. For  $q \neq 1$ ,  $\theta_{ss'}$  captures the statistical phase shift or exclusion deformation, thus enabling a continuous interpolation between classical parastatistics and topologically twisted quantum field theories. When  $q = 1$  so  $\epsilon_{ss'} = 0$ , one recovers the Green (parastatistics) limit.

### C. Graded projectors and exchange matrices $X_{(a,b)}$ for the Majorana tower

To implement transitions and symmetry-preserving operations between different  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded sectors in the Majorana Tower, we define graded projectors and exchange

TABLE III. Definition of the grade sectors for OAM-SAM ququart of Eq. (79). *Grade-changing gates:* Any  $\sigma$  flip on the  $\ell = 0$  pair  $\{|0, A\rangle, (0, 0)\}; \{|0, B\rangle, (0, 1)\}$  crosses the  $b$  grading. Throughout the ququart example, we assume the compensated sequence above unless a gate is explicitly labeled “ $\sigma$  flip (grade changing).”  $\sigma = +1$  for LCP and  $\sigma = -1$  for RCP.

| Mode       | OAM mode $\ell$ | $\sigma$ (SAM) | Grade $(a, b)$ |
|------------|-----------------|----------------|----------------|
| $\phi_1$   | $\ell = +1$     | +1 (LCP)       | (1, 0)         |
| $\phi_2$   | $\ell = -1$     | -1 (RCP)       | (1, 1)         |
| $\phi_3^+$ | $\ell = 0$      | +1 (A)         | (0, 0)         |
| $\phi_3^-$ | $\ell = 0$      | -1 (B)         | (0, 1)         |

matrices  $X_{(a,b)}$ . Each sector  $(a, b)$  corresponds to a unique combination of grading parity, typically associated with spin-like labels  $s$  via a map  $s \mapsto (a, b)_s$ .

We now define the graded projectors  $\mathcal{P}_{(a,b)}$ . Consider then  $\mathcal{P}_{(a,b)}$  that isolate fields or operators in a specific graded sector  $(a, b)$ ; then one writes  $\mathcal{P}_{(a,b)}\Psi = \delta_{(a,b),(a',b')}\Psi_{(a',b')}$ , where  $\Psi = \sum_{(a,b)} \Psi_{(a,b)}$  is a decomposition of the full field into its graded components. These projectors also satisfy the following condition,  $\mathcal{P}_{(a,b)}\mathcal{P}_{(a',b')} = \delta_{(a,b),(a',b')}\mathcal{P}_{(a,b)}$ , with  $\sum_{(a,b)} \mathcal{P}_{(a,b)} = \mathbb{I}$ .

The exchange matrix  $X_{(a,b)}$ , instead, encodes the graded transformation properties between sectors  $(a, b)$  and  $(a', b')$ . It acts on the graded tensor space via the following relationship:

$$X_{(a,b),(a',b')}\Psi_{(a,b)} \otimes \Psi_{(a',b')} = (-1)^{(a,b) \cdot (a',b')}\Psi_{(a',b')} \otimes \Psi_{(a,b)}, \quad (41)$$

with the graded dot product defined as  $(a, b) \cdot (a', b') = aa' + bb' \bmod 2$ . This ensures that exchanges between sectors with even total grading parity yield  $X_{(a,b)} = +1$  (bosoniclike behavior), while exchanges between sectors with odd total grading parity yield  $X_{(a,b)} = -1$  (fermioniclike behavior). In other words, the exchange matrix satisfies  $X_{(ab,a'b')} = (-1)^{(a,b) \cdot (a',b')}$ .

An example of such matrix is obtained for  $(a, b) \in \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ , denoting sectors assigned via the spin map  $s = \{0, 1/2, 1, 3/2\}$ , respectively. Then, the exchange matrix  $X_{(a,b)}$  can be written as follows:

$$X_{(a,b)} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad (42)$$

where rows and columns follow the order  $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$ . This matrix governs the signs encountered when exchanging graded field components in the Majorana Tower and ensures consistency with graded symmetry and parastatistics.

In the original infinite-component formalism introduced by Majorana, the relativistic field  $\Psi$  is expressed as an

infinite-dimensional spinor, where each component  $\Psi_s$  corresponds to a definite spin  $s$  of the spin map. When this Tower is embedded into a  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded paraparticle framework, each spin  $s$  is mapped to a grading label  $(a, b)_s$ , allowing the use of graded projectors and exchange matrices to manage sector-specific dynamics.

The full field then becomes  $\Psi = \sum_{s \in \mathbb{N}/2} \Psi_s = \sum_{(a,b)} \Psi_{(a,b)}$ , where  $\Psi_{(a,b)}$  denotes the component in the graded sector  $(a, b)$ . The projectors  $\mathcal{P}_{(a,b)}$  isolate each sector as  $\mathcal{P}_{(a,b)}\Psi = \sum_{s:(a,b)_s=(a,b)} \Psi_s$ , satisfying the orthogonality and completeness relations  $\mathcal{P}_{(a,b)}\mathcal{P}_{(c,d)} = \delta_{(a,b),(c,d)}\mathcal{P}_{(a,b)}$  and  $\sum_{(a,b)} \mathcal{P}_{(a,b)} = \mathbb{I}$ .

When two graded components  $\Psi_s$  and  $\Psi_{s'}$  interact or are exchanged, their transformation is governed by an exchange matrix  $X_{(a,b),(a',b')}$  acting as  $\Psi_s \otimes \Psi_{s'} \mapsto X_{(a,b),(a',b')} \Psi_{s'} \otimes \Psi_s$ , where  $X_{(a,b),(a',b')} = (-1)^{(a,b) \cdot (a',b')}$  is the graded parity factor. This defines how tensor products and interactions behave under sector exchanges, ensuring that bosonic and fermionic statistics are preserved or appropriately generalized.

We define the field as a direct sum over graded sectors  $\Psi = \sum_{(a,b)} \Psi_{(a,b)}$ , where  $\mathcal{P}_{(a,b)}\Psi = \Psi_{(a,b)}$ . Each component satisfies its own Majorana Dirac-like equation  $[\Gamma_{(a,b)}^\mu \partial_\mu - m/(s_{(a,b)} + 1/2)]\Psi_{(a,b)} = 0$ .

In the wave equation, the infinite-component Majorana field equation  $(\Gamma^\mu \partial_\mu - M\Sigma)\Psi = 0$  becomes a graded block structure. Specifically, it decomposes as

$$\left[ \sum_{(a,b)} \Gamma_{(a,b)}^\mu \partial_\mu \mathcal{P}_{(a,b)} - \sum_{(a,b)} M_{(a,b)} \mathcal{P}_{(a,b)} \right] \Psi = 0, \quad (43)$$

with each block obeying a sector-specific Dirac-like equation of the form  $[\Gamma_{(a,b)}^\mu \partial_\mu - M_{(a,b)}]\Psi_{(a,b)} = 0$ . The graded projectors isolate spin-dependent dynamics, while exchange matrices enforce braided symmetry between sectors. Together, they ensure the consistency of relativistic covariance, statistical symmetry, and graded quantum algebra in the Majorana Tower.

The Majorana's variational principle with graded Hopf algebra tools with action  $S$  gives the full equation over the graded field space, which becomes

$$\delta S = \delta \int d^4x \bar{\Psi}_{(a,b)} (\Gamma_{(a,b)}^\mu \partial_\mu + M_{(a,b)}) \Psi_{(a,b)} = 0, \quad (44)$$

with observables restricted to the  $(0, 0)$  sector to ensure hermiticity and measurable quantities. Thus, Majorana's equation in paraparticle form becomes a graded relativistic wave equation system, each component governed by a different trilinear algebra, yet unified by the mass-spin ratio and symmetry algebra. This formulation preserves covariance, accommodates arbitrary spin, and reveals a deeper algebraic structure underlying the spin-statistics connection. This aligns with the second quantization formalism introduced by Wang and Hazzard, where paraparticle creation and annihilation operators  $\hat{\psi}_{(i,a)}^\pm$  satisfy  $R$ -matrix-encoded trilinear commutation rules  $\hat{\psi}_{(i,a)}^\pm \hat{\psi}_{(j,b)}^\pm \pm R_{ab}^{cd} \hat{\psi}_{(j,c)}^\pm \hat{\psi}_{(i,d)}^\pm = 0$ , for which the symbol related only to  $\hat{\psi}$ ; “ $\pm$ ” here refer to creation/annihilation operators in general (instead of  $a^\dagger$  for the creation and  $a$  annihilation usual Dirac operators).

The Hamiltonian governing such fields becomes bilinear in the operators:

$$\hat{H} = \sum_{i,j,a} h_{ij} \hat{\psi}_{(i,a)}^+ \hat{\psi}_{(j,a)}^-, \quad (45)$$

with  $h_{ij} = h_{ji}^*$  to ensure Hermiticity. This Hamiltonian is diagonalizable using one or more canonical transformations of the paraparticle modes. This formulation yields distinct thermodynamic behavior from standard bosons or fermions due to the generalized exclusion rules; e.g., a single parafermion mode of order  $p$  can host at most  $p$  quanta, while a paraboson mode retains unlimited occupancy but with modified  $g^{(k)}$  correlations or admits only one paraparticle regardless of label.

Such behaviors reproduce Majorana's predicted mass-degenerate spin Towers, or spin tower with inverse mass spacing, while the graded structure accounts for algebraic distinctions among sectors. Hence, the explicit reformulation of Majorana's equation in a paraparticle setting connects infinite-spin relativistic fields to generalized quantum statistics, embedding relativistic covariance, spin symmetry, and parastatistics into a unified operator algebra framework.

Local Green components and vanishing spacelike brackets are so defined. For each Green index  $a = 1, \dots, p$ , the component field  $\phi_a(x)$  transforms in an irreducible  $(j, 0) \oplus (0, j)$  Lorentz representation and obeys the *standard* local (anti)commutation rule,

$$[\phi_a(x), \phi_a(y)]_\pm = 0, \quad \text{for } (x - y)^2 < 0, \quad (46)$$

where the upper sign is chosen for integer spin and the lower sign for half-integer spin. Summing over the internal Green index  $a$ , with  $+$  referred to parabosons and  $-$  to parafermions,

$$[\Phi(x), \Phi(y)]_\pm = \sum_{a=1}^p [\phi_a(x), \phi_a(y)]_\pm = 0, \quad (47)$$

with the condition for which  $(x - y)^2 < 0$  is satisfied, so the full paraparticle field  $\Phi(x)$  remains *local* in the Wightman sense. Equations (46) and (47) are the Greenberg-Messiah construction rewritten for our tower.

A well-defined CPT operator is guaranteed by the Dell'Antonio-Greenberg-Sudarshan theorem through locality and a positive spectrum holding for every component. An explicit realization is

$$\Theta \phi_a(t, \mathbf{x}) \Theta^{-1} = \eta \gamma^5 C \bar{\phi}_a(-t, \mathbf{x})^T, \quad (48)$$

with a phase  $\eta$ ,  $C$  is the charge-conjugation matrix, and where  $\Theta$  is antiunitary:  $\Theta i \Theta^{-1} = -i$  and  $\Theta P_\mu \Theta^{-1} = -P_\mu$ . Extending Eq. (48) linearly to  $\Phi(x)$  preserves Eqs. (46) and (47); hence, our Lorentz-covariant paraparticle tower satisfies microcausality and CPT invariance for all Green orders  $p$ . Obeying Lorentz covariance, fields transform in irreducible  $\text{SL}(2, C)$  representations and microcausality-field operators (or suitable graded versions of them) vanish outside the light cone. Microcausality-field operators (or suitable graded versions of them) vanish outside the light cone,  $[\phi_a(x), \phi_a(y)]_\pm = 0$  for  $(x - y)^2 < 0$ . Positive-energy vacuum with a stable ground state is ensured by the Majorana conditions satisfying Pauli-Lüders CPT under complete

Lorentz covariance. For relativistic fields, the only consistent assignments are (integer spin)  $\leftrightarrow$  parabosonic, (half-integer spin)  $\leftrightarrow$  parafermionic [55].

Let  $\Psi$  be an infinite-component spinor field  $\Psi = (\Psi_0, \Psi_{1/2}, \Psi_1, \Psi_{3/2}, \dots)^T$ . Each component  $\Psi_s$  has spin  $s$  and satisfies a Dirac-like equation. The general form of Majorana's wave equation is then  $(\Gamma^\mu \partial_\mu - M\Sigma)\Psi = 0$ , where  $\Gamma^\mu$  are infinite-dimensional generalizations of gamma matrices and  $\Sigma$  is a block-diagonal operator whose entries encode the spin-dependent mass  $m_s$ . Now one can assign each function  $\Psi_s$  to a sector of a  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded Lie superalgebra with paraboson statistics with integer spin  $s$  in the graded sector  $(0, 0)$ ,  $(1, 1)$  and half-integer spin for parafermions in graded sector  $(1, 0)$  and  $(0, 1)$ . In this way is defined the function  $\Psi = \sum_{(a,b)} \Psi_{(a,b)}$  for which is valid the usual relationship  $(a, b) \in \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ , each satisfying a graded Dirac-like equation in Eq. (21) with the graded commutation relations of Eq. (22).

In the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded framework, the braided coproduct structure introduced for the Majorana-paraparticle embedding naturally satisfies a generalized Yang-Baxter equation. For homogeneous operators  $\psi_{(a,b)}$  and  $\psi_{(c,d)}$  assigned to graded sectors, the braided coproduct is known to be defined as  $\Delta(\psi_{(a,b)}) = \psi_{(a,b)} \otimes \mathbb{I} + \sum_{(c,d)} R_{(a,b)}^{(c,d)} (\mathbb{I} \otimes \psi_{(c,d)})$ , where  $R_{(a,b)}^{(c,d)}$  denotes the braiding matrix elements encoding graded exchange symmetries. The generalized Yang-Baxter equation for the braiding matrix  $R$  reads  $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ , where  $R_{ij}$  acts on the  $i$ th and  $j$ th spaces of a threefold tensor product. In our construction, the  $R$ -matrix elements are explicitly functions of the grading scalar product, namely,

$R_{(a,b)}^{(c,d)} = (-1)^{(a,b) \cdot (c,d)} \delta_{(a,b)}^{(c,d)} + \theta_{(a,b),(c,d)}$ , with deformation parameter  $\theta_{(a,b),(c,d)}$  depending on a statistical phase  $q$  and spin-parity factors. Substituting this form into the Yang-Baxter identity and using the bilinearity of the scalar product modulo 2, one finds that the signs and phase deformations consistently match on both sides of the identity, thereby verifying that the braided coproduct structure defined here satisfies the generalized Yang-Baxter equation required for algebraic consistency of paraparticle fields. This guarantees that multiparticle states and operator compositions within the graded Majorana Tower respect coherent braiding and associativity properties, essential for physical realizations in structured light and quantum computation platforms. More details are in the Appendix.

Recent experimental progress has significantly strengthened the prospects for realizing parafermionic excitations in condensed matter systems. In particular, fractional quantum Hall edge states proximitized by superconductors have been proposed as promising hosts for non-Abelian anyons beyond Majorana fermions [56] (for the theoretical origin of fractional statistics in 2D, see Refs. [15–17]), and signatures of charge fractionalization reported in engineered Tomonaga-Luttinger liquids [57] and other recent experimental efforts [58–60] will be crucial in bridging the gap between theoretical predictions and their physical realization. The continued development of hybrid nanostructures, precision interferometry, and correlated edge state engineering promises to open avenues for observing and manipulating paraparticle modes in the near future.

#### IV. QUANTUM COMPUTING WITH OAM PHOTONS AND PARAPARTICLE FORMULATION

Let us now extend qubits and qudits properties with parastatistics. To this aim, we propose as example a photonic realization of paraparticle algebra using structured-light modes. Having established the algebraic and theoretical framework, we now explore its concrete experimental realization. Structured photonic systems, particularly those employing OAM modes, provide an ideal testbed for implementing the proposed paraparticle algebra and graded symmetries.

We start with the set of solutions of the Dirac-Majorana equation with infinite spin (the Majorana Tower)—and the correlated finite subsets of it [33]—in terms of paraparticle graded algebras for an application to quantum computing, discussing a couple of examples for a photonic realization, deriving selection rules, and giving theoretical examples of logical gates for quantum circuits. While previous studies explored paraparticle statistics theoretically or through digital quantum simulation, a direct photonic realization of graded paraparticle algebras has remained largely unexplored. Our approach bridges this divide by proposing a scalable either discrete or continuous-variable platform based on structured-light modes, offering a route to simulate exotic quantum statistics and implement paraparticle-inspired logical operations in photonic circuits with qudits. The continuous variable derives from an unbounded discrete set of integer values of the whole infinite Majorana Tower that represent excitations in a continuous-degree-of-freedom system, namely, the phase structure of the photonic wave front and the continuous rotation symmetry of the transverse optical field. Because OAM lives in a continuous Hilbert space [the representation space of the  $SO(2)$  or  $SU(2)$  rotation group], OAM platforms potentially can belong to the continuous-variable class of quantum systems as opposed to discrete-variable (DV) systems like qubits or qudits. Pure OAM states like those represented by Laguerre-Gaussian beams have a spectrum that is a numerably infinite set with  $\ell \in \mathbb{Z}$ , a discrete set modulo  $2\pi\ell$ , not continuum. The coherent superposition of OAM beams can give instead a continuous spectrum [61], recalling the building of continuum from natural number with Dedekind cuts [62].

While the full OAM Hilbert space constitutes an infinite-dimensional system often classified under continuous-variable quantum optics, truncating the Majorana Tower to a finite set of modes such as  $\ell = 0, \pm 1$ , it results in an effectively DV system when coherent superposition is fully exploited with continuous  $SU(2)$  transformations coupled with complex phase terms. In this regime, the photonic platform behaves as a qudit, and the paraparticle algebra is realized over a finite-dimensional DV Hilbert space rather than exploiting the full continuous angular momentum structure. This finite realization of the Majorana-paraparticle algebra can be built in an optical quantum circuit with the three OAM modes  $\ell = 0, \pm 1$  combined either with two polarization states of light indicated by, e.g.,  $\sigma = \pm 1$  or with the orthogonal physical waveguide modes characteristic of the optical circuit, here labeled  $A$  and  $B$  that can be the two orthogonal TE-like mode profiles of the  $\ell = 0$  manifold. In a rectangular waveguide, SAM can be coupled with OAM modes, such as  $\ell = 1$  with  $\sigma = 1$ , and

vice versa and for  $\ell = 0$ , either the two polarization modes  $\sigma = \pm 1$  or the two orthogonal waveguide modes  $A$  and  $B$ , depending on the geometry of the optical quantum circuit.

By mapping these four modes to the truncated spin sector of the Majorana Tower such as  $s = 1/2, 1, 3/2, 2$ , we obtain a minimal four-level system suitable for implementing a paraparticle-inspired ququart. While this setup does not capture the full infinite Majorana tower, it provides a physically realizable platform for simulating the essential graded algebraic structure and paraparticles using structured light.

Quasiparticles and paraparticles find deep analogies with phenomena hypothesized and then observed physical phenomena. An example is photons carrying OAM can manifest behaviors analogous to Majorana's mass-spin relationship [38]. Hypergeometric beams with OAM exhibit a group velocity  $v_g$  that is apparently less than the speed of light  $c$  in vacuum. This phenomenon arises due to the beam's geometry, leading to a projection effect where the measured group velocity along the propagation axis appears reduced. In this regime, the axial group velocity  $v_g$  follows a relationship analogous to Majorana's mass-spin formula, suggesting that the beam's OAM influences its effective propagation characteristics; the OAM modes  $\ell$  label different quantum states with distinct effective velocities or masslike quantities, forming a discrete spectrum analogous to spin  $s$  in Majorana's mass-spin relation, with perspectives in quantum computing (QC). While the reduced axial group velocity  $v_g < c$  in hypergeometric beams mimics an effective  $M \propto (\ell + \frac{1}{2})^{-1}$  scaling, no real rest mass is generated; the effect is purely geometrical and disappears in a comoving frame.

A similar thing occurs to OAM photons propagating through a resonant plasma [63,64] in which they acquire a Proca mass that depends on the OAM value. From a paraparticle perspective, this maps neatly onto systems where OAM-dependent excitations respect a mass spectrum indexed by internal quantum numbers and follow nonstandard exchange statistics. Thus, paraparticle fields could be constructed where creation/annihilation operators act on OAM modes and obey the generalized commutation relations of parastatistics, possibly with  $R$  matrices that encode angular momentum coupling rules.

These results open a path to interpreting photonic states with OAM as physical realizations of Majorana pseudoparticles that now we integrate with paraparticle theory. In the paraparticle framework, quanta obey generalized statistics governed by trilinear  $R$ -matrix commutation relations and  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded algebras. If each OAM mode is treated as a distinct species of quasiparticle, these can be embedded into graded sectors (e.g., OAM-even in bosonic sectors, OAM-odd in fermionic or parafermionic sectors), mimicking the statistical structure of parabosons and parafermions. Paraparticle algebra and  $R$ -matrix formalism are defined in terms of creation/annihilation operators  $\hat{\psi}_{(i,a)}^\pm$  that satisfy trilinear relations with  $R$  matrices,

$$\begin{aligned}\hat{\psi}_{(i,a)}^+ \hat{\psi}_{(j,b)}^+ &= \sum_{(c,d)} R_{(a,b)}^{(c,d)} \hat{\psi}_{(j,c)}^+ \hat{\psi}_{(i,d)}^+, \\ \hat{\psi}_{(i,a)}^- \hat{\psi}_{(j,b)}^- &= \sum_{(c,d)} R_{(b,a)}^{(d,c)} \hat{\psi}_{(j,c)}^- \hat{\psi}_{(i,d)}^-, \end{aligned}$$

$$\hat{\psi}_{(i,a)}^- \hat{\psi}_{(j,b)}^+ = \delta_{ij} \delta_{ab} - \sum_{(c,d)} R_{(a,b)}^{(c,d)} \hat{\psi}_{(j,c)}^+ \hat{\psi}_{(i,d)}^-, \quad (49)$$

including also

$$[\hat{\psi}_{(a,b)}^-, [\hat{\psi}_{(a',b')}^+, \hat{\psi}_{(a'',b'')}^-]] = \sum_{(c,d)} f_{(a,b),(a',b'),(a'',b'')}^{(c,d)} \hat{\psi}_{(c,d)}^-, \quad (50)$$

where the structure constants  $f^{(c,d)}$  derive from the associativity properties of the trilinear  $R$ -matrix algebra, following Govorkov's construction [19].

The Hamiltonian written in terms of these operators is then defined as

$$\hat{H} = \sum_{i,j,a} h_{ij} \hat{\psi}_{(i,a)}^+ \hat{\psi}_{(j,a)}^-, \quad (51)$$

which diagonalizes to

$$\hat{H} = \sum_k \epsilon_k \hat{n}_k, \quad \hat{n}_k = \hat{\psi}_{(k,a)}^+ \hat{\psi}_{(k,a)}^-, \quad (52)$$

with the spectrum that obeys the Majorana spin/mass condition  $\epsilon_k = m/(s_k + 1/2)$ . All this remains valid also for a finite subset of the Majorana Tower with a limited string of particle states.

### Paraparticles OAM photonic quasiparticles and Majorana states in quantum circuits

Having established the algebraic framework, we now turn to its application in photonic systems, where structured-light modes provide a natural testbed for paraparticle-inspired quantum logic. Recent research has explored the simulation of paraparticles in quantum information processing for physical phenomena like topological phases of matter. In particular, a study by Alderete *et al.* [65] demonstrated the simulation of paraparticle oscillators on a trapped-ion quantum computer whose Hamiltonian has been identified as an  $XY$  model. This can be translated in OAM photonic gates, encoding them as paraparticle states with the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded algebras and related formulations beyond the usual quantum photonic languages and standard circuit implementations.

The  $XY$  model is, in effect, a quantum spin chain with nearest-neighbor interactions only in the  $x$  and  $y$  components of the Pauli spin matrices  $\sigma^x, \sigma^y$ , with a parameter  $J$  that controls the strength of these interactions.

By mapping paraparticle states onto qubit registers, the authors successfully reproduced the dynamics of even-order parabosons, illustrating that paraparticle behavior can be digitally simulated on existing quantum hardware. Furthermore, in Refs. [7,8], it was provided a theoretical framework where paraparticles emerge naturally in solvable quantum spin models, suggesting their utility in developing robust qudit-based quantum architectures. These results indicate that paraparticles, with their generalized exchange statistics, could enhance error correction and information density in quantum computing systems. The one-dimensional quantum  $XY$  model is described by

$$H_{XY} = -J \sum_n (\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y), \quad (53)$$

where  $\sigma_n^x, \sigma_n^y$  are Pauli matrices at site  $n$  and  $J$  is the exchange coupling constant. The parameter  $J$  determines the strength and nature of the interaction. If  $J > 0$ , the system favors ferromagnetic alignment (neighboring spins tend to align). If  $J < 0$ , the system favors antiferromagnetic alignment (neighboring spins tend to antialign). Large values of  $|J|$  are related to stronger coupling between spins and typically a larger energy gap between ground and excited states. Equation (53) maps via Jordan-Wigner transformation to a system of massless fermions, which in the continuum limit is governed by a conformal field theory in  $1 + 1$  dimensions. The reduced density matrix of a half-infinite chain yields an entanglement Hamiltonian  $H_E = 2\pi \int_0^\infty x T_{(0,0)}(x) dx$ , equivalent to the modular Hamiltonian of a Rindler wedge, where  $T_{(0,0)}(x)$  is the energy density.

A fundamental point is that  $XY$  models can put a bridge between paraparticles with their parastatistics, including quantum computation to the basic fundamental mathematical concepts, indicating that the Majorana Tower either with Majorana particles or OAM photon quasiparticles can be written in terms of paraparticles and their statistics and graded algebras used for quantum computing.

## V. OAM PHOTONIC QUDITS AND PARASTATISTICS IN QUANTUM CIRCUITS

With this mathematical and algebraic formulation, one can implement a formalism based on paraparticle states for quantum computing in many different platforms. As already shown in the case of superconducting qubits, we now consider to expand this to different platforms and take as example room-temperature photonic quantum computing in configurations with OAM and polarization states that behave as sets of Majorana quasiparticles [38]. Photonic quantum computing provides a promising platform for scalable and low-decoherence quantum room-temperature information processing, particularly when leveraging structured-light modes such as orbital angular momentum. OAM modes, characterized by a helical phase structure and an unbounded Hilbert space, enable the realization of high-dimensional qudits, which naturally align with paraparticle frameworks that extend beyond conventional qubit systems. By encoding paraparticle states in superpositions of OAM modes, one can exploit the graded exchange symmetries intrinsic to paraparticles, leading to enriched entanglement structures and robust logical operations. This approach not only enhances information density but also offers ways to implement deterministic gates encoded into a single photon and implement parastatistics and other types of exotic quantum statistics in a photonic medium.

### A. Deterministic single-photon two-qubit gates

This mathematical formalism permits the construction of deterministic two- (or more) qubit deterministic gates that can be used to implement a controlled-NOT (CNOT) gate, which is one of the milestones in quantum computing.

Usually to have two qubits, one has to double the qubit state in different ways (path and other methods) and this mechanism works in a stochastic way unless making two photons interacting, e.g., in nonlinear media. A fundamen-

tal limitation of passive linear optics is that linear-optical elements alone (beam splitters/phase shifters plus photodetection) do not provide deterministic photon-photon interactions; hence, a deterministic universal entangling gate set requires additional resources (ancillas and feedforward) or measurement-induced effective nonlinearities.

To break this constraint, indistinguishability and measure are exploited: projecting some modes and applying classic corrections introduces an effective interaction that can be moved offline. An example is a gate via teleportation-based as in the Knill–Laflamme–Milburn (KLM). The Knill–Laflamme–Milburn protocol provides for offline preparation of a special entangled state (e.g.,  $|C_S\rangle$  of 4–8 photons). Then, one performs a Bell-state measurement between the data qubits and half of the resource applying optoelectronic feedforward ( $\pi$  phase or Pauli  $X$ ) on the rest of the photons. If the resource is ready, the CZ (or CNOT in dual-rail coding) always takes place on the logic qubits; the *alea* is transferred in the preparation phase, which can be repeated until it is successful, thus making the gate deterministic at a logical level [66]. Another way is the quasideterministic gate with few ancilla by Ralph and coworkers [67], which require a single ancilla photon plus photon-counting detectors using photon-number-resolving detectors (PNRDs) and quantum nondemolition (QND) measurements based on weak measurement. The single success is  $P \approx 0.25$ , but by chaining two attempts in parallel and using feedforward one brings  $P \rightarrow 1$  with linear overhead for each single gate. Alternatively, in measurement-based quantum computing (cluster-state QC), universality is obtained by preparing photonic cluster states offline using probabilistic fusion operations and then performing adaptive single-qubit measurements. Once the graph is available, each logical CZ is obtained with simple single-qubit measurements and classical corrections and again operational determinism at the expense of multiple ancilla photons. Recent demonstrations integrate delay-loop and quantum-dot sources for linear clusters of six photons on chips [68].

With linear optics alone, direct entangling is impossible, but it can be made deterministic at the logical level thanks to the methods already discussed whose performance today (2025) is limited mainly by losses and efficiency of the sources, not by switching times: with chips with a loss  $< 1$  dB and deterministic sources, the error threshold of the surface code is expected to be within demonstration in the next five years.

### B. Deterministic one-photon two-qubit deterministic gates in photonic platforms

A proof-of-principle optical circuit implementing grade-resolved spinorbit conversion has been disclosed in Refs. [69,70]. It exploits OAM and SAM encoding and, in some implementations, SAM–OAM classical entanglement [71], enabling deterministic single-photon logic without relying on optical nonlinearity or probabilistic two-photon interactions.

Single-photon two-qubit gates are realized when the two qubits are hosted by the same photon (e.g.,  $|\text{LCP/RCP}\rangle \otimes |\ell = \pm 1\rangle$ ), where LCP and RCP are the left- and right-circular polarizations. No interaction between photons is needed: linear-optics elements that couple the two degrees

of freedom imply that a deterministic unitary gate ( $p = 1$ ) is realized, an intraphoton CNOT (or CZ) that correlates polarization and OAM. Useful for preparing cluster states (e.g., the proposed 2D and 3D cluster states [67]), this architecture combines deterministic single-photon two-qubit operations with probabilistic fusion steps to scale up entanglement. Thus, unlike conventional linear-optics approaches—where two-qubit gates are inherently probabilistic, demand bulky interferometers, and scale poorly because they need extra ancilla photons and active feedforward—a SAM  $\otimes$  OAM platform delivers deterministic, on-chip gates within a single photon, eliminating postselection altogether. There is no intermediate measurement or probability of success: the gate is unit SU(4) performed with linear optics + controlled phase. Any leaks appear only as a channel error and not as gate failure.

The key advantages of this architecture is its modularity and a deterministic success probability ( $P = 1$ ) without postselection and the fact that the operation is intraphoton (it does not entangle distinct photons), so losses primarily affect transmission rather than gate success. For multiqubit algorithms, it is necessary to duplicate the degrees of freedom or use probabilistic fusion gates or higher OAM modes and OAM can propagate in multimode waveguides.  $Q$ -plates [72] and related spinorbit conversion setups (e.g., Refs. [69,70]) coherently couple SAM and OAM while conserving total angular momentum  $J$  (a Noether invariant).

### C. Isomorphism between $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_2^n \times \mathbb{Z}_2^n$ for high $\ell$ OAM values

For high values of the  $\ell$  OAM modes, we can use an alternative decomposition based on  $\mathbb{Z}_2^n \times \mathbb{Z}_2^n$  linked to the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -based formalism for qudits of dimension  $2^{n+1}$  and obtain the known OAM relationship with the product tensor describing the value  $2n + 1$  with  $n = |\ell|$ . Let  $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}_2^n$  and  $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{Z}_2^n$ . In this way, a generic element of the direct product is the pair  $(\mathbf{a}, \mathbf{b}) \in \mathbb{Z}_2^n \times \mathbb{Z}_2^n$ . Addition is always componentwise modulo two. We define a componentwise isomorphism through the map:

$$\Phi : \mathbb{Z}_2^n \times \mathbb{Z}_2^n \longrightarrow \prod_{i=1}^n (\mathbb{Z}_2 \times \mathbb{Z}_2),$$

$$\Phi(\mathbf{a}, \mathbf{b}) = ((a_1, b_1), (a_2, b_2), \dots, (a_n, b_n)), \quad (54)$$

where  $\Phi$  is bijective; i.e., each  $n$ -tuple of pairs has a unique preimage. Then  $\varphi$  is a group homomorphism  $\varphi : G \rightarrow H$ : for all  $(a, b), (a', b') \in \mathbb{Z}_2^n \times \mathbb{Z}_2^n$ ,  $\varphi((a, b) + (a', b')) = \varphi(a, b) + \varphi(a', b')$  (addition understood componentwise mod 2). For a word  $W = g_{i_1} \dots g_{i_k} \in G$ , the homomorphism acts multiplicatively:  $\varphi(w) = \varphi(g_{i_1}) \dots \varphi(g_{i_k})$ .

Hence, from this, one obtains

$$\mathbb{Z}_2^n \times \mathbb{Z}_2^n \cong \underbrace{(\mathbb{Z}_2 \times \mathbb{Z}_2) \times \dots \times (\mathbb{Z}_2 \times \mathbb{Z}_2)}_{n \text{ copies}}. \quad (55)$$

The physical interpretation for para-Majorana ququart blocks is then built as follows: for each index  $i$ , the ordered pair  $(a_i, b_i) \in \mathbb{Z}_2 \times \mathbb{Z}_2$  labels one independent ququart sector (e.g., helicity/OAM combinations of a structured photon). Enlarging  $n$  simply stacks these ququart blocks, while higher-dimensional para-Majorana codes are obtained by their braided tensor product  $V^{(1)} \otimes_R V^{(2)} \otimes_R \dots \otimes_R V^{(n)}$ , with

$R$  the braiding matrix.  $\mathbb{Z}_2^n \times \mathbb{Z}_2^n \cong \prod_{i=1}^n (\mathbb{Z}_2 \times \mathbb{Z}_2)$ , which means to have one ququart per index, tensor-braided for scaling.

In the simplest case, where OAM  $\ell = 0, \pm 1$  and polarization  $\sigma = \pm 1$  (related to SAM), one can and encode and manipulate a four-qudit state, ququart, or two qubits on a single photon. We now will focus on this particular case. The copropagation of spin and orbital modes in a submillimetre silicon trench guarantees phase stability that would otherwise require centimeter-scale balanced paths, while the same lithographic process furnishes mode-selective phase shifters and directional couplers with insertion losses below 1 dB. Encoding two logical qubits (or a ququart) in one particle doubles the computational density per photon and reduces circuit depth, allowing a surface-code-level operation count with at least an order-of-magnitude fewer resources than path- or time-bin-based schemes. In rectangular multimode waveguides, the coupling between SAM and OAM enables the realization of versatile photonic quantum gates. Single-qubit gates such as rotations around the Bloch sphere can be implemented via mode-selective phase shifters that impart controlled phase differences between coupled SAM-OAM states, effectively realizing operations like  $R_z(\theta)$  and  $R_x(\theta)$ . Beam splitter-like operations for OAM modes, achieved through specially designed directional couplers, allow for coherent superpositions necessary for Hadamard and generalized Fourier gates. Furthermore, the SAM-OAM coupling can be engineered to perform conditional logic by designing birefringent sections where the mode with  $\ell = +1$  and left-circular polarization evolves differently than  $\ell = -1$  and right-circular polarization, CNOT, and controlled-phase (CZ) gates can be implemented between OAM and SAM degrees of freedom, realizing deterministic single photon gates. In this architecture, the paraparticle-encoded qudits residing in different SAM-OAM sectors can be deterministically manipulated using integrated optical components, enabling robust and high-dimensional photonic quantum computation. In this case, we encode ququarts in the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded algebra.

In rectangular multimode optical waveguides, boundary conditions and symmetry-breaking effects can couple SAM and OAM, producing spin-orbit hybridized modes that are classically (nonseparably) correlated in SAM and OAM. Modes  $\phi_{(\ell, \sigma)}(x, y)$  in a rectangular waveguide, where  $\ell$  indexes the OAM mode and  $\sigma = \pm 1$  denotes right/left circular-polarization (SAM) or, equivalently,  $A$  and  $B$  waveguide modes for OAM  $\ell = 0$  with their different polarization states if applicable. The field can be expressed in the following way:

$$\Psi(x, y, z, t) = \sum_{(\ell, \sigma)} \hat{\psi}_{(\ell, \sigma)}(z, t) \phi_{(\ell, \sigma)}(x, y) e^{i(\beta_{(\ell, \sigma)} z - \omega t)}, \quad (56)$$

where  $\beta_{(\ell, \sigma)}$  is the propagation constant. The SAM-OAM coupling arises via birefringence or boundary effects and induces a hybrid mode basis:

$$|\ell, \sigma\rangle \rightarrow \sum_{(\ell', \sigma')} C_{(\ell, \sigma)}^{(\ell', \sigma')} |\ell', \sigma'\rangle. \quad (57)$$

Here, we adopt for the sake of simplicity the convention  $\sigma = +1$  for LCP and  $\sigma = -1$  for RCP, consistent with the spin

angular momentum sign. We identify this hybridization with a grading index  $(a, b) = (\ell \bmod 2, (1 - \sigma)/2)$ .

For  $\ell = 0$ , the two spin-orbit hybrid modes labeled by the helicity  $\sigma = \pm 1$  (right- and left-handed circular polarizations) possess grades  $(a, b) = (0, 0)$  and  $(0, 1)$ , respectively: the orbital part is even [ $a = 0$ ], while a flip of  $\sigma$  toggles the  $b$  component. A polarization rotator that reverses the helicity—e.g., a half-wave plate (HWP) with its fast axis at  $45^\circ$ —therefore implements the map  $(a, b) : (0, 0) \leftrightarrow (0, 1)$ , which is a pure  $b$ -grade operation. In the computational model that follows, such grade-changing gates are permitted only (1) when they are explicitly declared as interfaces between the even- and odd- $b$  sectors, or (2) when they are immediately followed by a compensating step that restores the original global  $\mathbb{Z}_2 \times \mathbb{Z}_2$  charge.

This maps structured-light modes into  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded sectors, enabling their interpretation as paraparticles. Operators  $\hat{\psi}_{(\ell, \sigma)}^\pm$  obey paraparticle algebra, with  $R$  matrix determined by the mode coupling matrix  $C_{(\ell, \sigma)}^{(\ell', \sigma')}$ . Here,  $\sigma$  is a binary mode index; it coincides with helicity only when the coupling vanishes.

## VI. PHOTONIC REALIZATION OF THE $\mathbb{Z}_2 \times \mathbb{Z}_2$ GREEN ALGEBRA

### A. Guided-mode basis

We extend the previous paraparticle results to photonic platforms. Consider as example a rectangular SiN waveguide engineered to support the four single-photon modes  $|\ell, \sigma\rangle \in \{|+1, L\rangle, |-1, R\rangle, |0, A\rangle, |0, B\rangle\}$ , where (1)  $\ell \in \{-1, 0, +1\}$  is the orbital-angular-momentum index and (2)  $\sigma \in \{L, R, A, B\}$  denotes either helicity ( $L, R$ ) or, for the  $\ell = 0$  subspace, two orthogonal transverse profiles ( $A, B$ ). Introduce canonical bosonic operators

$$[a_{\ell, \sigma}, a_{\ell', \sigma'}^\dagger] = \delta_{\ell \ell'} \delta_{\sigma \sigma'}, \quad [a_{\ell, \sigma}, a_{\ell', \sigma'}] = 0. \quad (58)$$

### B. Grade map and projectors

Define a  $\mathbb{Z}_2 \times \mathbb{Z}_2$  grade map

$$g : (\ell, \sigma) \mapsto (a, b), \quad a = \ell \bmod 2, \\ b = \begin{cases} 0, & \sigma = L \text{ or } A, \\ 1, & \sigma = R \text{ or } B, \end{cases} \quad (59)$$

so that the four physical modes populate the four group elements  $(a, b) \in \{(1, 0), (1, 1), (0, 0), (0, 1)\}$ .

For each grade, introduce the projector  $P_{(a, b)} = \sum_{\ell, \sigma : g(\ell, \sigma) = (a, b)} |\ell, \sigma\rangle \langle \ell, \sigma|$ . The graded creation/annihilation operators are then

$$\psi_{(a, b)}^\dagger := \sum_{\ell, \sigma} P_{(a, b)}^{\ell, \sigma} a_{\ell, \sigma}^\dagger, \quad \psi_{(a, b)} := (\psi_{(a, b)}^\dagger)^\dagger. \quad (60)$$

### C. Graded commutators

For two homogeneous operators  $X$  and  $Y$  of grades  $g(X) = (a, b)$  and  $g(Y) = (a', b')$ , define [48–50]

$$[X, Y]_\eta := XY - \eta^{g(X) \cdot g(Y)} YX, \quad \eta = -1, \\ g \cdot g' = aa' + bb' \pmod{2}. \quad (61)$$

Using the canonical relations (58), one finds the bilinear identities

$$[\psi_g, \psi_{g'}]_\eta = 0, \quad [\psi_g, \psi_{g'}^\dagger]_\eta = \delta_{g, g'} \mathbb{I}, \quad (62)$$

which reduce to bosonic (symmetric) or fermionic (antisymmetric) commutators according to  $g \cdot g' = 0$  or  $1$ .

### D. Trilinear Green relations

Green's order- $p$  paraparticle algebra is characterized by the nested commutator

$$[\psi_g, [\psi_{g'}^\dagger, \psi_{g''}]_\eta]_\eta = \frac{2}{p} (\delta_{g, g'} \psi_{g''} - \delta_{g, g''} \psi_{g'}). \quad (63)$$

In the present waveguide, each grade hosts a single physical mode, so  $p = 1$  and the prefactor is exactly 2 (parabosons  $\eta = +1$  and parafermions with  $\eta = -1$ ) [73].

Here, we provide an explicit demonstration that the proposed graded operators satisfy the Green-Volkov trilinear relations or the Yang-Baxter identity.

Green-Volkov trilinears in the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded Fock realization are obtained in the following way. Consider the canonical (anti)commutation relations, CAR and CCR. For a family of annihilation/creation operators indexed by a mode label  $i, j \in \mathcal{I}$ :

CAR (fermions): The canonical anticommutation relations are

$$\{c_i, c_j\} = 0, \quad \{c_i^\dagger, c_j^\dagger\} = 0, \quad \{c_i, c_j^\dagger\} = \delta_{ij} \mathbf{1}. \quad (64)$$

Instead, the CCR (bosons) are canonical commutation relations so defined,

$$[a_i, a_j] = 0, \quad [a_i^\dagger, a_j^\dagger] = 0, \quad [a_i, a_j^\dagger] = \delta_{ij} \mathbf{1}. \quad (65)$$

In the field-theoretic (continuous) version, these read

$$[\phi(\mathbf{x}), \phi(\mathbf{y})] = 0, \quad [\pi(\mathbf{x}), \pi(\mathbf{y})] = 0, \\ [\phi(\mathbf{x}), \pi(\mathbf{y})] = i \delta(\mathbf{x} - \mathbf{y}), \quad (66)$$

for bosons, and

$$\{\psi_\alpha(\mathbf{x}), \psi_\beta(\mathbf{y})\} = 0, \quad \{\psi_\alpha^\dagger(\mathbf{x}), \psi_\beta^\dagger(\mathbf{y})\} = 0, \\ \{\psi_\alpha(\mathbf{x}), \psi_\beta^\dagger(\mathbf{y})\} = \delta_{\alpha\beta} \delta(\mathbf{x} - \mathbf{y}), \quad (67)$$

for fermions.

The relation to the graded formalism is so obtained. In the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded setting, we use the bilinear bracket

$$[X, Y]_\eta := XY - \eta^{g(X) \cdot g(Y)} YX, \\ g(X) \cdot g(Y) = aa' + bb' \pmod{2}, \quad (68)$$

with  $\eta = -1$ . The graded bilinear identities

$$[\psi_g, \psi_{g'}]_\eta = 0, \quad [\psi_g, \psi_{g'}^\dagger]_\eta = \delta_{g, g'} \mathbf{1} \quad (69)$$

reduce sectorwise to the CCR (symmetric commutator) when  $g \cdot g' = 0$  and to the CAR (antisymmetric anticommutator) when  $g \cdot g' = 1$ .

Let  $G = \mathbb{Z}_2 \times \mathbb{Z}_2$  and let the homogeneous mode operators  $\psi_g$  and  $\psi_g^\dagger$  ( $g \in G$ ) be those defined in Eqs. (58)–(60), with graded bilinears given by Eq. (61), i.e.,  $[\psi_g, \psi_{g'}]_{\text{gr}} = 0$  and  $[\psi_g, \psi_{g'}^\dagger]_{\text{gr}} = \delta_{g, g'} \mathbf{1}$  [Eq. (62)]. Assume a Green order

$p \in \mathbb{N}$  implemented by “local Green components”  $\phi_g^{(\alpha)}$ ,  $\alpha = 1, \dots, p$ , such that for fixed  $g$ ,

$$[\phi_g^{(\alpha)}, \phi_g^{(\beta)\dagger}]_{\text{gr}} = \delta_{\alpha\beta} \mathbf{1}, \quad [\phi_g^{(\alpha)}, \phi_g^{(\beta)}]_{\text{gr}} = 0,$$

and operators in distinct grades  $g \neq g'$  graded commute. Define the (grade- $g$ ) mode by the  $p$ -averaged Green ansatz, i.e.,  $\psi_g := \frac{1}{p} \sum_{\alpha=1}^p \phi_g^{(\alpha)}$ .

Then, the following trilinear Green-Volkov identities hold sectorwise; for the parabosonic grades  $g \in \{(0, 0), (1, 1)\}$ , we write

$$\begin{aligned} [\psi_g, \{\psi_{g'}^\dagger, \psi_{g''}\}] &= \frac{2}{p} \delta_{g,g'} \psi_{g''}, \\ [\psi_g, \{\psi_{g'}^\dagger, \psi_{g''}^\dagger\}] &= \frac{2}{p} (\delta_{g,g'} \psi_{g''}^\dagger - \delta_{g,g''} \psi_{g'}^\dagger), \end{aligned} \quad (70)$$

and for the parafermionic grades,  $g \in \{(1, 0), (0, 1)\}$  one has

$$\begin{aligned} [\psi_g, [\psi_{g'}^\dagger, \psi_{g''}]] &= \frac{2}{p} \delta_{g,g'} \psi_{g''}, \\ [\psi_g, [\psi_{g'}^\dagger, \psi_{g''}^\dagger]] &= \frac{2}{p} (\delta_{g,g'} \psi_{g''}^\dagger + \delta_{g,g''} \psi_{g'}^\dagger). \end{aligned} \quad (71)$$

In particular, for the single-mode realization relevant here ( $p = 1$ ) the coefficient is 2, as it will be shown below.

To prove it, we detail the parabosonic case. The parafermionic case is analogous with commutators replaced by anticommutators in the outer bracket as in Eqs. (70) and (71). Fix a grade  $g$  that is bosonic, let  $g', g'' \in G$ . Using the Green decomposition  $\psi_g = \frac{1}{p} \sum_{\alpha} \phi_g^{(\alpha)}$ , bilinears from Eq. (62), and the graded Leibniz rule, we obtain

$$[\psi_g, \{\psi_{g'}^\dagger, \psi_{g''}\}] = \frac{1}{p^3} \sum_{\alpha, \beta, \gamma} [\phi_g^{(\alpha)}, \{\phi_{g'}^{(\beta)\dagger}, \phi_{g''}^{(\gamma)}\}].$$

If  $g \neq g'$  or  $g \neq g''$  and the three grades are mutually distinct, each summand vanishes by graded commutativity of different sectors. When  $g = g'$  and  $g''$  is arbitrary, we compute inside a single bosonic grade using the canonical relations (the same computation as in the one-mode paraboson algebra):

$$[\phi_g^{(\alpha)}, \{\phi_{g'}^{(\beta)\dagger}, \phi_{g''}^{(\gamma)}\}] = \delta_{g,g'} \cdot 2 \delta_{\alpha\beta} \phi_{g''}^{(\gamma)}.$$

Summing over  $\alpha, \beta$ , and  $\gamma$  gives

$$\frac{1}{p^3} \sum_{\alpha, \beta, \gamma} 2 \delta_{g,g'} \delta_{\alpha\beta} \phi_{g''}^{(\gamma)} = \frac{2}{p^2} \delta_{g,g'} \sum_{\gamma} \phi_{g''}^{(\gamma)} = \frac{2}{p} \delta_{g,g'} \psi_{g''},$$

which is Eq. (70). The two-creation identity (70) follows similarly from

$$[\phi_g^{(\alpha)}, \{\phi_{g'}^{(\beta)\dagger}, \phi_{g''}^{(\gamma)\dagger}\}] = 2 \delta_{g,g'} \delta_{\alpha\beta} \phi_{g''}^{(\gamma)\dagger} - 2 \delta_{g,g''} \delta_{\alpha\gamma} \phi_{g'}^{(\beta)\dagger}.$$

For parafermionic grades, the same computation goes through with the outer anticommutator, using the CAR in place of the CCR and the equal-grade identity  $\{c, [c^\dagger, \cdot]\} = 2(\cdot)$  inside the sum.

One could prefer to keep the bilinear graded bracket normalized as  $[\psi_g, \psi_{g'}^\dagger]_{\text{gr}} = \delta_{g,g'}$  [Eq. (62)] for all  $p$  and take the *root- $p$*  normalization  $\psi_g = \frac{1}{\sqrt{p}} \sum_{\alpha} \phi_g^{(\alpha)}$ , in which case the trilinear coefficient becomes 2 (independent of  $p$ ). In our physical platform, each grade is realized by a single guided

mode, i.e.,  $p = 1$ ; hence, all sectorwise trilinears reduce to the simple coefficient 2 used in Sec. VID.

#### Example with low-spin truncation: From Majorana blocks to $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded brackets

*Setup.* Consider the first four levels of the Majorana tower,  $s \in \{0, \frac{1}{2}, 1, \frac{3}{2}\}$ . Choose a parity-preserving bijection  $g : \{0, \frac{1}{2}, 1, \frac{3}{2}\} \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_2$  such that

$$p(s) := a_s + b_s \equiv 2s \pmod{2}, \quad \text{for } g(s) = (a_s, b_s),$$

so that integer (half-integer) spins lie in even (odd) total grade. A convenient explicit choice—equivalent to other assignments up to an automorphism of  $\mathbb{Z}_2 \times \mathbb{Z}_2$ —is

$$\begin{aligned} g(0) &= (0, 0), & g(\tfrac{1}{2}) &= (1, 0), \\ g(1) &= (1, 1), & g(\tfrac{3}{2}) &= (0, 1). \end{aligned} \quad (72)$$

Denote by  $\psi_s, \psi_s^\dagger$  the annihilation/creation operators in the sector  $g(s)$ . We use the graded bilinear bracket already introduced,

$$\begin{aligned} [X, Y]_{\text{gr}} &:= XY - (-1)^{g(X) \cdot g(Y)} YX, \\ g(X) \cdot g(Y) &:= aa' + bb' \pmod{2}, \end{aligned} \quad (73)$$

and the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  Green (trilinear) relations of order  $p = 1$  sectorwise.

*Bilinear relations (spin-statistics at the operator level).* With the parity choice in Eq. (72), the graded bilinear identities

$$[\psi_g, \psi_{g'}]_{\text{gr}} = 0, \quad [\psi_g, \psi_{g'}^\dagger]_{\text{gr}} = \delta_{g,g'} \mathbb{I}$$

specialize to the expected (anti)commutators:

$$\begin{aligned} [\psi_0, \psi_0^\dagger] &= 1 \quad (\text{bosonic for } s = 0), \\ \{\psi_{1/2}, \psi_{1/2}^\dagger\} &= 1 \quad (\text{fermionic for } s = \tfrac{1}{2}), \\ [\psi_1, \psi_1^\dagger] &= 1 \quad (\text{bosonic for } s = 1), \\ \{\psi_{3/2}, \psi_{3/2}^\dagger\} &= 1 \quad (\text{fermionic for } s = \tfrac{3}{2}), \\ [\psi_s, \psi_{s'}]_{\text{gr}} &= 0, \quad [\psi_s, \psi_{s'}^\dagger]_{\text{gr}} = 0 \quad (s \neq s'). \end{aligned} \quad (74)$$

Thus, the low-spin truncation reproduces, sector by sector, the spin-statistics sign encoded in the microcausality bracket of the tower fields.

*Trilinear Green-Volkov relations (order  $p = 1$ ).* Restricting the trilinear identities to each homogeneous sector yields, for the bosonic-grade blocks  $\{s = 0, 1\}$ ,

$$[\psi_s, \{\psi_s^\dagger, \psi_s\}] = 2 \psi_s, \quad s \in \{0, 1\}, \quad (75)$$

and, for the fermionic-grade blocks  $\{s = \frac{1}{2}, \frac{3}{2}\}$ ,

$$\{\psi_s, \{\psi_s^\dagger, \psi_s\}\} = 2 \psi_s, \quad s \in \{\tfrac{1}{2}, \tfrac{3}{2}\}. \quad (76)$$

Mixed-sector trilinears vanish when a Kronecker factor enforces grade mismatch. Equations (75) and (76) are the  $p = 1$  Green limits of the graded Green/Volkov algebra, consistent with Eqs. (58)–(60).

*Exchange-sign matrix.* The  $\mathbb{Z}_2 \times \mathbb{Z}_2$  exchange factor between two homogeneous sectors is  $X_{s,s'} := (-1)^{g(s) \cdot g(s')}$ . Or-

dering  $(s, s')$  as  $(0, \frac{1}{2}, 1, \frac{3}{2})$ , one finds

$$X_{s,s'} = \begin{pmatrix} + & + & + & + \\ + & - & - & + \\ + & - & + & - \\ + & + & - & - \end{pmatrix},$$

which matches the structure of the exchange matrix  $X$  given earlier [up to the row/column permutation induced by Eq. (72)]. This fixes all graded signs in tensor products without touching the intrasector bilinears.

*Photonic ququart identification.* Using the SAM/OAM map  $\varphi: (\ell, \sigma) \mapsto (a, b) = (\ell \bmod 2, (1 - \sigma)/2)$ , the four levels above identify one to one with the guided modes of Table III via

$$\begin{aligned} s = 0 \ (0, 0) &\leftrightarrow |\ell = 0, A\rangle, \\ s = \frac{1}{2} \ (1, 0) &\leftrightarrow |\ell = +1, \sigma = +1\rangle, \\ s = 1 \ (1, 1) &\leftrightarrow |\ell = -1, \sigma = -1\rangle, \\ s = \frac{3}{2} \ (0, 1) &\leftrightarrow |\ell = 0, B\rangle, \end{aligned}$$

so that the graded commutators (74) act exactly as in Eqs. (56)–(60) on the ququart creation/annihilation operators. In this four-level truncation, the Majorana tower blocks are in explicit one-to-one correspondence with  $\mathbb{Z}_2 \times \mathbb{Z}_2$  sectors; the operator content obeys the graded bilinear and trilinear relations with the correct spin-statistics signs, and the exchange phases are fixed by the  $X$  matrix. This provides a concrete template used later for the full photonic realization.

Field expansion is obtained by defining mode functions  $\varphi_{(a,b)}(x)$  for each sector and the paraparticle field reads

$$\Psi(x) = \sum_{(a,b) \in \mathbb{Z}_2^2} \psi_{(a,b)} \varphi_{(a,b)}(x) \quad (77)$$

and satisfies the graded Majorana equation introduced in the next section.

Equations (60)–(63) demonstrate that structured-light modes in a four-channel SAM-OAM waveguide realize a  $\mathbb{Z}_2 \times \mathbb{Z}_2$  Green paraparticle algebra of order  $p = 1$ . This closes the gap between the algebraic framework and the physical implementation.

Consider the Majorana Tower [33] with mass/spin relation-ship of Eq. (12). From that we write the unified operator field equation that describes the paraparticle-graded Majorana field equation mainly from Eqs. (20) and (21) and Eq. (43) for a generic operator becomes

$$\left[ \Gamma_{(a,b)}^\mu \partial_\mu + \frac{m}{s_{(a,b)} + \frac{1}{2}} \right] \hat{\psi}_{(a,b)}(x) = 0, \quad (78)$$

with  $\Gamma_{(a,b)}^\mu \equiv P_{(a,b)} \otimes \gamma^\mu$ ,  $P_{(a,b)} := |a, b\rangle\langle a, b|$ , and  $\gamma^0 = \sigma_z$ ,  $\gamma^1 = i\sigma_y$ , so that each graded sector  $(a, b)$  carries its own projector  $P_{(a,b)}$ , while the second tensor factor is the usual  $(1 + 1)$ -dimensional Dirac matrix  $(\gamma^0, \gamma^1)$ . With this definition,  $\Gamma_{(a,b)}^\mu$  acts like an ordinary gamma matrix on the spinor indices and as the identity inside the chosen  $\mathbb{Z}_2 \times \mathbb{Z}_2$  subspace, and Eq. (78) remains well defined for every grade.

Paraparticle formulation in rectangular multimode waveguides with SAM-OAM coupling generate qudit-based quan-

tum calculations on a single photon encoded in the paraparticle  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded algebras, in a different way to encode qubits and qudits. An illustrative example with ququarts is reported in the Appendix.

This results in classically entangled modes, where specific OAM values are restricted to certain polarization states. For instance, a typical configuration may allow the four states  $\ell = +1$  with LCP,  $\ell = -1$  with RCP, and  $\ell = 0$  with both polarizations that correspond to two different waveguide modes (A) and (B) that can also be  $\sigma = +1$  or  $\sigma = -1$ . We then assign to each mode a  $\mathbb{Z}_2 \times \mathbb{Z}_2$  grading based on OAM and SAM  $a = \ell \bmod 2$  and  $b = (1 - \sigma)/2$ , where  $\sigma = +1$  for LCP and  $-1$  for RCP. The graded sectors are thus defined as reported in Table III. Alternatively to polarization, in this photonic platform, one can include two distinct quasidegenerate  $\ell = 0$  spatial waveguide modes, labeled for convenience A and B, alongside and independently of the helicity index  $\sigma$  and the OAM modes  $\ell = +1$  and  $\ell = -1$ . This construction yields a ququart encoding composed of  $\{(\ell = +1, A), (\ell = -1, A), (\ell = 0, A), (\ell = 0, B)\}$ , which we map to successive spin sectors in the Majorana tower with corresponding graded labels  $(a, b)$ . The matrix  $\theta_{ss'}$  then specifies the allowed couplings and selection rules between these OAM-waveguide pairs: nonvanishing entries enable controlled interactions or quantum gates, while vanishing entries enforce symmetry-protected decoupling. In particular, the presence of two  $\ell = 0$  modes enables the simulation of higher-spin or ancillary sectors, offering additional flexibility for implementing paraparticle-inspired logic operations in photonic circuits.

The qudit encoding is obtained using graded paraparticle sectors: each mode  $\phi_{(a,b)}$  corresponds to a paraparticle operator  $\hat{\psi}_{(a,b)}$ , satisfying  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded algebra. A four-level qudit is formed by assigning logical basis states to distinct sectors:

$$\begin{aligned} |0\rangle &= \phi_{(0,0)} = (\ell = 0, A), \\ |1\rangle &= \phi_{(0,1)} = (\ell = 0, B), \\ |+\rangle &= \phi_{(1,0)} = (\ell = +1, \sigma = +1), \\ |-\rangle &= \phi_{(1,1)} = (\ell = -1, \sigma = -1). \end{aligned} \quad (79)$$

Logical operations are defined using graded exchange rules between the different  $\hat{\psi}_{(a,b)}$ :

$$\begin{aligned} \hat{\psi}_{(a,b)}^+ \hat{\psi}_{(a',b')}^+ &= (-1)^{(a,b) \cdot (a',b')} \hat{\psi}_{(a',b')}^+ \hat{\psi}_{(a,b)}^+ + \theta_{(a,b),(a',b')} \cdots \end{aligned} \quad (80)$$

The Hamiltonian governing the system is built from graded number operators:

$$\hat{H} = \sum_{(a,b)} \epsilon_{(a,b)} \hat{n}_{(a,b)}, \quad \hat{n}_{(a,b)} = \hat{\psi}_{(a,b)}^+ \hat{\psi}_{(a,b)}^-, \quad (81)$$

where  $\epsilon_{(a,b)}$  is the mode energy or effective mass determined by the waveguide geometry and the full field is assembled from mode-weighted operators such as

$$\Psi(x) = \sum_{(a,b)} \hat{\psi}_{(a,b)}(x) \phi_{(a,b)}(x). \quad (82)$$

This construction preserves both the relativistic structure and the statistical symmetries of the system.

### E. Quantum computation with SAM-OAM paraparticle modes

Paraparticles, which obey parastatistics beyond the standard Bose-Einstein and Fermi-Dirac statistics, offer a fundamentally different approach to encoding and manipulating quantum information. Here is a detailed breakdown of how they function and why they are potentially useful in quantum computing.

Their algebraic structure enables qubits as they are characterized by nontrivial exchange rules governed by trilinear commutation relations or braided tensor symmetries. Each paraparticle is defined within a graded sector of the type  $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_2$ , with creation/annihilation operators obeying the  $R$  matrix:

$$\hat{\psi}_{(j,a)}^+ \hat{\psi}_{(j,b)}^+ = \sum_{(c,d)} R_{(a,b)}^{(c,d)} \hat{\psi}_{(j,c)}^+ \hat{\psi}_{(j,d)}^+. \quad (83)$$

This structure allows for qudits that go beyond simple  $0 - 1$  logic, enabling richer computational states and graded exclusion principles (e.g., only a fixed number of identical paraparticles allowed in a state like paraqubits). Here, for the sake of simplicity we will focus more on qubit states.

Unlike fermions or bosons, paraparticles naturally avoid certain states due to their algebraic constraints. For instance, parabosons of order  $p$  allow up to  $p$  identical particles in the same state. Parafermions can only occupy graded antisymmetric states. This intrinsic structure acts like built-in error detection, as illegal states are algebraically forbidden. In principle, this allows quantum computers based on paraparticles to detect and correct errors without auxiliary encoding overhead, with a finite number of quantum states or through statistical methods implementing time series of data characterized by structured noise as in Nelson's quantum mechanics [74] adapted to the dynamics described by  $R$  discussed below.

Paraparticles can be implemented in 3D systems using algebraic braiding (i.e., exchange rules defined by  $R$  matrices rather than spatial motion). This makes them attractive for scalable quantum computing platforms as they can be mapped to multiqubit registers. Operators are translated into combinations of Pauli matrices or fermionic modes. Paraparticle oscillators and dynamics are then simulated using quantum hardware such as trapped ions, superconducting qubits, and photonics, as a parafermion oscillator can be encoded using a pair of qubits, with specially designed gates that emulate the graded commutation relations.

### F. Application to logical gates

Logical gate operations in SAM-OAM paraparticle systems can be implemented using  $\hat{n}_{(0,1)}$ , conditional control qubit in CNOT gates,  $\hat{E}_{(0,0),(0,1)}$  as logical  $X$  gate across SAM/waveguide-mode states at  $\ell = 0$ , and finally,  $\hat{E}_{(1,1),(1,0)}$  to swap OAM states with preserved polarization. These operator structures form the algebraic foundation for deterministic, symmetry-enforced quantum logic in structured photonic systems.

This architecture supports deterministic operation, since all logic is implemented via engineered mode coupling, grading symmetries, and classical interference and is based on qudits. It is scalable using integrated photonic circuits, and grading prevents logical errors via selection rules and paras-

TABLE IV. Mode encoding and graded symmetry. The label “wg” stands for general waveguide mode.

| Qubit   | Mode                 | Grading<br>$\mathbb{Z}_2 \times \mathbb{Z}_2$ | Logical<br>States        |
|---------|----------------------|-----------------------------------------------|--------------------------|
| Control | $\ell = 0$ ,         | $(0, b)$                                      | $ 0\rangle_C = (0, 0)$ , |
|         | wg $/\sigma = \pm 1$ | $(0, b)$                                      | $ 1\rangle_C = (0, 1)$   |
| Target  | $\ell = \pm 1$ ,     | $(1, a)$                                      | $ 0\rangle_T = (1, 1)$ , |
|         | $\sigma = \pm 1$     | $(1, b)$                                      | $ 1\rangle_T = (1, 0)$   |

tatistical exclusion. The SAM-OAM-graded path-entangled system offers a robust and compact framework for realizing three-qubit Toffoli logic gates. The unique combination of angular momentum encoding, spatial mode entanglement, and graded symmetry allows for nonclassical quantum logic to be implemented deterministically in a photonic platform.

A quantum computer can be designed using the SAM-OAM entangled modes as qudits or logical qubits following certain simple rules. Logical states are encoded in  $(a, b)$ -graded sectors and in this configuration they correspond to the SAM-OAM modes of the Majorana quasiparticle with  $\ell = \pm 1$  and  $\ell = 0$ . Quantum gates are implemented via exchange operations, parity-sensitive phase shifters, and optical interference. Entanglement is achieved using beam splitters and mode-selective couplers that preserve grading, and the readout is performed by spatial and polarization-resolved detectors.

The logical gate operations include the following:

$$\begin{aligned} X &: \hat{\psi}_{(0,0)} \leftrightarrow \hat{\psi}_{(0,1)}, \\ Z &: \text{Phase shift } \exp(i\pi) \text{ on } (0, 1), \\ CZ &: \exp(i\pi \hat{n}_{(0,1)} \hat{n}_{(1,1)}). \end{aligned}$$

This graded quantum architecture enables interesting robust and deterministic quantum logic gates with topological protection provided by parastatistical exclusion and symmetry constraints.

The deterministic CNOT gate in SAM-OAM-graded paraparticle systems represents the first step for setting up how a paraparticle formalism can be used in quantum circuits. In this case, we present a deterministic implementation of the CNOT gate using the physical quantities spin-orbital angular momentum (SAM-OAM) entangled photonic modes that are found in rectangular multimode waveguides. This architecture can exploit the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded symmetry of paraparticles, in which each mode is uniquely identified by its OAM, SAM, and possibly its spatial path.

Mode encoding and graded symmetry are so obtained: each photonic mode is classified by its  $(\ell, \sigma)$  pairing, with the grading defined by  $a = \ell \bmod 2$  and  $b = (1 - \sigma)/2$ , where  $\sigma = +1$  for left-circular and  $-1$  for right-circular polarization. Logical qubits are then encoded as in Table IV,

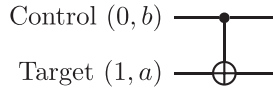
The CNOT gate applies an  $X$  operation (bit flip) on the target qubit conditional on the control being in the  $|1\rangle_C$  state. This is implemented by using the graded occupation number  $\hat{n}_{(0,1)}$  to detect control parity, and then executing a conditional swap of the target between  $(1, 0)$  and  $(1, 1)$ . One then has to enforce exchange symmetry using the matrix  $X_{(a,b),(a',b')} = (-1)^{(a,b) \cdot (a',b')}$ .

TABLE V. Logical qubits encoded in SAM-OAM-path modes; “wg” stands for general waveguide mode.

| Qubit                | Mode                                             | Grading<br>$\mathbb{Z}_2 \times \mathbb{Z}_2$ | Logical<br>states                                |
|----------------------|--------------------------------------------------|-----------------------------------------------|--------------------------------------------------|
| Qubit 1<br>(control) | $\ell = 0$ , Path $A_p$<br>wg / $\sigma = \pm 1$ | $(0, b)$<br>$(0, b)$                          | $ 0\rangle_C = (0, 0)$<br>$ 1\rangle_C = (0, 1)$ |
| Qubit 2<br>(control) | $\ell = \pm 1$ , Path $A_p$<br>$\sigma = \pm 1$  | $(1, a)$                                      | Two graded<br>sectors                            |
| Qubit 3<br>(target)  | SAM-OAM<br>in path $A_p/B_p$                     | varies                                        | Flip if both<br>controls = 1                     |

Deterministic operation and photonic design are realized using mode-resolved integrated waveguides for routing and encoding, control of phase shifting, and interferometric designs. In this way is ensured stable logic under grading.

The quantum circuit representation with the paraparticle  $(a, b)$ -graded sectors is shown with a schematic of a deterministic CNOT gate using SAM-OAM-graded photonic modes:



with control qubit  $(0, b) \rightarrow (\ell = 0, \sigma = \pm 1 \vee A, B)$  and target qubit lies in sectors  $(1, a) \rightarrow (\ell = \pm 1, \sigma = \pm 1)$ .

The control qubit is encoded in a  $\ell = 0$  mode with circular polarization. The target qubit, with  $\ell = \pm 1$ , is routed through an interferometric path that swaps its logical state only when the control is in the fermionic sector  $(0, 1)$ . The operation is deterministic via a  $\pi$ -phase shifter and interferometric routing. The SAM-OAM-graded framework provides deterministic logic gates by leveraging sector-specific occupation rules and topologically protected mode interactions. The CNOT gate demonstrates how single-photon logic can be both scalable and physically realizable using structured light and graded symmetry.

A further step is to realize with paraparticle formulation the Toffoli gate. In this case, we use the quantum computational degrees of freedom of SAM-OAM paraparticles and path entanglement. Thus, we design a Toffoli (CCNOT) gate within a photonic platform based on structured modes in rectangular multimode waveguides, where SAM and OAM are entangled and graded by a  $\mathbb{Z}_2 \times \mathbb{Z}_2$  symmetry. The implementation leverages paraparticle grading and path entanglement to deterministically execute a three-qubit controlled logic operation.

The logical qubits are encoded in SAM-OAM-path modes as reported in the following table (Table V). Each mode combination  $(\ell, \sigma)$  is mapped to a sector in the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  grading, and logical operations depend on exchange and interference rules between these sectors.

The Toffoli gate executes a controlled-controlled-NOT, flipping the target qubit if and only if both control qubits are in the logical 1 state:

$$\begin{aligned}
 |c_1 = 0, c_2 = *, t\rangle &\rightarrow |c_1 = 0, c_2 = *, t\rangle, \\
 |c_1 = 1, c_2 = 1, t\rangle &\rightarrow |c_1 = 1, c_2 = 1, t \oplus 1\rangle,
 \end{aligned}$$

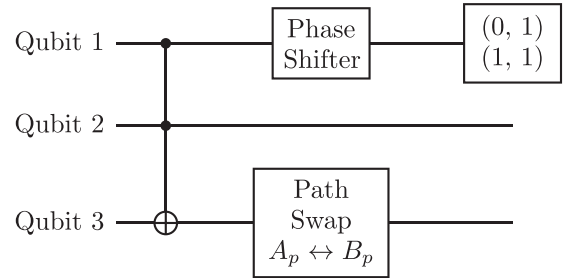
where the symbol “\*” means “any value” that is either 0 or 1. Path-entangled realization is included to realize the Toffoli

TABLE VI. Grading, operator assignment, and mode-operator mapping.

| Mode     | $\ell$ | $\sigma$    | Grade $(a, b)$ | Operator                 |
|----------|--------|-------------|----------------|--------------------------|
| $\phi_1$ | 0A     | $+1 \vee A$ | $(0, 0)$       | $\hat{\psi}_{(0,0)}^\pm$ |
| $\phi_2$ | 0      | $-1 \vee B$ | $(0, 1)$       | $\hat{\psi}_{(0,1)}^\pm$ |
| $\phi_3$ | +1     | +1          | $(1, 0)$       | $\hat{\psi}_{(1,0)}^\pm$ |
| $\phi_4$ | -1     | -1          | $(1, 1)$       | $\hat{\psi}_{(1,1)}^\pm$ |

CCNOT gate. The target qubit is encoded in spatial path modes ( $A_p/B_p$ ). Conditional operations on the control qubits (graded sectors) induce a  $\pi$ -phase shift or exchange, flipping the path of the target photon. This can be realized by a combination of OAM-polarization mode converters and phase shifters. To implement the other part of the circuit, one inserts interferometric elements (e.g., Mach-Zehnder) with graded switches and in this way are defined graded exchange operators enforcing  $(-1)^{(a,b) \cdot (a',b')}$  behavior.

The Toffoli gate with path entanglement and graded SAM-OAM modes is here schematized:



Two control qubits are encoded in polarization-OAM states in Path  $A_p$ . The target qubit is routed through interferometric paths. If both control inputs are in the logical 1 sector, a  $\pi$ -phase shift and destructive interference switch the target from Path A to Path B, implementing a NOT operation. Qubit 1:  $(0, b_1)$ , Qubit 2:  $(1, a_2)$ , Qubit 3: Path  $A_p/B_p$ . Notes: Path mode of the target encodes logical state. Only sectors  $(0, 1)$  and  $(1, 1)$  trigger interference path swap. Conditional  $\pi$ -phase shift and graded parity ensure deterministic action. Truth tables are reported in the Appendix.

### G. Operator structure of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded superalgebra

In this section, we present explicit operator constructions for the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded superalgebra, using paraparticle operators associated with SAM-OAM modes in rectangular waveguides.

Grading and operator assignment are made by labeling OAM and SAM as follows. Each SAM-OAM mode is labeled by orbital angular momentum  $\ell$  and polarization  $\sigma = \pm 1$  or general waveguide modes ( $A$  and  $B$ ). The grading, as is known, is defined by  $a = \ell \bmod 2$  and  $b = (1 - \sigma)/2$  and gives rise to the following mode-operator mapping reported in Table VI. The graded commutation relations are obtained using the scalar product  $(a, b) \cdot (a', b') = aa' + bb' \bmod 2$ ; the graded commutator for homogeneous operators is

$$\hat{\psi}_{(a,b)}^\pm \hat{\psi}_{(a',b')}^\pm = (-1)^{(a,b) \cdot (a',b')} \hat{\psi}_{(a',b')}^\pm \hat{\psi}_{(a,b)}^\pm. \quad (84)$$

The graded commutation relations are enlisted in the Appendix. Bilinear and Hamiltonian operators are then defined in terms of number and exchange operators,

$$\hat{n}_{(a,b)} = \hat{\psi}_{(a,b)}^+ \hat{\psi}_{(a,b)}^-, \quad (85)$$

$$\hat{E}_{(a,b),(a',b')} = \hat{\psi}_{(a,b)}^+ \hat{\psi}_{(a',b')}^-. \quad (86)$$

These obey the graded Lie algebra:

$$\begin{aligned} & [\hat{E}_{(a,b),(a',b')}, \hat{E}_{(c,d),(c',d')}] \\ &= \delta_{(a',b'),(c,d)} \hat{E}_{(a,b),(c',d')} - \delta_{(a,b),(c',d')} \hat{E}_{(c,d),(a',b')}. \end{aligned} \quad (87)$$

This structure is a nontrivial, reductive algebra that mimics the Lie algebra  $\mathfrak{gl}_N$ , which consists of all  $N \times N$  complex matrices with the commutator as the Lie bracket. This algebraic equivalence occurs within each graded sector and controls sector-preserving logical gates and exchange transformations. Thus,  $\mathfrak{gl}_N$  in second quantization, e.g., in paraparticle systems like Ref. [7], arises as the algebra of bilinear operators  $E_{i,j} = \sum_a \hat{\psi}_{(i,a)}^+ \hat{\psi}_{(a,b)}^-$  with dimension  $N^2$  with the commutation relations  $[\hat{E}_{i,j}, \hat{E}_{k,l}] = \delta_{jk} \hat{E}_{i,l} - \delta_{il} \hat{E}_{k,j}$ , which matches the structure constants of  $\mathfrak{gl}_N$ . This algebra, also written as  $\mathfrak{gl}(N, \mathbb{C}) = \mathfrak{gl}_{\mathbb{N}}(\mathbb{C})$ , describes internal symmetry transformations (e.g., flavor symmetry, mode mixing) and serves as a closure condition to ensure that paraparticle field theories are self-consistent and covariant. It contains the simple traceless subalgebra  $\mathfrak{sl}_N \subset \mathfrak{gl}_N$ .

An example of a model Hamiltonian for OAM-SAM coupled ququarts is here introduced to describe the dynamics of this system in rectangular waveguides. The ququart basis is so defined  $|0\rangle \equiv |L, 0\rangle$ ,  $|1\rangle \equiv |R, 0\rangle$ ,  $|2\rangle \equiv |L, +1\rangle$ , and  $|3\rangle \equiv |R, +1\rangle$ , where  $L/R$  denote left- and right-circular polarizations and  $\ell = 0, +1$  is the OAM quantum number.

The effective Hamiltonian is of the type

$$\hat{H} = \sum_{i=0}^3 \omega_i \hat{n}_i + \sum_{i \neq j} J_{ij} \hat{a}_i^\dagger \hat{a}_j + \sum_{i,j} \Delta_{ij} \hat{a}_i \hat{a}_j + \text{H.c.}, \quad (88)$$

where  $\omega_i$  are the mode frequencies,  $\hat{a}_i^\dagger$  and  $\hat{a}_i$  are the creation and annihilation operators, respectively, for each hybrid mode,  $J_{ij}$  are coherent hopping amplitudes between modes (due to spin-orbit coupling or induced perturbations), and  $\Delta_{ij}$  represent parametric couplings or nonlinear interactions.

Explicitly, for the cyclic  $\mathbb{Z}_4$  parafermion model, we can set

$$\hat{H} = \sum_{k=0}^3 \omega \hat{n}_k + g \sum_{k=0}^3 (\hat{a}_k^\dagger \hat{a}_{k+1} + \text{H.c.}), \quad (89)$$

with cyclic boundary conditions  $\hat{a}_{k+4} \equiv \hat{a}_k$ , where  $g$  is the coupling strength implementing the  $\tau$  shift operator.

In the context of rectangular waveguides, the OAM-SAM hybrid modes experience birefringent and geometric phase shifts that can be described by

$$\hat{H}_\sigma = \sum_{k=0}^3 \phi_k \hat{n}_k, \quad (90)$$

where  $\phi_k$  introduces the phase pattern associated with the clock operator  $\sigma_c$ :

$$\phi_0 = 0, \quad \phi_1 = \frac{\pi}{2}, \quad \phi_2 = \pi, \quad \phi_3 = \frac{3\pi}{2}. \quad (91)$$

This Hamiltonian captures the essential features of parafermionic dynamics in the photonic platform and provides a basis for quantitative predictions and experimental design.

Although our work extensively discusses the algebraic structures underlying the  $\mathbb{Z}_N$  parafermionic systems and their potential realization in photonic platforms, an explicit formulation of the model Hamiltonian or tight-binding lattice description has not been fully presented. To address this, we now outline a representative Hamiltonian that captures the essential features of the coupled OAM-SAM ququart system in rectangular waveguides.

The tight-binding Hamiltonian can be written as

$$H = \sum_j \left[ \omega_j a_j^\dagger a_j + \sum_{\langle j,k \rangle} J_{jk} a_j^\dagger a_k + \sum_{\langle j,k \rangle} \lambda_{jk} a_j^\dagger \sigma_z a_k \right], \quad (92)$$

where  $a_j^\dagger$  and  $a_j$  are, respectively, the creation and annihilation operators at site  $j$ ,  $\omega_j$  denotes the on-site energy,  $J_{jk}$  the nearest-neighbor coupling, and  $\lambda_{jk}$  encodes the OAM-SAM spin-orbit coupling, with  $\sigma_z$  acting on the polarization subspace (e.g., left- and right-circular polarizations).

For the parafermionic sector, we can introduce generalized  $\mathbb{Z}_N$  operators  $\alpha_j$  obeying

$$\alpha_j^N = 1, \quad \alpha_j \alpha_k = e^{2\pi i/N} \alpha_k \alpha_j \quad (j < k), \quad (93)$$

which decompose under the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  grading as

$$\alpha_j = \tau_j \sigma_j, \quad (94)$$

where  $\tau_j$  and  $\sigma_j$  represent commuting  $\mathbb{Z}_2$ -graded operators. A worked detailed matrix example for  $N = 4$  is included in Appendix A.

## H. Photonic realization of para-Majorana modes

An explicit proposal for realizing high-order para-Majorana modes in photonic systems can be obtained through nonlinear media or by exploiting the properties of classical entanglement between SAM and OAM with optical discontinuities or  $q$ -plates.

The first setup involves a one-dimensional coupled waveguide lattice or an array of coupled optical resonators and nonlinear media. Each lattice site can support photonic modes with degrees of freedom, such as polarization, orbital angular momentum, or frequency bins, which encode the paraparticle degrees of freedom. We consider an effective Hamiltonian of the form

$$\begin{aligned} H = \sum_j & [t a_j^\dagger a_{j+1} + t^* a_{j+1}^\dagger a_j \\ & + \Delta (a_j a_{j+1} + a_{j+1}^\dagger a_j^\dagger) + U (a_j^\dagger a_j)^n], \end{aligned} \quad (95)$$

where  $a_j^\dagger$  and  $a_j$  are photonic creation and annihilation operators at site  $j$ ,  $t$  is the nearest-neighbor hopping amplitude,  $\Delta$  is a parametric pairing term induced via four-wave mixing or external drive, and  $U (a_j^\dagger a_j)^n$  is a nonlinear interaction

term, such as Kerr-type or cross-phase-modulation, with  $n \geq 2$ . This nonlinear term is essential to simulate parastatistics beyond standard quadratic Majorana models with the usual techniques.

The system can be engineered to exhibit a synthetic  $\mathbb{Z}_N$  symmetry, where  $N$  determines the order of the paraparticles. Properly designed coupling phases and drive configurations can create synthetic gauge fields, emulating the symmetry algebra required for para-Majorana excitations. The edge sites of the lattice can host localized para-Majorana zero modes, analogous to the zero-energy modes in topological superconducting wires.

Experimental observables in such systems may include the detection of zero-energy edge states via transmission or reflection spectroscopy, observation of fractionalized photonic excitations through interference experiments, and potential non-Abelian braiding signatures via dynamic tuning of coupling parameters.

The linear-optics emulation via SAM-OAM coupling, instead of invoking genuine optical nonlinearities, is useful to recall that the *algebra* of para-Majorana modes can already be built up with *linear* devices that couple spin (SAM) and OAM of light.

(1) *Mode assignment.* Choose the four elementary grades of the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  algebra as  $\{|L, m\rangle, |R, m\rangle, |L, m+1\rangle, |R, m+1\rangle\}$ , they correspond to  $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$ , where “ $L/R$ ” denotes left-/right-hand circular polarization and  $m \in \mathbb{Z}$  is the OAM index. Single-mode operators  $a_{(a,b)}^{(\dagger)}$  acting on these four subspaces therefore inherit the required graded commutation relations.

(2) *Realization with  $q$ -plates and wave plates.* A birefringent  $q$ -plate acts as the mode converter

$$U_q|\sigma, m\rangle = |\bar{\sigma}, m + 2q\sigma\rangle, \quad \sigma = \pm 1, \quad (96)$$

flipping the SAM while adding  $2q\sigma$  units of OAM. The same conversion can be engineered in integrated chips by programmable phase discontinuities that locally modify the waveguide birefringence [70].

In the abstract algebra notation, one uses in this subspace  $b$ , which is no longer unitary, and defines the nested bracket

$$[b, \{b^\dagger, b\}] = 2b + \mathcal{O}(\phi^2), \quad (97)$$

which is the optical analog of Green’s trilinear identity for parabosons of order  $p = 2$ . In an optical setup we cannot implement  $b^\dagger$  directly, so we replace it by a small, unitary spin rotation  $V_\phi = e^{-i(\phi/2)\sigma_z}$  like that of an HWP whose optic axis is rotated by  $\phi/2$ . For small  $\phi$ , we expand  $V_\phi \simeq \mathbf{1} - \frac{i\phi}{2}\sigma_z$  and define the grade-projected ladder operator  $b := P_{(1,0)}U_q$ , where  $P_{(1,0)}$  projects onto the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  sector  $(a, b) = (1, 0)$ .  $P_{(1,0)}$  is the grade projector that retains only the states with odd OAM index,  $\ell \equiv 1 \pmod{2}$ , putting all other SAM-OAM components to zero.

Expand for a tiny HWP angle  $\phi$ , using  $V_\phi = \mathbf{1} - \frac{i\phi}{2}\sigma_z + \mathcal{O}(\phi^2)$ , which implies  $\{V_\phi, b\} = 2b - \frac{i\phi}{2}\{\sigma_z, b\} + \mathcal{O}(\phi^2)$ .

Because the SAM-OAM ladder satisfies  $\{\sigma_z, b\} = 2b$ , the leading term in the anticommutator is  $(2 - \phi)b$ . Substituting into the commutator, we obtain

$$[b, \{V_\phi, b\}] = 2 \sin \phi b = 2b + \mathcal{O}(\phi^3), \quad (97a)$$

where the sine emerges after resumming the exact exponential series (for  $\phi \lesssim \pi/2$ , one has  $\sin \phi = \phi - \phi^3/6 + \dots$ ). If we choose  $\phi = \pi/2$ , the prefactor is  $2 \sin(\pi/2) = 2$  and Eq. (97a) exactly reproduces Eq. (97). For arbitrary small  $\phi$ , Eq. (97a) approaches Eq. (97) up to a constant that can be absorbed into the normalization of  $b$ .

Hence,

$$[b, \{V_\phi, b\}] \longrightarrow [b, \{b^\dagger, b\}]$$

once the unitary surrogate  $V_\phi$  is expanded (or the HWP is set to  $\phi = \pi/2$ ). The optical bracket therefore implements the same Green trilinear algebra, but with equipment that is available in a purely linear photonic platform. In this way, the SAM-OAM toolbox ( $q$ -plates, wave plates, spatial-light modulators, and beam splitters) can emulate the para-Majorana algebra through linear mode conversion and interference, i.e., without requiring intrinsic photon-photon interaction. These fully controllable linear-optics elements therefore provide a practical route to test para-Majorana physics in photonic lattices before introducing true nonlinear media for deterministic multiphoton gates.

We also note that the  $\mathbb{Z}_N$ -graded parafermionic algebra for the grade group for one mode can, under suitable conditions, be reduced to a  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded superalgebra structure (see Appendixes B and C). This is achieved by decomposing the parafermionic generators into pairs of operators  $(\chi_i, \eta_i)$ , each graded under an independent  $\mathbb{Z}_2$  symmetry. The combined grading is then given by the bidegree  $(p_i, q_i)$ , where  $p_i, q_i \in \mathbb{Z}_2$ , and the commutation relations generalize to  $X_i X_j = (-1)^{p_i p_j + q_i q_j} X_j X_i$ . This decomposition preserves the essential algebraic features while enabling the construction of a superalgebraic framework, which may be particularly useful for connecting para-Majorana models to supersymmetric or superconformal extensions.

As an explicit example, consider the  $\mathbb{Z}_4$  parafermion algebra generated by the clock and shift matrices:

$$\sigma_c = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad (98)$$

satisfying  $\tau \sigma_c = i \sigma_c \tau$ . The operator  $\tau$  shifts the computational basis cyclically,  $\tau|k\rangle = |k+1 \pmod{4}\rangle$ ; hence, the name shift matrix. Conversely,  $\sigma_c$  leaves each basis vector in place but multiplies it by the phase  $\omega^k$  with equally spaced arguments  $\omega = e^{2\pi i/4} = i$ , i.e., by the “ticks” on a four-hour dial, so it is dubbed the clock matrix. Together, they satisfy the Weyl commutation rule  $\tau \sigma_c = \omega \sigma_c \tau$ , the defining relation of the  $\mathbb{Z}_4$  parafermion algebra.

These generators can be decomposed, with some caveats, into a  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded algebra by mapping  $\sigma_c$  and  $\tau$  onto Pauli matrices acting on two qubits or on a ququart from the product between two qubits,  $\sigma_c \rightarrow \sigma_z \otimes \mathbb{I}$ ,  $\tau \rightarrow \mathbb{I} \otimes \sigma_x$ , where  $\sigma_z$  and  $\sigma_x$  are Pauli operators. This decomposition splits the parafermionic generator into  $\psi_j \sim \chi_j \eta_j$ , where  $\chi_j$  and  $\eta_j$  are operators graded under the two independent  $\mathbb{Z}_2$  symmetries. Such a decomposition is permitted for  $N = 4$ , but not for prime  $N$ , due to the absence of a direct product decomposition of the underlying cyclic group. More precisely, the cyclic

group  $\mathbb{Z}_4 = \{0, 1, 2, 3\}$  is not isomorphic to the direct product  $\mathbb{Z}_2 \times \mathbb{Z}_2$ : It has an element of order 4, whereas every nonzero element of  $\mathbb{Z}_2 \times \mathbb{Z}_2$  has order 2. What is true is that the additive abelian group  $(\mathbb{Z}_4, +)$  becomes a two-dimensional vector space over the field  $\mathbb{Z}_2$  once we regard “ $2x = 0$ ” as scalar multiplication by 0 in  $\mathbb{Z}_2$ . In that  $\mathbb{Z}_2$ -module sense, we can write  $\mathbb{Z}_4 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ , but this identification forgets the full group structure (e.g.,  $1 + 1 = 2$  in  $\mathbb{Z}_4$ , whereas  $(1, 0) + (1, 0) = (0, 0)$  in  $\mathbb{Z}_2 \times \mathbb{Z}_2$ ). We stress that our decomposition  $\mathbb{Z}_4 \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_2$  is therefore a linear decomposition over  $\mathbb{Z}_2$ , not a group isomorphism.  $\mathbb{Z}_2 \times \mathbb{Z}_2$  is the Cartesian product of two copies of the two-element cyclic  $\mathbb{Z}_2 = \{0, 1\}$  with componentwise addition modulo 2,  $(a, b) + (c, d) = (a+c \bmod 2, b+d \bmod 2)$  and  $(a, b), (c, d) \in \{0, 1\} \times \{0, 1\}$  with the usual already defined four components. It is like exploiting the properties of a ququart as a tensorial product of two qubits, neglecting in this way all the additional properties of a ququart. Encoding a single physical ququart into two qubits keeps the same four-dimensional Hilbert space but trades away its native one-shot  $U(4)$  control requiring longer entangling gate sequences but permits in photonic platforms single-photon two qubit gates of deterministic type.

Ququarts provide a natural physical platform for realizing the  $\mathbb{Z}_4$  parafermion algebra. The generalized clock and shift operators acting on a ququart match the  $\mathbb{Z}_4$  relations  $\sigma_c^4 = \tau^4 = 1$ ,  $\sigma_c \tau = i \tau \sigma_c$ , where  $\sigma_c$  and  $\tau$  are the generalized Pauli operators. Furthermore, the ququart Hilbert space  $\mathcal{H}_4$  can be factorized as  $\mathcal{H}_2 \otimes \mathcal{H}_2$ , allowing the  $\mathbb{Z}_4$  grading to be decomposed into a  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded superalgebra. This provides flexibility in experimental implementations, either by directly manipulating ququart degrees of freedom or by encoding them in two entangled qubits.

A consistent two-qubit factorization of the  $\mathbb{Z}_4$  clock-shift pair is obtained with a clock operator; in the computational basis  $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ , the matrix  $\sigma_c = \text{diag}(1, i, -1, -i)$  is reproduced exactly by a Pauli  $Z$  on the *first* qubit and an  $S$  (“phase”) gate on the second qubit  $\sigma_c = Z \otimes S$ , with  $S := \text{diag}(1, i)$  and no extra global phase is needed.

The shift operator  $\tau$ , defined as adding one modulo four to the binary word  $|q_1 q_0\rangle$ , is implemented by the two-gate circuit  $\tau = X^{(0)} \text{CNOT}_{(0 \rightarrow 1)}$ , where the CNOT uses the least-significant qubit  $q_0$  as control and  $q_1$  as target, and  $X^{(0)}$  flips  $q_0$ . Acting on the basis states,  $|00\rangle \rightarrow |01\rangle \rightarrow |10\rangle \rightarrow |11\rangle \rightarrow |00\rangle$ . The resulting commutation rule is so given. Because  $Z$  commutes with the control of the CNOT and  $SX = iXS$ , one finds  $\tau \sigma_c = (X \otimes I)(\text{CNOT})(Z \otimes S) = i(Z \otimes S)(X \otimes I)(\text{CNOT}) = i \sigma_c \tau$ , preserving the Weyl relation  $\tau \sigma_c = i \sigma_c \tau$ .

Hence, the “clock” operator factorizes as a local phase  $Z \otimes S$ , while the “shift” operator is an entangling gate  $X^{(0)} \text{CNOT}_{(0 \rightarrow 1)}$ ; together, they provide an exact two-qubit realization of the  $\mathbb{Z}_4$  parafermion algebra.

An example of experimental realization of parafermionic ququarts using left- and right-circular polarizations combined with OAM modes can be simply summarized as follows. The ququart basis can be defined as  $\{|L, 0\rangle, |R, 0\rangle, |L, +1\rangle, |R, +1\rangle\}$ , where  $|L\rangle$  and  $|R\rangle$  denote circular polarization states and  $\ell$  is the OAM quantum number, or even  $\{|L, +1\rangle, |L, -1\rangle, |R, +1\rangle, |R, -1\rangle\}$  (if, instead, SAM and OAM are decoupled, one obtains a larger

Hilbert space with six independent states of a qudit with dimension  $d = 6$ ). The clock operator  $\sigma$  applies combined polarization- and OAM-dependent phase shifts, while the shift operator  $\tau$  cycles states across the four-dimensional basis. Shift operator  $\tau$  is defined by the action  $\tau|0\rangle = |1\rangle$ ,  $\tau|1\rangle = |2\rangle$ ,  $\tau|2\rangle = |3\rangle$ , and  $\tau|3\rangle = |0\rangle$ .

The operators satisfy  $\sigma_c^4 = \tau^4 = \mathbb{I}$ , where

$$\sigma_c^4 = \text{diag}(1, 1, 1, 1), \quad \tau^4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (99)$$

The key commutation relation is  $\tau \sigma_c = i \sigma_c \tau$ . Explicit calculation yields

$$\tau \sigma_c = \begin{pmatrix} 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \sigma_c \tau = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -1 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad (100)$$

which satisfy  $\tau \sigma_c = i \sigma_c \tau$ .

From the  $\mathbb{Z}_4$  clock and shift operators, we now decompose the four-dimensional space as a tensor product of two qubits,  $\mathcal{H}_4 \cong \mathcal{H}_2 \otimes \mathcal{H}_2$ , with basis  $|0\rangle \equiv |0\rangle_A |0\rangle_B$ ,  $|1\rangle \equiv |0\rangle_A |1\rangle_B$ ,  $|2\rangle \equiv |1\rangle_A |0\rangle_B$ , and  $|3\rangle \equiv |1\rangle_A |1\rangle_B$ .

In this basis, define two independent  $\mathbb{Z}_2$ -grading operators  $P = \sigma_z \otimes \mathbb{I}$  and  $Q = \mathbb{I} \otimes \sigma_z$ , where  $\sigma_z$  is the Pauli- $z$  matrix. Explicit matrix forms are

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (101)$$

The combined grading labels are  $|0\rangle \rightarrow (0, 0)$ ,  $|1\rangle \rightarrow (0, 1)$ ,  $|2\rangle \rightarrow (1, 0)$ , and  $|3\rangle \rightarrow (1, 1)$ . We can now rewrite  $\sigma$  and  $\tau$  as  $\sigma = PQ$ ;  $\tau$  is the cyclic permutation operator, where the cyclic  $\tau$  couples the two  $\mathbb{Z}_2$  sectors. This decomposition shows explicitly how the  $\mathbb{Z}_4$  algebra factors into two commuting  $\mathbb{Z}_2$ -grading sectors, providing a concrete realization of the  $\mathbb{Z}_4 \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_2$  reduction.

An optional effective Hamiltonian for the dynamics can be also

$$H = \omega \sum_{k=0}^3 \sigma_c^k + g \sum_{k=0}^3 \tau^k, \quad (102)$$

where  $\omega$  and  $g$  are experimentally tunable parameters.

These operations can be implemented in free space using wave plates, liquid crystal phase shifters,  $q$ -plates, and spatial-light modulators. Experimental observables include tests of the  $\mathbb{Z}_4$  algebra, cyclic state evolution, and coherence between polarization and OAM degrees of freedom, offering a promising platform for photonic parafermionic simulations.

For  $\ell = 0$  modes alone, quarter-wave plate (QWP) toggles the grade between  $(0, 0)$  and  $(0, 1)$ . Our default gate set therefore treats QWPs as legitimate grade-changing single-qubit rotations, or replaces an isolated QWP by the compensated sequence  $\text{QWP}(45^\circ) \rightarrow \text{HWP}(0^\circ) \rightarrow \text{QWP}(45^\circ)$ , which

TABLE VII. Internal qubit decomposition of the photonic ququart and deterministic one-photon devices that address them.

| Grade bit | Physical DOF      | Logical states                                         | Loss-free devices                               |
|-----------|-------------------|--------------------------------------------------------|-------------------------------------------------|
| $Q_a$     | OAM parity        | $\{ 0\rangle = a = 0,  1\rangle = a = 1\}$             | Dove prism in Sagnac, $q = \frac{1}{2}q$ -plate |
| $Q_b$     | helicity $\sigma$ | $\{ 0\rangle = \sigma = +1,  1\rangle = \sigma = -1\}$ | HWP stack QWP-HWP-QWP                           |

realizes the same physical polarization flip and returns the state to its original grade, thereby preserving sector-restricted logic when required.

Rectangular waveguides, instead, provide another alternative platform for realizing OAM-SAM coupled ququarts. In this setting, the hybridization of SAM (associated with circular polarization) and OAM arises due to the broken cylindrical symmetry of the rectangular cross section, leading to spin-orbit coupled modes. The ququart basis  $\{|L, 0\rangle, |R, 0\rangle, |L, +1\rangle, |R, +1\rangle\}$  maps directly onto the parafermionic algebra, where the clock operator  $\sigma$  corresponds to applying OAM- and SAM-dependent phase shifts, achievable via birefringence or geometric phase elements, and the shift operator  $\tau$  represents cyclic transitions between the hybrid modes, implemented through controlled spin-orbit coupling perturbations such as asymmetric tapers, strain, or nanostructured gratings. This architecture enables the exploration of  $\mathbb{Z}_4$  algebraic relations and topological dynamics in an integrated photonic platform, offering a promising route toward parafermionic quantum simulation and potentially topological quantum computation.

Relevant theoretical and experimental works supporting this approach, including the use of spinorbit-coupled ququarts in rectangular waveguides, build on studies of spinorbit interactions in nanophotonic systems [75–78] and related developments in topological/parafermionic photonics [79–82]. Notably, the integration of SAM-OAM coupling in guided-wave platforms and the demonstration of topological edge states in photonic lattices provide critical tools for realizing  $\mathbb{Z}_4$  parafermionic models in photonic systems.

### I. Deterministic gate set with the $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded ququart

Unlike standard linear-optical processors that need two-photon interference and postselection, our  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded ququart platform implements a universal gate set deterministically with passive, loss-free spin-orbit devices, thereby removing the exponential heralding overhead. Encoding two logical qubits inside a *single* photon also doubles the Hilbert-space dimension per particle, giving higher information density and a smaller component count while the built-in grade symmetry provides intrinsic error suppression.

In this case, we can build two internal qubits inside one photon. The four logical basis states defined in Eq. (79) may be viewed as *two* qubits carried by the same photon,  $Q_a := a = \ell \bmod 2$  and  $Q_b := b = (1 - \sigma)/2$ , where  $\ell \in \{-1, 0, +1\}$  is the OAM and  $\sigma = \pm 1$  the helicity. Table VII lists the mapping together with off-the-shelf optical elements

TABLE VIII. Grade sectors for the spin-orbit ququart. Left (right) circular polarization corresponds to  $\sigma = +1$  ( $-1$ ). The grade bits are  $a = \ell \bmod 2$  and  $b = (1 - \sigma)/2$ .

| Mode       | OAM $\ell$ | SAM $\sigma$ | Grade $(a, b)$ |
|------------|------------|--------------|----------------|
| $\phi_1$   | +1         | +1 (LCP)     | (1, 0)         |
| $\phi_2$   | -1         | -1 (RCP)     | (1, 1)         |
| $\phi_3^A$ | 0          | +1 (LCP)     | (0, 0)         |
| $\phi_3^B$ | 0          | -1 (RCP)     | (0, 1)         |

that act deterministically on each qubit. The corresponding grade sectors are summarized in Table VIII

The primitive single-qubit rotations can be defined in the following way:

(1) *On  $Q_b$  (SAM)*: a HWP set at  $45^\circ$  realizes the Pauli  $X_b$ , while a QWP-HWP-QWP sequence spans arbitrary  $SU(2)_b$  rotations.

(2) *On  $Q_a$  (OAM parity)*: a  $90^\circ$ -rotated Dove prism inside a polarization-insensitive Sagnac interferometer applies  $Z_a$ ; a  $q = 1/2q$ -plate tuned to  $\delta = \pi$  swaps  $\ell = 0 \leftrightarrow \pm 1$  and realizes  $X_a$ .

All these operations are passive; no ancilla photons or post-selection is required. Deterministic two-qubit entangling gate is so realized on the degrees of freedom of a single photon. Set a  $q = 1q$ -plate tuned to retardation;  $\delta = \pi$  couples the two degrees of freedom as [72,83]

$$\mathcal{QP}_\pi : |a\rangle_{Q_a} |b\rangle_{Q_b} \mapsto |a \oplus b\rangle_{Q_a} |b\rangle_{Q_b}, \quad (103)$$

which is a controlled-NOT with  $Q_b$  the control and  $Q_a$  the target. Prepending or postpending a HWP swaps the control and target roles.

An alternative all-glass implementation employs a polarization Sagnac interferometer with an embedded  $45^\circ$  Dove prism, demonstrated by Fiorentino and Wong [84] as a fully deterministic CNOT between polarization and momentum of a single photon. The gate set

$$\mathcal{G}_{\text{det}} = \{X_b, H_b, Z_a, H_a, \text{CNOT}_{b \rightarrow a}\} \quad (104)$$

(where  $H$  denotes the Hadamard rotation on the relevant qubit) is universal for  $SU(4)$  on the ququart and contains only deterministic, single-photon devices.

Grade-conservation rule: operations that flip exactly one grade bit, e.g.,  $X_b, X_a$ , or the above CNOT, change the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  charge. Our computational model therefore requires either explicit declaration of such gates as “grade interfaces” or a compensating inverse later in the circuit so that the net global grade is conserved. Because every gate in  $\mathcal{G}_{\text{det}}$  is loss-free, inserting such compensators does not reduce determinism or fidelity.

Deterministic interphoton entangling gates can be realized by combining the single-photon CNOT above with a polarization Sagnac that couples one photon’s  $Q_b$  to the path qubit of a second photon, or a semiconductor spin-photon interface providing a giant optical nonlinearity. Both preserve the global  $\mathbb{Z}_2 \times \mathbb{Z}_2$  charge and extend the present scheme toward scalable photonic paraparticle logic. For convenience, the symbols and notation used in this section are summarized in Table IX.

TABLE IX. Summary of symbols and notation used with the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded algebraic framework.

| Symbol                                        | Meaning                                                                        | Notes                                                                                                                 |
|-----------------------------------------------|--------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------|
| $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_2$ | Graded index for Lie algebra sectors                                           | Indices like $a, b \in \{0, 1\}$ label bosoniclike/fermioniclike sectors                                              |
| $\hat{\psi}_{(a,b)}^\pm$                      | Creation (+)/annihilation (−) operator in graded sector $(a, b)$               | Operators satisfy trilinear commutation/anticommutation relations                                                     |
| $\Psi_{(a,b)}$                                | Component of the infinite-dimensional Majorana wavefunction in sector $(a, b)$ | Part of graded decomposition of $\Psi$                                                                                |
| $\Gamma_{(a,b)}^\mu$                          | Graded gamma matrices in sector $(a, b)$                                       | Generalization of standard gamma matrices                                                                             |
| $M_{(a,b)}$                                   | Effective mass term in sector $(a, b)$                                         | $M_{(a,b)} = M(s_{(a,b)} + 1/2)$                                                                                      |
| $(a, b) \cdot (a', b')$                       | Scalar product between graded indices                                          | Defined as $aa' + bb' \pmod{2}$                                                                                       |
| $[X, Y]$                                      | Graded commutator                                                              | $XY - (-1)^{(a,b) \cdot (a',b')} YX$                                                                                  |
| $\Delta(\hat{\psi}_{(a,b)})$                  | Braided coproduct in graded Hopf algebra                                       | $\hat{\psi}_{(a,b)} \otimes I + \sum R_{(c,d)}^{(a,b)} I \otimes \hat{\psi}_{(c,d)}$                                  |
| $R_{(c,d)}^{(a,b)}$                           | Braiding matrix coefficients                                                   | Encode exchange statistics; satisfy Yang-Baxter equation                                                              |
| $\theta_{ss'}$                                | Deformation parameter for exchange statistics                                  | $\theta_{ss'} = \epsilon_{ss'}(q^{s+s'} - 1)$ , $ q  = 1$                                                             |
| $P_{(a,b)}$                                   | Projector onto graded sector $(a, b)$                                          | $P_{(a,b)}\Psi = \Psi_{(a,b)}$ , $\sum_{(a,b)} P_{(a,b)} = I$                                                         |
| $X_{(a,b),(a',b')}$                           | Exchange matrix for graded sectors                                             | $(-1)^{(a,b) \cdot (a',b')}$                                                                                          |
| $n_{(a,b)}$                                   | Number operator in graded sector                                               | $n_{(a,b)} = \hat{\psi}_{(a,b)}^+ \hat{\psi}_{(a,b)}^-$                                                               |
| $E_{(a,b),(a',b')}$                           | Exchange operator between graded sectors                                       | $E_{(a,b),(a',b')} = \hat{\psi}_{(a,b)}^+ \hat{\psi}_{(a',b')}^-$                                                     |
| $H$                                           | Hamiltonian operator                                                           | $H = \sum_{(a,b)} \epsilon_{(a,b)} n_{(a,b)}$                                                                         |
| $s_{(a,b)}$                                   | Spin associated with sector $(a, b)$                                           | $s = 0 \rightarrow (0, 0)$ , $s = 1/2 \rightarrow (1, 0)$ , $s = 1 \rightarrow (1, 1)$ , $s = 3/2 \rightarrow (0, 1)$ |

### J. Universality of the deterministic gate set $G_{\text{det}}$

We define the logical ququart  $\{|L, 0\rangle, |R, 0\rangle, |L, +1\rangle, |R, +1\rangle\}$ , and it is isomorphic to two qubits  $Q_a \otimes Q_b$ , with the identification  $|\ell \pmod{2}\rangle_{Q_a} \otimes |(1-\sigma)/2\rangle_{Q_b}$ . In this basis, the deterministic gate set reads (subscripts indicate the qubit on which the single-qubit operator acts.)  $G_{\text{det}} = \{X_b, H_b\} \cup \{Z_a, H_a\} \cup \{\text{CNOT}_{b \rightarrow a}\}$ .

*Step 1: local controllability.* The pairs  $\{H, Z\}$  and  $\{H, X\}$  each generate  $\text{SU}(2)$  because  $ZH = e^{\frac{\pi}{2}i(\sigma_x + \sigma_z)/\sqrt{2}}$  and together with  $H$  they form a basis of the Lie algebra  $\mathfrak{su}(2)$ . Hence  $\{Z_a, H_a\}$  gives arbitrary single-qubit rotations on  $Q_a$ , and  $\{X_b, H_b\}$  does the same on  $Q_b$ .

*Step 2: entangling power of  $\text{CNOT}_{b \rightarrow a}$ .* Write a two-qubit gate in its Cartan (KAK) decomposition normal form

$$U = (k_1 \otimes k_2) e^{i(c_1 \sigma_x \otimes \sigma_x + c_2 \sigma_y \otimes \sigma_y + c_3 \sigma_z \otimes \sigma_z)} (k_3 \otimes k_4).$$

The triplet  $(c_1, c_2, c_3) \in [0, \pi/2]^3$  lives in the Weyl chamber; if any component is nonzero, the gate is entangling.  $\text{CNOT}_{b \rightarrow a}$  is locally equivalent to  $\exp(\frac{i\pi}{2} \sigma_z \otimes \sigma_x)$ ; its Cartan coordinates are  $(\frac{\pi}{2}, 0, 0)$ , so it lies on the chamber edge of perfect entanglers.

*Step 3: Brylinski-Brylinski criterion.* A gate set is universal for  $\text{SU}(4)$  iff it contains a dense subgroup of local operations  $\text{SU}(2) \otimes \text{SU}(2)$  and at least one entangling two-qubit gate. These conditions are satisfied by steps 1 and 2; therefore,  $G_{\text{det}}$  is universal.

*Constructive three-CNOT decomposition.* For completeness, we give an explicit synthesis of an arbitrary  $U \in \text{SU}(4)$  in terms of elements of  $G_{\text{det}}$ . Following the algorithm of Vatan-Williams [85], one writes

$$U = (k_1 \otimes k_2) \text{CNOT}_{b \rightarrow a} (k_3 \otimes k_4) \circ \text{CNOT}_{b \rightarrow a} (k_5 \otimes k_6) \text{CNOT}_{b \rightarrow a} (k_7 \otimes k_8), \quad (105)$$

where each  $k_j \in \text{SU}(2)$  can be compiled with at most three gates from  $\{H, Z\}$  (on  $Q_a$ ) or  $\{H, X\}$  (on  $Q_b$ ) using Euler angles. The total depth is therefore three CNOT + 24 single-qubit rotations, matching the optimal gate counts known from the standard two-qubit model.

A remark on grade conservation: as  $X_b$  flips  $b$  while preserving  $a$ , and  $Z_a$  flips the phase of  $a$  while preserving  $b$ , every gate changes either zero or exactly one  $\mathbb{Z}_2 \times \mathbb{Z}_2$  charge bit. Circuits that must return to a fixed global grade can therefore insert a compensating gate without affecting universality or determinism.

## VII. PHOTONIC MODES AND ALGEBRAIC GENERATORS CORRESPONDENCE

### A. SAM-OAM modes as graded generators

We now summarize in a few points a dictionary of correspondences between the photonic SAM-OAM modes and the generators of the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded algebra here used, making the correspondence operational.

(1) *Grading map and projectors.* On the guided-mode basis  $\{|\ell, \sigma\rangle\}$ , we use the grading

$$g : (\ell, \sigma) \mapsto (a, b) := (\ell \pmod{2}, (1 - \sigma)/2) \quad (106)$$

and define the sector projectors

$$P_{(a,b)} := \sum_{(\ell, \sigma) : g(\ell, \sigma) = (a,b)} |\ell, \sigma\rangle \langle \ell, \sigma|. \quad (107)$$

(2) *Mode operators and matrix units.* Graded creation/annihilation operators are

$$\psi_{(a,b)}^\dagger := \sum_{(\ell, \sigma)} P_{(a,b)}^{(\ell, \sigma)} a_{\ell, \sigma}^\dagger, \quad \psi_{(a,b)} := (\psi_{(a,b)}^\dagger)^\dagger \quad (108)$$

and obey the graded bilinear relations in the main text. From these, we form the number and matrix-unit generators

$$N_{(a,b)} := \psi_{(a,b)}^\dagger \psi_{(a,b)}, \quad E_{(a,b);(c,d)} := \psi_{(a,b)}^\dagger \psi_{(c,d)}. \quad (109)$$

$E_{g,h}$  implement grade-preserving transitions ( $g = h$ ) and intersector maps ( $g \neq h$ ); together with  $N_g$  they generate the operator algebra that acts on the ququart.

(3) *Grade-parity (charge) operators.* The two commuting  $\mathbb{Z}_2$  “charges” are measured by

$$\begin{aligned} \Pi_a &:= (-1)^{N_a}, & N_a &:= \sum_{b=0,1} N_{(1,b)}, \\ \Pi_b &:= (-1)^{N_b}, & N_b &:= \sum_{a=0,1} N_{(a,1)}. \end{aligned} \quad (110)$$

Operationally,  $\Pi_a$  is OAM parity (even/odd  $\ell$ );  $\Pi_b$  is SAM (helicity) or the  $\ell = 0$  waveguide-mode bit.

(4) *Physical single-photon realizations of generators.* We realize the elementary grade-flip unitaries as follows:

$U_a$ : flip OAM parity  $a \mapsto a \oplus 1 \rightarrow q$ -plate ( $q = \frac{1}{2}, \delta = \pi$ ) or Dove-Sagnac parity gate;

$U_b$ : flip SAM/wg bit  $b \mapsto b \oplus 1 \mapsto$  HWP at  $45^\circ$  or  $A \leftrightarrow B$  mode swap for  $\ell = 0$ .

On the ququart,  $\{U_a, U_b\}$  generate the Pauli group on the two internal grade bits. Controlled versions (e.g.,  $\text{CNOT}_{b \rightarrow a}$ ) are implemented with a  $q = 1$  plate at  $\delta = \pi$  or with a polarization Sagnac + Dove prism, as detailed later in this section.

(5) *Exchange-sign (braid) operator.* The graded sign in the bracket,  $[X, Y]_{\text{gr}} = XY - (-1)^{g(X) \cdot g(Y)} YX$ , is encoded interferometrically by a two-arm Mach-Zehnder in which one arm applies a  $\pi$  phase conditioned on the grade bit. Let  $R_{g,h}$  denote the unit that applies  $(-1)^{g \cdot h}$ ; then, a sequence  $U_g$  followed by  $U_h$  in one arm and the reversed order in the other arm with  $R_{g,h}$  realizes an operational test of the graded sign (next subsection).

## B. Experimental observables that reveal the graded structure

We now list concrete, grade-resolved observables, and minimal protocols.

(1) *(O1) Grade-parity measurements.* Directly measure  $\langle \Pi_a \rangle$  and  $\langle \Pi_b \rangle$  by demultiplexing OAM parity and SAM (or A/B) and counting in the corresponding output ports. This reads out the two  $\mathbb{Z}_2$  charges on the ququart.

(2) *(O2) Commutation-phase interferometer [witness of  $(-1)^{g \cdot h}$ ].* This is articulated as follows: Build a balanced Mach-Zehnder. In arm 1, implement  $U_g$  and then  $U_h$ ; in arm 2, implement  $U_h$  and then  $U_g$ ; insert a conditional  $\pi$  phase  $R_{g,h}$  so that the two paths interfere with relative phase  $\phi_{g,h} = \pi(g \cdot h)$ . With an input state  $|\psi\rangle$ , the output port intensities are

$$I_{\pm} = \frac{1}{2}(1 \pm \cos \phi_{g,h}) = \begin{cases} \frac{1}{2}(1 \pm 1), & g \cdot h = 0, \\ \frac{1}{2}(1 \mp 1), & g \cdot h = 1, \end{cases} \quad (111)$$

so an *odd* grade inner product swaps the bright/dark ports. This is a direct measurement of the graded commutation sign.

(3) *(O3) Selection-rule suppression (off-diagonal blocks).* Drive a specific  $E_{(a,b);(c,d)}$  via a mode-selective coupler. Transitions forbidden by the grading (or by the device’s symmetry) are suppressed; allowed transitions show Rabi oscillations.

Comparing transfer visibilities across all four sectors reconstructs the block structure dictated by  $P_{(a,b)}$ .

(4) *(O4) Trilinear Green-Volkov bracket witness.* Use the sequence implementing the operational bracket  $[b, \{V_\phi, b\}]$ , where  $b := P_{(1,0)}U_{q=1/2}$  and  $V_\phi$  is a small spin rotation (HWP angle  $\phi/2$ ). Measuring the output amplitude as a function of  $\phi$  yields  $2 \sin \phi$ ; setting  $\phi = \pi/2$  gives 2, i.e., the parabolic trilinear coefficient. This realizes an experimental witness of the Green identity.

(5) *(O5) Deterministic grade-resolved CNOT truth table.* Run process tomography on the single-photon  $\text{CNOT}_{b \rightarrow a}$  ( $q$ -plate with  $q = 1$ ,  $\delta = \pi$ , or the Dove-Sagnac design) and verify (1) logical truth table and (2) global grade accounting (grade-flip count is odd for CNOT, so the net grade must be restored by a compensating gate later in the circuit).

(6) *(O6) Clock/shift algebra test.* Implement the  $\mathbb{Z}_4$  clock  $\sigma_c$  (mode-dependent phases:  $0, \pi/2, \pi, 3\pi/2$ ) and cyclic shift  $\tau$  (nearest-neighbor coupling) on the four-mode manifold. Verify  $\tau \sigma_c = i \sigma_c \tau$  and  $\tau^4 = \sigma_c^4 = \mathbb{1}$  by Ramsey-style sequences; this is the “generator” view complementary to the matrix units  $E_{g,h}$ .

## C. Minimal photonic circuits (from Sec. VI)

*Commutation-phase interferometer.*

- (1) Input: any ququart basis state (e.g.,  $|0, A\rangle$ ).
- (2) Arm 1: HWP( $45^\circ$ ) ( $U_b$ ), then  $q$ -plate ( $q = \frac{1}{2}, \delta = \pi$ ) ( $U_a$ ).
- (3) Arm 2:  $q$ -plate, then HWP in the opposite order.
- (4) Insert a grade-conditional  $\pi$  phase (e.g., a polarization  $\pi$  shifter active only for  $b=1$ ) to realize  $R_{(a,b);(a',b')}$ .
- (5) Recombine; detect the two output ports. Odd  $g \cdot h$  flips the bright port relative to even  $g \cdot h$ .

*C2. Green-Volkov trilinear bracket witness.*

- (1) Prepare an odd- $a$  sector (e.g.,  $|\ell = +1, L\rangle$ ).
- (2) Apply  $b := P_{(1,0)}U_{q=1/2}$ ; then, a small spin rotation  $V_\phi$  (HWP at  $\phi/2$ ); then  $b$  again.
- (3) Measure the amplitude in the target sector versus  $\phi$ . Fit  $A(\phi) \propto 2 \sin \phi$ ; at  $\phi = \pi/2$ , the prefactor equals 2.

*Device notes.*  $U_a$  may be implemented either with a  $q = \frac{1}{2}$  plate at  $\delta = \pi$  ( $\ell$  parity flip) or a Dove prism inside a polarization-insensitive Sagnac (OAM parity gate).  $U_b$  is an HWP at  $45^\circ$  (or an A/B mode swap for  $\ell = 0$ ). A  $q = 1$  plate at  $\delta = \pi$  couples SAM and OAM and realizes  $\text{CNOT}_{b \rightarrow a}$ . Conditional phases for  $R_{g,h}$  are realized as grade-dependent birefringent sections or interferometric phase inserts.

## D. Summary

The following table summarizes the dictionary and the observables.

These observables certify the graded structure at the single-photon, linear-optics level and are sufficient to validate the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  algebra experimentally on the SAM-OAM ququart.

## VIII. ERROR CORRECTION WITH PARAPARTICLES AND FRACTIONAL NELSON’S QUANTUM MECHANICS

Error correction can be realized in a way that preserves all the set of qubits, through the repetition of the calculation and statistical analysis of the results obtained so far. Fractional

| Generator (algebra)                | Photonic realization                                                 |
|------------------------------------|----------------------------------------------------------------------|
| $N_{(a,b)}$ (number in sector)     | grade-resolved demultiplexing and counting                           |
| $E_{(a,b);(c,d)}$ (matrix unit)    | mode-selective couplers/ $q$ -plates/MZIs                            |
| $\Pi_a, \Pi_b$ (grade charges)     | OAM parity (Dove-Sagnac) and SAM/A/B analyzers                       |
| $U_a$ (flip $a$ )                  | $q = \frac{1}{2}$ plate at $\delta = \pi$ or Dove-Sagnac parity gate |
| $U_b$ (flip $b$ )                  | HWP(45°) or A/B waveguide swap at $\ell = 0$                         |
| $R_{g,h}((-1)^{g \cdot h})$        | grade-conditional $\pi$ phase in MZI                                 |
| Green bracket $[b, \{V_\phi, b\}]$ | $b = P_{(1,0)}U_{q=1/2}$ ,<br>$V_\phi$ = small HWP rotation          |

Nelson's quantum mechanics extends the standard stochastic quantization picture by letting the underlying particle paths follow Lévy flights rather than Brownian motion, so the dynamical fractional Lévy order  $\alpha$  becomes a tunable parameter that continuously sweeps through bosonic, fermionic, and intermediate paraparticle statistics. Embedding our spin-orbit photonic ququarts (or higher-dimensional qudits) in such a fractional medium therefore gives a hardware-level control parameter for both the graded spectrum and the braid phase, enabling deterministic gate synthesis while naturally modeling the power-law, non-Markovian noise that dominates integrated-photonics platforms.

The adapting of Nelson's Quantum Mechanics to Paraparticles is obtained through generalized statistics extending the usual boson/fermion dichotomy, with trilinear commutation or anticommutation relations rather than bilinear ones. To integrate paraparticles into Nelson's formalism, the standard Wiener process must generalize to accommodate nontrivial statistical exchanges, potentially via algebraic deformations of Brownian motions governed by  $R$ -matrix quantization [86] and graded Lie algebras.

Nelson's stochastic formulation of quantum mechanics provides an elegant interpretation of quantum dynamics through underlying stochastic processes, described mathematically by forward and backward stochastic differential equations [87]. The standard form of these equations is given by  $dX(t) = b_+(X(t), t)dt + dW_+(t)$  and backward in time,  $dX(t) = b_-(X(t), t)dt + dW_-(t)$ , where  $W_\pm(t)$  represent standard Wiener processes, and the quantum potential emerges naturally through the difference of these drift terms. They represent standard Brownian motions. Replacing Brownian trajectories in Nelson's stochastic quantization by Lévy flights produces a fractional Schrödinger (or Majorana) equation whose order  $\alpha \in (1, 2]$  continuously interpolates between bosonic, fermionic, and paraparticle exchange statistics [88,89].

In our graded model, the same index  $\alpha$  can be mapped onto the paraparticle order  $p$  (and hence onto the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  grade), providing a *single tunable parameter* that controls both the spectrum [Eq. (12)] and the braided  $R$  matrix.

From an experimental viewpoint, fractional dynamics emerges naturally in photonics whenever long-range evanescent coupling or nonlocal metasurfaces are engineered: recent optical platforms already realize the fractional Schrödinger equation with unit efficiency and no postselection [90]. Embedding our graded ququart in such a medium therefore lets one dial the effective order  $\alpha$ —hence the paraparticle statistics—*in situ*. This adds a hardware knob for deterministic gate synthesis and offers built-in modeling of power-law, non-Markovian noise, which is the dominant decoherence channel in heterogeneous integrated photonics.

In short, the fractional Nelson formalism supplies both a seamless mathematical bridge between different paraparticle orders and an experimentally accessible control parameter that can be exploited for robust, deterministic gate design in spin-orbit photonic circuits.

In the fractional Schrödinger/Majorana equation, the Laplacian is replaced by its Riesz-fractional power, with the fractional order symbol  $\alpha$ :

$$i\hbar\partial_t\psi = \frac{\hbar^\alpha}{2m} D_\alpha(-\Delta)^{\alpha/2}\psi + V\psi, \quad 1 < \alpha \leq 2, \quad (112)$$

so the real number  $\alpha$  is the *order* of the derivative and  $D_\alpha$  the related scale factor. For  $\alpha = 2$ , Eq. (112) reduces to the ordinary Schrödinger equation; lowering  $\alpha$  from 2 toward 1 stretches the derivative into a nonlocal operator with power-law tails [88,89].

In Nelson's stochastic interpretation, the same  $\alpha$  is the stability index of the underlying Lévy flights: a Brownian path ( $\alpha = 2$ ) acquires heavy-tailed step lengths  $P(\ell) \propto \ell^{-(1+\alpha)}$  when  $1 < \alpha < 2$ . Within our graded model, we identify  $\alpha$  with the paraparticle order  $p$ , turning it into a continuous hardware “knob” that sweeps smoothly from bosonic statistics ( $\alpha = 2$ ) through parafermionic regimes down toward the fermionic limit ( $\alpha \rightarrow 1$ ). Photonic platforms that engineer long-range evanescent coupling already realize Eq. (112) optically and allow *in situ* tuning of  $\alpha$ , providing deterministic, grade-preserving control over both the spectrum [Eq. (12)] and the braid phase [90].

Besides the deterministic potential  $V$ , the quantum potential  $Q(x, t)$  emerges naturally as

$$Q(x, t) = \frac{\hbar^2}{2m} \frac{\Delta\sqrt{\rho(x, t)}}{\sqrt{\rho(x, t)}}. \quad (113)$$

Then, one defines the graded probability and measure. The probability measures must now be generalized to graded probability measures. Expectation values involve graded trace operations of the type  $\langle O \rangle = \text{Tr}_{\text{grad}}[\rho O]$  of a graded observable  $O$  with graded density matrix  $\rho$ . The quantum potential  $Q(x, t)$  must adapt to paraparticle symmetry, now arising from graded algebraic potentials.

In graded quantum systems, the definition of expectation value  $\langle O \rangle$  of an observable  $O$  in a density matrix  $\rho$  depends on the graded trace, which is defined as

$$\text{Tr}_{\text{grad}}[X] = \sum_{(a,b)} (-1)^{\epsilon(a,b)} \text{Tr}[X_{(a,b)}], \quad (114)$$

with  $\epsilon(a, b)$  assigning the appropriate sign factor over sectors. This construction ensures that both the probability measures

and the quantum potential  $Q(x, t)$  correctly incorporate the paraparticle symmetry and graded algebraic structure.

Fractional Brownian motion (fBm) introduces long-range temporal correlations characterized by the Hurst parameter  $H = \alpha/2 \in (0, 1)$ . Its covariance structure is defined by

$$\mathbb{E}[B_H(t)B_H(s)] = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}). \quad (115)$$

Replacing standard Brownian increments with fractional increments, the fractional dynamics becomes

$$dX_H(t) = b_H(X_H(t), t)dt + dB_H(t). \quad (116)$$

The fractional Schrödinger equation associated with these dynamics is then expressed as follows:

$$i\hbar \partial_t \psi(x, t) = -D(-\Delta)^H \psi(x, t) + V(x)\psi(x, t), \quad (117)$$

where the fractional Laplacian  $(-\Delta)^H$  is defined via Fourier transform

$$(-\Delta)^H f(x) = \mathcal{F}^{-1}\{|k|^{2H} \mathcal{F}\{f\}(k)\}(x). \quad (118)$$

Consequently, the fractional quantum potential is expressed in terms of the Hurst's exponent.

$$Q_H(x, t) = -\frac{\hbar^2}{2m} \frac{(-\Delta)^H \sqrt{\rho(x, t)}}{\sqrt{\rho(x, t)}}, \quad (119)$$

to which can be added also a deterministic function  $V$ .

### A. Fractional-graded stochastic dynamics and calculus

Combining fractional Brownian motion with paraparticle algebra yields fractional-Gaussian stochastic differential equations

$$d\hat{X}_H(t) = \hat{b}_H(\hat{X}_H(t), t)dt + d\hat{B}_H(t), \quad (120)$$

where the increments  $d\hat{B}_H(t)$  obey graded fractional relations of the following type, which are quadratic variation of graded fBm.

$$dB_{(a)}^{(H)}(t) dB_{(b)}^{(H)}(t') = \delta_{ab} \gamma_H(t - t') dt dt', \quad (121)$$

where  $\gamma_H(\tau) := H(2H - 1)$  denotes the fractional correlation structure and  $|\tau|^{2H-2}$ ,  $\tau := t - t'$ .

The covariance kernel  $\gamma_H(\tau)$  is even,  $\gamma_H(\tau) = \gamma_H(-\tau)$ , reflecting time-reversal symmetry of the noise correlations. Then follows the *Brownian limit*  $H \rightarrow \frac{1}{2}$  for which  $\gamma_H(\tau) \rightarrow \delta(\tau)$ , recovering the local Itô rule.

*Long memory* is achieved for  $H > \frac{1}{2}$ ;  $\gamma_H(\tau) \propto |\tau|^{2H-2}$  is integrable but nonlocal, producing the  $1/f^{2H-1}$  spectral density characteristic of fractional Brownian dynamics.

The integration of fractional-graded stochastic processes is defined via generalized Itô-Skorokhod integrals [91]

$$\int_0^T \hat{F}(t) d\hat{B}_H(t) := \lim_{|\Pi| \rightarrow 0} \sum_j \hat{F}(t_j^*) (\hat{B}_H(t_{j+1}) - \hat{B}_H(t_j)), \quad (122)$$

with algebraic definitions ensuring graded commutations.

The quantum potential in this fractional-graded framework of Eq. (119) is defined via fractional-graded functional calculus, maintaining paraparticle algebra consistency. Fractional Brownian increments  $dB_H(t)$  employed in Eqs. (120) and

(121) are defined through the Itô-Skorokhod integral framework. Specifically, the increments satisfy Eq. (115) involving the Hurst parameter  $H$ . All integrals involving fractional increments are assumed to satisfy standard conditions of integrability and adaptedness required by fractional stochastic calculus.

Explicit fractional-graded Nelson equations are formulated by defining the forward and backward equations,

$$d\hat{X}_H^+(t) = \hat{b}_H^+(\hat{X}_H^+(t), t)dt + d\hat{B}_H^+(t),$$

$$d\hat{X}_H^-(t) = \hat{b}_H^-(\hat{X}_H^-(t), t)dt + d\hat{B}_H^-(t).$$

These equations lead to a fractional and graded Schrödinger-type equation respecting paraparticle statistics. This combined fractional-graded approach introduces non-Markovian, algebraically rich quantum dynamics, providing a theoretical foundation for experimentally realizable quantum statistical phenomena in structured photonic systems and other condensed matter analogs.

### B. Structured photonic modes and Nelson's formalism

We extend Nelson's fractional-graded quantum mechanics to structured photonic systems, specifically focusing on optical quantum circuits utilizing OAM. Structured photonic modes naturally provide a robust experimental platform for realizing the theoretical constructs introduced above.

Consider optical modes characterized by their SAM and OAM. These structured photonic states can be described mathematically as

$$\Psi_{\ell, \sigma}(\mathbf{r}, t) = \psi_{\ell, \sigma}(z, t) \phi_{\ell, \sigma}(x, y) e^{i(\beta_{\ell, \sigma} z - \omega t)}, \quad (123)$$

where  $\ell$  denotes OAM,  $\sigma = \pm 1$  indicates polarization states or other independent waveguide modes, and  $\beta_{\ell, \sigma}$  is the propagation constant.

Fractional-graded quantum dynamics of OAM modes given by Nelson's fractional-graded stochastic framework translates into the structured photonic context by treating each OAM mode as Majorana pseudoparticle corresponding to a paraparticle state with fractional-graded stochastic dynamics,

$$d\hat{\psi}_{\ell, H}(z, t) = \hat{b}_{\ell, H}(z, t)dt + d\hat{B}_{\ell, H}(z, t), \quad (124)$$

where  $d\hat{B}_{\ell, H}(z, t)$  is a graded fractional increment satisfying the fractional-graded algebra defined earlier.

### C. Quantum potential and mode coupling

The fractional graded quantum potential in the context of structured photonic OAM modes becomes

$$\hat{Q}_{\ell, H}(z, t) = -\frac{\hbar^2}{2m} \frac{(-\partial_z^2)^H_{\text{graded}} \sqrt{\hat{\rho}_{\ell}(z, t)}}{\sqrt{\hat{\rho}_{\ell}(z, t)}}. \quad (125)$$

This potential governs the nonlinear coupling and interactions among various OAM modes. The operator  $(-\partial_z^2)^H$  is understood as the Riesz fractional derivative (or fractional Laplacian), a symmetric and nonlocal generalization of the standard Laplacian. It is defined via its Fourier representation:

$$\mathcal{F}[(-\partial_z^2)^H f(z)](k) = |k|^{2H} \mathcal{F}[f](k),$$

where  $H \in (0, 1]$  denotes the fractional order. This operator naturally encodes long-range correlations and anomalous dispersion, features essential to the graded quantum dynamics considered here respecting the algebraic rules of  $\mathbb{Z}_2 \times \mathbb{Z}_2$  sectoral structure of the field  $\hat{\rho}_\ell(z, t)$ , which originates from the SAM-OAM decomposition of the photonic paraparticle modes. Each density field  $\hat{\rho}_{(a,b)}$  belongs to a specific graded sector labeled by  $(a, b)$ , and the dynamics governed by  $\hat{Q}_{(a,b)}$  obey graded commutation relations and selection rules. These rules enforce algebraic constraints such as sector-preserving evolution and graded exclusion, making the potential sector-dependent and algebraically constrained. Thus, the fractional operator acts not only nonlocally in space but also coherently within the graded algebraic framework. Alternative definitions such as the Caputo or Riemann-Liouville derivatives are not suitable in this spatial context, as they are asymmetric and tailored for time-fractional processes. Quantum gates with fractional-graded OAM modes are implemented using and implementing quantum logic gates within optical waveguides by exploiting the fractional-graded dynamics of structured modes. A deterministic CNOT gate, for instance, can be realized by using graded paraparticle statistics encoded in SAM-OAM coupled modes, leading to stable and robust quantum logic operations of the type

$$|\ell, \sigma\rangle \rightarrow \sum_{(\ell', \sigma')} C_{(\ell', \sigma')}^{(\ell, \sigma)} |\ell', \sigma'\rangle, \quad (126)$$

with the coefficients  $C_{(\ell', \sigma')}^{(\ell, \sigma)}$  governed by fractional-graded quantum dynamics.

Structured photonic systems, such as multimode integrated waveguides and fiber-based circuits, represent viable experimental platforms for these fractional-graded quantum dynamics. The inherent robustness of structured-light modes, combined with fractional quantum dynamics, offers a powerful avenue for scalable and error-resilient quantum computing architectures.

Future research could focus on explicit experimental validations, numerical simulations of fractional-graded dynamics in integrated photonic circuits, and further theoretical developments toward realizing robust quantum information processing and quantum communication protocols using structured optical modes. Nelson's fractional-graded formalism adapted to structured photonic circuits can indeed be employed to test components of photonic circuits and implement quantum error correction.

#### D. Fractional-graded diagnostic and correction in photonic quantum circuits

Photonic quantum circuits, like those discussed in the patents, represent a promising platform for scalable and robust quantum information processing. Here, we introduce a general framework based on fractional-graded algebras to perform diagnostic and error correction tasks, leveraging the interplay between graded symmetries and fractional dynamics to enhance fault tolerance in paraparticle-encoded photonic systems.

Let  $\Psi_{\ell, \sigma}(x, t)$  be the field representing a structured-light mode in a quantum photonic circuit, where  $\ell \in \mathbb{Z}$  denotes the OAM and  $\sigma = \pm 1$  the SAM. Each mode is embedded

in a  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded algebraic structure labeled by  $(a, b) = (\ell \bmod 2, \frac{1-\sigma}{2})$ .

Each mode evolves according to a fractional stochastic differential equation with graded paraparticle statistics:

$$d\hat{X}_{\ell, \sigma}(t) = \hat{b}_{\ell, \sigma}(t) dt + d\hat{B}_{\ell, \sigma}^{(H)}(t), \quad (127)$$

where  $\hat{b}_{\ell, \sigma}(t)$  is a mode-dependent drift operator and  $d\hat{B}_{\ell, \sigma}^{(H)}(t)$  is a graded fractional Brownian increment with Hurst index  $H \in (0, 1)$  satisfying

$$\begin{aligned} d\hat{B}_a^{(H)}(t) d\hat{B}_b^{(H)}(t') + (-1)^{(a,b)} d\hat{B}_b^{(H)}(t') d\hat{B}_a^{(H)}(t) \\ = 2\delta_{ab}\gamma_H(t-t') dt dt'. \end{aligned} \quad (128)$$

#### E. Quantum field and density evolution

Let  $\rho_{\ell, \sigma}(x, t) = |\Psi_{\ell, \sigma}(x, t)|^2$  be the probability density of the structured mode. Then, the total density evolves as  $\partial_t \rho + \nabla \cdot (\rho v) = 0$ , with  $v = \frac{1}{2}(b_+ + b_-)$ . The effective quantum potential that governs diagnostic behavior is given by Eq. (129).

$$Q_{\ell, \sigma}^{(H)}(x, t) = -\frac{\hbar^2}{2m} \frac{(-\Delta)^H \sqrt{\rho_{\ell, \sigma}(x, t)}}{\sqrt{\rho_{\ell, \sigma}(x, t)}}, \quad (129)$$

where  $(-\Delta)^H$  denotes the fractional Laplacian acting in space. Diagnostic signature and correction condition is then expanded as follows: from Eq. (129), the deviation  $\Delta Q^{(H)}$  between theoretical and measured quantum potentials defines a diagnostic operator as in Ref. [92],  $\Delta Q_{\ell, \sigma}^{(H)} = Q_{\ell, \sigma}^{(H)} - Q_{\ell, \sigma}^{(H), \text{expected}}$ , and the correction term  $\hat{U}_{\text{corr}}$  is triggered when the deviation exceeds a noise threshold  $\epsilon$ ,  $\|\Delta Q_{\ell, \sigma}^{(H)}\| > \epsilon$ . In this case, one defines a conditional control unitary  $\hat{U}_{\text{corr}}$  acting on sector  $(a, b)$  such that  $\hat{U}_{\text{corr}} = \exp(-i\theta_{(a,b)} \hat{n}_{(a,b)})$  and  $\hat{n}_{(a,b)} = \hat{\psi}_{(a,b)}^\dagger \hat{\psi}_{(a,b)}$ , where  $\theta_{(a,b)}$  is dynamically adjusted based on  $\Delta Q_{(a,b)}^{(H)}(t)$ .

The complete system evolution within the circuit then becomes

$$\hat{H}_{\text{total}} = \hat{H}_{\text{OAM}} + \hat{H}_{\text{SAM}} + \hat{Q}^{(H)} + \hat{U}_{\text{corr}}(\Delta Q), \quad (130)$$

governing how diagnostic deviations influence active feedback correction during real-time photonic quantum computation or with *a posteriori* selection from multiple runs.

The central insight of this work is that paraparticle algebras, which have traditionally been studied as abstract mathematical structures, can find natural realizations within structured-light systems. Specifically, spin-orbit coupled photonic modes, such as those carrying both SAM and OAM, inherently possess the degrees of freedom and exchange statistics required to realize  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded algebraic structures.

The photonic modes considered here are not merely carriers of quantum information; rather, they act as effective realizations of the algebraic elements themselves. For example, the OAM degree of freedom, accessible through Laguerre-Gaussian or Bessel beams, can encode parafermionic excitations, while the coupling between SAM and OAM mediates the graded commutation relations central to the algebra. This mapping obviates the need for condensed matter or atomic systems, offering a purely photonic, room-temperature platform for realizing exotic statistics.

From a dynamical perspective, the use of fractional Brownian motion in the stochastic framework introduces a natural description of environmental noise and decoherence. Unlike conventional Markovian approaches, this formalism accounts for memory effects, which are known to play a significant role in realistic optical systems. Nelson's stochastic mechanics, originally formulated for nonrelativistic quantum systems, provides a conceptual analog for describing emergent interactions and nonlocal effects in the photonic context and even the presence of a coupled deterministic mechanism described by the Wold theorem. The classical Wold decomposition (for stationary processes) is so defined. For any zero-mean, covariance-stationary stochastic process  $\{X_t\}_{t \in \mathbb{Z}}$  with finite variance, Wold's theorem states [93,94] then

$$X_t = \underbrace{\sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}}_{\text{(i) stochastic MA}(\infty)} + \underbrace{\eta_t}_{\text{(ii) deterministic}}, \quad \sum_{j=0}^{\infty} \psi_j^2 < \infty, \quad (131)$$

where  $\{\varepsilon_t\}$  is a white-noise sequence ( $\mathbb{E} \varepsilon_t = 0$ ,  $\text{var} \varepsilon_t = \sigma_\varepsilon^2$ ),  $\psi_0 = 1$ , and  $\eta_t$  is perfectly predictable from its own past. Equation (131) underpins the entire autoregressive moving-average (ARMA)/autoregressive integrated moving-average (ARIMA) modeling framework [95], truncating the infinite moving-average yields a finite-order ARMA approximation whose parameters can be estimated consistently by maximum likelihood or Whittle's method.

From the other side, the fractional extension (including long-memory processes and fBm) start with the definition of fractional Brownian motion  $B_H(t)$ , ( $0 < H < 1$ ), which is nonstationary, but its increments  $X_k = B_H(k+1) - B_H(k)$  form a stationary, long-memory process called fractional Gaussian noise (FGN). Granger and Joyeux [96] and Hosking [97] showed that FGN admits a Wold-type representation with slowly decaying moving-average coefficients,

$$X_k = \sum_{j=0}^{\infty} \psi_j^{(H)} \varepsilon_{k-j}, \quad \psi_j^{(H)} \sim j^{H-\frac{3}{2}}, \quad (132)$$

or, equivalently, an ARFIMA(0,  $d$ , 0) representation with  $d = H - \frac{1}{2}$ . This fractional moving-average expansion supports both maximum-likelihood-Whittle estimation of the Hurst exponent  $H$  [98] and efficient FGN/fBm simulation via finite-order ARMA truncations or FFT-based convolution kernels [99,100].

A continuous Wold-Volterra analog also exists; *fBm* that can be written as a causal Volterra integral of standard Brownian motion  $W(t)$ , namely,  $B_H(t) = \int_{-\infty}^t K_H(t, s) dW(s)$ , with kernel given by  $K_H(t, s) = \frac{1}{\Gamma(H+\frac{1}{2})} [(t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}]$ . This kernel form is the starting point for fractional Kalman filtering, rough-path analysis, and optical analogs of fractional quantum mechanics.

In summary, Eq. (131) and its fractional counterpart Eq. (132) provide a unified moving-average language that covers both short- and long-memory regimes, furnishing the statistical and computational backbone for applications ranging from ARMA control loops to Lévy-flight optical lattices.

Taken together, these theoretical insights suggest that a tabletop experimental setup involving spatial-light modulators,  $q$ -plates, beam splitters, and single-photon detectors could serve as a testbed for exploring the phenomenology of graded paraparticles. Importantly, the proposed architecture points toward scalable photonic quantum computing schemes that transcend the limitations of qubit-based architectures by leveraging qudits with rich internal structure. While challenges remain—including mode purity, optical loss, and error correction protocols—the theoretical foundation developed here provides a promising roadmap for experimental exploration at the intersection of algebraic quantum theory and modern photonic technology.

## IX. CONCLUSIONS

We have developed a comprehensive theoretical framework embedding Majorana's infinite-component relativistic wave equations into the algebraic formalism of paraparticles, explicitly employing  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded Lie algebras and advanced  $R$ -matrix quantization techniques. Our approach systematically associates spin-dependent mass spectra with well-defined graded sectors characterized by generalized quantum statistics, resulting in a unified relativistic quantum field theory rigorously consistent with Majorana's original mass-spin relation. This unification effectively bridges disparate areas within theoretical physics, suggesting possible future paraparticle-based quantum computers beyond qubits, using graded qudits and potential advances in quantum error correction by exploiting graded symmetries and mathematical tools of fractal stochastic processes related to Nelson's quantum mechanics.

Furthermore, we extended our formalism toward practical applications. In particular, an integrated platform based on Pancharatnam-Berry metasurfaces has been proposed in our recent patent filings (Refs. [69,70]). These filings illustrate how structured photonic systems, especially spin-orbit-coupled modes in optical waveguides, can exhibit classical entanglement that mirrors the algebraic structures of paraparticles. These findings open experimentally viable pathways to simulate exotic quantum statistics utilizing structured light. Our detailed exploration highlights profound interrelations among orbital angular momentum, spin angular momentum, and parastatistics, offering impactful insights for fundamental research and practical quantum information applications.

Future investigations prompted by our framework may focus on detailed experimental realizations in integrated photonic circuits, extensive numerical simulations to rigorously validate theoretical predictions, and further theoretical expansions to include interacting systems and gauge symmetries. The algebraic infrastructure developed herein lays a strong foundation for advancing quantum computational architectures and innovative quantum communication technologies, thus fostering significant interdisciplinary collaborations and technological breakthroughs. Moreover, starting from Refs. [7,8], the digital simulation architectures here proposed could serve as a benchmarking tool for the algebraic dynamics of paraparticle modes in photonic platforms, enabling a complementary validation pathway between discrete quantum simulations and continuous structured-light implementations.

Other results presented here establish a rigorous theoretical framework for the realization of parafermionic excitations in OAM-SAM coupled systems, unifying the algebraic structure of  $\mathbb{Z}_N$  parafermions with physically implementable models. The mathematical analysis demonstrates the compatibility of graded algebraic symmetries with photonic architectures, providing a foundation for further theoretical developments and opening directions in the exploration of non-Abelian quasiparticles.

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No other authors have competing interests to declare.

### DATA AVAILABILITY

No data were created or analyzed in this study.

### APPENDIX A: MATHEMATICAL TOOLS

This Appendix provides a brief guide to the mathematical structures underlying the main results of this work, including Hopf algebras, graded algebras, braided tensor categories, and quantum groups. For readers unfamiliar with these topics, we recommend several accessible references: the foundational treatments by Majid [101], Kassel [102], and Klimyk and Schmüdgen [103]; the categorical perspective in Etingof *et al.* [104]; the theory of Lie superalgebras in Scheunert [105]; and the more physics-oriented introductions in Majid's *Quantum Groups Primer* [106] and Pressley-Segal's *Loop Groups* [107]. Together, these works provide a comprehensive starting point for understanding the algebraic techniques used in quantum field theory, statistical physics, and quantum information.

### APPENDIX B: BRAIDED COPRODUCT AND BRAIDED TENSOR PRODUCT

A braided tensor product is a generalization of the ordinary tensor product that includes braiding, a structured rule for swapping or exchanging elements from two algebraic sectors. A braided coproduct is a mathematical operation that generalizes the usual notion of a coproduct in algebra used to define how algebraic structures behave under tensor products but now includes a braiding (or twist) that captures nontrivial exchange symmetries. In the context of graded or quantum algebras, especially those involving paraparticles, anyons, or quantum groups, the braided coproduct ensures that interchanging two particles or operators does not simply commute or anticommute, but follows more complex rules governed by a braiding matrix (often called an  $R$  matrix). A field-theoretic formulation of such braided structures is provided by braided quantum field theory [108]. In this Appendix, we provide a detailed exposition of the concepts of *braided coproduct* and *braided tensor product* as applied to the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded

Hopf algebraic framework used to describe paraparticles and Majorana fields.

Standard vs braided coproducts are obtained from Hopf algebras. Let  $\mathcal{H}$  be a Hopf algebra. A standard coproduct is a map

$$\Delta : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}, \quad (\text{B1})$$

which satisfies the coassociativity condition  $(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta$ . In symmetric (bosonic) systems, the coproduct of an operator  $\psi$  typically takes the form  $\Delta(\psi) = \psi \otimes \mathbb{I} + \mathbb{I} \otimes \psi$ .

However, in paraparticle or anyonic systems, particle exchange follows a *nontrivial* rule. The *braided coproduct* introduces a deformation via an  $R$  matrix:

$$\Delta_B(\psi_{(a,b)}) = \psi_{(a,b)} \otimes \mathbb{I} + \sum_{(c,d)} R_{(a,b)}^{(c,d)} (\mathbb{I} \otimes \psi_{(c,d)}), \quad (\text{B2})$$

where  $R_{(a,b)}^{(c,d)}$  encodes the braiding symmetry and grading structure.

#### 1. The braiding matrix and exchange and Yang-Baxter equation

The  $R$ -matrix components are chosen to respect the graded symmetry. For  $\mathbb{Z}_2 \times \mathbb{Z}_2$  grading, the basic exchange rule between two elements  $\psi_{(a,b)}$  and  $\psi_{(a',b')}$  is defined by  $\psi_{(a,b)} \otimes \psi_{(a',b')} \mapsto (-1)^{(a,b) \cdot (a',b')} \psi_{(a',b')} \otimes \psi_{(a,b)}$ , where the scalar product is  $(a,b) \cdot (a',b') = aa' + bb' \pmod{2}$ . This rule is implemented by the braiding operator  $\mathcal{R}$ ,  $\mathcal{R}(\psi_{(a,b)} \otimes \psi_{(a',b')}) = R_{(a,b)}^{(a',b')} (\psi_{(a',b')} \otimes \psi_{(a,b)})$ . In deformed or quantum settings, the braiding coefficient may contain a deformation parameter  $q \in \mathbb{C}$  with  $|q| = 1$ , leading to  $R_{(a,b)}^{(a',b')} = (-1)^{(a,b) \cdot (a',b')} + \theta_{(a,b),(a',b')}$ , with  $\theta_{(a,b),(a',b')} = \epsilon_{(a,b),(a',b')} (q^{s+s'} - 1)$ , for associated spins  $s$  and  $s'$  and sign  $\epsilon$  determined by the parity.

*The coherence via the Yang-Baxter equation:* to ensure consistency and associativity in multiparticle states, the braiding map must satisfy the Yang-Baxter equation  $\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$ . Here,  $\mathcal{R}_{ij}$  acts on the  $i$ th and  $j$ th components of a triple tensor product space  $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$ . This ensures that the total braiding is associative and well defined regardless of the order of exchanges and that multiphoton entanglement remains consistent under sector exchange, a critical requirement for physical implementation in waveguide-based photonic circuits.

#### 2. Definition of braided tensor product

The braided tensor product of two algebra elements  $A \in \mathfrak{g}_{(a,b)}$  and  $B \in \mathfrak{g}_{(a',b')}$  is defined by  $A \otimes B \mapsto (-1)^{(a,b) \cdot (a',b')} B \otimes A$ , which is a generalization of the symmetric (+1) or antisymmetric (−1) rules. This braiding is reflected in  $A \otimes B = \mathcal{R}(B \otimes A)$ , with  $\mathcal{R}$  being the braiding operator defined by the  $R$  matrix  $\mathcal{R}(B \otimes A) = \sum_{(c,d)} R_{(a,b)}^{(c,d)} (B_{(c,d)} \otimes A_{(a,b)})$ .

*Example in  $\mathbb{Z}_2 \times \mathbb{Z}_2$  sectors.* Let  $\psi_{(0,1)}$  and  $\psi_{(1,1)}$  be two paraparticle field operators. Their braided tensor product satisfies  $\psi_{(0,1)} \otimes \psi_{(1,1)} = -\psi_{(1,1)} \otimes \psi_{(0,1)}$ , because  $(0,1) \cdot (1,1) = 1 \Rightarrow (-1)^1 = -1$ . However, if both operators are in bosoniclike sectors, say  $\psi_{(0,0)}$  and

$\psi_{(1,1)}$ , then  $\psi_{(0,0)} \otimes \psi_{(1,1)} = \psi_{(1,1)} \otimes \psi_{(0,0)}$ , because  $(0,0) \cdot (1,1) = 0 \Rightarrow (-1)^0 = +1$ .

*Physical relevance:* The braided coproduct and tensor product define how paraparticle fields combine in multimode Fock space. They encode exotic statistics beyond fermionic or bosonic behavior and result critical in simulating or realizing paraparticle logic in photonic systems or quantum circuits. What is relevant is that they ensure algebraic consistency through the Yang-Baxter identity, guaranteeing that logical gates and quantum entanglement remain well defined under sector transitions.

These structures underlie the operator algebra governing the interaction of Majorana-like fields embedded into  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded sectors, allowing the construction of multiqubit logical operations in structured photonic systems.

*Green-Volkov (GV)/Yang-Baxter demonstration:*

*Lemma (explicit Yang-Baxter equation (YBE) check for  $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ ).* Let  $R$  act on homogeneous sectors by  $R(\psi_{(a,b)} \otimes \psi_{(c,d)}) = [(-1)^{(a,b) \cdot (c,d)} + \theta_{(a,b),(c,d)}] \psi_{(c,d)} \otimes \psi_{(a,b)}$ , where  $(a,b) \cdot (c,d) = ac + bd \pmod{2}$  and the deformation  $\theta$  is additive in each slot (equivalently, a two-cocycle representative). Then,  $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$  on  $V^{\otimes 3}$ .

*Proof.* Acting on  $\psi_{(a,b)} \otimes \psi_{(c,d)} \otimes \psi_{(e,f)}$ ,  $R_{12}R_{13}R_{23}$  produces the overall phase  $(-1)^{(a,b) \cdot (c,d) + (a,b) \cdot (e,f) + (c,d) \cdot (e,f)}$  times the swapped tensor, whereas  $R_{23}R_{13}R_{12}$  produces  $(-1)^{(c,d) \cdot (e,f) + (a,b) \cdot (e,f) + (a,b) \cdot (c,d)}$  times the same tensor. Since the dot product is bilinear mod 2, the exponents coincide. The deformation phases multiply as  $\theta_{(a,b),(c,d)} + \theta_{(a,b),(e,f)} + \theta_{(c,d),(e,f)}$  on the left and  $\theta_{(c,d),(e,f)} + \theta_{(a,b),(e,f)} + \theta_{(a,b),(c,d)}$  on the right; additivity (or the two-cocycle condition) makes them equal. ■

## APPENDIX C: GRADED HOPF ALGEBRA FRAMEWORK

*Definition (braided Hopf algebra of graded creation/annihilation modes).* Let  $G = \mathbb{Z}_2 \times \mathbb{Z}_2$  with the bicharacter  $\chi((a,b), (c,d)) = (-1)^{ac+bd}$  and let  $P_g$  be the central projectors of  $\mathbb{C}[G]$ . Denote by  $\mathcal{F}$  the unital algebra generated by homogeneous  $\{\hat{\psi}_g^\pm\}_{g \in G}$  with degree  $|\hat{\psi}_g^\pm| = g$  and braided bracket  $[X, Y]_\chi := XY - \chi(|X|, |Y|)YX$ . The *graded paraparticle mode algebra* admits two equivalent presentations:

(1) *Canonical (pairwise) form* used for the  $p = 1$  limit and for device-level modeling,

$$[\hat{\psi}_g^-, \hat{\psi}_h^+]_\chi = \delta_{g,h} \mathbf{1}, \quad [\hat{\psi}_g^\pm, \hat{\psi}_h^\pm]_\chi = 0. \quad (\text{C1})$$

(2) *Green-Volkov (trilinear) form* used for genuine paraparticles of order  $p > 1$ , obtained by replacing Eq. (C1) with the  $\chi$ -graded Green-Volkov identities:

$$\begin{aligned} [\{\hat{\psi}_g^-, \hat{\psi}_h^+\}_\chi, \hat{\psi}_k^\pm]_\chi &= \pm 2 \delta_{h,k} \hat{\psi}_g^\pm, \\ [\{\hat{\psi}_g^\pm, \hat{\psi}_h^\pm\}_\chi, \hat{\psi}_k^\pm]_\chi &= 0, \end{aligned} \quad (\text{C2})$$

(with the obvious adjoint relations), which reduce to Green's paraboson/parafermion relations when  $G = \mathbb{Z}_2$ . In both cases, the (quasitriangular)  $G$ -graded Hopf structure is

$$\begin{aligned} \Delta(\hat{\psi}_g^\pm) &= \hat{\psi}_g^\pm \otimes \mathbf{1} + \mathbf{1} \otimes \hat{\psi}_g^\pm, \\ \epsilon(\hat{\psi}_g^\pm) &= 0, \quad S(\hat{\psi}_g^\pm) = -\hat{\psi}_g^\pm, \end{aligned} \quad (\text{C3})$$

with universal  $R$  matrix

$$R_\chi = \sum_{g,h \in G} \chi(g, h) P_g \otimes P_h, \quad c_\chi(x \otimes y) = R_\chi(y \otimes x). \quad (\text{C4})$$

Equations (C3) and (C4) ensure that the tensor product of two mode algebras is braided by  $\chi$ , so that the exchange of homogeneous operators is governed by the same sign rule used in the graded Clifford package.

*Physical role.* The coproduct in Eq. (C3) encodes *mode stacking*: On a two-register state, it creates/annihilates an excitation in either register, while the braiding  $c_\chi$  supplies the relative exchange phase. The counit evaluates vacuum amplitudes and the antipode implements the algebraic inverse (particle-hole conjugation for Majorana modes). The diagonal  $R_\chi$  in Eq. (C4) provides a compact representation of two-mode exchange gates: in the photonic implementation, it reduces to a controlled sign determined by the SAM/OAM parities of the two modes.

## APPENDIX D: APPLICATIONS AND EXAMPLES

### 1. Explicit model Hamiltonian for OAM-SAM coupled ququarts

To complement the algebraic framework developed in the main text, we provide here a schematic Hamiltonian that explicitly captures the essential features of the OAM-SAM coupled ququart system and illustrates how the  $\mathbb{Z}_4$  parafermionic symmetry emerges.

We consider a four-mode system defined by the basis states  $|0\rangle = |L, 0\rangle$ ,  $|1\rangle = |R, 0\rangle$ ,  $|2\rangle = |L, 1\rangle$ , and  $|3\rangle = |R, 1\rangle$ , where  $L/R$  denote left- and right-circular polarizations (SAM) and  $\ell = 0, 1$  are OAM quantum numbers. The general tight-binding Hamiltonian reads

$$\hat{H} = \sum_{j=0}^3 \omega_j a_j^\dagger a_j + \sum_{(j,k)} J_{jk} a_j^\dagger a_k + \sum_{(j,k)} \lambda_{jk} a_j^\dagger \sigma_z a_k + \text{H.c.}, \quad (\text{D1})$$

where  $a_j^\dagger$  and  $a_j$  are bosonic or fermionic creation and annihilation operators,  $\omega_j$  are mode frequencies,  $J_{jk}$  are nearest-neighbor coupling amplitudes between modes, and  $\lambda_{jk}$  encode spin-orbit (OAM-SAM) couplings, with  $\sigma_z$  acting on the polarization subspace.

For the parafermionic sector, we use the generalized clock and shift operators  $\sigma_c$  and  $\tau$  for which  $\sigma|k\rangle = e^{i\frac{\pi}{2}k}|k\rangle$ ,  $\tau|k\rangle = |k+1 \pmod{4}\rangle$ , satisfying  $\sigma_c^4 = \tau^4 = \mathbb{I}$ ,  $\tau\sigma_c = i\sigma_c\tau$ . These operators can be realized in the Hamiltonian through cyclic hopping terms:

$$\hat{H}_{\text{para}} = g \sum_{k=0}^3 a_k^\dagger a_{k+1} + \text{H.c.}, \quad (\text{D2})$$

with periodic boundary conditions  $a_{k+4} \equiv a_k$ . The clock operator  $\sigma_c$  can be implemented via mode-dependent phase shifts

$$\hat{H}_{\text{phase}} = \sum_{k=0}^3 \phi_k a_k^\dagger a_k, \quad \phi_k = \frac{\pi}{2}k. \quad (\text{D3})$$

The operators  $a_k^\dagger$  and  $a_k$  are, respectively, paraparticle creation and annihilation operators, associated with  $(a, b)$ -graded SAM-OAM photonic modes. Each index  $k = 0, 1, 2, 3$  corresponds to a distinct mode sector in the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded algebra, with the mapping  $a = \ell \pmod{2}$  and  $b = (1 - \sigma)/2$ .

These operators satisfy graded trilinear commutation relations and are not standard bosons or fermions.

The para-Hamiltonian  $\hat{H}_{\text{para}}$  in Eq. (D2) models a cyclic hopping process across these four sectors and can be physically realized using photonic lattices or waveguide arrays with mode converters that cyclically couple structured-light modes. In this context, OAM and SAM are manipulated using  $q$ -plates, spatial-light modulators, or directional couplers in a looped topology.

The graded phase Hamiltonian  $\hat{H}_{\text{phase}}$  in Eq. (D3) represents sector-dependent energy shifts or phase rotations. These can be implemented via programmable phase elements such as liquid crystal modulators or birefringent structures. The phase pattern  $\phi_k = \frac{\pi}{2}k$  is naturally compatible with the  $\mathbb{Z}_4$  clock operator structure and its reduction to  $\mathbb{Z}_2 \times \mathbb{Z}_2$  grading.

Together, these Hamiltonians provide an experimentally accessible route to simulate the algebraic properties of graded paraparticles in photonic platforms, enabling deterministic quantum logic respecting topological and symmetry constraints.

This explicit form connects the abstract algebraic symmetries to physically controllable parameters in photonic or condensed matter platforms, providing a basis for future experimental implementations. The graded phase factors follow from the multiplication rules of  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded algebras, as developed in works by Palev, Tolstoy, and Scheunert [105,109–112]. The deformation factor  $q^{|s-s'|}$  arises from the introduction of a quantum group deformation parameter, a technique pioneered by Majid [110] and others in the study of braided Hopf algebras. Recent contributions by Toppan [11] and collaborators have applied these structures to the embedding of paraparticles in a Majorana tower, providing the explicit algebraic form used in our numerical simulations. The coupling coefficients  $\theta_{ss'}$  are determined by the algebraic framework imposed on the Majorana tower, rather than fixed by the original Majorana mass-spin relation. Specifically, the graded phase factors  $(-1)^{(a,b)-(a',b')}$  arise from the multiplication rules of the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded algebra, while the deformation factor  $q^{|s-s'|}$  is introduced through the quantum group or braided Hopf algebra formalism. Together, these components define the algebraic coupling pattern between spin sectors. The explicit form of  $\theta_{ss'}$  thus depends on the theoretical choices made in the construction of the model and reflects the desired symmetry, braiding, and selection rules to be implemented in the physical or simulated system. The graded couplings express not only the static algebraic structure but also the dynamic control possibilities offered by the truncated Majorana tower framework. Leveraging these selection rules in photonic implementations enables targeted manipulation of spin sectors, allowing for the design of quantum gates and protocols that intrinsically respect paraparticle symmetries. Beyond fundamental interest, this framework points toward practical applications in photonic quantum computing, including sector-isolated logical operations, error suppression through symmetry protection, and the emulation of exotic quantum phases. Moreover, the photonic Majorana ququart architecture provides a promising experimental testbed to explore connections between graded algebras, quantum entanglement, and topologically inspired quantum information processing.

## 2. Graded sector-based CNOT and Toffoli gates

Here, we present the explicit truth tables for quantum logic gates operating over  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded sectors used in the Majorana-paraparticle framework. Control and target qubits are assigned sector labels  $(a, b)$ , with graded parity governing their transformations.

The CNOT gate flips the target qubit if and only if the control qubit is in the logical 1 sector. In the graded sector formalism, this corresponds to the control being in sector  $(0, 1)$  or  $(1, 1)$ , depending on encoding. The graded CNOT truth table is

| Control qubit | Target qubit (input) | Target qubit (output) | Operation |
|---------------|----------------------|-----------------------|-----------|
| $(0, 0)$      | $t$                  | $t$                   | No flip   |
| $(0, 1)$      | 0                    | 1                     | Flip      |
| $(0, 1)$      | 1                    | 0                     | Flip      |
| $(1, 0)$      | $t$                  | $t$                   | No flip   |
| $(1, 1)$      | 0                    | 1                     | Flip      |
| $(1, 1)$      | 1                    | 0                     | Flip      |

where  $t \in \{0, 1\}$  denotes the target qubit logical value.

Example of graded sector-based Toffoli (CCNOT) gate: The Toffoli gate (controlled-controlled-NOT or CCNOT) flips the target qubit if and only if both control qubits are simultaneously in their logical 1 sectors,

- (1) First control in sector  $(0, 1)$ , e.g., SAM—right circular polarization;
- (2) Second control in sector  $(1, 1)$ , e.g., OAM—odd mode with certain parity.

The graded Toffoli truth table then becomes

| Control qubit 1 | Control qubit 2 | Target (input) | Target (output) | Operation |
|-----------------|-----------------|----------------|-----------------|-----------|
| $(0, 0)$        | $(1, 0)$        | $A_p$          | $A_p$           | No flip   |
| $(0, 0)$        | $(1, 1)$        | $A_p$          | $A_p$           | No flip   |
| $(0, 1)$        | $(1, 0)$        | $A_p$          | $A_p$           | No flip   |
| $(0, 1)$        | $(1, 1)$        | $A_p$          | $B_p$           | Flip      |
| $(0, 1)$        | $(1, 1)$        | $B_p$          | $A_p$           | Flip      |
| $(0, 0)$        | $(1, 0)$        | $B_p$          | $B_p$           | No flip   |
| $(0, 0)$        | $(1, 1)$        | $B_p$          | $B_p$           | No flip   |
| $(0, 1)$        | $(1, 0)$        | $B_p$          | $B_p$           | No flip   |

where  $A_p$  denotes the logical “0” state (e.g., path  $A_p$ ), and  $B_p$  denotes the logical “1” state (say, path  $B_p$ ). Flip means toggling between  $A_p$  and  $B_p$  paths (logical NOT on the target). In both gates, the operation respects the underlying graded parity structure,  $X_{(a,b),(a',b')} = (-1)^{(a,b)-(a',b')}$ , ensuring that only specified graded sector combinations trigger target flips, preserving graded quantum symmetry.

A quantitative loss-and-fidelity budget for either free-space or integrated implementations can be obtained by inserting platform-specific transmission coefficients in the equations reported in this work. We defer such calculations to future experimental work.

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