Complete classification of integrability and nonintegrability for spin-$\frac{1}{2}$ chain with symmetric nearest-neighbor interaction

Mizuki Sanatani [1,*] Yuuya Chiba [2,†] and Naoto Shiraishi [1,‡]

[1] *Graduate School of Arts and Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro, Tokyo 153-8902, Japan*

[2] *Nonequilibrium Quantum Statistical Mechanics RIKEN Hakubi Research Team, RIKEN Cluster for Pioneering Research (CPR), 2-1 Hirosawa, Wako, Saitama 351-0198, Japan*

(Received 12 February 2026; accepted 9 June 2026; published 28 August 2026)

General spin-$\frac{1}{2}$ chains with symmetric nearest-neighbor interaction are studied. We rigorously prove that all spin models in this class, except for known integrable systems, are nonintegrable in the sense that they possess no nontrivial local conserved quantities. This result confirms that there are no missing integrable systems; i.e., integrable systems in this class are exactly those that are already known. In addition, this result excludes the possibility of intermediate systems that have a finite number of nontrivial local conserved quantities. Our findings support the expectation that integrable systems are exceptional in quantum many-body systems and most systems are nonintegrable.

DOI: 10.1103/k8xr-brs7

I. INTRODUCTION

In the study of quantum many-body systems, distinction between integrable and nonintegrable systems is crucial. Quantum integrable systems have Hamiltonians whose eigenvalues and eigenstates are exactly solvable [1–4]. Integrable systems have been studied for almost a century, and various models have been unveiled to be integrable [5–16]. Integrable systems are highly useful in that many important physical quantities can be calculated exactly. The solvability of integrable systems is thought to be a consequence of the presence of an infinite number of local conserved quantities [17,18]. Here, a local conserved quantity is a conserved quantity given by a sum of operators acting on a few spatially consecutive sites. To the best of our knowledge, all known integrable systems with local and shift-invariant Hamiltonians discovered to date possess infinite local conserved quantities [19,20].

On the other hand, unlike conventional macroscopic systems, integrable systems exhibit some anomalous behaviors. For instance, thermalization of isolated systems, a fundamental principle of thermodynamics, does not occur in integrable systems due to the obstruction of local conserved quantities [21–23], as observed in experiments using ultracold atoms [24–26]. In addition, other empirical laws including the Kubo formula in linear response theory and Fourier's law in thermal conduction do not apply to integrable systems [27–30]. Since these laws are empirically confirmed, it is believed that integrable systems are exceptional and almost all systems are nonintegrable in the sense of the absence of nontrivial conserved quantities [31,32]. Various numerical experiments support this belief. For instance, the distribution of the energy level spacings is qualitatively different between integrable and nonintegrable systems, and generic systems follow the distribution for the latter one (the Wigner-Dyson distribution) [33–37].

In spite of these strong expectations, ubiquitousness of nonintegrability has not yet been addressed from an analytical perspective. In contrast to vast literature on integrable systems, it has been considered to be difficult to prove nonintegrability. In fact, a rigorous proof of nonintegrability was elusive for a long time.

Recently, nonintegrability was rigorously proved in the XYZ model with Z magnetic field [32]. Triggered by this work, several models, the mixed-field Ising model [38], the PXP model [39], and the Heisenberg model with next-nearest-neighbor interaction [40], were proved to be nonintegrable. However, these studies treat nonintegrability of individual systems. To substantiate the expectation that most systems are inherently nonintegrable, it is necessary to prove nonintegrability of a broad class of systems.

In this paper, we demonstrate the ubiquitousness of nonintegrability within a general class of systems. Precisely, we consider general spin-$\frac{1}{2}$ chains with shift-invariant and parity-symmetric nearest-neighbor interactions. We rigorously prove that all systems except known integrable systems are indeed nonintegrable; i.e., they have no nontrivial local conserved quantity. This class of models in consideration includes many well-established integrable systems, such as the Heisenberg model and the transverse-field Ising model. Our result establishes that there is no further integrable system beyond known ones (see Fig. 1). In addition, our result also shows that any system has infinitely many local conserved quantities or no nontrivial local conserved quantity, and there is no

*Contact author: yamaguchi-q@g.ecc.u-tokyo.ac.jp

†Contact author: yuya.chiba@riken.jp

‡Contact author: shiraishi@phys.c.u-tokyo.ac.jp

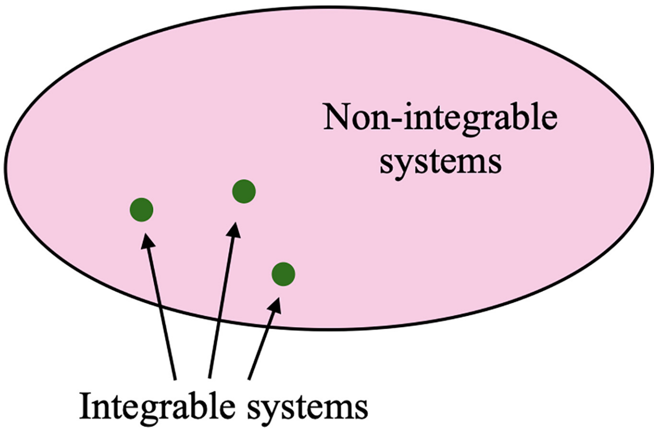

FIG. 1. A schematic of what we have proven in this article. All systems except for known integrable systems are rigorously proven to be nonintegrable, which dominates the space of possible models in this class.

intermediate system with a finite number of nontrivial local conserved quantities. This result supports the view that integrable systems are exceptional and almost all systems are nonintegrable.

II. SETUP AND MAIN RESULT

We consider a general spin-$\frac{1}{2}$ chain with N spins satisfying the following conditions: the Hamiltonian is translationally invariant, and the Hamiltonian consists only of parity-symmetric nearest-neighbor interaction terms and magnetic field terms. Precisely, the Hamiltonian in consideration is given by

$$H = \sum_{i=1}^{N} \left(\sum_{\alpha,\beta\in\{x,y,z\}} J_{\alpha\beta}\sigma_i^\alpha\sigma_{i+1}^\beta + \sum_{\alpha\in\{x,y,z\}} h_\alpha\sigma_i^\alpha \right), \quad (1)$$

with $J_{\alpha\beta} = J_{\beta\alpha}$ for any α and β. We identify site $N+1$ to site 1, meaning the periodic boundary condition. We abbreviate Pauli matrices σ_i^x, σ_i^y, and σ_i^z as X_i, Y_i, and Z_i, respectively.

In this article, we use the term *nonintegrable* to signify that the system has no nontrivial local conserved quantity. Here, we say that a quantity is *conserved* if it commutes with the Hamiltonian of the system, and a quantity is *local* if it can be written by a sum of operators that act on $O(1)$ consecutive sites. In particular, a quantity that can be written by a sum of operators on k consecutive sites is called a *k-support* quantity. Notice that locality means spatial locality of the range of interactions, not few-body interactions. For instance, $\sum_{i=1}^{N} X_iY_{i+4}$ is a 5-support local quantity (not 2-support), and $\sum_{i=1}^{N} Z_iZ_{i+N/2}$ is *not* a local quantity. A local conserved quantity with at least 3-support is referred to as nontrivial. If a system has an infinite number of local conserved quantities in the thermodynamic limit, we call this system *integrable*, and if it has a finite number of nontrivial local conserved quantities, we call this system *partially integrable* in correspondence with the terminology in studies of classical Hamiltonian systems [41].

Below, we complete the classification of the systems represented by Eq. (1) with a rigorous proof from the viewpoint of integrability. We prove that all systems represented by Eq. (1) are classified into the following two groups:

(i) *Integrable systems*: The XXX (Heisenberg) model [5], the XXZ model [7], the XYZ model [9], the XY model [6], the transverse-field Ising model [6], trivial classical models, and their global spin rotations.

(ii) *Nonintegrable systems*: All systems except the above.

This is the main result of this article. Notably, even at finely tuned magnetic field, no additional integrable points arise beyond the known families. Since the listed integrable models are already established in the literature, it suffices to prove nonintegrability for all remaining systems. We outline the argument here; full details are given in the companion paper [42].

III. IMPLICATIONS OF MAIN RESULT

Before going to the proof idea and the proof outline, we here highlight the meaning of our result. Our main result manifests the fact that there is no missing integrable system that is awaited to be discovered. By conventional approaches in the field of integrable systems, it is almost hopeless to prove that obtained integrable models are the complete list of integrable models and there is no oversight. Our nonintegrability proof indeed does this and proves that there are no more integrable systems in the form of Eq. (1).

This result also supports the expectation that integrable systems are highly exceptional systems and most generic systems are nonintegrable. Since nonintegrability is necessary to fulfill the linear response theory and thermalization, the ubiquitousness of nonintegrability is strongly believed. However, this belief is usually confirmed only by numerical simulations, and its rigorous mathematical foundation has been elusive. Our result serves as the first mathematical foundation of the above belief.

In addition, our result demonstrates the absence of a partially integrable system in this class, i.e., an intermediate system with finite nontrivial local conserved quantities. Although all known generic models have sufficiently many local conserved quantities or no nontrivial local conserved quantity, in principle, there may exist an intermediate system that has a few nontrivial local conserved quantities. This is an open problem in the research field of integrable systems [43]. Our result solves this problem in negative as long as the system is described in the form of Eq. (1).

IV. PROOF IDEA

In accordance with Refs. [32,38–40], we prove the absence of local conserved quantities by the following approach: We first expand a candidate of a local conserved quantity by strings of Pauli operators and the identity (X, Y, Z, and I). We call these strings l-support *Pauli strings* $\boldsymbol{A}_i^l = A_iA_{i+1}\cdots A_{i+l-1}$, where $A_j \in \{X, Y, Z, I\}$ $(j \neq i, i+l-1)$ and $A_i, A_{i+l-1} \in \{X, Y, Z\}$ are satisfied. Since the basis of k-support quantities is l $(\leqslant k)$-support Pauli strings (hereafter simply called l-support operators), any k-support quantity can

TABLE I. We classify integrability/nonintegrability of all systems expressed in Eq. (4). Any system in the form of Eq. (1) is reduced to Eq. (4). The asterisk in the table means that similar classifications apply to cases where Pauli X, Y, and Z are permuted.

Rank	Standard form ($J_X, J_Y, J_Z \neq 0$)	Detailed conditions	Integrability	Name and Reference
0	$\sum_{i=1}^{N} h_Z Z_i$	–	Integrable	Free spins
1	$\sum_{i=1}^{N}(J_Z Z_i Z_{i+1} + h_X X_i + h_Z Z_i)$	$h_X = 0$	Integrable	Longitudinal-field Ising model
		$h_X \neq 0, h_Z = 0$	Integrable	Transverse-field Ising model [6]
		$h_X \neq 0, h_Z \neq 0$	Nonintegrable	Proven in Ref. [38]
2	$\sum_{i=1}^{N} \begin{array}{l}(J_X X_i X_{i+1} + J_Y Y_i Y_{i+1}\\ +h_X X_i + h_Y Y_i + h_Z Z_i)\end{array}$	$(h_X, h_Y) = (0, 0)$	Integrable	XY model [6]
		$(h_X, h_Y) \neq (0, 0)$	Nonintegrable	Proven in this article
3	$\sum_{i=1}^{N} \begin{array}{l}(J_X X_i X_{i+1} + J_Y Y_i Y_{i+1} + J_Z Z_i Z_{i+1}\\ +h_X X_i + h_Y Y_i + h_Z Z_i)\end{array}$	$J_X = J_Y = J_Z$	Integrable	XXX model [5]
		$J_X = J_Y \neq J_Z$, $(h_X, h_Y) = (0, 0)$	Integrable	XXZ model [7]
		$J_X = J_Y \neq J_Z$, $(h_X, h_Y) \neq (0, 0)$	Nonintegrable	Proven in this article
		$J_X \neq J_Y, J_Y \neq J_Z, J_Z \neq J_X$, $(h_X, h_Y, h_Z) = (0, 0, 0)$	Integrable	XYZ model [9]
		$J_X \neq J_Y, J_Y \neq J_Z, J_Z \neq J_X$, $(h_X, h_Y, h_Z) \neq (0, 0, 0)$	Nonintegrable	Proven in this article (Special case: Ref. [32])

be expanded as

$$Q = \sum_{l=0}^{k} \sum_{i=1}^{N} \sum_{\boldsymbol{A}^l} q_{\boldsymbol{A}_i^l} \boldsymbol{A}_i^l, \tag{2}$$

where all coefficients $\{q_{\boldsymbol{A}_i^l}\}$ are real numbers. Moreover, since H is a 2-support quantity, the commutator of Q and H is an at most $(k+1)$-support quantity, which can be expanded as

$$\frac{1}{2i}[Q, H] = \sum_{l=0}^{k+1} \sum_{i=1}^{N} \sum_{\boldsymbol{B}^l} r_{\boldsymbol{B}_i^l} \boldsymbol{B}_i^l. \tag{3}$$

The conservation condition, $r_{\boldsymbol{B}_i^l} = 0$ for all $\boldsymbol{B}_i^l$, yields a system of linear equations of $\{q_{\boldsymbol{A}_i^l}\}$. The goal of the proof of nonintegrability is to show that these equations do not have solutions except for $q_{\boldsymbol{A}_i^k} = 0$ for all $\boldsymbol{A}_i^k$, which means that Q cannot be a k-support conserved quantity. Since we need to treat general k, we should find a systematic way to show $q_{\boldsymbol{A}_i^k} = 0$ for all $\boldsymbol{A}_i^k$.

We first simplify the Hamiltonian (1) by diagonalizing the interaction matrix $(J_{\alpha\beta})$ with the global spin rotation $\begin{pmatrix}X'\\Y'\\Z'\end{pmatrix} = R\begin{pmatrix}X\\Y\\Z\end{pmatrix}$. Through this, a general Hamiltonian (1) can be transformed into the following standard form:

$$H = \sum_{i=1}^{N} \begin{array}{l}(J_X X_i X_{i+1} + J_Y Y_i Y_{i+1} + J_Z Z_i Z_{i+1}\\ +h_X X_i + h_Y Y_i + h_Z Z_i).\end{array} \tag{4}$$

Notice that the integrability/nonintegrability does not change through this transformation. Hence, it suffices to treat only the standard form (4). Our complete classification for the standard form is summarized in Table I.

Note that the procedure for the nonintegrability proof depends on the rank of $(J_{\alpha\beta})$ (i.e., the number of nonzero entries in J_X, J_Y, J_Z). Below, we prove the absence of 3-support local conserved quantity in the case of rank 2 (i.e., $J_X, J_Y \neq 0$, and $J_Z = 0$) as an example (See the companion paper [42] for the full proof). Our proof for this case consists of two steps. In Step 1, we use the condition $\{r_{\boldsymbol{B}_i^{k+1}} = 0\}$ to eliminate most of coefficients $\{q_{\boldsymbol{A}_i^k}\}$. In Step 2, we show that the remaining coefficients are also zero by using the condition $\{r_{\boldsymbol{B}_i^k} = 0\}$.

V. PROOF FOR 3-SUPPORT CONSERVED QUANTITIES WITH RANK 2: STEP 1

To consider linear equations of $\{q_{\boldsymbol{A}_i^k}\}$ given by the conservation condition, we introduce a useful expression called *column expression*, with which we can grasp commutation relations visually. Let us take as an example a commutator between $\boldsymbol{A}_i^3 = Z_i Y_{i+1} X_{i+2}$ in Q and $Y_{i+2} Y_{i+3}$ in H generates $\boldsymbol{B}_i^4 = Z_i Y_{i+1} Z_{i+2} Y_{i+3}$ as

$$[Z_i Y_{i+1} X_{i+2}, Y_{i+2} Y_{i+3}] = +2i\, Z_i Y_{i+1} Z_{i+2} Y_{i+3}. \tag{5}$$

In the column expression, this commutation relation is expressed as

$$\begin{array}{ccccc} & Z_i & Y_{i+1} & X_{i+2} & \\ & & & Y_{i+2} & Y_{i+3} \\ \hline + & Z_i & Y_{i+1} & Z_{i+2} & Y_{i+3}. \end{array} \tag{6}$$

Here, the horizontal line represents commutation operation, and the horizontal positions of three operators reflect the spatial positions of the sites they act on. The coefficient $2i$ is dropped.

Since the coefficient of a 4-support operator $\boldsymbol{B}_i^4$ in $[Q, H]$ is zero due to the conservation condition, each 4-support operator $\boldsymbol{B}_i^4$ accompanies a linear equation of $\{q_{\boldsymbol{A}_i^3}\}$ derived from the commutation relation. For example, $\boldsymbol{B}_i^4 = Z_i Y_{i+1} Z_{i+2} Y_{i+3}$ is generated only by Eq. (6) and no other commutator generates $Z_i Y_{i+1} Z_{i+2} Y_{i+3}$, since the Hamiltonian does not have ZZ

interaction and thus we do not have the following form of commutator:

$$\begin{array}{ccccc} & & ? & ? & ? \\ & ? & ? & & \\ \hline + & Z_i & Y_{i+1} & Z_{i+2} & Y_{i+3}. \end{array} \tag{7}$$

The above fact directly means

$$J_Y q_{Z_iY_{i+1}X_{i+2}} = r_{Z_iY_{i+1}Z_{i+2}Y_{i+3}} = 0. \tag{8}$$

By similar arguments, we can show that $q_{Z_iA_{i+1}A_{i+2}} = 0$ for arbitrary A_{i+1} and A_{i+2}. Due to the inversion symmetry, Pauli strings with the reverse order of above also have zero coefficients: $q_{A_iA_{i+1}Z_{i+2}} = 0$ for arbitrary A_i and A_{i+1}.

We also observe that

$$\begin{array}{ccccc} & X_i & X & Y & \\ & & & X & X \\ \hline - & X_i & X & Z & X \end{array} \tag{9}$$

is the only contribution to $\boldsymbol{B}_i^4 = X_iX_{i+1}Z_{i+2}X_{i+3}$, since we do not have the following form of commutator:

$$\begin{array}{ccccc} & & ? & ? & ? \\ & X & X & & . \\ \hline - & X_i & X & Z & X \end{array} \tag{10}$$

This fact directly implies $q_{X_iX_{i+1}Y_{i+2}} = 0$. Moreover, by observing that

$$\begin{array}{ccccc} & X_i & I & Z & \\ & & & X & X \\ \hline + & X_i & I & Y & X \end{array} \tag{11}$$

is the only contribution to $\boldsymbol{B}_i^4 = X_iI_{i+1}Z_{i+2}X_{i+3}$ and considering similar commutators, we find $q_{X_iX_{i+1}A_{i+2}} = q_{X_iI_{i+1}A_{i+2}} = q_{Y_iY_{i+1}A_{i+2}} = q_{Y_iI_{i+1}A_{i+2}} = 0$ for arbitrary A_{i+2}.

Moreover, all contributions to $\boldsymbol{B}_i^4 = X_iY_{i+1}Z_{i+2}X_{i+3}$ are listed as

$$\begin{array}{ccccc} & X_i & Y & Y & \\ & & & X & X \\ \hline - & X_i & Y & Z & X \end{array} \quad \begin{array}{ccccc} & & Z_{i+1} & Y & X \\ & X & X & & \\ \hline + & X_i & Y & Y & X, \end{array} \tag{12}$$

which lead to

$$J_X q_{X_iY_{i+1}Y_{i+2}} = J_X q_{Z_{i+1}Y_{i+2}X_{i+3}}. \tag{13}$$

Recalling that $q_{Z_{i+1}Y_{i+2}X_{i+3}}$ has already been shown to be zero, we conclude that $q_{X_iY_{i+1}Y_{i+1}} = 0$. In a similar manner to above, we can show $q_{X_iY_{i+1}A_{i+2}} = 0$ as well as $q_{Y_iX_{i+1}A_{i+2}} = 0$ for arbitrary A_{i+2}.

In summary, $q_{\boldsymbol{A}_i^3} = 0$ holds for all $\boldsymbol{A}^3$ except for $\boldsymbol{A}^3 = XZX, XZY, YZX$, and YZY. In addition, we can show that the remaining coefficients satisfy the following linear relations:

$$\begin{aligned} \frac{q_{Y_iZ_{i+1}Y_{i+2}}}{J_Y} &= \frac{q_{X_{i+1}Z_{i+2}X_{i+3}}}{J_X} = \frac{q_{Y_{i+2}Z_{i+3}Y_{i+4}}}{J_Y} = \cdots, \\ q_{Y_iZ_{i+1}X_{i+2}} &= -q_{X_{i+1}Z_{i+2}Y_{i+3}} = q_{Y_{i+2}Z_{i+3}X_{i+4}} = \cdots. \end{aligned} \tag{14}$$

Hence, to prove the absence of 3-support conserved quantity, it suffices to show the coefficients of $Y_1Z_2Y_3$, $Y_2Z_3Y_4$, $Y_1Z_2X_3$, and $Y_2Z_3X_4$ as zero.

VI. PROOF FOR 3-SUPPORT CONSERVED QUANTITIES WITH RANK 2: STEP 2

We next focus on 3-support operators $\{\boldsymbol{B}_i^3\}$ in $[Q, H]$ and derive $q_{X_iZ_{i+1}Y_{i+2}} = 0$ and $q_{X_iZ_{i+1}X_{i+2}} = 0$ for general i, which include the case of $i = 1$ and 2. Applying a global spin rotation if necessary, without loss of generality, we assume $h_X \neq 0$.

Dropping all the contributions from $\boldsymbol{A}_i^3$ whose coefficients are shown to be zero in Step 1, we find that $\boldsymbol{B}_i^3 = Y_iY_{i+1}Y_{i+2}$ is generated only by

$$\begin{array}{cccc} & Y_i & Z & Y \\ & & X & \\ \hline + & Y_i & Y & Y, \end{array} \tag{15}$$

which directly implies $h_X q_{Y_iZ_{i+1}Y_{i+2}} = 0$, meaning

$$q_{Y_iZ_{i+1}Y_{i+2}} = 0. \tag{16}$$

Inserting Eq. (14) to this, we have

$$q_{X_iZ_{i+1}X_{i+2}} = 0. \tag{17}$$

Next, all commutators generating $\boldsymbol{B}_i^3 = Z_iZ_{i+1}X_{i+2}$ are

$$\begin{array}{cccc} & Y_i & Z & X \\ & X & & \\ \hline - & Z_i & Z & X \end{array} \quad \begin{array}{cccc} & X_i & Z & X \\ & Y & & \\ \hline + & Z_i & Z & X \end{array} \quad \begin{array}{cccc} & Z_i & Y & \\ & & X & X \\ \hline - & Z_i & Z & X, \end{array} \tag{18}$$

and all commutators generating $\boldsymbol{B}_{i-1}^3 = X_{i-1}Y_iY_{i+1}$ are

$$\begin{array}{cccc} & X_{i-1} & Z & Y \\ & & X & \\ \hline + & X_{i-1} & Y & Y \end{array} \quad \begin{array}{cccc} & & Z_i & Y \\ & X & X & \\ \hline + & X_{i-1} & Y & Y \end{array}, \tag{19}$$

which lead to

$$\begin{aligned} -h_X q_{Y_iZ_{i+1}X_{i+2}} + h_Y q_{X_iZ_{i+1}X_{i+2}} - J_X q_{Z_iY_{i+1}} &= 0, \\ h_X q_{X_{i-1}Z_iY_{i+2}} + J_X q_{Z_iY_{i+1}} &= 0. \end{aligned} \tag{20}$$

Summing these two relations and inserting Eq. (14) into it, we have

$$-2h_X q_{Y_iZ_{i+1}X_{i+2}} + h_Y q_{X_iZ_{i+1}X_{i+2}} = 0. \tag{21}$$

Plugging Eq. (17) into Eq. (21), we conclude that $q_{Y_iZ_{i+1}X_{i+2}} = 0$. This completes the proof of the absence of 3-support conserved quantities in rank 2 case.

VII. OUTLINE OF THE PROOF FOR THE GENERAL CASE

We here outline the proof of our main result. The complete proof is given in the companion paper [42].

The proof of the absence of k-support conserved quantities in rank 2 case is quite close to that for $k = 3$. In Step 1, we employ the condition $\{r_{\boldsymbol{B}_i^{k+1}} = 0\}$ and show that (1) $q_{\boldsymbol{A}_i^k} = 0$ holds except for $\boldsymbol{A} = XZZ\cdots ZZX$, $XZZ\cdots ZZY$, $YZZ\cdots ZZX$, and $YZZ\cdots ZZY$ and (2) some linear relations hold between the remaining coefficients. In Step 2, we employ the condition $\{r_{\boldsymbol{B}_i^k} = 0\}$ and show that all the remaining coefficients are zero.

In rank 1 case [38] and rank 3 case [32,42], the framework of showing $q_{\boldsymbol{A}_i^k} = 0$ for all $\boldsymbol{A}^k$ by first focusing on $\{r_{\boldsymbol{B}_i^{k+1}} = 0\}$ and then on $\{r_{\boldsymbol{B}_i^k} = 0\}$ is common to rank 2 case, while how to eliminate the coefficients $q_{\boldsymbol{A}_i^k}$ are quite different.

VIII. CONCLUSION AND DISCUSSION

We have performed a complete classification of integrability and nonintegrability for spin-$\frac{1}{2}$ chain with symmetric nearest-neighbor interaction, based on the number of local conserved quantities. All systems, other than known integrable systems, are rigorously proved to be nonintegrable. Precisely, it is established that systems are either integrable systems with infinite local conserved quantities or nonintegrable systems with no local conserved quantity, and there is no intermediate system with a finite number of local conserved quantities. Our result offers a rigorous confirmation of the expectation that most systems are nonintegrable.

We remark that a rank-by-rank analysis is essential. The proof often yields constraints of the form $J_\alpha q_{\mathcal{O}} = 0$ [e.g., Eq. (8)], which imply $q_{\mathcal{O}} = 0$ only if $J_\alpha \neq 0$. Hence, when the set of nonzero couplings changes, the implication becomes vacuous and the proof must be reformulated.

Notably, the integrable systems naturally emerge as exceptional cases in the proof of nonintegrability without resorting any existing techniques for integrable systems, including the Bethe ansatz and the Jordan-Wigner transformation to free fermions. Therefore, the nonintegrability proof method is expected to help the discovery of novel integrable systems without relying on specific tools for integrable systems. Also, the result in the companion paper [42], along with Refs. [38,44,45], shows that expansion of physical quantities and coefficient comparison are useful not only for proving nonintegrability, but also for construction of local conserved quantities of both integrable and nonintegrable systems.

In the class of models in consideration, the presence or absence of k-local conserved quantities for $3 \leqslant k \leqslant N/2$ follow the same fate, i.e., if a system has (respectively, does not have) 3-local conserved quantity, then this system has (does not have) k-local conserved quantities for any $4 \leqslant k \leqslant N/2$. A similar property is conjectured for more general classes of spin systems in Ref. [43], where for a broad class of spin chains with nearest-neighbor interactions, the presence or absence of 3-support conserved quantities is conjectured to determine the presence or absence of general k-support conserved quantities with $k \geqslant 4$. Clarifying whether this conjecture holds or does not hold in a more general class of systems is left as an important future problem.

ACKNOWLEDGMENTS

The authors are grateful to H. K. Park, A. Kuniba, C. Matsui, K. Minami, M. Kunimi, H. Tasaki, and H. Katsura for fruitful discussion. N.S. and Y.C. are supported by JST ERATO Grant No. JPMJER2302, Japan. Y.C. is also supported by Japan Society for the Promotion of Science KAKENHI Grant No. JP21J14313 and the Special Postdoctoral Researchers Program at RIKEN.

DATA AVAILABILITY

No data were created or analyzed in this study.

[1] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Elsevier, Amsterdam, 2016).

[2] M. Jimbo and T. Miwa, *Algebraic Analysis of Solvable Lattice Models*, CBMS Regional Conference Series in Mathematics, Vol. 85 (American Mathematical Society, Providence, RI, 1994).

[3] L. D. Faddeev, How algebraic Bethe ansatz works for integrable model, in *Symétries Quantiques*, edited by A. Connes *et al.*, Les Houches, 1995 (North-Holland, Amsterdam, 1998), pp. 149–219.

[4] M. Takahashi, *Thermodynamics of One-Dimensional Solvable Models* (Cambridge University Press, Cambridge, 1999).

[5] H. A. Bethe, Zur Theorie der Metalle. I. Eigenwerte und Eigenfunktionen der linearen Atomkette, Zeit. für Physik **71**, 205 (1931).

[6] E. Lieb, T. Schultz, and D. Mattis, Two soluble models of an antiferromagnetic chain, Ann. Phys. **16**, 407 (1961).

[7] C.-N. Yang and C.-P. Yang, One-dimensional chain of anisotropic spin-spin interactions. I. Proof of Bethe's hypothesis for ground state in a finite system, Phys. Rev. **150**, 321 (1966).

[8] E. H. Lieb and F.-Y. Wu, Absence of Mott transition in an exact solution of the short-range, one-band model in one dimension, Phys. Rev. Lett. **20**, 1445 (1968).

[9] R. Baxter, One-dimensional anisotropic Heisenberg chain, Phys. Rev. Lett. **26**, 834 (1971).

[10] Z. Maassarani, The *XXC* models, Phys. Lett. A **244**, 160 (1998).

[11] A. Y. Kitaev, Unpaired Majorana fermions in quantum wires, Phys. Usp. **44**, 131 (2001).

[12] Y. Yanagihara and K. Minami, Exact solution of a cluster model with next-nearest-neighbor interaction, Prog. Theor. Exp. Phys. **2020**, 113A01 (2020).

[13] B. Sutherland, Model for a multicomponent quantum system, Phys. Rev. B **12**, 3795 (1975).

[14] L. A. Takhtajan, The picture of low-lying excitations in the isotropic Heisenberg chain of arbitrary spins, Phys. Lett. A **87**, 479 (1982).

[15] H. Babujian, Exact solution of the one-dimensional isotropic Heisenberg chain with arbitrary spins S, Phys. Lett. A **90**, 479 (1982).

[16] M. N. Barber and M. T. Batchelor, Spectrum of the biquadratic spin-1 antiferromagnetic chain, Phys. Rev. B **40**, 4621 (1989).

[17] J.-S. Caux and J. Mossel, Remarks on the notion of quantum integrability, J. Stat. Mech.: Theory Exp. (2011) P02023.

[18] C. Gogolin and J. Eisert, Equilibration, thermalisation, and the emergence of statistical mechanics in closed quantum systems, Rep. Prog. Phys. **79**, 056001 (2016).

[19] M. Grabowski and P. Mathieu, Structure of the conservation laws in quantum integrable spin chains with short range interactions, Ann. Phys. **243**, 299 (1995).

[20] M. Fagotti and Fabian H. L. Essler, Reduced density matrix after a quantum quench, Phys. Rev. B **87**, 245107 (2013).

[21] M. Rigol, V. Dunjko, V. Yurovsky, and M. Olshanii, Relaxation in a completely integrable many-body quantum system: An

ab initio study of the dynamics of the highly excited states of 1D lattice hard-core bosons, Phys. Rev. Lett. **98**, 050405 (2007).

[22] B. Pozsgay, The generalized Gibbs ensemble for Heisenberg spin chains, J. Stat. Mech.: Theory Exp. (2013) P07003.

[23] L. Vidmar and M. Rigol, Generalized Gibbs ensemble in integrable lattice models, J. Stat. Mech.: Theory Exp. (2016) 064007.

[24] T. Kinoshita, T. Wenger, and D. S. Weiss, A quantum Newton's cradle, Nature (London) **440**, 900 (2006).

[25] T. Langen, S. Erne, R. Geiger, B. Rauer, T. Schweigler, M. Kuhnert, W. Rohringer, I. E. Mazets, T. Gasenzer, and J. Schmiedmayer, Experimental observation of a generalized Gibbs ensemble, Science **348**, 207 (2015).

[26] F. Reiter, F. Lange, S. Jain, M. Grau, J. P. Home, and Z. Lenarčič, Engineering generalized Gibbs ensembles with trapped ions, Phys. Rev. Res. **3**, 033142 (2021).

[27] P. Mazur, Non-ergodicity of phase functions in certain systems, Physica **43**, 533 (1969).

[28] M. Suzuki, Ergodicity, constants of motion, and bounds for susceptibilities, Physica **51**, 277 (1971).

[29] X. Zotos, F. Naef, and P. Prelovsek, Transport and conservation laws, Phys. Rev. B **55**, 11029 (1997).

[30] K. Saito, Strong evidence of normal heat conduction in a one-dimensional quantum system, Europhys. Lett. **61**, 34 (2003).

[31] T. Mori, T. N. Ikeda, E. Kaminishi, and M. Ueda, Thermalization and prethermalization in isolated quantum systems: A theoretical overview, J. Phys. B: At. Mol. Opt. Phys. **51**, 112001 (2018).

[32] N. Shiraishi, Proof of the absence of local conserved quantities in the XYZ chain with a magnetic field, Europhys. Lett. **128**, 17002 (2019).

[33] F. J. Dyson, Statistical theory of the energy levels of complex systems. I, J. Math. Phys. **3**, 140 (1962).

[34] E. P. Wigner, Characteristic vectors of bordered matrices with infinite dimensions I, in *The Collected Works of Eugene Paul Wigner: Part A: The Scientific Papers*, edited by A. S. Wightman (Springer-Verlag, Berlin/Heidelberg, 1993), pp. 524–540.

[35] M. Rigol, Breakdown of thermalization in finite one-dimensional systems, Phys. Rev. Lett. **103**, 100403 (2009).

[36] L. F. Santos and M. Rigol, Onset of quantum chaos in one-dimensional bosonic and fermionic systems and its relation to thermalization, Phys. Rev. E **81**, 036206 (2010).

[37] Y. Y. Atas, E. Bogomolny, O. Giraud, and G. Roux, Distribution of the ratio of consecutive level spacings in random matrix ensembles, Phys. Rev. Lett. **110**, 084101 (2013).

[38] Y. Chiba, Proof of absence of local conserved quantities in the mixed-field Ising chain, Phys. Rev. B **109**, 035123 (2024).

[39] H. K. Park and S. Lee, Graph-theoretical proof of nonintegrability in quantum many-body systems: Application to the PXP model, Phys. Rev. B **111**, L081101 (2025).

[40] N. Shiraishi, Absence of local conserved quantity in the Heisenberg model with next-nearest-neighbor interaction, J. Stat. Phys. **191**, 114 (2024).

[41] A. Goriely, Integrability, partial integrability, and nonintegrability for systems of ordinary differential equations, J. Math. Phys. **37**, 1871 (1996).

[42] M. Sanatani, Y. Chiba, and N. Shiraishi, companion paper, Proof of the absence of local conserved quantities in general spin-$\frac{1}{2}$ chains with symmetric nearest-neighbor interaction, Phys. Rev. B **114**, 094437 (2026).

[43] M. P. Grabowski and P. Mathieu, Integrability test for spin chains, J. Phys. A: Math. Gen. **28**, 4777 (1995).

[44] Y. Nozawa and K. Fukai, Explicit construction of local conserved quantities in the XYZ spin-1/2 chain, Phys. Rev. Lett. **125**, 090602 (2020).

[45] K. Fukai, All local conserved quantities of the one-dimensional Hubbard model, Phys. Rev. Lett. **131**, 256704 (2023).