

Spectral repulsion does not uniquely determine modal exchange in a deformed microwave billiard

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An avoided crossing is a familiar spectral signature of interaction between nearby modes, but the eigenvalue trajectory alone does not establish whether their modal identities are exchanged. We examine this question in a deformed microwave billiard by comparing finite element eigenvalues and fields. Across 21 screened adjacent level events, the spectral slope is compared with the boundary response obtained from the fields at the event center. For the 11 events whose rotation width can be resolved within the sampled interval, the boundary-derived and overlap-derived widths agree to within about 11%. One event behaves differently: The spectral model predicts substantial modal rotation, while the computed fields change very little and are instead reproduced by the boundary response. Another event shows that the mode that improves the eigenvalue model is not the mode that best restores the field subspace. Spectral repulsion alone is therefore not sufficient to determine modal exchange or the reference subspace required over a finite parameter range.

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I. INTRODUCTION

The evolution of eigenvalues and eigenfunctions under an external control parameter is a recurring problem in quantum and classical wave systems. When two nearby levels approach as a parameter is varied, their coupling can prevent an exact crossing and produce the characteristic level repulsion of an avoided crossing. Such parametric spectra have played an important role in studies of degeneracies, quantum chaos, and universal level dynamics [1–7]. In tunable microwave cavities, longitudinal symmetry breaking produces avoided mode crossings and modifies the mode structure at closest approach [8]. A quadrupole-billiard study has also shown wavefunction mixing near an avoided crossing and mode exchange across it [9]. These examples show that interaction, mixing, and exchange can accompany avoided crossings, but they do not imply that an eigenvalue trajectory alone uniquely establishes modal-identity exchange for every event. The eigenvalue trajectory contains only spectral information. A physical description of the interaction also concerns how the eigenfunctions evolve: Whether the two states exchange their modal identities and how many reference directions at the event center are needed to cover their evolution. These questions are related, but they are not equivalent. This general nonuniqueness is well established. Here, we examine it through the eigenfields of a specific parameter-dependent microwave billiard. Eigenvector responses to parameter variation have also been studied in chaotic and random matrix systems

[10,11], while analogous rotations of mode shapes occur in structural eigenvalue curve veering [12,13].

Microwave billiards provide a controlled setting in which these distinctions can be examined because, below the cutoff for higher vertical modes, a thin microwave cavity realizes the same scalar Helmholtz eigenvalue problem as a two-dimensional quantum billiard [14,15]. Direct measurements and numerical reconstructions of microwave billiard eigenfunctions make field-level comparisons accessible in the same platform [16–21]. Changes of a boundary or the position of an internal scatterer have been used to study parametric level dynamics and to distinguish perturbations that affect the wave field globally from those acting locally [22–26]. Other studies have demonstrated that eigenfunctions can change substantially in the vicinity of avoided crossings and have characterized this evolution through curvature, mode structure transfer, entropy, localization, and related field measures [27–30]. Related microwave billiard studies have also used controlled coupling between subsystems to connect measured spectra with effective few block descriptions [31]. These studies indicate that the field structure contains information beyond the eigenvalue spectrum. The remaining identification problem is therefore to determine whether a spectrum resembling an avoided crossing is accompanied by an actual exchange of modal identity and whether two reference directions are sufficient to describe the field evolution.

Here, we compare the spectral and field responses numerically in a two-dimensional Helmholtz model of a deformed microwave billiard containing two fixed scatterers and one movable scatterer. Finite element method (FEM) calculations resolve the eigenvalues and eigenfunctions as the movable scatterer is displaced [32]. A local two-level fit gives the event position, minimum gap, and relative spectral slope. The Hadamard shape derivative, evaluated from the fields at the event center, then gives the boundary response to motion of the scatterer [33,34]. Together with the minimum gap, it

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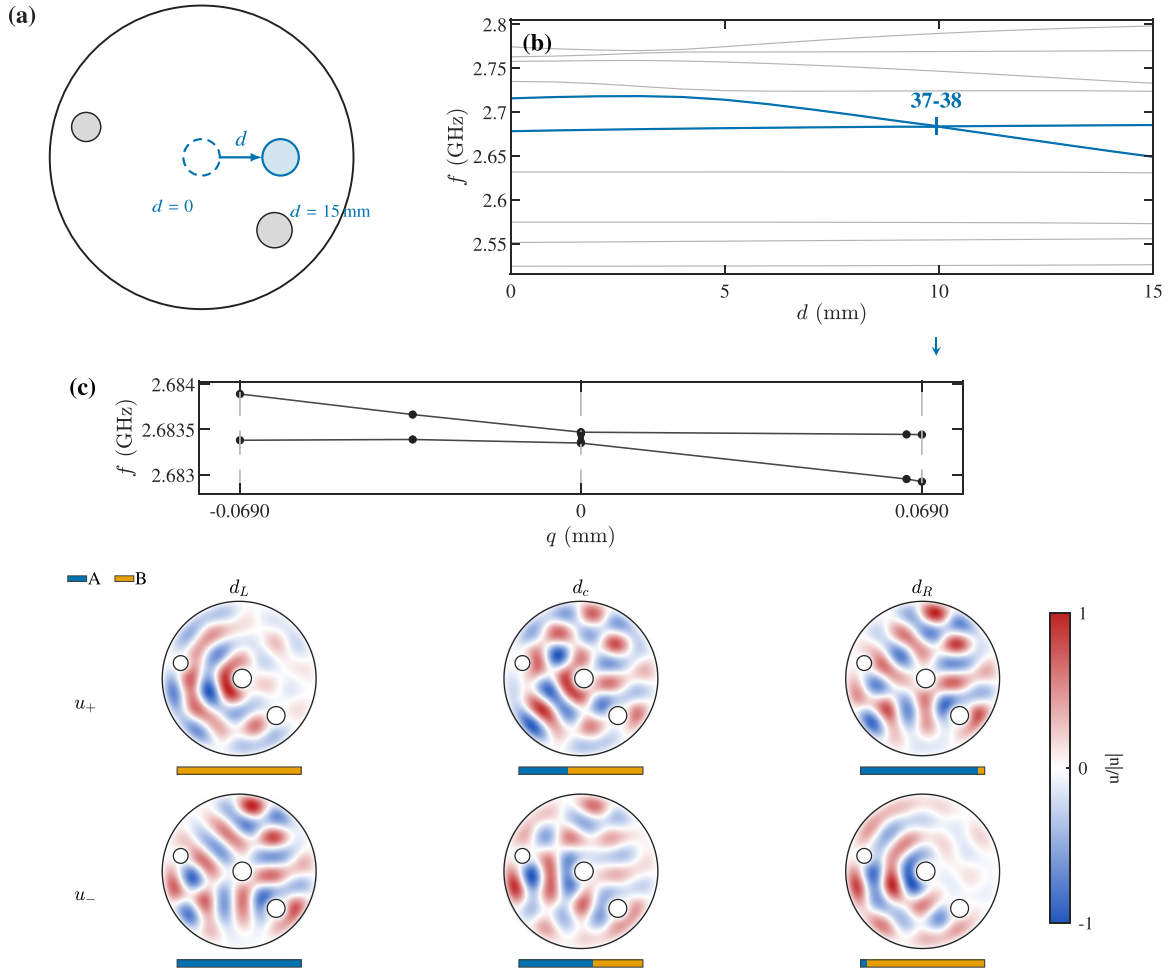


FIG. 1. Geometry and representative exchange for mode pair 37–38. (a) Billiard with two fixed scatterers and one movable scatterer; the two displayed movable scatterer positions are schematic. (b) FEM eigenfrequency trajectories of modes 33–42, with pair 37–38 highlighted and its event center marked. (c) Local spectrum and fields at d_L , d_c , and d_R . Black circles are FEM eigenfrequencies, gray segments are guides, and the arrow marks $g_{\min}$. Bars show squared overlaps with the left reference modes A (blue) and B (orange); each field map has independent amplitude normalization.

provides a local field-derived estimate of the scale over which modal rotation should occur. Modal overlaps, closely related to the modal assurance criterion (MAC) for tracking changing mode shapes [35–37], are then compared with this estimate; the estimate is not fitted to the overlap evolution. Projectors onto reference subspaces show how many directions at the event center are needed to cover the evolving target pair [38–40]. The spectrum, the boundary response, the modal overlaps, and the subspace projectors answer different questions. Their comparison identifies ordinary modal exchange, a spectrally nonunique case, and an event for which the spectral and field corrections point in different directions. These diagnostics are used to classify the field evolution of individual spectral events.

II. DEFORMED BILLIARD AND A REPRESENTATIVE MODAL EXCHANGE

The system shown in Fig. 1(a) is a two-dimensional circular billiard of radius $R = 250$ mm. Two fixed circular scatterers are centered at $(120, -120)$ and $(-190, 50)$ mm,

with radii 29 and 24 mm, respectively. A third circular scatterer of radius 30 mm is displaced horizontally, with center $(d, 0)$ and $0 \leq d \leq 15$ mm. For the cavity geometry considered here, the scalar field u_n represents the electric field component perpendicular to the billiard plane and satisfies the Dirichlet Helmholtz problem

$$-\nabla^2 u_n(\mathbf{r}; d) = k_n^2(d) u_n(\mathbf{r}; d), \quad u_n(\mathbf{r}; d)|_{\partial\Omega(d)} = 0, \quad (1)$$

with eigenfrequency $f_n(d) = ck_n(d)/(2\pi)$ [14,15,32]. The displacement shown between the two movable scatterer outlines in panel (a) is magnified schematically and does not represent the actual geometric scale.

Figure 1(b) places the selected 37–38 event in the wider spectral context of modes 33–42. Denoting the lower- and upper-frequency branches by $f_-(d)$ and $f_+(d)$, their instantaneous separation is $g(d) = f_+(d) - f_-(d)$. We define

$$d_c = \arg \min_d g(d), \quad g_{\min} = g(d_c). \quad (2)$$

For the 37–38 pair, the baseline global mesh H7 ($H_{\max} = 7$ mm) gives $d_c \simeq 9.934$ mm and $g_{\min} \simeq 0.12$ MHz. Panel

(c) magnifies this event using $q = d - d_c$. The field samples are taken at $q_L \simeq -0.069$ mm, $q_c = 0$, and $q_R \simeq +0.069$ mm. The local strip contains all five available FEM coordinates in this interval; only the three field sampling coordinates receive vertical guides. The two branches repel near d_c , but spectral repulsion alone does not establish modal exchange.

To test the field identity, we use the eigenmodes at the left sampling point as reference states, $\phi_A = u_-(d_L)$ and $\phi_B = u_+(d_L)$. After interpolation onto a common spatial grid, all fields are normalized such that $\langle u, u \rangle_Q = 1$, where $\langle u, v \rangle_Q = u^\dagger Q v$ and Q is the quadrature weight matrix on the common grid. The squared overlaps are

$$M_{\alpha s}(d) = |\phi_\alpha^\dagger Q u_s(d)|^2, \quad \alpha \in \{A, B\}, \quad s \in \{-, +\}. \quad (3)$$

This normalized mode shape correlation is closely related to the modal assurance criterion commonly used for mode identification and tracking [35,36]. The retained weight of branch s in the two-dimensional reference pair is

$$\rho_s(d) = M_{As}(d) + M_{Bs}(d) \quad (4)$$

and the displayed weights, normalized within the pair, are

$$w_\alpha^{(s)}(d) = \frac{M_{\alpha s}(d)}{\rho_s(d)}, \quad w_A^{(s)}(d) + w_B^{(s)}(d) = 1. \quad (5)$$

Components outside the reference pair are therefore not represented in the horizontal bars. For visualization, each field map is divided by its own maximum absolute amplitude, and the arbitrary global sign is chosen sequentially along each tracked branch to preserve visual continuity. Neither operation affects the squared overlaps. Because the reference modes A and B are defined at the finite sampling point d_L rather than as asymptotic diabatic states, their normalized weights are not required to be exactly equal at the spectral minimum gap position d_c . For the upper branch, (w_A, w_B) evolves from $(0,1)$ at d_L to $(0.400, 0.600)$ at d_c and to $(0.946, 0.054)$ at d_R . The lower branch shows the complementary evolution, $(1, 0) \rightarrow (0.600, 0.400) \rightarrow (0.054, 0.946)$. The minimum retained pair weight over the local sampling interval is $\rho_{\min} = \min_{d \in \mathcal{D}_{\text{loc}}, s \in \{-, +\}} \rho_s(d) > 0.9999$, where $\mathcal{D}_{\text{loc}}$ denotes the local overlap grid used for the event. Thus, the two branches remain confined to an essentially two-dimensional reference space while their dominant reference contents reverse. The 37–38 event therefore serves as a reference case in which spectral repulsion is accompanied by an almost complete exchange of modal identity within an essentially two-dimensional reference space.

III. SPECTRAL FORM AND MODAL DYNAMICS

The 37–38 event in Fig. 1 is a reference case in which level repulsion accompanies an almost complete exchange of modal identity. We now ask whether a similar spectrum necessarily implies the same field evolution. Figure 2 compares pair 137–138 from the secondary analysis batch with pair 139–140 from the primary batch. Panels (a) and (b) show their local eigenfrequency evolution. Motivated by the standard two-level description of avoided crossings [2,3,27,28],

the ordered eigenvalues are fitted in locally unfolded spectral units using

$$E_\pm^{\text{fit}}(q) = E_0 + bq + cq^2 \pm \sqrt{(a_{\text{fit}}q + \beta q^2)^2 + \left(\frac{g_{\min}}{2}\right)^2}, \quad (6)$$

where $E_0 + bq + cq^2$ describes the smooth spectral background shared by the two branches, $a_{\text{fit}}q + \beta q^2$ is their relative detuning, and β is its quadratic correction. This form is used as a local spectral description, not as a unique account of the field dynamics. Unless physical frequency units are stated explicitly, the spectral energies and $g_{\min}$ in the remainder of this section are in locally unfolded units. Figure 2 uses the same centered, locally unfolded vertical convention for both events. For pairs 137–138 and 139–140, the fits use the respective local windows $d \in [8, 10]$ mm and $d \in [4, 6]$ mm; the spectral points and gray curves in Figs. 2(a) and 2(b) are restricted to those windows. For panels (c) and (d), the fitted spectral Hamiltonian is evaluated over the full field interval $q \in [-1.5, 1.5]$ mm with the fixed left reference state. Outside the fitted spectral window, the spectral eigenvector curve is a continuation of the local Hamiltonian; neither it nor the boundary curve is fitted to the overlap data. To ask whether the spectrum selects this form, we compare Eq. (6) with two independently fitted smooth branches,

$$E_-^{(p)}(q) = \sum_{j=0}^p c_j^- q^j, \quad E_+^{(p)}(q) = \sum_{j=0}^p c_j^+ q^j, \quad p = 2, 3. \quad (7)$$

The curved avoided crossing form has seven parameters; the quadratic and cubic alternatives have six and eight. All models use the same ordered two branch data, equal point weights, and the same local window. We report the residual sum of squares, root mean square error (RMSE), the corrected Akaike information criterion (AICc) [41], and fivefold contiguous block cross-validation (CV) error [42]; details are provided in the Supplemental Material (SM) [43]. Because the FEM ordinates are deterministic, these are comparisons of descriptions, not stochastic model selection tests. We call the spectrum nonunique when a tested alternative is no worse than the avoided crossing form in both AICc and blocked CV RMSE; agreement in only one metric is reported as metric dependent. For pair 137–138, the avoided crossing form is preferred on the H7 data. In contrast, for pair 139–140 the branchwise cubic has lower AICc and lower CV RMSE on both H7 and H5 ($H_{\max} = 5$ mm). The avoided crossing form is therefore a useful local description of pair 139–140, but the spectral data do not uniquely fix its eigenvector dynamics over the sampled interval. The polynomial alternative is not a competing microscopic model; it tests whether the local eigenvalue data uniquely select the avoided crossing ansatz.

Equation (6) is the spectrum of the local two-state Hamiltonian

$$H_2(q) = E_{\text{bg}}(q)I + \begin{pmatrix} \Delta(q) & V \\ V & -\Delta(q) \end{pmatrix}, \quad (8)$$

where $E_{\text{bg}}(q) = E_0 + bq + cq^2$, $\Delta(q) = a_{\text{fit}}q + \beta q^2$, and $V = g_{\min}/2$. At $q = 0$, $\Delta(0) = 0$ and $\Delta'(0) = a_{\text{fit}}$. When the parameter derivative of H_2 is expressed in the instantaneous

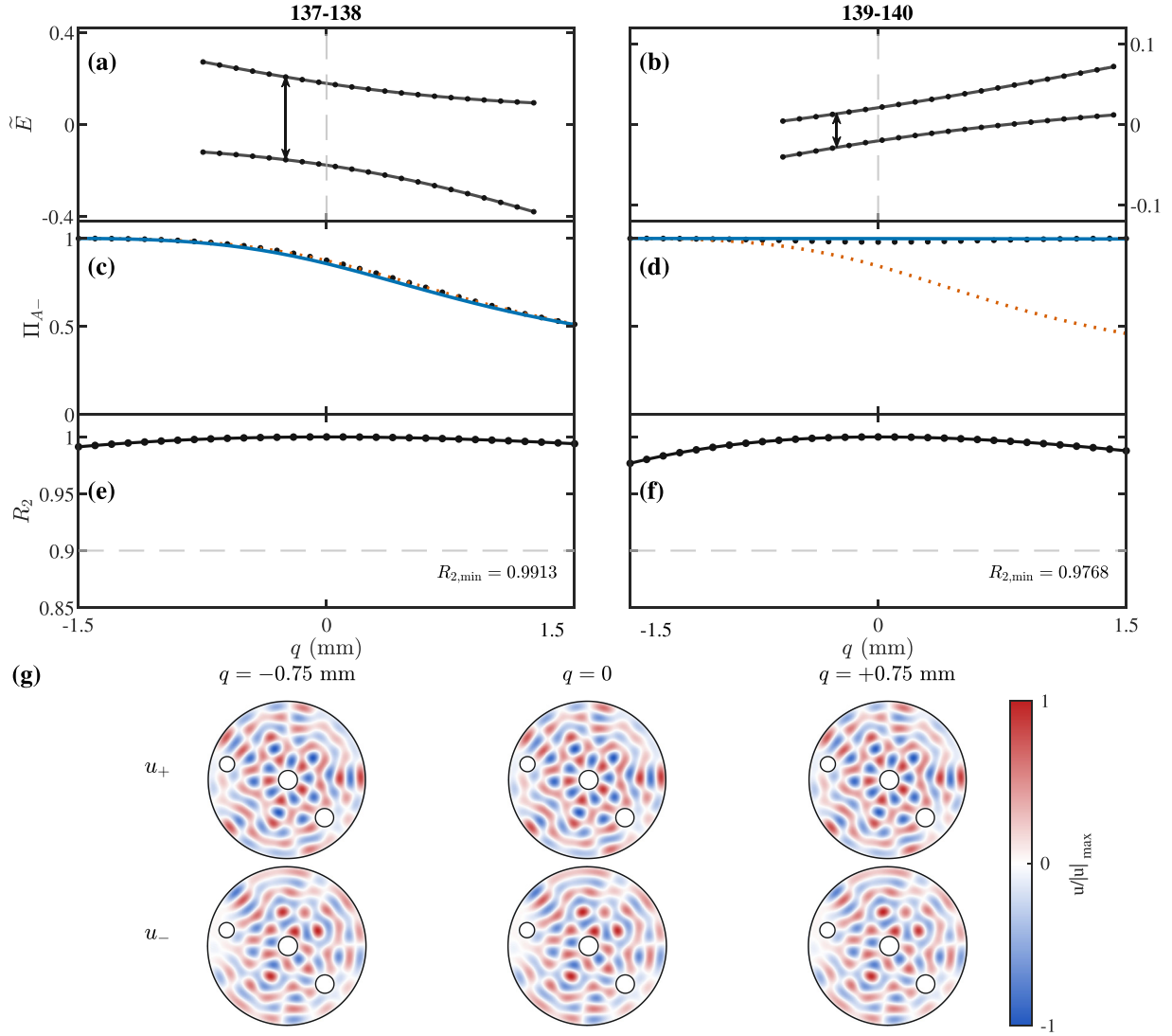


FIG. 2. Spectral and field diagnostics for 137–138 (left) and 139–140 (right). (a), (b) Locally unfolded FEM spectra (black circles) and local two-level fits (gray), with arrows marking $g_{\min}$. (c), (d) Overlap relative to the fixed reference, Π_{A-} : FEM values (black circles), spectral prediction (red dotted), and boundary-derived estimate (blue solid). (e), (f) Two-dimensional reference subspace coverage R_2 ; circles are FEM values, solid lines join samples, and the gray dashed line marks 0.90. (g) FEM field sequence for the energy-ordered branches of 139–140 at $q = -0.75, 0$, and $+0.75$ mm (corresponding to $d = 3.823\,46, 4.573\,46$, and $5.323\,46$ mm). Each field map is independently normalized by its maximum absolute amplitude, and the global sign is aligned sequentially along each branch. In contrast to the exchange maps in Fig. 1(c), the spatial patterns retain their dominant reference identity with no reference-content reversal over the displayed interval, consistent with panels (d) and (f).

eigenstate basis at the event center, its off-diagonal element therefore has magnitude $|\partial_q H_2(0)|_{-+} = |a_{\text{fit}}|$, where $-$ and $+$ denote the lower and upper instantaneous states. The boundary response can be calculated from the FEM fields with the Hadamard shape derivative. A local two-state interaction would then predict $|W_{12}| \simeq |a_{\text{fit}}|$. Here, W_{12} should not be confused with the gap coupling $V = g_{\min}/2$: The latter sets the minimum spectral gap, whereas W_{12} characterizes the off-diagonal response to the control parameter in the instantaneous basis at the event center. In this model, appreciable modal rotation occurs when the detuning becomes comparable to the coupling scale, $|\Delta(q)| \sim V$. Near the event center, $\Delta(q) \simeq a_{\text{fit}}q$, giving a characteristic displacement scale of order $g_{\min}/(2|a_{\text{fit}}|)$. Replacing the fitted slope by $|W_{12}|$ gives the boundary-derived rotation-width estimate used below. More

generally, a real two-state Hamiltonian can be written as

$$\begin{aligned} H(q) &= E_{\text{bg}}(q)I + x(q)\sigma_x + z(q)\sigma_z, \\ E_{\pm}(q) &= E_{\text{bg}}(q) \pm r(q), \quad r(q) = \sqrt{x(q)^2 + z(q)^2}. \end{aligned} \quad (9)$$

The spectrum fixes the radial function $r(q)$, whereas the eigenvectors depend on the direction $\theta(q) = \text{atan2}[x(q), z(q)]$. The particular ansatz in Eq. (8) sets $x = V$ and $z = \Delta(q)$. Thus, a good fit to the spectral radius alone does not fix this directional motion or the associated modal rotation. The boundary derivative at the event center supplies the missing local information about the eigenvector direction. For the pair at the event center, the Hadamard matrix is evaluated on the movable boundary Γ_m from FEM fields normalized with the FEM mass

matrix, as

$$\mathcal{H}_{ij} = \int_{\Gamma_m} v_n(\mathbf{r}) [\partial_n u_i(\mathbf{r})]^* \partial_n u_j(\mathbf{r}) ds, \quad v_n = \mathbf{e}_x \cdot \mathbf{n}_m. \quad (10)$$

Here, $\mathbf{n}_m$ points outward from the movable scatterer. This positive sign convention is equivalent to the conventional Dirichlet Hadamard formula with a minus sign when the outward normal of the billiard domain is used [33,34]. For horizontal translation, the normal derivatives are obtained by radial extrapolation from the boundary. The resulting matrix is converted to locally unfolded spectral units per millimeter with the two-dimensional Weyl derivative; the explicit conversion is provided in SM [43]. In the instantaneous center basis, the first-order boundary Hamiltonian is

$$H_{\text{inst}}(q) = \begin{pmatrix} E_-(0) & 0 \\ 0 & E_+(0) \end{pmatrix} + q \begin{pmatrix} W_{--} & W_{-+} \\ W_{-+} & W_{++} \end{pmatrix}. \quad (11)$$

The trace of the derivative shifts only the common background, while $W_{++} - W_{--}$ is a separate relative diagonal term. For a real symmetric Hadamard matrix, we denote the off-diagonal center basis element by $W_{12} \equiv W_{-+} = W_{+-}$. For the eigenvector-response curves obtained without fitting to the overlap data in Fig. 2, we distinguish the spectral predictor and the boundary proxy,

$$H_s(q) = V\sigma_x + \Delta_{\text{fit}}(q)\sigma_z, \quad \Delta_{\text{fit}}(q) = a_{\text{fit}}q + \beta q^2,$$

and, after a fixed $\pi/2$ rotation of the center basis,

$$H_b(q) = V\sigma_x + qW_{12}\sigma_z.$$

The latter is a minimal proxy for the rotation scale supplied by W_{12} ; it is not a complete first-order boundary Hamiltonian for the spectral trajectories over the sampled interval. In particular, $W_{++} - W_{--}$ remains an independently determined relative diagonal term in Eq. (11). Since a_{fit} and W_{12} are expressed in identical unfolded units, we define the slope ratio and the rotation scale estimated from the boundary response as

$$R_a = \frac{|W_{12}|}{|a_{\text{fit}}|}, \quad \ell_{\text{pred}} = \frac{g_{\text{min}}}{2|W_{12}|}. \quad (12)$$

$R_a \simeq 1$ indicates agreement between the two derivative scales. Pair 137–138 gives $R_a \simeq 1.00$, whereas pair 139–140 gives $R_a \simeq 0.048$. Its boundary response is therefore only about 5% of the relative slope inferred from the spectral form. The avoided crossing description does not provide the field-evolution scale for this event; the boundary response remains consistent with the weak modal evolution.

Panels (c) and (d) examine modal identity with the Q -weighted overlap, defined in Eq. (3). The lower-frequency eigenmode at the left edge, $\phi_A = u_-(q_L)$, with $q_L = -1.5$ mm, defines the fixed reference state. Using the overlap in Eq. (3), the normalization is across the two instantaneous target branches for the single fixed state A , rather than across the A/B reference states for each branch as in Eq. (5). The displayed weights are

$$\begin{aligned} \Pi_{As}(q) &= \frac{M_{As}(q)}{M_{A-}(q) + M_{A+}(q)}, \\ \Pi_{A-}(q) + \Pi_{A+}(q) &= 1. \end{aligned} \quad (13)$$

Because A is the instantaneous eigenmode at the finite left boundary q_L , rather than an asymptotic diabatic state, equality of Π_{A-} and Π_{A+} is not required to occur at the spectral minimum gap position $q = 0$. When the observation interval contains the turnover, the fixed reference overlap gives a characteristic width ℓ_{MAC} according to the reporting rule in SM [43]. In panel (c), the upper branch A weight rises from nearly zero to 0.488 38 and the lower branch weight falls from nearly unity to 0.511 62. Pair 137–138 therefore rotates substantially, although the plotted interval does not show a complete identity exchange. In panel (d), the corresponding weights at the right end point are 3.66×10^{-5} and 0.999 963. Pair 139–140 shows essentially no rotation over the same interval. Its unconstrained width fit places $\delta \simeq -1.64$ mm outside the $[-1.5, 1.5]$ mm window, while a window-constrained fit reaches the boundary and the curve has no turnover. The width is therefore not identifiable and neither ℓ_{MAC} nor R_ℓ is reported for this event. Panel (g) shows the corresponding field maps to Fig. 1(c). From $q = -0.75$ mm through $q = 0$ to $q = +0.75$ mm, the upper and lower spatial patterns of 139–140 retain their dominant reference identities and do not show the reference-content reversal seen for 37–38. The retained identities agree with the weak fixed-reference evolution in panel (d) and the high rank-2 coverage in panel (f).

The spectral predictor $H_s(q)$ and boundary proxy $H_b(q)$ yield eigenvector-response curves without fitting to the overlap data. For pair 139–140, the spectral predictor $\Delta(q) = a_{\text{fit}}q + \beta q^2$ has an overlap RMSE of 0.2686 and predicts a right-end-point lower-branch weight of 0.459. The boundary model $\Delta(q) = W_{12}q$ has an RMSE of 0.010 71 and gives 0.9973, in agreement with the FEM value of 0.999 96. The boundary-derived curve reproduces the weak rotation seen in the fields. The spectrally fitted two-state form does not. Neither behavior is universal, and the local spectral pattern alone does not distinguish the corresponding field evolution. Panels (e) and (f) test whether each instantaneous target pair remains in a two-dimensional reference space defined at $q = 0$. With $\tilde{\Phi}_2$ denoting the Q -orthonormalized reference basis at $q = 0$ and $\tilde{\Psi}_2(q)$ the instantaneous target pair basis, the normalized Frobenius overlap can be interpreted as the mean squared cosine of the principal angles between the two subspaces in the Q -weighted inner product [38–40]. The agreement is

$$R_2(q) = \frac{1}{2} \|\tilde{\Phi}_2^\dagger Q \tilde{\Psi}_2(q)\|_F^2, \quad R_{2,\text{min}} = \min_q R_2(q). \quad (14)$$

Panel (e) gives $R_{2,\text{min}} = 0.991 31$ for pair 137–138, whereas panel (f) gives $R_{2,\text{min}} = 0.976 81$ for pair 139–140. Both target pairs are well represented by a two-dimensional reference space over the common interval, although only pair 137–138 rotates substantially and has matching derivative scales. High coverage alone does not imply rotation or exchange. Numerical convergence tests and the extended calculation for pair 137–138 are provided in SM [43].

IV. BOUNDARY COUPLING AND MODAL WIDTH DIAGNOSTICS

Figure 3 compares the relative slope of the local spectral form with the boundary response to displacement. For

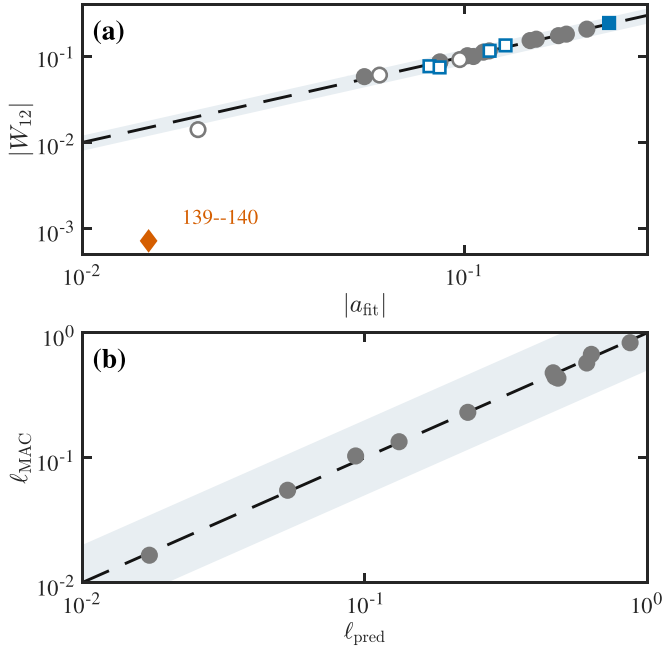


FIG. 3. Boundary coupling and modal width diagnostics for 21 events. (a) $|W_{12}|$ vs $|a_{\text{fit}}|$; the line denotes equality and the band marks $0.8 \leq R_a \leq 1.2$. Filled circles and squares denote the two predeclared analysis batches, open symbols denote insufficient range (IR) events, and the orange diamond denotes 139–140. (b) ℓ_{pred} vs ℓ_{MAC} for the 11 identifiable (ID) events; the line denotes equality and the band marks $0.5 \leq R_\ell \leq 2$. B (bounded), NI (not identifiable), and IR events have no reported modal width.

each event, the eigenvalue trajectories determine d_c , $g_{\min}$, and a_{fit} . The center fields determine W_{12} through Eq. (10), and hence ℓ_{pred} . The event list and the provenance of the two predeclared analysis batches are specified in SM [43]. Across the 21 screened events, the spectral and boundary derivative scales are closely matched for the great majority of cases. Pair 139–140 is the clear exception, with $R_a \simeq 0.048$, while pair 142–143 gives the smaller deviation $R_a \simeq 0.70$. The 139–140 result establishes a strong mismatch between the spectral slope scale and the boundary response.

The width comparison requires an additional identifiability decision. We classify a width as identifiable only when a retained pointwise curve is available, the fitted center is interior, a constrained fit does not reach the window boundary, multistart solutions are stable, the width spans at least three samples, and the observation interval contains a modal turnover. The full algorithm and numerical thresholds are provided in SM [43]. Eleven events pass, one is bounded, two are not identifiable, and seven have insufficient range or retained data. Only the 11 identifiable events appear as ordinary width points in panel (b), and their R_ℓ values range from 0.898 to 1.11. Pair 139–140 and pair 142–143 are not identifiable; pair 99–100 is bounded; and the remaining seven are reported as insufficient range. In particular, no modal width or R_ℓ is assigned to pair 139–140. The complete values and status reasons are provided in SM [43].

The five-event batch contributes to the slope comparison but has no retained pointwise overlap histories for a uniform width screen. Its values for each event are listed in SM [43].

V. REFERENCE SUBSPACE COVERAGE OVER A FINITE WINDOW

The derivative scales and width status in Fig. 3 do not determine whether a prescribed basis at the event center covers the evolving target pair. We assess this question with a reference subspace over the sampled window. Let $\tilde{\Psi}_2(q)$ be the Q -orthonormalized basis of the two instantaneous target modes tracked by identity, and let $\tilde{\Phi}_m$ be an m -dimensional Q -orthonormalized reference basis constructed from eigenmodes at the event center $q = 0$. This generalizes the rank-2 metric in Eq. (14) to an m -dimensional basis [38–40]. Its coverage is

$$R_m(q) = \frac{1}{2} \|\tilde{\Phi}_m^\dagger Q \tilde{\Psi}_2(q)\|_F^2, \quad R_{m,\min} = \min_q R_m(q). \quad (15)$$

The instantaneous target space remains two dimensional; the normalization factor is fixed by that target space, so $R_m = 1$ when both target directions are fully contained in the reference space. We use $R_{m,\min} \geq 0.90$ as the coverage criterion over the sampled window. Five cases were selected from the original end-point-retention results: the three secondary events with low end-point retention (131–132, 97–98, and 92–93), the secondary slope-consistent event 137–138, and the primary spectrally nonunique event 139–140. Pair 107–108 remains in the secondary slope comparison of Fig. 3, but its end-point retention is 0.906 79 and it carries the limited-range flag; it is not included in this analysis. Panels (a) and (b) test the two-dimensional reference space. The minimum coverages are 0.871 07, 0.961 60, 0.970 52, 0.991 31, and 0.976 81 for mode pairs 131–132, 97–98, 92–93, 137–138, and 139–140, respectively. Four of the five events remain above the stated threshold throughout the displayed interval, whereas pair 131–132 falls below it. On the baseline H7 mesh and the present 31-point window, two directions at the event center do not reach the chosen coverage for pair 131–132. Pair 139–140 nevertheless has high two-dimensional coverage despite the negligible rotation in Fig. 2. This metric asks whether the target pair lies in a specified field subspace, not whether the modes exchange identity within that subspace.

Panel (c) examines larger reference spaces for pair 131–132. Adding mode 134, which ranks first by the candidate criterion, raises $R_{m,\min}$ from 0.871 07 to 0.882 78. Adding mode 130 next gives the four-direction center-reference basis $\{131, 132, 134, 130\}$ and increases the minimum coverage to 0.971 98. In the *post hoc* sequence chosen by coverage, adding mode 130 directly gives $R_{3,\min} = 0.950 55$, whereas adding mode 134 alone reaches only 0.882 78. Mode 130 therefore provides the missing field subspace direction among the bases shown in panel (c). The $\{131, 132, 130\}$ basis is chosen *post hoc* from the coverage over the full window and is used only to explain this result. The candidate criterion and the mode-133 check are provided in SM [43]. These results refer to the present H7 calculation and sampled window; they do not establish an intrinsic reference dimension. Figure 5 tests the effective models built from these bases.

For a selected basis $\mathcal{B}$ of modes at the center, each model in Fig. 5 uses the first-order form at the exact event center. Following standard first-order perturbation theory [44],

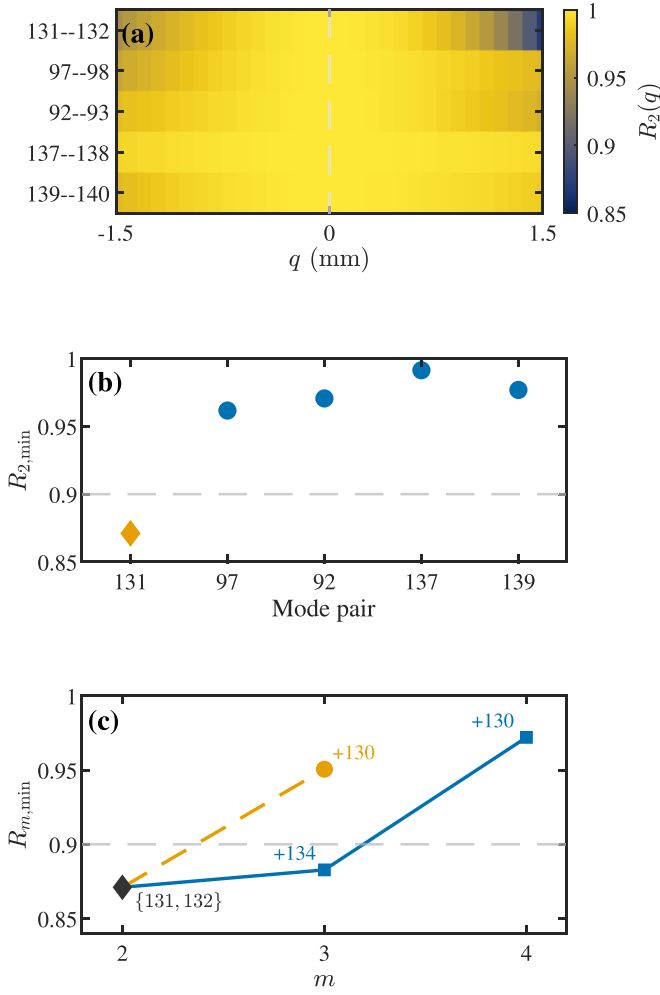


FIG. 4. Coverage by a reference subspace over a finite window. (a) Two-dimensional coverage R_2 for the five labeled mode pairs; the dashed line marks the event center. (b) Corresponding minima $R_{2,\min}$; the orange diamond is 131–132 and blue circles are the other pairs. (c) Minimum coverage for the displayed bases at the event center for 131–132. Blue squares trace the sequence ranked by the candidate criterion and orange circles the *post hoc* coverage sequence; labels identify the added mode. Horizontal dashed lines mark 0.90.

we use

$$H_{\text{eff}}^{(\mathcal{B})}(q) = H_0^{(\mathcal{B})} + qW^{(\mathcal{B})}, \quad q = d - d_c, \quad (16)$$

where $H_0^{(\mathcal{B})}$ and $W^{(\mathcal{B})}$ are matrices evaluated at the exact event center and restricted to the chosen basis. The four bases are $\{131, 132\}$, $\{131, 132, 134\}$, $\{131, 132, 130\}$, and $\{131, 132, 134, 130\}$. All models are evaluated on the same 31 displacement samples and against the same FEM target doublet tracked by identity; no model parameter is refitted over the full window for Fig. 5. At $q=0$, the two model eigenstates with the largest one-to-one projections onto reference modes 131 and 132 define the model target doublet. The doublet is then tracked toward positive and negative q by maximum-weight one-to-one eigenvector-overlap assignment. Energy ordering is used only to label the two selected eigenvalues.

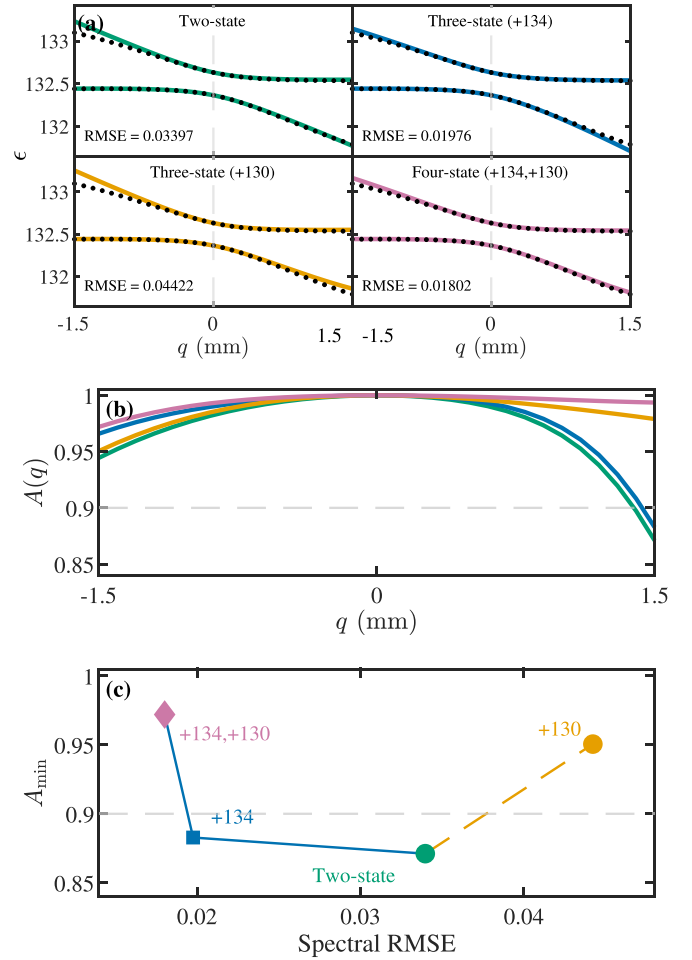


FIG. 5. Fixed effective models for pair 131–132. (a) FEM eigenvalues (black circles) and model predictions: two-state (green), +134 (blue), +130 (orange), and +134, +130 (purple). The annotations give spectral RMSE. (b) Projector agreement $A(q)$ for the same models. (c) Minimum projector agreement vs spectral RMSE; the blue path follows the candidate ranking and the orange path the *post hoc* basis sequence. Horizontal dashed lines mark 0.90.

Panel (a) compares the two target eigenvalues from each model with the FEM trajectories. The normalized spectral RMSE uses the local mean spacing defined in SM [43]. The two-state model gives $\text{RMSE} = 0.033\,97$. Adding mode 134 reduces this value to 0.019 76, a reduction of approximately 41.8%, and improves both target trajectories. In contrast, adding mode 130 alone increases the error to 0.044 22, so the *post hoc* three-state basis selected by the coverage criterion does not improve the target spectrum. Of the two three-state extensions shown, only the +134 basis improves the target spectrum substantially. The four-state model gives the smallest error among the fixed bases tested, 0.018 02, only slightly below the +134 result.

Panels (b) and (c) evaluate the corresponding target field subspaces. Let $\tilde{\Psi}_{2,\mathcal{B}}^{\text{model}}(q)$ and $\tilde{\Psi}_2^{\text{FEM}}(q)$ denote the Q -orthonormalized model and FEM target bases. Their agreement $A_{\mathcal{B}}(q)$ uses the same normalized Frobenius projector overlap as Eq. (15) [38–40], with the reference basis at the event center replaced by the model target basis; we write

$A_{B,\min} = \min_q A_B(q)$. This quantity tests the fields produced by the model; it is distinct from the coverage of a reference basis at the event center in Fig. 4. The two-state model gives $A_{\min} = 0.871\,07$. Adding mode 134 raises it to $0.882\,78$, still below the 0.90 threshold. Adding mode 130 directly instead raises it to $0.950\,53$, while the four-state model reaches $0.971\,98$. The two three-state extensions affect different aspects of the result. Mode 134 mainly corrects the eigenvalue trajectories and only weakly improves the field subspace. Mode 130 does little for the spectrum by itself, but recovers most of the missing field component. Among the fixed bases, only $\{131, 132, 134, 130\}$ gives both the lowest spectral RMSE and a minimum projector agreement above 0.90 . Its improvement over $+134$ is modest. The labels “two-state,” “three-state,” and “four-state” refer only to the fixed bases tested on this window, not to an intrinsic target-space dimension.

VI. CONCLUSION

Modal exchange, weak rotation, and more complicated multistate evolution can all occur for interacting modes. Our results show that the eigenvalue trajectory alone does not distinguish among these behaviors for an individual event in the deformed microwave billiard. We distinguish them by comparing spectral fits with moving-boundary response, fixed-reference overlaps, and subspace projectors. Pair 139–140 is one example. The avoided crossing ansatz is spectrally nonunique on both H7 and H5 meshes and predicts substantial rotation that is absent from the FEM fields. The boundary derivative instead reproduces the weak evolution.

Across the 21 screened events, the spectral and boundary derivative scales agree closely in most cases. The 11

identifiable widths over the sampled interval give $0.898 \leq R_\ell \leq 1.11$; no width is assigned when the overlap history does not contain sufficient information.

Coverage by a reference subspace asks a separate question. For pair 131–132, mode 134 mainly improves the target spectrum, while the *post hoc* mode 130 restores the field subspace. Their combination is a descriptive result for the tested bases, not a mesh-independent reference dimension. Eigenvalue repulsion constrains the spectral scale of an interaction, but the fields are needed to determine modal rotation and the reference subspace required over a finite parameter range.

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DATA AVAILABILITY

The processed numerical data, figure source data, values for each event, and analysis scripts supporting this work will be deposited in a public repository upon publication. Until then, these materials and additional FEM field data are available from the corresponding author upon reasonable request.

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