

Transdimensional dynamics of Kerr cavity solitons

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We study the bifurcation and stability of radial symmetric solitons in one- (1D), two- (2D), and three-dimensional (3D) damped-driven Kerr cavity systems. We employ a dimension-parameterized one-dimensional framework, by imposing symmetry, where the dimension of the systems can be adjusted by a real parameter, enabling us to analyze the impact of the dimensionality increase, in a continuous fashion, on the system's dynamics. Our results applied to Kerr cavity solitons unveil how key bifurcations, such as Hopf or exponential instabilities, appear in the system and reveal that the soliton solutions tend to become more and more unstable in systems with increased dimensions. Consequently, the stability region for 2D solitons is much narrower than in 1D, and all 3D solitons are unstable against collapse in the 3D damped-driven nonlinear Schrödinger equation. Our results and methods may be applied to other physically relevant 1D, 2D, 3D systems under symmetric conditions.

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Solitons are wave packets that maintain their shape while propagating through nonlinear media, appearing in different physical contexts such as Bose-Einstein condensates, plasmas, hydrodynamics, biology, and optics [1–5], to cite a few. In optical conservative systems, solitons form due to a balance between nonlinearity and dispersion or diffraction. This balance occurs in one- (1D), two- (2D), and three-dimensional (3D) systems, respectively, giving rise to a plethora of possible solitary waves, ranging from, e.g., temporal, spatial, and spatiotemporal solitons, which in the case of 3D are called light bullets (LB) [6–9]. Purely conservative spatiotemporal solitons have only been experimentally observed in scant occasions, in the form of dynamical wave packets which survive over a few dispersion or diffraction lengths only [10–12]. In open or dissipative systems, where loss and external energy sources play a key role [5], solitons tend to become robust attractors and are referred to as dissipative solitons (DS): examples include DS in mode-locked lasers [13–15] and multimode Fabry–Perot microresonators [16]. Compared to their 1D counterparts [17,18], spatiotemporal solitons exhibit more complex dynamics, owing to their higher-dimensional nature.

Exploring changes in system dimensionality across different scientific fields has often led to the discovery of new phenomena and performance improvements. Examples include nanophotonics and plasmonics [19,20], cosmology [21],

Fermi gases [22], geosciences [23], Casimir effects [24,25], and quantum networks [26,27]. Similarly, in the field of DS, the reduction of 2D spatial Kerr cavity solitons (CS) [28,29] to 1D temporal CS [17,18,30–32] has facilitated the development of coherent optical frequency combs, unlocking the potential of a new technology. Conversely, exploring higher dimensions has led to the prediction of more complex forms of solitons and patterns in 2D [29,33–37] and in 3D [38–45].

The dynamics of multidimensional Kerr CS are described by the driven and damped nonlinear Schrödinger equation (NLSE), that was originally studied in plasma physics [46,47]. This equation is adaptable across various dimensions, as illustrated by the spatial Lugiato-Lefever equation [48] and the 1D temporal Haelterman-Trillo-Wabnitz equation [49–51]. Localized states in these systems, which involve bistability induced by nonlinearity, have been extensively theoretically studied [52–56]. While an attempt to explore 3D bulk optical Kerr cavities, incorporating both diffraction and dispersion [57], claimed stable LB, a thorough stability analysis was absent, which is crucial to elucidate the presence of bifurcations in the solutions and to verify the LB's stability. Moreover, in higher-dimensional systems the dynamics may undergo significant changes, such as wave collapse—a long-standing challenge in systems with a cubic nonlinearity in both conservative and dissipative systems [3,58–60]. Until now, how these phenomena evolve with increasing dimensionality remains an open question.

In this work, we analyze the bifurcation structure of the damped-driven NLSE as the dimension parameter, introduced in Refs. [6,61], changes continuously. In this way, we can trace the onset of characteristic higher-dimensional dynamics associated with the cubic nonlinearity [3,58,59]. In particular, we reveal how dimensionality impacts the existence and

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stability of multidimensional CS within the framework of the damped-driven NLSE, covering settings from 1D to 3D. By reducing all models to a single 1D equation with an added parameter, we are able to elucidate the distinct characteristics of radial symmetric CS, both within specific dimensions and across transdimensional regimes. Our findings show that the stability region for 2D CS is significantly reduced when compared to 1D CS, in agreement with previous 2D studies [29]. In 3D, although localized solutions (LS) exist, they are all unstable, and prone to wave collapse, unless additional constraints are imposed, as previously noted [35,42].

By neglecting the angular (azimuthal and polar) degrees of freedom, the driven and damped NLSE reads as

$$\frac{\partial A}{\partial t} = i \left[\frac{D-1}{r} \frac{\partial A}{\partial r} + \frac{\partial^2 A}{\partial r^2} \right] A + i|A|^2 A - (1 + i\delta)A + P, \quad (1)$$

where t is the propagation time, r stands for, e.g., the propagation coordinate in 1D systems or the radial coordinates in 2D and 3D systems with cylindrical and spherical symmetry, respectively. δ is the cavity detuning and P is the amplitude of the external driving.

This model adopts a dimension-parametrized framework in which $D = 1, 2, 3$ correspond to 1D, 2D, and 3D systems, respectively. The parameter D modulates the differential operator and serves as a continuous variable to explore the impact of dimensionality on the system's dynamics. While noninteger values of D may lack a direct physical meaning, the reduced 1D NLSE still supports soliton-like solutions enabling a systematic tracking of their stability variations within a constrained subset of the support domain. This mathematical extension provides valuable insights into the evolution of bifurcation behaviors as the system transitions between well-defined physical dimensions. The regime $D > 3$ is beyond the scope of the present study, but may offer meaningful insights in future work if relevant physical analogs are identified. To distinguish between physical and nonphysical solutions, we refer to LS for $D \neq 1, 2, 3$, whereas CS specifically denote cases with physically meaningful integer dimensions $D = 1, 2, 3$. Other similar approaches have been explored in fractal NLSE studies [62–64] and fractional nonlinear-wave physics models [65].

Figure 1(i) illustrates the case of a stable 1D CS for parameters $P = 1.21$ and $\delta = 1.7$. By progressively increasing D , we compute new LS through the `pde2path` [66]. As shown in Fig. 1(a), the peak amplitude $\max(|A|)$ of these LS increases with D . For $D = 2$ and 3, we extend the radial profiles $A(r)$ to their full distributions and plot their 1D cross sections $A(x, y = 0)$ and $A(x = 0, y = \tau)$ in Figs. 1(ii),(iii). Profiles of the 2D CS $|A(x, y)|$ and 3D CS $|A(x, y, \tau)|$ are represented in Figs. 1(iv) and 1(vi), while the cross-section of the 3D CS $|A(x, y = 0, \tau)|$ is displayed in Fig. 1(v). As shown in Figs. 1(i), 1(ii), 1(iii), increasing D leads to higher peak values of CS, whereas the background field amplitudes remain unchanged.

A linear stability analysis (LSA) reveals that the stability of the radially symmetric LS is compromised as D increases [cf. Fig. 1(a)]. In the transdimensional regime between $D = 1$ and 2, the LS develop an oscillatory instability, known as Hopf

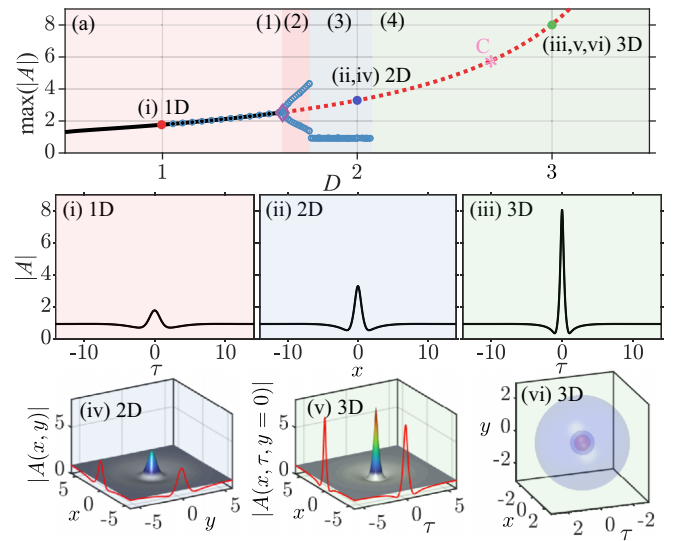


FIG. 1. (a) Bifurcation diagram, in the form $\max(|A|)$ vs D , of LS with $\delta = 1.7$, $P = 1.21$. Stable and unstable LS predicted by LSA are shown as solid black and red dashed curves, respectively, and verified by simulations of Eq. (1), where blue circles indicate the maximum and minimum values of $|A(r = 0)|$. The dynamical regimes include: (1) stable LS, (2) breathing LS, (3) breathers decaying into HS, and (4) wave collapse. The profiles of the 1D, 2D, and 3D solitons, marked by (i),(ii),(iii) at $D = 1, 2, 3$ in (a), are plotted vs r in (i),(ii),(iii). Panels (iv),(v),(vi) reconstruct the 2D profile of panel (ii), the cross of the bullet in (iii) and its full 3D profile.

bifurcation (H) [67]. Between $D = 2$ and 3, the instability becomes a purely exponential one, as the two eigenvalues associated with the internal modes [68] collapse onto the real axis, giving birth to a purely exponential instability [69]. We mark this instability C. The LSA is validated by numerical simulations of Eq. (1). Our results show that stable breathing solutions are restricted to region 2 [see Fig. 1(a)]; beyond this, the final state converges to the homogeneous state (HS), while in region 4, wave collapse occurs (see examples in Supplemental Material [69]). Notably, wave collapse occurs before reaching point C, as the C value is determined for stationary solutions. Once an LS enters the breathing process, transient field intensities exceed those of the stationary state, potentially triggering wave collapse. A video illustrating the simulation of radial field profiles for various values of the dimension parameter D is available in [70]. In what follows, we will unfold the structure of these multiple bifurcations and explore the existence of stable high-dimensional CS.

Figures 2(a)–2(c) show examples of bifurcation diagrams of the soliton peak $\max(|A|)$, when varying δ for $D = 1, 2, 3$ in Fig. 1(a). In Fig. 2(a), the stable HS branch (solid gray curve) bifurcates at the branching point B_1 , where it intersects with the unstable HS (dashed gray) branch and the unstable CS (dashed red) branch. The latter connects to the stable CS branch at the fold point F_1 . When $D = 2$ [cf. Fig. 2(b)], solitons feature higher peak-intensity levels, and the branch between F_1 and F_2 extends over a larger δ range ($1.68 < \delta < 2.32$), although the stable CS only exist in a narrow range ($\Delta\delta \approx 10^{-3}$) in between the Hopf bifurcation H and F_1 (see inset). For $D = 3$, CS undergo a similar structure,

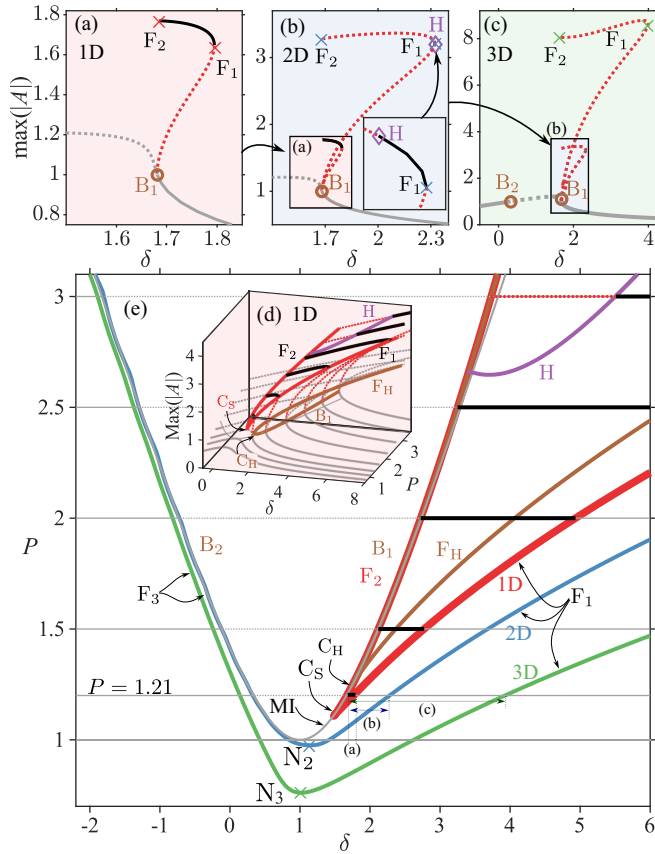


FIG. 2. (a),(b),(c) Bifurcation diagrams for CS and HS represented by $\max(|A|)$ versus δ for $D = 1, 2, 3$ dimensions, respectively, with a fixed driving value $P = 1.21$. (d) Bifurcation diagram vs δ for different P when $D = 1$. Stable and unstable HSs are shown as solid and dashed gray curves, respectively, separated by fold bifurcations (B_1 and F_H), indicated with brown curves. Stable and unstable CS are represented by solid black and red dotted curves, respectively. Two fold bifurcations (F_1, F_2) for CS are shown as solid red curves, and Hopf bifurcations H are marked by purple curves. Cusp bifurcations are labeled as C_S for CS and C_H for HS. (e) Phase diagram of P versus δ , highlighting regions divided by fold and Hopf bifurcations. For the 2D and 3D cases, fold bifurcations (F_1, F_3) are represented by blue and green curves, respectively, in contrast to the 1D case.

with branches extending over an even larger δ region [$\delta \in (1.68, 4)$].

To further illustrate the system's stability, we analyze the evolution of both fold (F) and Hopf (H) bifurcations. Starting from the 1D cases, as depicted in Fig. 2(d), we explore the bifurcation diagram vs δ for both HS and CS for different driving values P . The HS undergo Turing or modulational instability (MI) (see gray line) for $P > 1$, and a cusp bifurcation C_H at $(P, \delta) = (1.24, \sqrt{3})$, which leads to the appearance of two bifurcation curves, F_H and B_1 . For CS, the fold (F_1, F_2) and Hopf (H) bifurcations, depicted by red solid and purple curves in Fig. 2(d), are replotted in Fig. 2(e) as a phase diagram. F_2 for CS shares the same (P, δ) values as B_1 for HS. The point C_S where F_1 and F_2 collide and disappear corresponds to the cusp bifurcation at $(P, \delta) = (1.1, 1.48)$. 1D

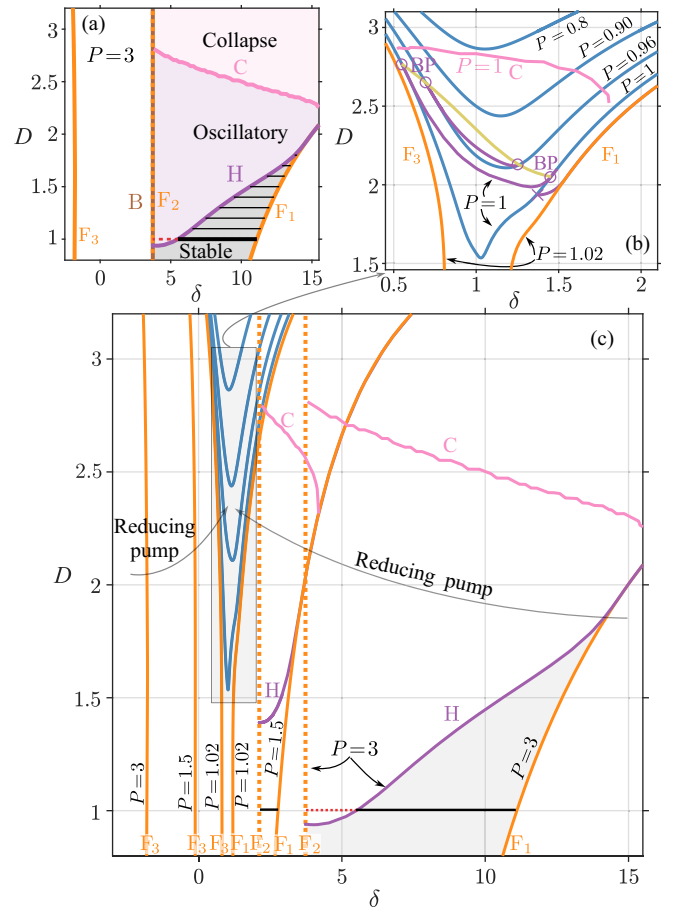


FIG. 3. Phase diagram showing different bifurcations (marked by F_1, F_2, F_3 for fold and H for Hopf) when varying D and δ in (a) $P = 3$, (b) $0.8 \leq P \leq 1$, and (c) $0.8 \leq P \leq 3$.

CS are confined to the region between F_1 and F_2 , while 1D breathers occur between H and F_2 .

We extended our analysis to the fold bifurcation of 2D and 3D CS [see blue and green curves in Fig. 2(e)]. For P values above C_S , 2D and 3D CS solution branches behave as their 1D counterparts: CS are limited by F_1 and F_2 , as depicted in Figs. 2(b), 2(c). However, this configuration changes drastically below C_S , where 2D and 3D CS still exist. At the MI threshold, near $P = 1$, the existence of CS may now extend until reaching a new fold bifurcation F_3 . This point evolves into F_1 through the point N_2 (P, δ) = (1.148, 0.976) for 2D, and N_3 (P, δ) = (1.016, 0.761) for 3D, respectively. Unlike the 1D scenario, CS can now reach negative values of detuning, owing to the broad existence of stable patterns [33,36,71].

The transition from 1D to 3D bifurcation structures is analyzed by gradually increasing D , as shown in Fig. 3(a) for $P = 3$. The evolution of F_1 and F_2 is traced from 1D CS, while F_3 is identified from higher-dimensional states. Localized solutions exist within the region bounded by F_1 and F_3 , with their stability governed by Hopf bifurcation H (see eigenvalue analysis in the Supplemental Material [69]). The positions of F_2 and F_3 remain nearly constant in δ , whereas F_1 and H shift toward higher δ as D increases. This shift leads to a compression of the stable regions (black solid lines) and an

expansion of the unstable regions, separated by H, indicating a progressive reduction in soliton stability with increasing D . At driving value $P = 3$, 2D CS are unstable, and for $D > 2$, a bifurcation associated with the C point (solid pink curve) emerges, moving upward in the diagram as δ decreases.

To identify regions of stable LS in higher dimensions, we analyzed the evolution of bifurcations for reduced driving amplitudes, as shown in Fig. 3(c). Reducing the driving can stabilize LS between H and F_1 , as observed in 1D [56] and 2D [29], or may shift the C point above $D = 3$. Figure 3(c) compares bifurcation F_1 , F_2 , F_3 , H for $P = 3, 1.5, 1.02$. At $P = 1.5$, the unstable region for $D = 1$ vanishes as H shifts to higher D , but the stable region at $D = 2$ remains narrow. The instability C continues to dominate at higher D . Further reducing P to 1.02 brings F_1 and F_3 closer around $\delta = 1$. For $P \leq 1$ (blue curves), F_2 disappears, and F_1 merges with F_3 , as illustrated by the blue curves for $P = 1, 0.96, 0.90, 0.8$ in Fig. 3(c), with a detailed view in Fig. 3(b).

As P decreases, the CS regions between the two folds shift to higher dimensions. This is demonstrated for $P = 1, 0.96, 0.9$ in Fig. 3(b), where the folds move toward larger D values, indicating the absence of 2D CS. The minimum driving value required for 2D CS existence is $P = 0.975$. For 3D CS, the existence region similarly shrinks as P decreases from 1 to 0.8, eventually disappearing entirely for $P < 0.761$. This means at lower driving powers, LS are restricted to higher dimensions; however, their stability must be further examined.

In fact, low driving levels are insufficient to stabilize 3D CS. As P decreases from 1 to 0.9, the two branching points [labeled BP in Fig. 3(b)] from which the Hopf lines bifurcate move closer together until they collide, causing the Hopf curves to vanish for $P < 0.93$. This suggests that, in higher dimensions, LS may persist but remain unstable for $P < 0.93$. For example, at $P = 0.9$ and $P = 0.8$, CS exist only for $D = 3$ but remain unstable. Additionally, the C instability, occurring between 2D and 3D and represented by the pink curve, persists as P decreases (see Supplementary Material [69]). At $P = 0.8$, all CS in Fig. 3(b) exhibit instability. This analysis indicates radially symmetric stable 3D CS cannot exist.

We perform numerical simulations to validate the predicted stability of 2D and 3D CS. For $D = 2$, Fig. 4(a) presents bifurcation diagrams versus δ for $P = 1, 0.99, 0.98$. At $P = 1$, the CS branch forms a closed loop, disconnected from HS. Loading CS as initial solutions, numerical simulations reveal the CS stability by plotting peak values $\max(|A(x, y = 0, t)|)$ [blue dots in Fig. 4(a)]. The upper branch transitions through four regimes: (i) stable CS, (ii) 2D breathers, (iii) stable CS, and (iv) quasiperiodic 2D patterns, as demonstrated in Supplementary Material [69] and videos [70]. Examples of (i) 2D CS and (ii) 2D breathers are shown in Figs. 4(i), 4(ii), with their temporal evolution depicted in Fig. 4(e).

If the LS predicted by the LSA along the radial direction are unstable, it implies that the CS in the full-dimensional system are unstable. However, the LSA is restricted to radial perturbations and does not account for nonradial angular instabilities. To address this limitation, we perform numerical simulations in their original coordinates to verify the stability of these solutions. As revealed in Fig. 4(a), the pattern-formation regime (iv), highlighted in pink, is unstable due to

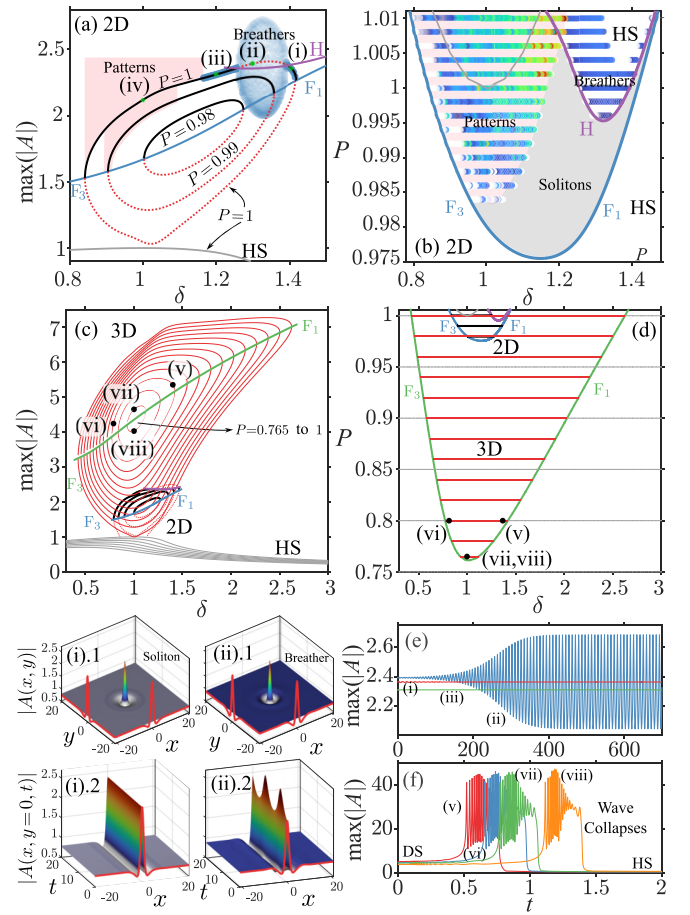


FIG. 4. (a) Bifurcation diagram showing $\max(|A|)$ of 2D CS vs δ for driving values $P = 1, 0.99, 0.98$. Loading CS solutions [i.e., at (i)–(iv)] from continuation, the final states are obtained by numerical simulations and are represented by blue dots for $P = 1$: (i), (iii) solitons ($\delta = 1.4, 1.2$); (ii) breathers ($\delta = 1.3$); (iv) patterns ($\delta = 1$). Panels (i), (ii) show the time evolution of (i), (ii). The time evolutions of the peak values $\max(|A|)$ for the case (i), (ii), (iii) is depicted in (e). (b) Phase diagram summarizing the four final stable states for 2D CS. (c), (d) Bifurcation and phase diagrams versus δ for 3D CS, as P varies from 1 to 0.765, in comparison with 2D diagrams. LSA and DNS show that all 3D CS are unstable. Time evolutions of four examples marked at (v)–(viii) in (c), (d) are shown in (f).

angular instabilities. At $P = 0.99$, the pattern region diminishes, and the breather region disappears. By $P = 0.98$, only stable 2D solitons remain, consistent with LSA predictions. A synthesis of continuation and simulation results is presented in Fig. 4(b), where color-coded dots represent the standard deviation of the peak amplitude $|A|$ (values increasing from blue, green, yellow to red). Reduced fluctuations are observed in patterned regions at lower detuning values. Within the breather zone, some breathers transition into HS, while stable 2D CS exist within the shallow gray region.

For 3D CS, as predicted in Fig. 3(b), stable 3D CS are not feasible under the current conditions. Nonetheless, numerical simulations were performed for scenarios most likely to support stability, specifically at very low driving values. The bifurcation and phase diagrams for P ranging from 1 to 0.765 are presented in Figs. 4(c), 4(d), alongside their 2D

counterparts for comparison. The red curves confirm that all 3D CS remain unstable across these conditions. The time evolution of four 3D CS examples, labeled (vi)–(viii) in Figs. 4(c), 4(d), is shown in Fig. 4(f). As expected, the 3D CS undergo rapid spatiotemporal wave collapse before decaying into HS (see video [70] for detailed time evolution).

In conclusion, we have investigated the stability of multidimensional solitons by analyzing the bifurcation structures of solutions through the continuous variation of the parameter D . Our study was applied to the driven-damped NLSE, which describes the dynamics of scalar fields in cubic nonlinear media, including optical systems. By employing dimension reduction and continuation methods, we unveiled the existence and stability of CS across different dimensions. Although noninteger values of D may not correspond to physical dimensions, treating D as a continuous parameter provides useful insight into how stability and bifurcation properties evolve between lower

and higher dimensions ($1 \leq D \leq 3$). Our findings indicate that the stability of these states decreases very strongly with increasing dimensionality, so that while stable 1D and 2D CS do exist, stable 3D solitons do not, even in the limit of very low driving values. Because of the universality of the driven and damped NLSE, we anticipate that the results of our approach may have significant ramifications in a variety of different physical settings.

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The authors declare no conflicts of interest.

Data availability. The data that support the findings of this article are openly available [70].

- [1] T. Dauxois and M. Peyrard, *Physics of Solitons* (Cambridge University Press, 2006).
- [2] B. A. Malomed, D. Mihalache, F. Wise, and L. Torner, Spatiotemporal optical solitons, *J. Opt. B* **7**, R53 (2005).
- [3] Y. V. Kartashov, G. E. Astrakharchik, B. A. Malomed, and L. Torner, Frontiers in multidimensional self-trapping of nonlinear fields and matter, *Nat. Rev. Phys.* **1**, 185 (2019).
- [4] B. A. Malomed, *Multidimensional Solitons* (AIP Publishing, Melville, New York, 2022).
- [5] N. N. Akhmediev and A. Ankiewicz, *Dissipative Solitons : From Optics to Biology and Medicine* (Springer Science & Business Media, 2008).
- [6] Y. Silberberg, Collapse of optical pulses, *Opt. Lett.* **15**, 1282 (1990).
- [7] Y. S. Kivshar and G. P. Agrawal, *Optical Solitons: From Fibers to Photonic Crystals*, edited by Y. S. Kivshar and G. P. Agrawal (Academic Press, Burlington, 2003), pp. 1–540.
- [8] G. Fibich and B. Ilan, Optical light bullets in a pure Kerr medium, *Opt. Lett.* **29**, 887 (2004).
- [9] Y. Sun, P. Parra-Rivas, G. P. Agrawal, T. Hansson, C. Antonelli, A. Mecozzi, F. Mangini, and S. Wabnitz, Multimode solitons in optical fibers: a review, *Photon. Res.* **12**, 2581 (2024).
- [10] S. Minardi, F. Eilenberger, Y. V. Kartashov, A. Szameit, U. Röpke, J. Kobelke, K. Schuster, H. Bartelt, S. Nolte, L. Torner, F. Lederer, A. Tünnermann, and T. Pertsch, Three-dimensional light bullets in arrays of waveguides, *Phys. Rev. Lett.* **105**, 263901 (2010).
- [11] W. H. Renninger and F. W. Wise, Optical solitons in graded-index multimode fibres, *Nat. Commun.* **4**, 1719 (2013).
- [12] P. Panagiotopoulos, P. Whalen, M. Kolesik, and J. V. Moloney, Super high power mid-infrared femtosecond light bullet, *Nat. Photonics* **9**, 543 (2015).
- [13] J. Javaloyes, Cavity light bullets in passively mode-locked semiconductor lasers, *Phys. Rev. Lett.* **116**, 043901 (2016), 1506.02917.
- [14] S. V. Gurevich and J. Javaloyes, Spatial instabilities of light bullets in passively-mode-locked lasers, *Phys. Rev. A* **96**, 023821 (2017).
- [15] L. G. Wright, D. N. Christodoulides, and F. W. Wise, Spatiotemporal mode-locking in multimode fiber lasers, *Science* **358**, 94 (2017).
- [16] M. Nie, K. Jia, Y. Xie, S. Zhu, Z. Xie, and S.-W. Huang, Synthesized spatiotemporal mode-locking and photonic flywheel in multimode mesoresonators, *Nat. Commun.* **13**, 6395 (2022).
- [17] F. Leo, S. Coen, P. Kockaert, S. P. Gorza, P. Emplit, and M. Haelterman, Temporal cavity solitons in one-dimensional kerr media as bits in an all-optical buffer, *Nat. Photonics* **4**, 471 (2010).
- [18] T. Herr, V. Brasch, J. D. Jost, C. Y. Wang, N. M. Kondratiev, M. L. Gorodetsky, and T. J. Kippenberg, Temporal solitons in optical microresonators, *Nat. Photonics* **8**, 145 (2014).
- [19] A. Boltasseva and V. M. Shalaev, Transdimensional photonics, *ACS Photonics* **6**, 1 (2019).
- [20] I. V. Bondarev, H. Mousavi, and V. M. Shalaev, Transdimensional epsilon-near-zero modes in planar plasmonic nanostructures, *Phys. Rev. Res.* **2**, 013070 (2020).
- [21] A. Riotto, E. W. Kolb, and J. Lesgourgues, Transdimensional physics and inflation, *Phys. Rev. D* **66**, 083512 (2002).
- [22] M. G. Endres, Transdimensional equivalence of universal constants for fermi gases at unitarity, *Phys. Rev. Lett.* **109**, 250403 (2012).
- [23] M. Sambridge, K. Gallagher, A. Jackson, and P. Rickwood, Trans-dimensional inverse problems, model comparison and the evidence, *Geophys. J. Int.* **167**, 528 (2006).
- [24] M. Bordag, U. Mohideen, and V. M. Mostepanenko, New developments in the Casimir effect, *Phys. Rep.* **353**, 1 (2001).
- [25] R. S. Decca, D. López, E. Fischbach, G. L. Klimchitskaya, D. E. Krause, and V. M. Mostepanenko, Novel constraints on light elementary particles and extra-dimensional physics from the Casimir effect, *The European Physical Journal C* **2007** 51:4 51, 963 (2007).
- [26] Y. Zheng, C. Zhai, D. Liu, J. Mao, X. Chen, T. Dai, J. Huang, J. Bao, Z. Fu, Y. Tong, X. Zhou, Y. Yang, B. Tang, Z. Li, Y. Li, Q. Gong, H. K. Tsang, D. Dai, and J. Wang, Multichip multidimensional quantum networks with entanglement retrievability, *Science* **381**, 221 (2023).

- [27] J. Wang, S. Paesani, Y. Ding, R. Santagati, P. Skrzypczyk, A. Salavrakos, J. Tura, R. Augusiak, L. Mančinska, D. Bacco, D. Bonneau, J. W. Silverstone, Q. Gong, A. Acín, K. Rottwitt, L. K. Oxenløwe, J. L. O'Brien, A. Laing, and M. G. Thompson, Multidimensional quantum entanglement with large-scale integrated optics, *Science* **360**, 285 (2018).
- [28] S. Barland, J. R. Tredicce, M. Brambilla, L. A. Lugiato, S. Balle, M. Giudici, T. Maggipinto, L. Spinelli, G. Tissoni, T. Knödl, M. Miller, and R. Jäger, Cavity solitons as pixels in semiconductor microcavities, *Nature (London)* **419**, 699 (2002).
- [29] W. J. Firth, G. K. Harkness, A. Lord, J. M. McSloy, D. Gomila, and P. Colet, Dynamical properties of two-dimensional Kerr cavity solitons, *J. Opt. Soc. Am. B* **19**, 747 (2002).
- [30] F. Leo, L. Gelens, P. Emplit, M. Haelterman, and S. Coen, Dynamics of one-dimensional Kerr cavity solitons, *Opt. Express* **21**, 9180 (2013).
- [31] C. Godey, I. V. Balakireva, A. Coillet, and Y. K. Chembo, Stability analysis of the spatiotemporal lugiato-lefever model for Kerr optical frequency combs in the anomalous and normal dispersion regimes, *Phys. Rev. A* **89**, 063814 (2014).
- [32] T. J. Kippenberg, A. L. Gaeta, M. Lipson, and M. L. Gorodetsky, Dissipative Kerr solitons in optical microresonators, *Science* **361**, eaan8083 (2018).
- [33] D. Gomila, A. Jacobo, M. A. Matías, and P. Colet, Phase-space structure of two-dimensional excitable localized structures, *Phys. Rev. E* **75**, 026217 (2007).
- [34] Y. Kartashov, M. Gorodetsky, A. Kudlinski, and D. Skryabin, Two-dimensional nonlinear modes and frequency combs in bottle microresonators, *Opt. Lett.* **43**, 2680 (2018).
- [35] C. Milián, Y. V. Kartashov, D. V. Skryabin, and L. Torner, Clusters of cavity solitons bounded by conical radiation, *Phys. Rev. Lett.* **121**, 103903 (2018).
- [36] S. B. Ivars, Y. V. Kartashov, L. Torner, J. A. Conejero, and C. Milián, Reversible self-replication of spatiotemporal Kerr cavity patterns, *Phys. Rev. Lett.* **126**, 063903 (2021).
- [37] S. B. Ivars, Y. V. Kartashov, P. F. de Córdoba, J. A. Conejero, L. Torner, and C. Milián, Photonic snake states in two-dimensional frequency combs, *Nat. Photon.* **17**, 767 (2023).
- [38] M. Tlidi and P. Mandel, Three-dimensional optical crystals and localized structures in cavity second harmonic generation, *Phys. Rev. Lett.* **83**, 4995 (1999).
- [39] K. Staliunas, Three-dimensional turing structures and spatial solitons in optical parametric oscillators, *Phys. Rev. Lett.* **81**, 81 (1998).
- [40] M. Tlidi, M. Haelterman, and P. Mandel, 3D patterns and pattern selection in optical bistability, *Europhys. Lett.* **42**, 505 (1998).
- [41] M. Tlidi, M. Haelterman, and P. Mandel, Three-dimensional structures in diffractive and dispersive nonlinear ring cavities, *Quantum Semiclass. Opt.* **10**, 869 (1998).
- [42] Y. Sun, P. Parra-Rivas, C. Milián, Y. V. Kartashov, M. Ferraro, F. Mangini, R. Jauberteau, F. R. Talenti, and S. Wabnitz, Robust three-dimensional high-order solitons and breathers in driven dissipative systems: A Kerr cavity realization, *Phys. Rev. Lett.* **131**, 137201 (2023).
- [43] Y. Sun, P. Parra-Rivas, M. Ferraro, F. Mangini, and S. Wabnitz, Dynamics of dissipative structures in coherently-driven Kerr cavities with a parabolic potential, *Chaos, Solitons Fractals* **176**, 114064 (2023).
- [44] Y. Sun, P. Parra-Rivas, F. Mangini, and S. Wabnitz, Multidimensional localized states in externally driven Kerr cavities with a parabolic spatiotemporal potential: A dimensional connection, *Chaos, Solitons Fractals* **183**, 114870 (2024).
- [45] Q. Tang, Y. Zhang, Y. V. Kartashov, and C. Milián, Reconfigurable topological dissipative light bullets, [arXiv:2410.05981](https://arxiv.org/abs/2410.05981).
- [46] G. J. Morales and Y. C. Lee, Ponderomotive-force effects in a nonuniform plasma, *Phys. Rev. Lett.* **33**, 1016 (1974).
- [47] D. J. Kaup and A. C. Newell, Theory of nonlinear oscillating dipolar excitations in one-dimensional condensates, *Phys. Rev. B* **18**, 5162 (1978).
- [48] L. A. Lugiato and R. Lefever, Spatial dissipative structures in passive optical systems, *Phys. Rev. Lett.* **58**, 2209 (1987).
- [49] M. Haelterman, S. Trillo, and S. Wabnitz, Dissipative modulation instability in a nonlinear dispersive ring cavity, *Opt. Commun.* **91**, 401 (1992).
- [50] M. Haelterman, S. Trillo, and S. Wabnitz, Additive-modulation-instability ring laser in the normal dispersion regime of a fiber, *Opt. Lett.* **17**, 745 (1992).
- [51] S. Wabnitz, Suppression of interactions in a phase-locked soliton optical memory, *Opt. Lett.* **18**, 601 (1993).
- [52] N. N. Rozanov and G. V. Khodova, Autosolitons in bistable interferometers, *Opt. Spectrosc.* **65**, 449 (1988).
- [53] N. N. Rosanov, A. V. Fedorov, and G. V. Khodova, Effects of spatial distributivity in semiconductor optical bistable systems, *Phys. Status Solidi B* **150**, 545 (1988).
- [54] N. N. Rosanov and G. V. Khodova, Diffractive autosolitons in nonlinear interferometers, *J. Opt. Soc. Am. B* **7**, 1057 (1990).
- [55] N. N. Rosanov, *Spatial Hysteresis and Optical Patterns* (Springer, Berlin, Heidelberg, 2002).
- [56] I. Barashenkov and Y. S. Smirnov, Existence and stability chart for the ac-driven, damped nonlinear Schrödinger solitons, *Phys. Rev. E* **54**, 5707 (1996).
- [57] S. S. Gopalakrishnan, K. Panajotov, M. Taki, and M. Tlidi, Dissipative light bullets in Kerr cavities: Multistability, clustering, and rogue waves, *Phys. Rev. Lett.* **126**, 153902 (2021).
- [58] L. Bergé, Wave collapse in physics: principles and applications to light and plasma waves, *Phys. Rep.* **303**, 259 (1998).
- [59] E. A. Kuznetsov and F. Dias, Bifurcations of solitons and their stability, *Phys. Rep.* **507**, 43 (2011).
- [60] V. E. Zakharov and E. A. Kuznetsov, Solitons and collapses: two evolution scenarios of nonlinear wave systems, *Phys.-Usp.* **55**, 535 (2012).
- [61] S. McCalla and B. Sandstede, Snaking of radial solutions of the multi-dimensional Swift–Hohenberg equation: A numerical study, *Physica D* **239**, 1581 (2010).
- [62] A. Khalili Golmankhaneh, S. Pellis, and M. Zingales, Fractal Schrödinger equation: Implications for fractal sets, *J. Phys. A: Math. Theor.* **57**, 185201 (2024).
- [63] J. Sun, Fractal modification of Schrödinger equation and its fractal variational principle, *Therm. Sci.* **27**, 2029 (2023).
- [64] W. Tang, Y. Wang, and Z. Li, Numerical simulation of fractal wave propagation of a multi-dimensional nonlinear fractional-in-space Schrödinger equation, *Phys. Scr.* **98**, 045205 (2023).
- [65] B. A. Malomed, Basic fractional nonlinear-wave models and solitons, *Chaos: Interdiscip. J. Nonlinear Sci.* **34**, 022102 (2024).

- [66] H. Uecker, *Numerical Continuation and Bifurcation in Non-linear PDEs* (Society for Industrial and Applied Mathematics, Philadelphia, PA, 2021).
- [67] J. E. Marsden and M. McCracken, *The Hopf bifurcation and its applications* (Springer Science & Business Media, 2012), Vol. 19.
- [68] D. V. Skryabin, Energy of the soliton internal modes and broken symmetries in nonlinear optics, *J. Opt. Soc. Am. B* **19**, 529 (2002).
- [69] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/k54w-s7lg> for details on model reduction, numerical discretization methods, implementation with the path-continuation software pde2path, bifurcation and stability analyses, eigenvalue studies, and radial numerical simulations.
- [70] Y. Sun, Visual materials for the paper Transdimensional Dynamics of Kerr Cavity Solitons, Zenodo (2025), doi: [10.5281/zenodo.15427863](https://doi.org/10.5281/zenodo.15427863). These video supplements present 2D simulations under varying detuning, highlighting the emergence of stationary and breathing 2D solitons, along with static and chaotic pattern formation. It also features 3D simulations demonstrating wave collapse, and radial-profile simulations across different values of the dimension parameter D .
- [71] M. Tlidi, R. Lefever, and P. Mandel, Pattern selection in optical bistability, *Quantum Semiclass. Opt.* **8**, 931 (1996).