

Fractonic superfluids. III. Hybridizing higher moments

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Fractonic superfluids are featured by the interplay of spontaneously broken charge symmetry and mobility constraints on single-particle kinematics due to the conservation of higher moments, such as dipoles, angular charge moments, and quadrupoles. Building on prior studies by Yuan *et al.* [*Phys. Rev. Res.* **2**, 023267 (2020)] and Chen *et al.* [*Phys. Rev. Res.* **3**, 013226 (2021)], we study a class of fractonic superfluids, termed *hybrid fractonic superfluids* (HFS), in which bosons of multiple species interact while moment hybridization is conserved. We explore the consequences of hybridization via two model series: *Model Series A*, conserving total moments of the same order across species, and *Model Series B*, conserving total moments of different orders. In Model Series A, we analyze dipole moment hybridization and extend the discussion to higher-order moments, examining the ground state, Goldstone modes, correlation functions, and so on. We compute the minimal spatial dimensions, where the total charge symmetry begins to get partially broken via particle-hole condensation, leading to true off-diagonal long-range order. In Model Series B, we focus on HFS with hybrid dipole-quadrupole conservation. For both series, we introduce Bose-Hubbard-type lattice models that reduce to either of both series in the weak Hubbard interaction regime. We perform a mean-field analysis on the global phase diagram and discuss experimental realizations in strongly tilted optical lattices via a third-order perturbation theory. This work, alongside prior studies, completes a trilogy on fractonic superfluids, uncovering symmetry-breaking physics emerging from higher moment conservation, leaving various promising studies for future investigation.

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I. INTRODUCTION

In recent years, fracton topological order has been gaining increasing attention. Exactly solvable lattice models of fracton topological order are typically described by fracton codes, such as the three-dimensional X-cube model [1–20]. Topological excitations in fracton codes are commonly referred to as fractonic excitations, including fractons, subdimensional particles (e.g., lineons and planeons), and more intricate spatially extended excitations that are subject to mobility restriction [13–15]. In contrast to conventional topological orders, such as the toric code model, where topological excitations can be freely moved to any location via string operators, fractonic excitations in fracton topological order exhibit mobility restrictions. For instance, a single fracton cannot move freely along any directions, while a single lineon can only move along a straight line. Due to these mobility restrictions, fracton codes are believed to provide robust fault-tolerant quantum memory [2–5]. The exploration of fracton topological order

has also stimulated studies on subsystem symmetry-protected topological order (SPT) [21–32], substantially enlarging the territory of SPT phases.

Another active research direction involves treating fractons as constituent particles of many-body systems, where mobility restrictions arise from the conservation of higher-order moments [33–91]. Such systems are referred to as *fractonic many-body systems*. It is well known that conventional total charge conservation is associated with a global U(1) symmetry. Similarly, these new conserved quantities formed by higher moments are associated with new U(1) symmetries. Upon gauging, the resulting gauge degrees of freedom are symmetric tensors; therefore, these new symmetries are sometimes alternatively referred to as “higher-rank symmetries.” To sum up, this line of research has evolved into a platform for exploring new possibility of symmetries, stimulating interests and ideas from dynamics, hydrodynamics, gravity, and other domains [49–67].

On the other hand, spontaneous symmetry breaking (SSB) is a fundamental concept spanning a broad range of research areas in theoretical physics, from the Standard Model of particle physics to the Ginzburg-Landau theory of second-order phase transitions. In condensed matter physics, superfluids serve as canonical examples of systems with SSB, where the full charge U(1) symmetry group is spontaneously broken, and off-diagonal long-range order (ODLRO) is established. A celebrated phenomenon in this context is the Kosterlitz-Thouless topological phase transition, which occurs at finite

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temperature in two dimensions (2D). The study of SSB, particularly in the context of continuous global symmetries, has long been associated with important topics such as Goldstone mode counting, the Mermin-Wagner theorem concerning the stability of ODLRO at finite temperatures, and the infrared divergences induced by quantum fluctuations at zero temperature. In recent years, growing interest in generalized symmetries [92,93] has led to significant advances across various subfields of theoretical physics. Within the framework of generalized symmetries, long-range entangled states, which do not fall under the conventional classification of SSB phases, are now interpreted as SSB phases of higher-(form) symmetries. As the fracton physics is evolving rapidly, it is timely and important to investigate SSB physics in fracton systems. More specifically, for fractonic many-body quantum systems, with their exotic conserved quantities and corresponding new symmetries, key questions arise: *What are the physical consequences of the SSB of such new symmetries? What are the resulting phases of matter?* Motivated by these questions, a series of exotic SSB phases of matter, termed *fractonic superfluids*, were proposed in Ref. [43] and have been explored in details in Refs. [43–48]. Interestingly, symmetry breaking in the presence of dipole conservation has become a rapidly developing topic in the literature of theoretical physics, e.g., Refs. [50–54,91] where the notions of fracton superfluids and dipole superfluids are introduced.

In fractonic superfluids, the constituent bosons that are condensed are either completely immobile [43] or movable only along fixed directions [44–47]. Due to the similar restriction of mobility, these particles are also referred to as fractons and lineons respectively by borrowing the terminology in fracton topological order and fracton codes. For simplicity, dipole moments, which are first-order moments, are often used in concrete model studies. The mobility restriction on fractons can be realized by imposing conservation of dipole moments $\hat{Q}^{(i)} = \int d^d \mathbf{x} \hat{\rho} x_i$ [43]. Additionally, the conservation of angular charge moments (abbreviated as “angular moments”), $\hat{Q}_{a,b} = \int d^d \mathbf{x} (\hat{\rho}_a x_b - \hat{\rho}_b x_a)$, was explored in Refs. [44–47] to capture the partial restriction of mobility. Here, $\hat{\rho}$ represents the density operator, and the indices a, b label the species of bosons. The total particle number conservation (i.e., “charge” conservation) is implicitly assumed when a higher-moment conservation law is considered.

In Refs. [43,44], minimal models of fractonic superfluids are constructed and analyzed across various aspects, including symmetry transformations, Noether currents, ground-state wave functions, Gross-Pitaevskii equations, hydrodynamics, and vortex configurations. Through low-energy effective field theories, it has been shown that conservation of higher moments tends to induce strong quantum fluctuations, making the establishment of ODLRO more challenging under higher-moment conservation. Specifically, in systems with charge and dipole moment conservation, charge symmetry breaking occurs only in spatial dimensions $d \geq 3$ at zero temperature. Recently, Ref. [73] demonstrated more generally that charge symmetry breaking is forbidden at finite temperature even in four dimensions as long as dipole conservation is imposed. In Ref. [45], continuity equations and Navier-Stokes-type equations were derived from the Euler-Lagrange equations of fractonic superfluids. Critical velocity

fields and density currents were obtained, leading to a Landau-type criterion. Reference [46] examined the emergence of higher-rank symmetries and angular moment conservation at low energies through renormalization group analysis, providing insights into their experimental realization. Meanwhile, Ref. [47] investigated finite-temperature phase transitions in fractonic superfluids, showing that the topological vortices (symmetry defects) proposed in Ref. [44] form a hierarchical structure of bound states, resulting in hierarchical confinement-deconfinement transitions upon increasing temperature.

Bose-Hubbard-type lattice models incorporating higher-moment conservation are also of significant interest, as they extend the conventional Bose-Hubbard model, which realizes the superfluid phase in the weak interaction regime. Dipolar Bose-Hubbard lattice models have been constructed and studied extensively in various spatial dimensions, yielding profound physical insights (e.g., Refs. [58,74–78]). In the small- U limit, these lattice models at long wavelengths reduce to the many-boson model of fractonic superfluids discussed in Ref. [43].

In the fractonic superfluid systems and associated Bose-Hubbard lattice models discussed above, the formulation of conserved quantities plays a crucial role in shaping the low-energy physics. Different conserved quantities can lead to entirely distinct outcomes in the interplay between SSB and fracton physics, influencing phenomena such as vortex configurations, the Mermin-Wagner theorem, finite-temperature phase diagrams, and hydrodynamics. Therefore, it is worthwhile to explore new conserved quantities that substantially extend beyond straightforward generalizations of previously studied cases.

Following the research on fractonic superfluids [43,44], in this paper, we continue to consider spontaneous symmetry breaking and other aspects in higher-moment-conserving systems. We introduce the concept of *hybrid fractonic superfluids* (HFS) by imposing the conservation of *hybrid* moments. HFS represent a class of higher-moment symmetry-breaking phases characterized by hybrid moments, which are constructed from the moments of bosons belonging to different species. Therefore, HFS models inherently involve at least two distinct species of bosons. We systematically investigate HFS and their realizations in Bose-Hubbard lattice models. Specifically, we focus on two representative series of HFS (referred to as *Model Series A* and *Model Series B*), distinguished by the mechanisms of moment hybridization. Notably, previous studies [44–47] on angular moments can also be literally regarded as a special subclass of fractonic superfluids with hybrid moments and could be categorized as *Model Series C* if needed. Given the profound physical implications uncovered in these prior studies, which likely represent only the tip of the iceberg of hybridization, it is imperative to systematically explore the broader impacts of hybridization, a key motivation for this work. This article, together with Refs. [43,44], forms a trilogy of fractonic superfluids. From the trilogy as well as the extended studies in Refs. [45–48], we attempt to uncover SSB physics of new symmetries brought by new conservation laws in quantum many-body systems.

Leaving technical details and coherent discussions in the main text, here we summarize key facts of Model Series A,

TABLE I. Comparison of patterns of charge symmetry breaking in four different model realizations with two species of bosons in two spatial dimensions at $T = 0$. The specific analysis of the $[d, 1, 2, 2]$ model is given in Sec. III, and the hybrid dipole-quadrupole moment conserving model is discussed in Sec. IV. $U(1)_{1,C}$ and $U(1)_{2,C}$ denote $U(1)$ symmetries associated with the particle number (charge) conservation of first and second species of bosons, respectively. $U(1)_{-,C}$ denote the *relative* charge symmetry associated with the particle number *difference* of the two species of bosons.

Superfluid models with two species of bosons	Two decoupled conventional superfluids	Two decoupled fractonic superfluids [43] with separate dipole conservations	$[d, 1, 2, 2]$ model in Model Series A	Hybrid dipole-quadrupole conserving model in Model Series B
Higher-moment conservation	N/A	Dipole Moments of Two Species	Hybrid Dipole Moments	Hybrid Dipole-Quadrupole Moments
Broken charge symmetry	$U(1)_{1,C} \otimes U(1)_{2,C}$	N/A	$U(1)_{-,C}$	$U(1)_{1,C}$
ODLRO	✓	×	✓	✓
Order parameter	$\langle \hat{\Phi}_1(\mathbf{x}) \rangle, \langle \hat{\Phi}_2(\mathbf{x}) \rangle$	N/A	$\langle \hat{\Phi}_1(\mathbf{x}) \hat{\Phi}_2^\dagger(\mathbf{x}) \rangle$	$\langle \hat{\Phi}_1(\mathbf{x}) \rangle$

Model Series B, and the Bose-Hubbard-type lattice model, respectively.

(i) In Model Series A, we consider the hybridization of moments of the *same* order moments constituted by bosons of *different* species. Arbitrary models in Model Series A can be represented by a tuple $[d, N, m, m']$, where d is the spatial dimension, N is the order of the moments participating in the hybridization, m is the total number of species of bosons, and m' denotes the number of species of bosons involved in the hybridization. We specifically study the $[d, 1, 2, 2]$ model of Model Series A in which total dipole moments are conserved. In this model, we examine the ground states, Goldstone modes, Noether currents, and correlation functions. We find that one of the dispersion relations is linear. In two spatial dimensions, the correlation function $\langle \hat{\Phi}_1^\dagger(\mathbf{x}) \hat{\Phi}_2(\mathbf{x}) \hat{\Phi}_2^\dagger(\mathbf{0}) \hat{\Phi}_1(\mathbf{0}) \rangle$ saturates to a constant at long distances. As a result, the system undergoes the spontaneous breaking of *relative* charge symmetry (denoted as $U(1)_{-,C}$) that is associated with the conservation of particle number *difference* of different species of bosons, thus supporting true ODLRO. In other words, the spontaneous breaking of relative charge symmetry in $[d, 1, 2, 2]$ model is triggered by nonzero order parameter $\langle \hat{\Phi}_1(\mathbf{x}) \hat{\Phi}_2^\dagger(\mathbf{x}) \rangle$. These results are also extended to arbitrary $[d, N, m, m']$ models, and partial spontaneous breaking of charge symmetry occurs in spatial dimensions $d = N + 1$.

(ii) In Model Series B, we consider the hybridization of moments of *different* order constituted by bosons of *different* species. Except for the order requirement, the other settings are the same as those in Model Series A. We specifically study an example with conservation of hybridization of dipole moments of species $\hat{\rho}_1 = \hat{\Phi}_1^\dagger \hat{\Phi}_1$ and quadrupole moments of species $\hat{\rho}_2 = \hat{\Phi}_2^\dagger \hat{\Phi}_2$, and also find one of the dispersion relations is linear. In two spatial dimensions, the correlation function $\langle \hat{\Phi}_2^\dagger(\mathbf{x}) \hat{\Phi}_2(\mathbf{0}) \rangle$ decays in a power law, and the correlation function $\langle \hat{\Phi}_1^\dagger(\mathbf{x}) \hat{\Phi}_1(\mathbf{0}) \rangle$ saturates to a constant at long distances. Therefore, the spontaneous breaking of charge symmetry of bosons of species 1 occurs, and the system has a true ODLRO. This spontaneous breaking in Model Series B is characterized by the order parameter $\langle \hat{\Phi}_1(\mathbf{x}) \rangle$. A summary of comparison between different model realizations with two species of bosons is given in Table I.

(iii) As the third piece of the trilogy of fractonic superfluids, Model Series A and Model Series B presented in this

paper incorporate hybridization of moments and significantly enrich the physics of fractonic superfluids. For instance, the hybridization influences the value of critical dimensions for ODLRO of charge operators (see Sec. IIIC for more details). While spontaneous breaking of charge symmetry in fractonic superfluids with dipole conservation [43] provides prototypical case study of the generalized Mermin-Wagner theorem [71,73], the hybrid moment conservation presented in this work provides a new platform for generalized Mermin-Wagner theorems.

(iv) We further construct Bose-Hubbard-type lattice models whose small Hubbard interaction energy (U) limit recovers Model Series A and Model Series B at long wavelengths. To obtain the global phase diagram of lattice models, we perform a mean-field approximation for $[d, 1, 2, 2]$ lattice model (i.e., the Hubbard model corresponding to the HFS labeled by $[d, 1, 2, 2]$). The mean-field results indicate that the introduction of hybridization brings exotic intermediate phases between Mott insulator (MI) phase and HFS phase. It will be valuable to study these newly introduced Bose-Hubbard models by using large-scale numerical simulations, such as quantum Monte Carlo and tensor network. Finally, we also discuss the realization of $[d, 1, 2, 2]$ lattice model in *strongly tilted optical lattices* by means of a third-order perturbation theory. A summary of comparison between Bose-Hubbard-type models with different conservation laws is given in Table II.

The remaining part of this paper is organized as follows. In Sec. II, previously studied fractonic superfluids are first reviewed in brief. Then the idea of hybridization is introduced in order to construct two series of minimal models (Model Series A and B) in the continuum space. The two series of models are studied respectively in Secs. III and IV. In Sec. V, Bose-Hubbard-type lattice models are constructed. Section VI provides a summary of this paper and offers additional perspectives for future research.

II. CONSTRUCTION OF HAMILTONIAN OF HYBRID MOMENTS

In this section, we start with a review of fractonic superfluids without hybridization [43]. Then, by introducing two or more species of bosons in the system, we subsequently

TABLE II. Comparison of three different Bose-Hubbard-type models. This table illustrates the conserved quantities and order parameters of these lattice models. The order parameters are applied to characterize exotic intermediate phases between MI (Mott insulating phase) and HFS (hybrid fractonic superfluid).

Lattice model	Conserved quantities	Order parameter
Bose-Hubbard model (Ref. [94])	Charge	$\langle \hat{b}_i \rangle$
Dipolar Bose-Hubbard model (Ref. [74])	Charge and dipole moment	$\langle \hat{b}_i \rangle$ and $\langle \hat{b}_{i+\hat{x}_i}^\dagger \hat{b}_i \rangle$
Lattice Model Series A [Eq. (77)]	Charge and hybrid dipole-dipole moment	$\langle \hat{b}_{a,i} \rangle$ and $\langle \hat{b}_{a,i+\hat{x}_i}^\dagger \hat{b}_{a,i} \rangle$ and $\langle \hat{b}_{1,i}^\dagger \hat{b}_{2,i} \rangle$

develop two distinct series of models with hybridization. During the procedure of model construction, we introduce the definition of conserved quantities and associated symmetry transformations on boson operators.

A. Review of fractonic superfluids

Initially, the concept of fractonic superfluids was introduced in single-component boson systems where multipole moments are conserved. We set $\hat{\Phi}^\dagger(\mathbf{x})$ and $\hat{\Phi}(\mathbf{x})$ to be bosonic creation and annihilation operators and $\hat{\rho}(\mathbf{x}) = \hat{\Phi}^\dagger(\mathbf{x})\hat{\Phi}(\mathbf{x})$ is a density operator. The creation and annihilation operators satisfy the standard commutation relations $[\hat{\Phi}(\mathbf{x}), \hat{\Phi}^\dagger(\mathbf{y})] = \delta^d(\mathbf{x} - \mathbf{y})$, where $\mathbf{x} = (x_1, \dots, x_d)$ is a spatial coordinate and we neglect it in the following for symbol convenience. We formulate the charges related to the multipole moments systematically as

$$\hat{Q}_{n,d}^{(i)} = \int d^d \mathbf{x} \hat{\rho} \mathcal{M}_{n,d}^{(i)}, \quad (1)$$

where $\mathcal{M}_{n,d}^{(i)}$ denotes the monomials obtained by multiplying the spatial rectangular coordinates with the coefficients equal to 1. The notation n stands for the number of multiplied spatial rectangular coordinates in $\mathcal{M}_{n,d}^{(i)}$, and d stands for spatial dimensions. Each $\hat{Q}_{n,d}^{(i)}$ denotes one component of the n -order moment in d spatial dimensions. The notation i denotes a index of the monomial for n th-order moments in the lexicographic ordering,¹ and the index i is listed from 1 to $C_{d-1+n}^n = \frac{(d-1+n)!}{(d-1)!n!}$.

In addition, the conservation of charge $\hat{Q} = \int d^d \mathbf{x} \hat{\rho}$ is also required. Given the maximal order of the monomials as N , we can construct a symmetry group $\mathcal{G}$ with an element

$$U = \exp \left[-i \left(\lambda \hat{Q} + \sum_{n=1}^N \sum_{i=1}^{C_{d-1+n}^n} \lambda_{n,d}^{(i)} \hat{Q}_{n,d}^{(i)} \right) \right] \quad (2)$$

¹In the lexicographic ordering, the arrangement of monomials obtained by multiplying the same number of x 's follows the rule: for two monomials, the order of x_1 is firstly compared, with the monomial with the greater order of x_1 coming first. If two monomials have the same order of x_1 , compare the order of x_2 , and so on. For instance, in two spatial dimensions, we can sort the monomials of quadrupole moments as $\mathcal{M}_{2,2}^{(1)} = x_1^2$, $\mathcal{M}_{2,2}^{(2)} = x_1 x_2$, and $\mathcal{M}_{2,2}^{(3)} = x_2^2$. In three spatial dimensions, we can sort the monomials of quadrupole moments as $\mathcal{M}_{2,3}^{(1)} = x_1^2$, $\mathcal{M}_{2,3}^{(2)} = x_1 x_2$, $\mathcal{M}_{2,3}^{(3)} = x_1 x_3$, $\mathcal{M}_{2,3}^{(4)} = x_2^2$, $\mathcal{M}_{2,3}^{(5)} = x_2 x_3$, and $\mathcal{M}_{2,3}^{(6)} = x_3^2$.

which leads to the following transformation on boson operators:

$$\hat{\Phi}' = U \hat{\Phi} U^\dagger = \hat{\Phi} e^{i(\lambda + \sum_n \sum_i C_{d-1+n}^n \lambda_{n,d}^{(i)} \mathcal{M}_{n,d}^{(i)})}. \quad (3)$$

Here, $\lambda, \lambda_{n,d}^{(i)} \in \mathbb{R}$ are group parameters. Hereafter, this symmetry will be alternatively called a higher-rank symmetry since the symmetry leads to higher-rank tensor gauge fields upon gauging. The kinetic term $\mathcal{H}_0$ is no longer Gaussian, e.g., the Hamiltonian constructed in the Appendix of Ref. [43]:

$$\begin{aligned} \mathcal{H}_0 = & \sum_{i_1, \dots, i_{N+1}}^d K^{i_1 \dots i_{N+1}} (\hat{\Phi}^\dagger)^{N+1} (\nabla_{i_1 \dots i_{N+1}} \log \hat{\Phi}^\dagger) \\ & \times \hat{\Phi}^{N+1} (\nabla_{i_1 \dots i_{N+1}} \log \hat{\Phi}), \end{aligned} \quad (4)$$

where $\nabla_{i_1 \dots i_n} = \partial_{i_1} \dots \partial_{i_n}$ and symmetric coefficients $K^{i_1 \dots i_{N+1}}$. The operator $\log \hat{\Phi}$ is a very useful notation to simplify the Hamiltonian (see Appendix of Ref. [43]). For convenience, we set the coefficients $K^{i_1 \dots i_{N+1}} > 0$. We can apply the potential term $V(\hat{\Phi}^\dagger, \hat{\Phi}) = -\mu \hat{\Phi}^\dagger \hat{\Phi} + \frac{g}{2} \hat{\Phi}^\dagger \hat{\Phi}^\dagger \hat{\Phi} \hat{\Phi}$, where μ is the chemical potential and $g > 0$ describes onsite repulsive interaction. A fractonic superfluid phase can be reached for a positive chemical potential.

To get an understanding on the fractonic superfluid phase, we can look at the following two examples. When the dipole moments are conserved, the kinetic term takes the form as

$$\begin{aligned} \mathcal{H}_0 = & \sum_{i,j}^d K^{ij} (\hat{\Phi}^\dagger \partial_i \partial_j \hat{\Phi}^\dagger - \partial_i \hat{\Phi}^\dagger \partial_j \hat{\Phi}) \\ & \times (\hat{\Phi} \partial_i \partial_j \hat{\Phi} - \partial_i \hat{\Phi} \partial_j \hat{\Phi}). \end{aligned} \quad (5)$$

The Hamiltonian is more complicated when the quadrupole moments are conserved:

$$\begin{aligned} \mathcal{H}_0 = & \sum_{i,j,k}^d K^{ijk} (\hat{\Phi}^{\dagger 2} \partial_i \partial_j \partial_k \hat{\Phi}^\dagger - \hat{\Phi}^\dagger \partial_i \hat{\Phi}^\dagger \partial_j \partial_k \hat{\Phi}^\dagger \\ & - \hat{\Phi}^\dagger \partial_j \hat{\Phi}^\dagger \partial_i \partial_k \hat{\Phi}^\dagger - \hat{\Phi}^\dagger \partial_k \hat{\Phi}^\dagger \partial_i \partial_j \hat{\Phi}^\dagger \\ & + 2 \partial_i \hat{\Phi}^\dagger \partial_j \hat{\Phi}^\dagger \partial_k \hat{\Phi}^\dagger) (\hat{\Phi}^2 \partial_i \partial_j \partial_k \hat{\Phi} - \hat{\Phi} \partial_i \hat{\Phi} \partial_j \partial_k \hat{\Phi} \\ & - \hat{\Phi} \partial_j \hat{\Phi} \partial_i \partial_k \hat{\Phi} - \hat{\Phi} \partial_k \hat{\Phi} \partial_i \partial_j \hat{\Phi} + 2 \partial_i \hat{\Phi} \partial_j \partial_k \hat{\Phi}). \end{aligned} \quad (6)$$

For the two examples, the dynamics of a single particle are strongly restricted while a bound state becomes free when the cooperative motions are required to conserve multipole moments: dipole moments for Hamiltonian in Eq. (5) and quadrupole moments for Eq. (6). The corresponding fractonic superfluid phase then has one Goldstone mode with a high-order dispersion. For a general N as the largest order of the monomials, the effective theory describes the Goldstone mode

with the dispersion relation $\omega \propto |\mathbf{k}|^{N+1}$. Systems characterized by Goldstone mode with dispersion relation $\omega \propto |\mathbf{k}|^{N+1}$ exhibit ODLRO in spatial dimensions $d \geq N + 2$, as shown in Ref. [43]. Therefore, this system breaks its charge symmetry in spatial dimensions $d \geq N + 2$. The spontaneous breaking of charge symmetry in fractonic superfluids manifests as a consequence of the generalized Mermin-Wagner theorem [71,73].

B. Construction of Model Series A denoted as $[d, N, m, m']$

Next, we begin to construct models with hybridization of moments. We first focus on Model Series A in which hybrid moments are formed by moments of the same order. The general construction looks very tedious. The readers who are more interested in concrete models can directly read from Eq. (15).

Consider a system with m species of bosons, the operators $\hat{\Phi}_a^\dagger(\mathbf{x})$ and $\hat{\Phi}_a(\mathbf{x})$ create and annihilate a boson of species a , respectively, and satisfy the commutation relations $[\hat{\Phi}_a(\mathbf{x}), \hat{\Phi}_b^\dagger(\mathbf{y})] = \delta_{ab}\delta^d(\mathbf{x} - \mathbf{y})$. In such a system, we consider the following conserved quantities in Eqs. (7)–(10):

(i) The charges of bosons of species a are conserved (the range of a is $1 \leq a \leq m$):

$$\hat{Q}_a = \int d^d \mathbf{x} \hat{\rho}_a. \quad (7)$$

(ii) The hybrid moments defined below are conserved. N is the order of the moments which participate in the hybridization (the range of the label i is $1 \leq i \leq C_{d-1+N}^N$):

$$\hat{Q}_{\text{mix}}^{(i)} = \int d^d \mathbf{x} \sum_{a=1}^{m'} \hat{\rho}_a \mathcal{M}_{N,d}^{(i)}. \quad (8)$$

(iii) All the moments whose orders are lower than N are conserved (the ranges of a , n , and i are $1 \leq a \leq m$, $1 \leq n \leq N - 1$, and $1 \leq i \leq C_{d-1+n}^n$, respectively):

$$\hat{Q}_{a,n,d}^{(i)} = \int d^d \mathbf{x} \hat{\rho}_a \mathcal{M}_{n,d}^{(i)}. \quad (9)$$

(iv) The moments of bosons of species $m' + 1 \leq a \leq m$ do not participate in the hybridization. Their N th-order moments are independently conserved (the ranges of a and i are $m' + 1 \leq a \leq m$, $1 \leq i \leq C_{d-1+N}^N$):

$$\hat{Q}_{a,N,d}^{(i)} = \int d^d \mathbf{x} \hat{\rho}_a \mathcal{M}_{N,d}^{(i)}. \quad (10)$$

Compared with the separate conservation of the N th-order moments for each species of bosons, the number of conserved quantities in the present hybridization case decreases. That means freeing up some mobility of bosons. We can see this better in the corresponding symmetry. We denote a group $\mathcal{G}$ generated by the conserved quantities [Eqs. (7)–(10)], and an

element of the group

$$U = \exp \left[-i \sum_{a=1}^m \left(\lambda_a \hat{Q}_a + \sum_n^{N-1} \sum_i^{C_{d-1+n}^n} \lambda_{a,n}^{(i)} \hat{Q}_{a,n,d}^{(i)} \right) - i \sum_i^{C_{d-1+N}^N} \lambda_{\text{mix}}^{(i)} \hat{Q}_{\text{mix}}^{(i)} - i \sum_{a=m'+1}^m \sum_i^{C_{d-1+N}^N} \lambda_{a,N}^{(i)} \hat{Q}_{a,N,d}^{(i)} \right] \quad (11)$$

leads to the following transformation on boson operators:

$$\begin{aligned} \hat{\Phi}'_a &= U \hat{\Phi}_a U^\dagger \\ &= \hat{\Phi}_a \exp \left\{ i \left[\lambda_a + \sum_n^{N-1} \sum_i^{C_{d-1+n}^n} \lambda_{a,n}^{(i)} \mathcal{M}_{n,d}^{(i)} + \sum_{b=1}^{m'} \delta_{ab} \right. \right. \\ &\quad \times \left. \sum_i^{C_{d-1+N}^N} \lambda_{\text{mix}}^{(i)} \mathcal{M}_{N,d}^{(i)} + \sum_{b=m'+1}^m \delta_{ab} \sum_i^{C_{d-1+N}^N} \lambda_{a,N}^{(i)} \mathcal{M}_{N,d}^{(i)} \right] \Big\}, \end{aligned} \quad (12)$$

where $\lambda_a, \lambda_{a,n}^{(i)}, \lambda_{\text{mix}}^{(i)} \in \mathbb{R}$ are group parameters. The group $\mathcal{G}$ can be written as the direct product of $U(1)$ groups,

$$\begin{aligned} \mathcal{G} &= \left(\bigotimes_a^m U(1)_{a,C} \right) \otimes \left(\bigotimes_a^m \bigotimes_{n,i} U(1)_{a,n,i} \right) \\ &\quad \otimes \left(\bigotimes_i U(1)_{\text{mix-A},i} \right) \otimes \left(\bigotimes_{a=m'+1}^m \bigotimes_i U(1)_{a,N,i} \right), \end{aligned} \quad (13)$$

where group elements of $U(1)_{a,C}$, $U(1)_{a,n,i}$, and $U(1)_{\text{mix-A},i}$ are, respectively, labeled by $\lambda_a, \lambda_{a,n}^{(i)}$, and $\lambda_{\text{mix}}^{(i)}$. The notation $\bigotimes \cdots U(1) \cdots$ here denotes the direct product of multiple corresponding $U(1)$ groups.

A Hamiltonian satisfying this conservation law needs to commute with the conserved quantities [Eqs. (7)–(10)]. The minimal Hamiltonian can be written as

$$\begin{aligned} \mathcal{H}_0 &= \sum_a^m \sum_{i_1, \dots, i_{N+1}}^d K_a^{i_1 \dots i_{N+1}} (\hat{\Phi}_a^\dagger)^{N+1} (\nabla_{i_1, \dots, i_{N+1}} \log \hat{\Phi}_a^\dagger) \\ &\quad \times \hat{\Phi}_a^{N+1} (\nabla_{i_1, \dots, i_{N+1}} \log \hat{\Phi}_a) + \sum_{a \neq b}^{m'} \sum_{i_1, \dots, i_N}^d \Gamma_{a,b}^{i_1 \dots i_N} (\hat{\Phi}_a^\dagger)^N \\ &\quad \times (\hat{\Phi}_b^\dagger)^N [(\nabla_{i_1, \dots, i_N} \log \hat{\Phi}_a^\dagger) - (\nabla_{i_1, \dots, i_N} \log \hat{\Phi}_b^\dagger)] \\ &\quad \times \hat{\Phi}_a^N \hat{\Phi}_b^N [(\nabla_{i_1, \dots, i_N} \log \hat{\Phi}_a) - (\nabla_{i_1, \dots, i_N} \log \hat{\Phi}_b)]. \end{aligned} \quad (14)$$

For convenience, we also set $K_a^{i_1 \dots i_{N+1}} > 0$, $\Gamma_{a,b}^{i_1 \dots i_N} > 0$. In addition, $\Gamma_{a,b}^{i_1 \dots i_N} = \Gamma_{b,a}^{i_1 \dots i_N}$ is needed.

In the construction above, the spatial dimensions d , the order N , the number of boson species involved in hybridization m' , and the total number of boson species m are tunable for Model Series A. A tuple $[d, N, m, m']$ can be used to uniquely denote one example of Model Series A.

With the above preparation, we begin to construct the simplest example of Model Series A, denoted as $[d, 1, 2, 2]$ model. $[d, 1, 2, 2]$ model can be regarded as a system in

which the total dipole moments are conserved along with the conservation of the charges of each species,

$$\hat{Q}_a = \int d^d \mathbf{x} \hat{\rho}_a \quad (a = 1, 2), \quad (15)$$

$$\hat{Q}_{\text{mix}}^{(i)} = \int d^d \mathbf{x} (\hat{\rho}_1 + \hat{\rho}_2) x_i \quad (i = 1, \dots, d). \quad (16)$$

The total symmetry group $\mathcal{G}$ can be written as

$$\mathcal{G} = (\text{U}(1)_{1,C} \otimes \text{U}(1)_{2,C}) \otimes \left(\bigotimes_i \text{U}(1)_{\text{mix-A},i} \right). \quad (17)$$

An element of the group $\mathcal{G}$ is given by

$$U = \exp \left[-i \left(\lambda_1 \hat{Q}_1 + \lambda_2 \hat{Q}_2 + \sum_i \lambda_{\text{mix}}^{(i)} \hat{Q}_{\text{mix}}^{(i)} \right) \right]. \quad (18)$$

And the transformation on field operators can be written as

$$\hat{\Phi}'_a = U \hat{\Phi}_a U^\dagger = \hat{\Phi}_a \exp \left[i \left(\lambda_a + \sum_i \lambda_{\text{mix}}^{(i)} x_i \right) \right]. \quad (19)$$

To satisfy this conservation law, the minimal Hamiltonian of $[d, 1, 2, 2]$ model with an energy potential term is

$$\begin{aligned} \mathcal{H} = & \sum_a \sum_{i,j} K_a^{ij} (\hat{\Phi}_a^\dagger \partial_i \partial_j \hat{\Phi}_a^\dagger - \partial_i \hat{\Phi}_a^\dagger \partial_j \hat{\Phi}_a^\dagger) \\ & \times (\hat{\Phi}_a \partial_i \partial_j \hat{\Phi}_a - \partial_i \hat{\Phi}_a \partial_j \hat{\Phi}_a) \\ & + \sum_{a \neq b} \sum_i \Gamma_{a,b}^i (\hat{\Phi}_a^\dagger \partial_i \hat{\Phi}_b^\dagger - \hat{\Phi}_b^\dagger \partial_i \hat{\Phi}_a^\dagger) \\ & \times (\hat{\Phi}_a \partial_i \hat{\Phi}_b - \hat{\Phi}_b \partial_i \hat{\Phi}_a) + V(\hat{\Phi}^\dagger, \hat{\Phi}), \end{aligned} \quad (20)$$

where $V(\hat{\Phi}^\dagger, \hat{\Phi}) = \sum_a^2 -\mu_a \hat{\Phi}_a^\dagger \hat{\Phi}_a + \frac{g}{2} \hat{\Phi}_a^\dagger \hat{\Phi}_a^\dagger \hat{\Phi}_a \hat{\Phi}_a$, and the coefficients satisfy $K_a^{ij} > 0$, $\Gamma_{a,b}^i > 0$, $K_a^{ij} = K_a^{ji}$, $\Gamma_{a,b}^i = \Gamma_{b,a}^i$, and $g > 0$. We can also derive the corresponding conserved quantities and Noether currents. For the conserved charge $\hat{Q}_a = \int d^d \mathbf{x} \hat{\Phi}_a^\dagger \hat{\Phi}_a = \int d^d \mathbf{x} \hat{\rho}_a$, the current is given by ($i = 1, \dots, d$):

$$\begin{aligned} \hat{J}_i^a = & i \sum_j K_a^{ij} \partial_j [\hat{\Phi}_a^{\dagger 2} (\hat{\Phi}_a \partial_i \partial_j \hat{\Phi}_a - \partial_i \hat{\Phi}_a \partial_j \hat{\Phi}_a) - \text{H.c.}] \\ & + 2i \sum_{b \neq a} \Gamma_{a,b}^i [(\hat{\rho}_a \hat{\Phi}_b \partial_i \hat{\Phi}_b^\dagger - \hat{\rho}_b \hat{\Phi}_a \partial_i \hat{\Phi}_a^\dagger) - \text{H.c.}] \end{aligned} \quad (21)$$

For the hybrid moments $\hat{Q}_{\text{mix}}^{(j)} = \int d^d \mathbf{x} (\hat{\rho}_1 x_j + \hat{\rho}_2 x_j) = \int d^d \mathbf{x} \hat{\rho}_{\text{mix}}^{(j)}$, the current is given by ($i = 1, \dots, d$):

$$\begin{aligned} \hat{D}_i^{(j)} = & x_j \hat{J}_i^1 + x_j \hat{J}_i^2 - i \sum_a^2 K_a^{ij} \\ & \times [\hat{\Phi}_a^{\dagger 2} (\hat{\Phi}_a \partial_i \partial_j \hat{\Phi}_a - \partial_i \hat{\Phi}_a \partial_j \hat{\Phi}_a) - \text{H.c.}] \end{aligned} \quad (22)$$

The $\Gamma_{a,b}^i$ term is the emerging term from the hybridization of moments. This Hamiltonian density [Eq. (20)] is the simplest example of HFS in the Model Series A. The effective field theory of this Hamiltonian will be studied in Sec. III.

C. Construction of Model Series B

In the construction of Model Series B, in a system with m species of bosons, we consider the hybridization of moments of different orders from different species. Although general formulas in Model Series B are expected to be more complex than Model Series A, let us present conserved quantities below for completeness (a concrete example will be provided shortly):

(i) The charges of bosons of species a are conserved (the range of a is $1 \leq a \leq m$):

$$\hat{Q}_a = \int d^d \mathbf{x} \hat{\rho}_a, \quad (23)$$

(ii) The hybrid moments defined below are conserved:

$$\hat{Q}_{\text{mix}}^{(i_1) \dots (i_{m'})} = \int d^d \mathbf{x} \sum_a^{m'} \hat{\rho}_a \mathcal{M}_{N_a, d}^{(i_a)} \ell^{N_{m'} - N_a}. \quad (24)$$

Here, m' denotes the number of species of bosons involved in the hybridization. We use N_a to denote the orders of the moments of the species a which participate in the hybridization. For briefness, we let N_a increase as a increase, i.e., $N_1 < N_2 < \dots < N_{m'}$. ℓ with a dimension of length denotes the coefficient by which moments of different orders are converted to each other. The introduction of ℓ ensures that the quantities in $\hat{Q}_{\text{mix}}^{(i_1) \dots (i_{m'})}$ have the same dimensions. For each $\hat{Q}_{\text{mix}}^{(i_1) \dots (i_{m'})}$, the index i_a can be arbitrarily taken in the range $1 \leq i_a \leq C_{d-1+N_a}^{N_a}$, but cannot be repeated with other $\hat{Q}_{\text{mix}}^{(i_1) \dots (i_{m'})}$. The number of the conserved quantities $\hat{Q}_{\text{mix}}^{(i_1) \dots (i_{m'})}$ is $C_{d-1+N_1}^{N_1}$.

(iii) All the moments of species a whose orders are lower than N_a are conserved (the ranges of a , n , and i are $1 \leq a \leq m'$, $1 \leq n \leq N_a - 1$, and $1 \leq i \leq C_{d-1+n}^n$, respectively):

$$\hat{Q}_{a,n,d}^{(i)} = \int d^d \mathbf{x} \hat{\rho}_a \mathcal{M}_{n,d}^{(i)}. \quad (25)$$

(iv) The components of N_a -th-order moments which do not participate in the hybridization are conserved:

$$\hat{Q}_{a,N_a,d}^{(i_a)} = \int d^d \mathbf{x} \hat{\rho}_a \mathcal{M}_{N_a,d}^{(i_a)}. \quad (26)$$

Due to the fact that the numbers of components of different order moments are not equal in two spatial dimensions and above, some components of the N_a -th-order moments do not participate in the hybridization, and they are required to be conserved in Model Series B. The index i_a takes all $1 \leq i_a \leq C_{d-1+N_a}^{N_a}$ that are not taken in Eq. (24). The range of a is $1 \leq a \leq m'$.

The above conserved quantities look apparently more tedious than Model Series A. In the following, we directly move to concrete model construction. In order to present a concrete model as simple as possible, we assume $m' = m$ and fix the coefficient m and N_a , e.g., $m = 2$, $N_1 = 1$, and $N_2 = 2$. In a system with two species of bosons, we mix dipole moments of bosons of species 1 and quadrupole moments of bosons of species 2. For convenience, the conserved quantities are given by

$$\hat{Q}_a = \int d^d \mathbf{x} \hat{\rho}_a \quad (a = 1, 2), \quad (27)$$

$$\hat{Q}_2^{(i)} = \int d^d \mathbf{x} \hat{\rho}_2 x_i \quad (i = 1, \dots, d), \quad (28)$$

$$\hat{Q}_{\text{mix}}^{i,ii} = \int d^d \mathbf{x} (\hat{\rho}_1 x_i \ell + \hat{\rho}_2 x_i^2) \quad (i = 1, \dots, d), \quad (29)$$

$$\hat{Q}_2^{ij} = \int d^d \mathbf{x} \hat{\rho}_2 x_i x_j \quad (i, j = 1, \dots, d, \quad i \neq j). \quad (30)$$

We denote a group $\mathcal{G}$ generated by the conserved quantities [Eqs. (27)–(30)], and an element of the group

$$U = \exp \left[-i \left(\lambda_1 \hat{Q}_1 + \lambda_2 \hat{Q}_2 + \sum_i \lambda_2^{(i)} \hat{Q}_2^i + \sum_i \lambda_{\text{mix}}^{(i)} \hat{Q}_{\text{mix}}^{i,ii} + \sum_{i \neq j} \lambda_2^{(ij)} \hat{Q}_2^{ij} \right) \right] \quad (31)$$

leads to the following transformations on field operators:

$$\hat{\Phi}'_1 = U \hat{\Phi}_1 U^\dagger = \hat{\Phi}_1 \exp \left[i \left(\lambda_1 + \sum_i \lambda_{\text{mix}}^{(i)} x_i \ell \right) \right], \quad (32)$$

$$\hat{\Phi}'_2 = U \hat{\Phi}_2 U^\dagger = \hat{\Phi}_2 \exp \left[i \left(\lambda_2 + \sum_i \lambda_2^{(i)} x_i + \sum_i \lambda_{\text{mix}}^{(i)} x_i^2 + \sum_{i \neq j} \lambda_2^{(ij)} x_i x_j \right) \right], \quad (33)$$

where $\lambda_1, \lambda_2, \lambda_2^{(i)}, \lambda_{\text{mix}}^{(i)}, \lambda_2^{(ij)} \in \mathbb{R}$ are group parameters. The group $\mathcal{G}$ can be written as the direct product of U(1) groups,

$$\mathcal{G} = \left(\bigotimes_a \text{U}(1)_{a,C} \right) \otimes \left(\bigotimes_i \text{U}(1)_{2,i} \right) \otimes \left(\bigotimes_i \text{U}(1)_{\text{mix-B},i} \right) \otimes \left(\bigotimes_{i \neq j} \text{U}(1)_{2,\{i,j\}} \right), \quad (34)$$

where group elements of $\text{U}(1)_{a,C}$, $\text{U}(1)_{2,i}$, $\text{U}(1)_{\text{mix-B},i}$, and $\text{U}(1)_{2,\{i,j\}}$ are, respectively, labeled by $\lambda_a, \lambda_2^{(i)}, \lambda_{\text{mix}}^{(i)}, \lambda_2^{(ij)}$.

A Hamiltonian satisfying this conservation law needs to commute with conserved quantities [Eqs. (27)–(30)]. We can write the minimal Hamiltonian of Model Series B with the potential terms

$$\begin{aligned} \mathcal{H} = & \sum_{i,j}^d K_1^{ij} (\hat{\Phi}_1^\dagger \partial_i \partial_j \hat{\Phi}_1^\dagger - \partial_i \hat{\Phi}_1^\dagger \partial_j \hat{\Phi}_1^\dagger) (\hat{\Phi}_1 \partial_i \partial_j \hat{\Phi}_1 - \partial_i \hat{\Phi}_1 \partial_j \hat{\Phi}_1) \\ & + \sum_{i,j,k}^d K_2^{ijk} (\hat{\Phi}_2^{\dagger 2} \partial_i \partial_j \partial_k \hat{\Phi}_2^\dagger - \hat{\Phi}_2^\dagger \partial_i \hat{\Phi}_2^\dagger \partial_j \partial_k \hat{\Phi}_2^\dagger \\ & - \hat{\Phi}_2^\dagger \partial_j \hat{\Phi}_2^\dagger \partial_i \partial_k \hat{\Phi}_2^\dagger - \hat{\Phi}_2^\dagger \partial_k \hat{\Phi}_2^\dagger \partial_i \partial_j \hat{\Phi}_2^\dagger + 2 \partial_i \hat{\Phi}_2^\dagger \partial_j \hat{\Phi}_2^\dagger \partial_k \hat{\Phi}_2^\dagger) \\ & \times (\hat{\Phi}_2^2 \partial_i \partial_j \partial_k \hat{\Phi}_2 - \hat{\Phi}_2 \partial_i \hat{\Phi}_2 \partial_j \partial_k \hat{\Phi}_2 - \hat{\Phi}_2 \partial_j \hat{\Phi}_2 \partial_i \partial_k \hat{\Phi}_2 \\ & - \hat{\Phi}_2 \partial_k \hat{\Phi}_2 \partial_i \partial_j \hat{\Phi}_2 + 2 \partial_i \hat{\Phi}_2 \partial_j \hat{\Phi}_2 \partial_k \hat{\Phi}_2) \\ & + \sum_i^d \Gamma_{1,2}^i (\ell \hat{\Phi}_1^\dagger \hat{\Phi}_1^\dagger \partial_i^2 \hat{\Phi}_2^\dagger - \ell \hat{\Phi}_1^\dagger \partial_i \hat{\Phi}_2^\dagger \partial_i \hat{\Phi}_2^\dagger - 2 \partial_i \hat{\Phi}_1^\dagger \hat{\Phi}_2^{\dagger 2}) \\ & \times (\ell \hat{\Phi}_1 \hat{\Phi}_2^2 \partial_i^2 \hat{\Phi}_2 - \ell \hat{\Phi}_1 \partial_i \hat{\Phi}_2 \partial_i \hat{\Phi}_2 - 2 \partial_i \hat{\Phi}_1 \hat{\Phi}_2^2) \\ & + V(\hat{\Phi}^\dagger, \hat{\Phi}). \end{aligned} \quad (35)$$

For convenience, the coefficients need to satisfy $K_1^{ij} = K_1^{ji} > 0$, $K_2^{ijk} = K_2^{ikj} = K_2^{jik} = K_2^{kji} = K_2^{kij} = K_2^{jki} > 0$, $\Gamma_{1,2}^i > 0$, and $g > 0$. The $\Gamma_{1,2}^i$ terms are the emerging terms from the hybridization of moments. In the same way as Model Series A, we can also obtain Noether currents of Model Series B. The $\Gamma_{1,2}^i$ terms liberate part of the boson mobility restriction. The effective field theory of this Hamiltonian is performed in Sec. IV.

III. HYBRID FRACTONIC SUPERFLUIDS: MODEL SERIES A

In the preceding section, we have constructed the minimal Hamiltonians with conservation of hybrid moments. In the present and next sections, we subsequently study physical properties of Model Series A and Model Series B. In this section, we study physical properties of Model Series A. We start our analysis with the $[d, 1, 2, 2]$ model and then directly generalize to arbitrary $[d, N, m, m']$ models.

A. Classical ground-state wave functions

Classically, the energy density $\mathcal{E}$ of the $[d, 1, 2, 2]$ model in Eq. (20) has the form as

$$\begin{aligned} \mathcal{E} = & \sum_a^2 \sum_{i,j}^d K_a^{ij} |\phi_a \partial_i \partial_j \phi_a - \partial_i \phi_a \partial_j \phi_a|^2 \\ & + \sum_{a \neq b}^2 \sum_i^d \Gamma_{a,b}^i |\phi_a \partial_i \phi_b - \phi_b \partial_i \phi_a|^2 \\ & + \sum_a^2 -\mu_a |\phi_a|^2 + \frac{g}{2} |\phi_a|^4 \end{aligned} \quad (36)$$

which is obtained by replacing the non-Hermitian operators $\hat{\Phi}_a$ in Eq. (20) by their coherent-state eigenvalue $\phi_a \in \mathbb{C}$. Considering the chemical potential $\mu_1 > 0$, $\mu_2 > 0$, at $|\phi_a| = \sqrt{\rho_{a0}} = \sqrt{\frac{\mu_a}{g}}$, the potential reaches the minimal value. Then, the energy density reduces to $\mathcal{E} = \sum_a^2 \sum_{i,j}^d K_a^{ij} \rho_{a0}^2 (\partial_i \partial_j \theta_a)^2 + \sum_{a \neq b}^2 \sum_i^d \Gamma_{a,b}^i \rho_{10} \rho_{20} (\partial_i \theta_a - \partial_i \theta_b)^2 - \sum_a^2 \frac{\mu_a^2}{2g}$. As the coefficients $K_a^{ij} > 0$ and $\Gamma_{a,b}^i > 0$, to make these terms reach the minimal value, we need $\partial_i \partial_j \theta_a = 0$ and $\partial_i \theta_a = \partial_i \theta_b$ for any a and b , which leads to the classical ground state $|\text{GS}_\beta^{\{\theta_{a0}\}}\rangle = \bigotimes_{\mathbf{x}} |\text{GS}_\beta^{\{\theta_{a0}\}}\rangle_{\mathbf{x}}$. Here $|\text{GS}_\beta^{\{\theta_{a0}\}}\rangle_{\mathbf{x}} = \frac{1}{C} \prod_a^2 \exp[\sqrt{\rho_{a0}} e^{i(\theta_{a0} + \sum_i \beta_i x_i)} \hat{\Phi}_a^\dagger(\mathbf{x})] |0\rangle$. C is the normalization factor, and $|0\rangle$ is the quantum vacuum state. $\beta = \{\beta_1, \dots, \beta_d\}$ has the dimension of momentum. A significant feature of this ground state is the formation of ODLRO. The correlation functions are given by $\langle \hat{\Phi}_a^\dagger(\mathbf{x}) \hat{\Phi}_b(\mathbf{0}) \rangle = \sqrt{\rho_{a0} \rho_{b0}} e^{i(\theta_{b0} - \theta_{a0} - \sum_i \beta_i x_i)}$, $\langle \hat{\Phi}_a^\dagger(\mathbf{x}) \hat{\Phi}_b(\mathbf{x}) \hat{\Phi}_a(\mathbf{0}) \hat{\Phi}_b^\dagger(\mathbf{0}) \rangle = \rho_{a0} \rho_{b0}$, and $\langle \hat{\Phi}_a^\dagger(\mathbf{x}) \hat{\Phi}_b^\dagger(\mathbf{x}) \hat{\Phi}_a(\mathbf{0}) \hat{\Phi}_b(\mathbf{0}) \rangle = \rho_{a0} \rho_{b0} e^{-2i \sum_i \beta_i x_i}$. The zero in bold font $\mathbf{0}$ denotes the coordinate origin. For the hybrid fractonic superfluid phase in the $[d, 1, 2, 2]$ model, the order parameters are defined as the expectation values on the ground state: $\langle \hat{\Phi}_1(\mathbf{x}) \rangle = \sqrt{\rho_{10}} e^{i(\theta_{10} + \sum_i \beta_i x_i)}$, $\langle \hat{\Phi}_2(\mathbf{x}) \rangle = \sqrt{\rho_{20}} e^{i(\theta_{20} + \sum_i \beta_i x_i)}$, and $\langle \hat{\Phi}_1(\mathbf{x}) \hat{\Phi}_2^\dagger(\mathbf{x}) \rangle = \sqrt{\rho_{10} \rho_{20}} e^{i(\theta_{10} - \theta_{20})}$. After condensation, the Noether currents [Eqs. (21) and (22)]

become

$$\hat{\rho}_a = \rho_{a0},$$

$$\hat{f}_i^a = -2 \sum_j^d K_a^{ij} \rho_{a0}^2 \partial_i \partial_j^2 \hat{\theta}_a + 4\Gamma_{1,2}^i \rho_{10} \rho_{20} (\partial_i \hat{\theta}_1 + \partial_i \hat{\theta}_2), \quad (37)$$

and

$$\hat{\rho}_{\text{mix}}^{(j)} = (\rho_{10} + \rho_{20}) x_j,$$

$$\hat{\mathcal{D}}_i^{(j)} = x_j \hat{f}_i^1 + x_j \hat{f}_i^2 + 2 \sum_a^2 K_a^{ij} \rho_{a0}^2 \partial_i \partial_j \hat{\theta}_a. \quad (38)$$

B. Goldstone modes and quantum fluctuations

Quantum fluctuations tend to destroy true ODLRO. In this subsection, we introduce the effect of quantum fluctuations. Considering quantum fluctuations, we have $\hat{\Phi}_a(\mathbf{x}) = \sqrt{\rho_{a0} + \hat{f}_a(\mathbf{x})} e^{i\hat{\theta}_a(\mathbf{x})}$, where $\hat{f}_a(\mathbf{x}, t) \ll \rho_{a0}$ are the quantum fluctuations of the density fields, and $\hat{\theta}_a$ is the quantum fluctuation of the phase fields. The commutation relation between $\hat{\theta}_a$ and $\hat{f}_a$ is given by $[\hat{\theta}_a(\mathbf{x}), \hat{f}_b(\mathbf{y})] = -i\delta^{(d)}(\mathbf{x} - \mathbf{y})\delta_{ab}$. Therefore, the canonical momentum operator conjugate to $\hat{\theta}_a$ is given by $\hat{\pi}_a = -\hat{f}_a$, satisfying $[\hat{\theta}_a(\mathbf{x}), \hat{\pi}_b(\mathbf{y})] = i\delta_{ab}\delta^{(d)}(\mathbf{x} - \mathbf{y})$. To the second order, we can derive the effective Hamiltonian

$$\mathcal{H} = \sum_a^2 \left\{ \frac{g}{2} \hat{\pi}_a^2 + \sum_{i,j}^d K_a^{ij} \left[\rho_{a0}^2 (\partial_i \partial_j \hat{\theta}_a)^2 + \frac{(\partial_i \partial_j \hat{\pi}_a)^2}{4} \right] \right. \\ \left. + \sum_{b \neq a}^2 \sum_i^d \Gamma_{a,b}^i \left[\frac{(\rho_{a0} \partial_i \hat{\pi}_b - \rho_{b0} \partial_i \hat{\pi}_a)^2}{4\rho_{a0}\rho_{b0}} \right. \right. \\ \left. \left. + \rho_{a0}\rho_{b0} (\partial_i \hat{\theta}_b - \partial_i \hat{\theta}_a)^2 \right] \right\}. \quad (39)$$

The time evolutions of $\hat{\theta}_a$ and $\hat{\pi}_a$ are given by

$$\dot{\hat{\theta}}_a = -i \left[\hat{\theta}_a, \int d^d \mathbf{x} \mathcal{H} \right] = g \hat{\pi}_a + \frac{1}{2} \sum_{i,j}^d K_a^{ij} \partial_i^2 \partial_j^2 \hat{\pi}_a \\ - \sum_{b \neq a}^2 \sum_i^d \Gamma_{a,b}^i \left(\frac{\rho_{b0}}{\rho_{a0}} \partial_i^2 \hat{\pi}_a - \partial_i^2 \hat{\pi}_b \right), \quad (40)$$

$$\dot{\hat{\pi}}_a = -i \left[\hat{\pi}_a, \int d^d \mathbf{x} \mathcal{H} \right] = -2 \sum_{i,j}^d K_a^{ij} \rho_{a0}^2 \partial_i^2 \partial_j^2 \hat{\theta}_a \\ + 2 \sum_{b \neq a}^2 \sum_i^d \Gamma_{a,b}^i \rho_{a0} \rho_{b0} (\partial_i^2 \hat{\theta}_a - \partial_i^2 \hat{\theta}_b). \quad (41)$$

Considering the isotropic case $K_a^{ij} = \frac{\kappa}{2}$, $\Gamma_{a,b}^i = \frac{\Gamma}{2}$, we define the coherence length

$$\xi_c = \pi \sqrt{\frac{2\Gamma}{g}}. \quad (42)$$

In the long-wavelength limit, i.e., $|\mathbf{k}| \ll 2\pi\xi_c^{-1}$, we have $g\hat{\pi}_a \gg -\sum_{b \neq a}^2 \sum_i^d \Gamma_{a,b}^i \partial_i^2 \hat{\pi}_a$, and thus, with finite ρ_{b0}/ρ_{a0} , the last term in Eq. (40) is negligible. In addition, the term $\frac{1}{2} \sum_{i,j}^d K_a^{ij} \partial_i^2 \partial_j^2 \hat{\pi}_a$ with higher-order gradient is also negligible. Thus, we can safely make the approximation $\hat{\pi}_a = \hat{\theta}_a/g$. Plugging this approximation into Eq. (39), and excluding the higher derivative terms of $\hat{\theta}_a$ and $\hat{\theta}_b$, we can obtain the following effective Hamiltonian:

$$\mathcal{H} = \sum_a^2 \left[\frac{g}{2} \hat{\pi}_a^2 + \frac{\kappa \rho_{a0}^2}{2} \sum_{i,j}^d (\partial_i \partial_j \hat{\theta}_a)^2 \right. \\ \left. + \sum_{b \neq a}^2 \sum_i^d \frac{\Gamma \rho_{a0} \rho_{b0}}{2} (\partial_i \hat{\theta}_a - \partial_i \hat{\theta}_b)^2 \right]. \quad (43)$$

In the momentum space, the above Hamiltonian can be further expressed in the following compact form:

$$H = \frac{1}{2(2\pi)^d} \int d^d \mathbf{k} \psi^\dagger(\mathbf{k}) M(\mathbf{k}) \psi(\mathbf{k}), \quad (44)$$

where we set $\psi(\mathbf{k}) = [\hat{\pi}_1(\mathbf{k}) \ \hat{\pi}_2(\mathbf{k}) \ \hat{\theta}_1(\mathbf{k}) \ \hat{\theta}_2(\mathbf{k})]^T$, $\psi^\dagger(\mathbf{k}) = [\hat{\pi}_1(-\mathbf{k}) \ \hat{\pi}_2(-\mathbf{k}) \ \hat{\theta}_1(-\mathbf{k}) \ \hat{\theta}_2(-\mathbf{k})]$, $M(\mathbf{k}) = \text{diag}[M_1(\mathbf{k}), M_2(\mathbf{k})]$. The matrices $M_1(\mathbf{k}), M_2(\mathbf{k})$ are given by $M_1(\mathbf{k}) = \text{diag}(g, g)$ and

$$M_2(\mathbf{k}) = \begin{bmatrix} \kappa \rho_{10}^2 |\mathbf{k}|^4 + 2\Gamma \rho_{10} \rho_{20} |\mathbf{k}|^2 & -2\Gamma \rho_{10} \rho_{20} |\mathbf{k}|^2 \\ -2\Gamma \rho_{10} \rho_{20} |\mathbf{k}|^2 & \kappa \rho_{20}^2 |\mathbf{k}|^4 + 2\Gamma \rho_{10} \rho_{20} |\mathbf{k}|^2 \end{bmatrix}.$$

In the momentum space, $\hat{\theta}_a(\mathbf{k})$ and $\hat{\pi}_b(\mathbf{k}')$ satisfy the commutation relation $[\hat{\theta}_a(\mathbf{k}), \hat{\pi}_b(\mathbf{k}')] = i(2\pi)^d \delta_{ab} \delta^{(d)}(\mathbf{k} + \mathbf{k}')$. The derivation of the Hamiltonian in Eq. (44) is given in Appendix A 1.

Next, we apply the Bogoliubov transformation to send the above Hamiltonian into a diagonal form. For this purpose, we introduce a transformation matrix $T(\mathbf{k})$ which changes the basis from $\psi(\mathbf{k})$ into $\phi(\mathbf{k})$ defined as $\phi(\mathbf{k}) = [\hat{\alpha}(\mathbf{k}) \ \hat{\beta}(\mathbf{k}) \ \hat{\alpha}^\dagger(-\mathbf{k}) \ \hat{\beta}^\dagger(-\mathbf{k})]^T = T(\mathbf{k})\psi(\mathbf{k})$. Here $\hat{\alpha}(\mathbf{k})$ and $\hat{\beta}(\mathbf{k})$ are annihilation operators of two independent quasiparticle modes, satisfying the following commutation relations: $[\hat{\alpha}(\mathbf{k}), \hat{\alpha}^\dagger(\mathbf{k}')] = [\hat{\beta}(\mathbf{k}), \hat{\beta}^\dagger(\mathbf{k}')] = (2\pi)^d \delta^{(d)}(\mathbf{k} - \mathbf{k}')$, $[\hat{\alpha}(\mathbf{k}), \hat{\alpha}(\mathbf{k}')] = [\hat{\beta}(\mathbf{k}), \hat{\beta}(\mathbf{k}')] = [\hat{\alpha}(\mathbf{k}), \hat{\beta}(\mathbf{k}')] = [\hat{\alpha}^\dagger(\mathbf{k}), \hat{\beta}^\dagger(\mathbf{k}')] = 0$. The detailed expression of $T(\mathbf{k})$ is given in Appendix A 2. As a result, the Hamiltonian in Eq. (44) is transformed into

$$H = \frac{1}{2(2\pi)^d} \phi^\dagger(\mathbf{k}) [T^{-1}(\mathbf{k})]^\dagger M(\mathbf{k}) T^{-1}(\mathbf{k}) \phi(\mathbf{k}), \quad (45)$$

where the matrix $[T^{-1}(\mathbf{k})]^\dagger M(\mathbf{k}) T^{-1}(\mathbf{k}) = \text{diag}[D_1(\mathbf{k}), D_2(\mathbf{k}), D_3(\mathbf{k}), D_4(\mathbf{k})]$ is a diagonal matrix

with the diagonal elements

$$D_1(\mathbf{k}) = D_3(\mathbf{k}) = \left\{ \frac{g}{2} [\kappa(\rho_{10}^2 + \rho_{20}^2)|\mathbf{k}|^4 + 4\Gamma\rho_{10}\rho_{20}|\mathbf{k}|^2 - \sqrt{\kappa^2(\rho_{10}^2 - \rho_{20}^2)|\mathbf{k}|^8 + 16\Gamma^2\rho_{10}^2\rho_{20}^2|\mathbf{k}|^4}] \right\}^{\frac{1}{2}}, \quad (46)$$

$$D_2(\mathbf{k}) = D_4(\mathbf{k}) = \left\{ \frac{g}{2} [\kappa(\rho_{10}^2 + \rho_{20}^2)|\mathbf{k}|^4 + 4\Gamma\rho_{10}\rho_{20}|\mathbf{k}|^2 + \sqrt{\kappa^2(\rho_{10}^2 - \rho_{20}^2)|\mathbf{k}|^8 + 16\Gamma^2\rho_{10}^2\rho_{20}^2|\mathbf{k}|^4}] \right\}^{\frac{1}{2}}. \quad (47)$$

Then, we can obtain the following Hamiltonian formed by the creation and annihilation operators of Goldstone modes:

$$H = \frac{1}{(2\pi)^d} \int d^d\mathbf{k} D_1(\mathbf{k}) \hat{\alpha}^\dagger(\mathbf{k}) \hat{\alpha}(\mathbf{k}) + D_2(\mathbf{k}) \hat{\beta}^\dagger(\mathbf{k}) \hat{\beta}(\mathbf{k}), \quad (48)$$

where the normal ordering is applied. The derivation of this Hamiltonian is in Appendix A 3. The dispersion relations are

$$\omega_1 = D_1(\mathbf{k}) \approx \sqrt{\frac{g}{2} \kappa (\rho_{10}^2 + \rho_{20}^2) |\mathbf{k}|^2}, \quad (49)$$

$$\omega_2 = D_2(\mathbf{k}) \approx \sqrt{4g\Gamma\rho_{10}\rho_{20}} |\mathbf{k}|. \quad (50)$$

C. Correlation functions

To check if the ground state spontaneously breaks charge symmetry, we need to calculate correlation functions² in the presence of quantum fluctuations. In the present case, *two-operator* correlation functions can be written as

$$\begin{aligned} \langle \hat{\Phi}_1^\dagger(\mathbf{x}) \hat{\Phi}_1(\mathbf{0}) \rangle &= \rho_{10} e^{-\frac{1}{2} \langle [\hat{\theta}_1(\mathbf{x}) - \hat{\theta}_1(\mathbf{0})]^2 \rangle} \\ &= \rho_{10} e^{\langle \hat{\theta}_1(\mathbf{x}) \hat{\theta}_1(\mathbf{0}) \rangle - \langle \hat{\theta}_1^2(\mathbf{0}) \rangle}, \\ \langle \hat{\Phi}_2^\dagger(\mathbf{x}) \hat{\Phi}_2(\mathbf{0}) \rangle &= \rho_{20} e^{-\frac{1}{2} \langle [\hat{\theta}_2(\mathbf{x}) - \hat{\theta}_2(\mathbf{0})]^2 \rangle} \\ &= \rho_{20} e^{\langle \hat{\theta}_2(\mathbf{x}) \hat{\theta}_2(\mathbf{0}) \rangle - \langle \hat{\theta}_2^2(\mathbf{0}) \rangle}, \\ \langle \hat{\Phi}_1^\dagger(\mathbf{x}) \hat{\Phi}_2(\mathbf{0}) \rangle &= \sqrt{\rho_{10}\rho_{20}} e^{-\frac{1}{2} \langle [\hat{\theta}_1(\mathbf{x}) - \hat{\theta}_2(\mathbf{0})]^2 \rangle} \\ &= \sqrt{\rho_{10}\rho_{20}} e^{\langle \hat{\theta}_1(\mathbf{x}) \hat{\theta}_2(\mathbf{0}) \rangle - \frac{1}{2} \langle \hat{\theta}_1^2(\mathbf{0}) \rangle - \frac{1}{2} \langle \hat{\theta}_2^2(\mathbf{0}) \rangle}. \end{aligned}$$

The *four-operator* correlation functions can be written as

$$\begin{aligned} \langle \hat{\Phi}_1^\dagger(\mathbf{x}) \hat{\Phi}_2(\mathbf{x}) \hat{\Phi}_1(\mathbf{0}) \hat{\Phi}_2^\dagger(\mathbf{0}) \rangle &= \rho_{10}\rho_{20} \exp [\langle \hat{\theta}_1(\mathbf{x}) \hat{\theta}_1(\mathbf{0}) \rangle + \langle \hat{\theta}_2(\mathbf{x}) \hat{\theta}_2(\mathbf{0}) \rangle \\ &\quad + 2\langle \hat{\theta}_1(\mathbf{0}) \hat{\theta}_2(\mathbf{0}) \rangle - \langle \hat{\theta}_1^2(\mathbf{0}) \rangle - \langle \hat{\theta}_2^2(\mathbf{0}) \rangle - 2\langle \hat{\theta}_1(\mathbf{x}) \hat{\theta}_2(\mathbf{0}) \rangle], \\ \langle \hat{\Phi}_1^\dagger(\mathbf{x}) \hat{\Phi}_2^\dagger(\mathbf{x}) \hat{\Phi}_1(\mathbf{0}) \hat{\Phi}_2(\mathbf{0}) \rangle &= \rho_{10}\rho_{20} \exp [\langle \hat{\theta}_1(\mathbf{x}) \hat{\theta}_1(\mathbf{0}) \rangle + \langle \hat{\theta}_2(\mathbf{x}) \hat{\theta}_2(\mathbf{0}) \rangle \\ &\quad - 2\langle \hat{\theta}_1(\mathbf{0}) \hat{\theta}_2(\mathbf{0}) \rangle - \langle \hat{\theta}_1^2(\mathbf{0}) \rangle - \langle \hat{\theta}_2^2(\mathbf{0}) \rangle + 2\langle \hat{\theta}_1(\mathbf{x}) \hat{\theta}_2(\mathbf{0}) \rangle]. \end{aligned}$$

²The correlation functions calculated in this paper are regularized by the coherence length ξ_c .

As usual, the long-distance behaviors of correlation functions reflect certain order parameters of SSB phases. Here, $\langle \hat{\Phi}_1^\dagger(\mathbf{x}) \hat{\Phi}_1(\mathbf{0}) \rangle$, $\langle \hat{\Phi}_1^\dagger(\mathbf{x}) \hat{\Phi}_2(\mathbf{x}) \hat{\Phi}_2^\dagger(\mathbf{0}) \hat{\Phi}_1(\mathbf{0}) \rangle$, and $\langle \hat{\Phi}_1^\dagger(\mathbf{x}) \hat{\Phi}_2^\dagger(\mathbf{x}) \hat{\Phi}_1(\mathbf{0}) \hat{\Phi}_2(\mathbf{0}) \rangle$ correspond to the order parameters $\langle \hat{\Phi}_1(\mathbf{x}) \rangle$, $\langle \hat{\Phi}_1(\mathbf{x}) \hat{\Phi}_2^\dagger(\mathbf{x}) \rangle$, and $\langle \hat{\Phi}_1(\mathbf{x}) \hat{\Phi}_2(\mathbf{x}) \rangle$, respectively. Using the equations above, we can calculate two-operator and four-operator correlation functions in different spatial dimensions. The calculation results of correlation functions are shown in Table III. The calculation details are in Appendix B. In these correlation functions, we set $c_1 = \frac{2\sqrt{g\kappa(\rho_{10}^2 + \rho_{20}^2)}}{g}$, $c_2 = \frac{8\sqrt{g\Gamma\rho_{10}\rho_{20}}}{g}$. The coherence length $\xi_c = \pi\sqrt{\frac{2\Gamma}{g}}$ is given by Eq. (42). During the computation, we require that $|\mathbf{x}|$ is much larger than the coherence length ξ_c but much smaller than the system size L .

For the later convenience, we express the total charge symmetry in two equivalent ways:

$$\mathcal{G}_C = U(1)_{1,C} \otimes U(1)_{2,C} = U(1)_{+,C} \otimes U(1)_{-,C}, \quad (51)$$

where $U(1)_{+,C}$ corresponds to the conserved total charges $\hat{Q}_+ = \int d^d\mathbf{x} (\hat{\rho}_1 + \hat{\rho}_2)$ of the whole system of two species of bosons, and the *relative charge symmetry* $U(1)_{-,C}$ corresponds to the conserved relative charges (i.e., charge difference) $\hat{Q}_- = \int d^d\mathbf{x} (\hat{\rho}_1 - \hat{\rho}_2)$.

We can see that in one spatial dimension, the correlation functions $\langle \hat{\Phi}_1^\dagger(\mathbf{x}) \hat{\Phi}_1(\mathbf{0}) \rangle$ and $\langle \hat{\Phi}_1^\dagger(\mathbf{x}) \hat{\Phi}_2^\dagger(\mathbf{x}) \hat{\Phi}_1(\mathbf{0}) \hat{\Phi}_2(\mathbf{0}) \rangle$ decay exponentially. The correlation function $\langle \hat{\Phi}_1^\dagger(\mathbf{x}) \hat{\Phi}_2(\mathbf{x}) \hat{\Phi}_2^\dagger(\mathbf{0}) \hat{\Phi}_1(\mathbf{0}) \rangle$ decays in a power law. This shows that there is no spontaneous breaking of charge symmetry in one spatial dimension.

In two spatial dimensions, the correlation functions $\langle \hat{\Phi}_1^\dagger(\mathbf{x}) \hat{\Phi}_1(\mathbf{0}) \rangle$ and $\langle \hat{\Phi}_1^\dagger(\mathbf{x}) \hat{\Phi}_2^\dagger(\mathbf{x}) \hat{\Phi}_1(\mathbf{0}) \hat{\Phi}_2(\mathbf{0}) \rangle$ decay in a power law. The correlation function $\langle \hat{\Phi}_1^\dagger(\mathbf{x}) \hat{\Phi}_2(\mathbf{x}) \hat{\Phi}_2^\dagger(\mathbf{0}) \hat{\Phi}_1(\mathbf{0}) \rangle$ saturates to a constant at long distances. According to the commutation relations

$$\begin{aligned} [\hat{\Phi}_1(\mathbf{x}) \hat{\Phi}_2^\dagger(\mathbf{x}), \hat{Q}_+] &= 0, \\ [\hat{\Phi}_1(\mathbf{x}) \hat{\Phi}_2^\dagger(\mathbf{x}), \hat{Q}_-] &= 2\hat{\Phi}_1(\mathbf{x}) \hat{\Phi}_2^\dagger(\mathbf{x}), \end{aligned} \quad (52)$$

the system breaks the relative charge symmetry $U(1)_{-,C}$, and has a true ODLRO. In three spatial dimensions and above, all the correlation functions mentioned above are constants at long distances. This means that the charge symmetry is completely broken in three spatial dimensions.

In summary, the charge symmetry is not broken in one spatial dimension, and is completely broken in three spatial dimensions and above. In two spatial dimensions, a partial breaking of the charge symmetry occurs.

D. Arbitrary $[d, N, m, m']$ models of Model Series A

In this subsection, we generalize the above analysis to $[d, N, m, m']$ models of Model Series A with the conserved quantities [Eqs. (7)–(10)]. The corresponding Hamiltonian density is Eq. (14). In this case, we focus on an isotropic coupling constant $K_a^{i_1 \dots i_{N+1}} = \frac{\kappa}{2}$, $\Gamma_{a,b}^{i_1 \dots i_N} = \frac{\Gamma}{2}$. For simplicity, the chemical potential $\mu_a = \mu > 0$ is assumed, resulting in classical degenerate ground states with finite uniform density distribution $\rho_a = \rho_0$. We can also derive an effective theory for the quantum fluctuation field $\hat{\theta}_a$ after

TABLE III. Correlation functions of the $[d, 1, 2, 2]$ model at zero temperature. This table illustrates the broken charge symmetry of the $[d, 1, 2, 2]$ model and the corresponding order parameters for $d = 1, 2, 3$. The detailed analysis is provided in Sec. III C. In Sec. III D, we generalize these results to arbitrary $[d, N, m, m']$ models. The parameters $c_1, c_2, \rho_{10}, \rho_{20}$, and ξ_c are positive real constants, with ξ_c specifically denoting the coherence length.

d	$\langle \hat{\Phi}_1^\dagger(\mathbf{x}) \hat{\Phi}_1(\mathbf{0}) \rangle$	$\langle \hat{\Phi}_1^\dagger(\mathbf{x}) \hat{\Phi}_2(\mathbf{0}) \rangle$	$\langle \hat{\Phi}_1^\dagger(\mathbf{x}) \hat{\Phi}_2(\mathbf{x}) \hat{\Phi}_2^\dagger(\mathbf{0}) \hat{\Phi}_1(\mathbf{0}) \rangle$	$\langle \hat{\Phi}_1^\dagger(\mathbf{x}) \hat{\Phi}_2^\dagger(\mathbf{x}) \hat{\Phi}_1(\mathbf{0}) \hat{\Phi}_2(\mathbf{0}) \rangle$	Broken charge symmetry	Order parameter
1	$\rho_{10} e^{-\frac{ \mathbf{x} }{2c_1} \left(\frac{ \mathbf{x} }{\xi_c}\right)^{-\frac{1}{\pi c_2}}}$	0	$\rho_{10} \rho_{20} \left(\frac{ \mathbf{x} }{\xi_c}\right)^{-\frac{4}{\pi c_2}}$	$\rho_{10} \rho_{20} e^{-\frac{2}{c_1} \mathbf{x} }$	N/A	N/A
2	$\rho_{10} \left(\frac{ \mathbf{x} }{\xi_c}\right)^{-\frac{2}{\pi c_1}} e^{-\frac{1}{2\pi c_2 \xi_c}}$	$\sqrt{\rho_{10} \rho_{20}} \left(\frac{ \mathbf{x} }{\xi_c}\right)^{-\frac{2}{\pi c_1}} e^{-\frac{1}{2\pi c_2 \xi_c}}$	$\rho_{10} \rho_{20} e^{-\frac{2}{\pi c_2 \xi_c}}$	$\rho_{10} \rho_{20} \left(\frac{ \mathbf{x} }{\xi_c}\right)^{-\frac{2}{\pi c_1}}$	$U(1)_{-,C}$	$\langle \hat{\Phi}_1(\mathbf{x}) \hat{\Phi}_2^\dagger(\mathbf{x}) \rangle$
3	$\rho_{10} e^{-\frac{1}{4\pi c_1 \xi_c}} e^{-\frac{1}{c_2 \xi_c^2}}$	$\sqrt{\rho_{10} \rho_{20}} e^{-\frac{1}{4\pi c_1 \xi_c}} e^{-\frac{1}{c_2 \xi_c^2}}$	$\rho_{10} \rho_{20} e^{-\frac{4}{c_2 \xi_c^2}}$	$\rho_{10} \rho_{20} e^{-\frac{1}{\pi c_1 \xi_c}}$	$U(1)_{1,C} \otimes U(1)_{2,C}$	$\langle \hat{\Phi}_1(\mathbf{x}) \rangle, \langle \hat{\Phi}_2(\mathbf{x}) \rangle$

condensation,

$$\mathcal{H} = \sum_a^m \left[\frac{g}{2} \hat{\pi}_a^2 + \frac{\kappa \rho_0^{N+1}}{2} (\nabla^{N+1} \hat{\theta}_a)^2 \right] + \frac{\Gamma \rho_0^{2N}}{2} \sum_{a \neq b} \sum_{i_1, \dots, i_N}^m (\partial_{i_1} \dots \partial_{i_N} \hat{\theta}_a - \partial_{i_1} \dots \partial_{i_N} \hat{\theta}_b)^2. \quad (53)$$

Through the same processes, we can obtain the Hamiltonian in the momentum space

$$H = \frac{1}{2(2\pi)^d} \int d^d \mathbf{k} \psi^\dagger(\mathbf{k}) M(\mathbf{k}) \psi(\mathbf{k}), \quad (54)$$

where

$$\psi(\mathbf{k}) = [\hat{\pi}_1(\mathbf{k}) \quad \dots \quad \hat{\pi}_m(\mathbf{k}) \quad \hat{\theta}_1(\mathbf{k}) \quad \dots \quad \hat{\theta}_m(\mathbf{k})]^T, \quad (55)$$

$$M(\mathbf{k}) = \text{diag}(M_1(\mathbf{k}), M_2(\mathbf{k}), M_3(\mathbf{k})). \quad (56)$$

The matrix $M_1(\mathbf{k})$ is a $m \times m$ diagonal matrix whose diagonal elements are g . The diagonal elements of the $m' \times m'$ matrix $M_2(\mathbf{k})$ are $[\kappa \rho_0^{N+1} |\mathbf{k}|^{2N+2} + 2(m' - 1) \Gamma \rho_0^{2N} |\mathbf{k}|^{2N}]$, and all the off-diagonal elements are $[-2 \Gamma \rho_0^{2N} |\mathbf{k}|^{2N}]$. $M_3(\mathbf{k})$ is a diagonal $(m - m') \times (m - m')$ matrix whose diagonal elements are $[\kappa \rho_0^{N+1} |\mathbf{k}|^{2N+2}]$. By diagonalizing this Hamiltonian, we end up with the dispersion relations

$$\omega_a = \sqrt{\kappa g \rho_0^{N+1} |\mathbf{k}|^{N+1}} \quad (1 \leq a \leq m - m' + 1), \quad (57)$$

$$\omega_a = \sqrt{\kappa g \rho_0^{N+1} |\mathbf{k}|^{2N+2} + 2m' g \Gamma \rho_0^{2N} |\mathbf{k}|^{2N}} \approx \sqrt{2m' g \Gamma \rho_0^{2N} |\mathbf{k}|^N} \quad (m - m' + 2 \leq a \leq m). \quad (58)$$

The two-operator correlation function can be written as

$$\begin{aligned} \langle \hat{\Phi}_a^\dagger(\mathbf{x}) \hat{\Phi}_a(\mathbf{0}) \rangle &= \rho_0 e^{-\frac{1}{2} \langle (\hat{\theta}_a(\mathbf{x}) - \hat{\theta}_a(\mathbf{0}))^2 \rangle} \\ &= \rho_0 e^{\langle \hat{\theta}_a(\mathbf{x}) \hat{\theta}_a(\mathbf{0}) \rangle - \langle \hat{\theta}_a^2(\mathbf{0}) \rangle}. \end{aligned} \quad (59)$$

The four-operator correlation functions can be written as

$$\begin{aligned} \langle \hat{\Phi}_a^\dagger(\mathbf{x}) \hat{\Phi}_b(\mathbf{x}) \hat{\Phi}_a(\mathbf{0}) \hat{\Phi}_b^\dagger(\mathbf{0}) \rangle &= \rho_0^2 \exp [\langle \hat{\theta}_a(\mathbf{x}) \hat{\theta}_a(\mathbf{0}) \rangle + \langle \hat{\theta}_b(\mathbf{x}) \hat{\theta}_b(\mathbf{0}) \rangle + 2 \langle \hat{\theta}_a(\mathbf{0}) \hat{\theta}_b(\mathbf{0}) \rangle \\ &\quad - \langle \hat{\theta}_a^2(\mathbf{0}) \rangle - \langle \hat{\theta}_b^2(\mathbf{0}) \rangle - 2 \langle \hat{\theta}_a(\mathbf{x}) \hat{\theta}_b(\mathbf{0}) \rangle], \end{aligned} \quad (60)$$

$$\begin{aligned} \langle \hat{\Phi}_a^\dagger(\mathbf{x}) \hat{\Phi}_b^\dagger(\mathbf{x}) \hat{\Phi}_a(\mathbf{0}) \hat{\Phi}_b(\mathbf{0}) \rangle &= \rho_0^2 \exp [\langle \hat{\theta}_a(\mathbf{x}) \hat{\theta}_a(\mathbf{0}) \rangle + \langle \hat{\theta}_b(\mathbf{x}) \hat{\theta}_b(\mathbf{0}) \rangle - 2 \langle \hat{\theta}_a(\mathbf{0}) \hat{\theta}_b(\mathbf{0}) \rangle \\ &\quad - \langle \hat{\theta}_a^2(\mathbf{0}) \rangle - \langle \hat{\theta}_b^2(\mathbf{0}) \rangle + 2 \langle \hat{\theta}_a(\mathbf{x}) \hat{\theta}_b(\mathbf{0}) \rangle]. \end{aligned} \quad (61)$$

If the species of bosons satisfy the conditions $1 \leq a, b \leq m'$ and $a \neq b$, the exponent of e in Eq. (60) is an integral of a function which is proportional to $(1/|\mathbf{k}|^N)$. The correlation function $\langle \hat{\Phi}_a^\dagger(\mathbf{x}) \hat{\Phi}_b(\mathbf{x}) \hat{\Phi}_a(\mathbf{0}) \hat{\Phi}_b^\dagger(\mathbf{0}) \rangle$ saturates to a constant in spatial dimensions $d \geq N + 1$. For the correlation functions $\langle \hat{\Phi}_a^\dagger(\mathbf{x}) \hat{\Phi}_a(\mathbf{0}) \rangle$ and $\langle \hat{\Phi}_a^\dagger(\mathbf{x}) \hat{\Phi}_b^\dagger(\mathbf{x}) \hat{\Phi}_a(\mathbf{0}) \hat{\Phi}_b(\mathbf{0}) \rangle$, in the exponent of e , the integrand is the sum of a function proportional to $(1/|\mathbf{k}|^N)$ and another function proportional to $(1/|\mathbf{k}|^{N+1})$. The two correlation functions saturate to a constant only in dimensions $d \geq N + 2$. This implies a partial spontaneous breaking of charge symmetry in spatial dimensions $d = N + 1$. The system breaks the relative charge symmetry of the bosons involved in the hybridization.

IV. HYBRID FRACTONIC SUPERFLUIDS: MODEL SERIES B

In this section, we focus on the details in Model Series B. Based on the Hamiltonian [Eq. (35)] of a hybrid dipole-quadrupole conserving model, we present the ground state along with its Goldstone modes and calculate correlation functions to study the minimal spatial dimension in which this model breaks charge symmetry and has a true ODLRO.

A. Classical ground-state wave functions

Classically, the energy density $\mathcal{E}$ of the Hamiltonian [Eq. (35)] has the form as

$$\begin{aligned} \mathcal{E} &= \sum_{i,j}^d K_1^{ij} |\phi_1 \partial_i \partial_j \phi_1 - \partial_i \phi_1 \partial_j \phi_2|^2 \\ &\quad + \sum_{i,j,k}^d K_2^{ijk} |\phi_2^2 \partial_i \partial_j \partial_k \phi_2 - \phi_2 \partial_i \partial_j \partial_k \phi_2 \\ &\quad - \phi_2 \partial_j \phi_2 \partial_i \partial_k \phi_2 - \phi_2 \partial_k \phi_2 \partial_i \partial_j \phi_2 + 2 \partial_i \phi_2 \partial_j \phi_2 \partial_k \phi_2|^2 \\ &\quad + \sum_i^d \Gamma_{1,2}^i |\ell \phi_1 \phi_2 \partial_i^2 \phi_2 - \ell \phi_1 \partial_i \phi_2 \partial_i \phi_2 - 2 \partial_i \phi_1 \phi_2^2|^2 \\ &\quad - \mu_1 |\phi_1|^2 - \mu_2 |\phi_2|^2 + \frac{g}{2} |\phi_1|^4 + \frac{g}{2} |\phi_2|^4. \end{aligned} \quad (62)$$

If $\mu_1, \mu_2 > 0$, at $|\phi_1| = \sqrt{\rho_{10}} = \sqrt{\frac{\mu_1}{g}}$, $|\phi_2| = \sqrt{\rho_{20}} = \sqrt{\frac{\mu_2}{g}}$, the potential reaches the minimal value. Then, the

energy density reduces to $\mathcal{E} = \sum_{i,j}^d K_1^{ij} \rho_{10}^2 (\partial_i \partial_j \theta_1)^2 + \sum_{i,j,k}^d K_2^{ijk} \rho_{20}^3 (\partial_i \partial_j \partial_k \theta_2)^2 + \sum_i^d \Gamma_{1,2}^i \rho_{10} \rho_{20}^2 (2\partial_i \theta_1 - \ell \partial_i^2 \theta_2)^2 - \frac{\mu_1^2}{2g} - \frac{\mu_2^2}{2g}$. To minimize the kinetic terms, we need to enforce three conditions: $\partial_i \partial_j \theta_1 = 0$, $\partial_i \partial_j \partial_k \theta_2 = 0$, and $2\partial_i \theta_1 - \ell \partial_i^2 \theta_2 = 0$ for any i, j , and k . The first two conditions are satisfied by $\theta_1 = \theta_{10} + \sum_i^d \beta_i^{(1)} x_i$ and $\theta_2 = \theta_{20} + \sum_{i,j}^d \beta_{ij}^{(2)} x_i x_j$. And the last condition gives another limit of $\beta_i^{(1)}$ and $\beta_{ij}^{(2)}$: $2\beta_i^{(1)} - \ell \beta_{ii}^{(2)} = 0$. Thus, we can give the ground state

$$|\text{GS}_{\beta^{(1)}, \beta^{(2)}}\rangle = \bigotimes_{\mathbf{x}} \frac{1}{C} \exp \left[\sqrt{\rho_{10}} e^{i(\theta_{10} + \sum_i^d \beta_i^{(1)} x_i)} \hat{\Phi}_1^\dagger(\mathbf{x}) \right] \times \exp \left[\sqrt{\rho_{20}} e^{i(\theta_{20} + \sum_{i,j}^d \beta_{ij}^{(2)} x_i x_j)} \hat{\Phi}_2^\dagger(\mathbf{x}) \right] |0\rangle. \quad (63)$$

This ground state also has the formation of ODLRO. We can calculate correlation functions in the classical level,

$$\langle \hat{\Phi}_1^\dagger(\mathbf{x}) \hat{\Phi}_1(\mathbf{0}) \rangle = \rho_{10} e^{-i \sum_i^d \beta_i^{(1)} x_i}, \quad (64)$$

$$\langle \hat{\Phi}_2^\dagger(\mathbf{x}) \hat{\Phi}_2(\mathbf{0}) \rangle = \rho_{20} e^{-i \sum_{i,j}^d \beta_{ij}^{(2)} x_i x_j}. \quad (65)$$

The order parameter can be determined with finite expectation value on the ground state

$$\langle \hat{\Phi}_1(\mathbf{x}) \rangle = \sqrt{\rho_{10}} e^{i(\theta_{10} + \sum_i^d \beta_i^{(1)} x_i)}, \quad (66)$$

$$\langle \hat{\Phi}_2(\mathbf{x}) \rangle = \sqrt{\rho_{20}} e^{i(\theta_{20} + \sum_{i,j}^d \beta_{ij}^{(2)} x_i x_j)}. \quad (67)$$

B. Goldstone modes and quantum fluctuations

Considering quantum fluctuations, we have $\hat{\Phi}_a(\mathbf{x}) = \sqrt{\rho_{a0} + \hat{f}_a(\mathbf{x})} e^{i\hat{\theta}_a(\mathbf{x})}$, where $\hat{f}_a(\mathbf{x}) \ll \rho_{a0}$. Through the same processes in Model Series A, we can obtain the effective Hamiltonian under the long-wave approximation

$$\begin{aligned} \mathcal{H} = & \frac{g}{2} \hat{\pi}_1^2 + \frac{g}{2} \hat{\pi}_2^2 + \frac{\kappa_1 \rho_{10}^2}{2} \sum_{i,j}^d \hat{\theta}_1 \partial_i^2 \partial_j^2 \hat{\theta}_1 \\ & - \frac{\kappa_2 \rho_{20}^3}{2} \sum_{i,j,k}^d \hat{\theta}_2 \partial_i^2 \partial_j^2 \partial_k^2 \hat{\theta}_2 + \frac{\Gamma \rho_{10} \rho_{20}^2}{2} \sum_i^d (\ell^2 \hat{\theta}_2 \partial_i^4 \hat{\theta}_2 \\ & - 4\hat{\theta}_1 \partial_i^2 \hat{\theta}_1 + 2\ell \hat{\theta}_1 \partial_i^3 \hat{\theta}_2 - 2\ell \hat{\theta}_2 \partial_i^3 \hat{\theta}_1), \end{aligned} \quad (68)$$

where we set $K_1^{ij} = \frac{\kappa_1}{2}$, $K_2^{ijk} = \frac{\kappa_2}{2}$, and $\Gamma_{1,2}^i = \frac{\Gamma}{2}$, $\forall i, j, k$, that is, all the elements of the coefficient tensors K_1^{ij} , K_2^{ijk} , and $\Gamma_{1,2}^i$ are $\frac{\kappa_1}{2}$, $\frac{\kappa_2}{2}$, and $\frac{\Gamma}{2}$, respectively. $\hat{\pi}_a = -\hat{f}_a$ are the conjugate momentum densities of $\hat{\theta}_a$. Continuing the derivation process, we can finally obtain the regularized quantized and diagonalized Hamiltonian,

$$H = \frac{1}{(2\pi)^d} \int d^d \mathbf{k} D_1(\mathbf{k}) \hat{\alpha}^\dagger(\mathbf{k}) \hat{\alpha}(\mathbf{k}) + D_2(\mathbf{k}) \hat{\beta}^\dagger(\mathbf{k}) \hat{\beta}(\mathbf{k}), \quad (69)$$

where the normal ordering is applied. The operators in Model Series B above still satisfy the same commutation relations as in Model Series A. The dispersion relations are

$$\omega_1 = D_1(\mathbf{k}), \quad \omega_2 = D_2(\mathbf{k}). \quad (70)$$

The dispersion relations of Model Series B are complex and anisotropic and are given in Appendix (C17). We can only approximate the dispersion relations for given spatial dimensions. One of the dispersion relations is linear. To leading order of $|\mathbf{k}|$, the dispersion relation is isotropic:

$$\omega_1 \approx \sqrt{4g\Gamma \rho_{10} \rho_{20}^2} |\mathbf{k}|. \quad (71)$$

In one spatial dimension, the other one of the dispersion relations is given by

$$\omega_2 \approx \sqrt{\kappa_2 g \rho_{20}^3 + \frac{\kappa_1 g \rho_{10}^2 \ell^2}{4}} |\mathbf{k}|^3. \quad (72)$$

In two or more spatial dimensions, ω_2 is anisotropic and possesses a dispersion of second order. Since the derivation process is similar to that of Model Series A, we put the detailed derivation of Model Series B in Appendix C. The derivation of the dispersion relations in different spatial dimensions is given in Appendix D.

C. Correlation functions

To check the effect of hybridization in Model Series B on ODLRO, we calculate correlation functions. Unlike Model Series A, we only need two-operator correlation functions to characterize the spontaneous breaking of charge symmetry in Model Series B instead of four-operator correlation functions. The two-operator correlation functions can be written as

$$\begin{aligned} \langle \hat{\Phi}_1^\dagger(\mathbf{x}) \hat{\Phi}_1(\mathbf{0}) \rangle &= \rho_{10} e^{-\frac{1}{2} \langle (\hat{\theta}_1(\mathbf{x}) - \hat{\theta}_1(\mathbf{0}))^2 \rangle} \\ &= \rho_{10} e^{\langle \hat{\theta}_1(\mathbf{x}) \hat{\theta}_1(\mathbf{0}) \rangle - \langle \hat{\theta}_1^2(\mathbf{0}) \rangle}, \end{aligned} \quad (73)$$

$$\begin{aligned} \langle \hat{\Phi}_2^\dagger(\mathbf{x}) \hat{\Phi}_2(\mathbf{0}) \rangle &= \rho_{20} e^{-\frac{1}{2} \langle (\hat{\theta}_2(\mathbf{x}) - \hat{\theta}_2(\mathbf{0}))^2 \rangle} \\ &= \rho_{20} e^{\langle \hat{\theta}_2(\mathbf{x}) \hat{\theta}_2(\mathbf{0}) \rangle - \langle \hat{\theta}_2^2(\mathbf{0}) \rangle}. \end{aligned} \quad (74)$$

Then, we can calculate the correlation function of the two species of bosons. Due to the anisotropy, we can just give a general trend of the correlation functions in two or more spatial dimensions. The calculation results of correlation functions are shown in Table IV. The details of the analytical calculations and numerical calculations are in Appendix D. In these correlation functions, we set $c_{11} = \frac{4\Gamma \rho_{10} \rho_{20}}{\sqrt{g\Gamma \rho_{10}}}$, $c_{21} = \frac{4\sqrt{\kappa_1 \rho_{10}^2 \ell^2 + 4\kappa_2 \rho_{20}^3}}{\sqrt{g\ell^2}}$, and the coherence length $\xi_c = 2\pi \sqrt{\frac{\Gamma \rho_{20}}{g\rho_{10}}}$.

Through the calculations, we confirm that the correlation function $\langle \hat{\Phi}_1^\dagger(\mathbf{x}) \hat{\Phi}_1(\mathbf{0}) \rangle$ decays in a power law in one spatial dimension and saturates a constant in two or more spatial dimensions. The correlation function $\langle \hat{\Phi}_2^\dagger(\mathbf{x}) \hat{\Phi}_2(\mathbf{0}) \rangle$ decays exponentially in one spatial dimension, decays in a power law in two spatial dimensions, and saturates a constant in three or more spatial dimensions. The constant behaviors of these two correlation functions represent the spontaneous breaking of $U(1)_{1,C}$ symmetry and $U(1)_{2,C}$ symmetry, respectively. Therefore, this system has a true ODLRO in two spatial dimensions and above. Compared with dipole-conserving and quadrupole-conserving fractonic superfluid models with no

TABLE IV. Correlation functions of the hybrid dipole-quadrupole moment conserving model [Eq. (35)] at zero temperature with quantum fluctuations. This table illustrates the broken charge symmetry of this model and the corresponding order parameters at $d = 1, 2, 3$. The analysis is detailed in Sec. IV C. The parameters c_{11} , c_{21} , ρ_{10} , and ξ_c are positive real constants, with ξ_c specifically denoting the coherence length.

d	$\langle \hat{\Phi}_1^\dagger(\mathbf{x}) \hat{\Phi}_1(\mathbf{0}) \rangle$	$\langle \hat{\Phi}_2^\dagger(\mathbf{x}) \hat{\Phi}_2(\mathbf{0}) \rangle$	Broken charge symmetry	Order parameter
1	$\rho_{10} \left(\frac{ \mathbf{x} }{\xi_c} \right)^{-\frac{c_{11}+c_{21}}{\pi c_{11}c_{21}}}$	0	N/A	N/A
2	$\rho_{10} e^{-\frac{c_{11}+c_{21}}{2\pi c_{11}c_{21}\xi_c}}$	Power-law decay	$U(1)_{1,C}$	$\langle \hat{\Phi}_1(\mathbf{x}) \rangle$
3	Constant	Constant	$U(1)_{1,C} \otimes U(1)_{2,C}$	$\langle \hat{\Phi}_1(\mathbf{x}) \rangle, \langle \hat{\Phi}_2(\mathbf{x}) \rangle$

hybridization, the spontaneous breaking of the charge symmetry $U(1)_{1,C}$ and $U(1)_{2,C}$ occurs in lower spatial dimensions. This decrease in dimension comes from the hybridization of conserved quantities as well as the coupling of the two species of bosons.

V. BOSE-HUBBARD LATTICE MODELS

Just as conventional superfluids can be realized in the small- U phase of conventional Bose-Hubbard lattice models, a similar Bose-Hubbard-type lattice model can be constructed, whose low-energy physics in the small- U limit reduces to the low-energy effective field theory of fractonic superfluids [43,74]. In the preceding sections, we have studied the HFS phases of matter in continuum space.

In this section, we will construct lattice models that (i) satisfy the aforementioned conservation laws and (ii) reduce to continuum models in Model Series A and B in the small- U limit. Therefore, unless otherwise specified, we will use *Lattice Model Series A* (*Lattice Model Series B*) to denote the Bose-Hubbard lattice models that, in the small- U limit, reduce to continuum models in Model Series A (Model Series B). Upon completing lattice construction, in this section we will also present a third-order perturbation theory to demonstrate how to realize lattice models in strongly tilted optical lattices.

A. Construction of Lattice Model Series

We specifically study the lattice model corresponding to the $[d, 1, 2, 2]$ model shown in Eq. (20). We put all fields on a square lattice and use difference operators to replace all spatial derivatives:

$$\partial_i \hat{\Phi}_a(\mathbf{x}) \longrightarrow \frac{1}{\eta} (\hat{b}_{a,\mathbf{i}+\hat{x}_i} - \hat{b}_{a,\mathbf{i}}), \quad (75)$$

$$\partial_i \partial_j \hat{\Phi}_a(\mathbf{x}) \longrightarrow \frac{1}{\eta^2} (\hat{b}_{a,\mathbf{i}+\hat{x}_i+\hat{x}_j} - \hat{b}_{a,\mathbf{i}+\hat{x}_i} - \hat{b}_{a,\mathbf{i}+\hat{x}_j} + \hat{b}_{a,\mathbf{i}}), \quad (76)$$

where η is the lattice constant. $\hat{x}_i$ and $\hat{x}_j$ ($i, j = 1, 2, \dots, d$) denote spatial unit vectors. $\mathbf{i} = \eta^{-1}(x_1, \dots, x_d)$ represent lattice coordinates. By applying the above substitution rules, we can straightforwardly obtain the following lattice model that conserves quantities in Eqs. (15) and (16):

$$H_{\text{lat A}} = H_{\text{hop}} + H_{\text{onsite}}, \quad (77)$$

where

$$H_{\text{hop}} = \sum_a^2 \left[\sum_{\mathbf{i}, \mathbf{j}} -t_a^{(ij)} \hat{b}_{a,\mathbf{i}+\hat{x}_i+\hat{x}_j}^\dagger \hat{b}_{a,\mathbf{i}}^\dagger \hat{b}_{a,\mathbf{i}+\hat{x}_i} \hat{b}_{a,\mathbf{i}+\hat{x}_j} + \sum_{b \neq a}^2 \sum_{\mathbf{i}, \mathbf{i}} -t_{ab}^{(ij)} \hat{b}_{b,\mathbf{i}+\hat{x}_i}^\dagger \hat{b}_{a,\mathbf{i}+\hat{x}_i} \hat{b}_{a,\mathbf{i}}^\dagger \hat{b}_{b,\mathbf{i}} \right] + \text{H.c.}, \quad (78)$$

$$H_{\text{onsite}} = \sum_a^2 \left[-\mu \sum_{\mathbf{i}} \hat{n}_{a,\mathbf{i}} + \frac{U}{2} \sum_{\mathbf{i}} \hat{n}_{a,\mathbf{i}} (\hat{n}_{a,\mathbf{i}} - 1) \right]. \quad (79)$$

The $t_a^{(ij)}$ terms and the $t_{ab}^{(ij)}$ terms ($t_a^{(ij)} = t_a^{(ji)} > 0$, $t_{ab}^{(i)} = t_{ba}^{(i)} > 0$) correspond to the $K_{a,b}^{ij}$ terms and the $\Gamma_{a,b}^i$ terms in the $[d, 1, 2, 2]$ model shown in Eq. (20), respectively. These hopping processes are illustrated in Figs. 1(a)–1(c) in the case of two-dimensional square lattice.

Next, we discuss the construction of the lattice model corresponding to the continuum model described by Eq. (35). Evidently, Model Series B is significantly more complex than Model Series A, and this complexity extends to the construction of its lattice model. For simplicity, we neglect the K_2^{ijk} terms. We further set $\ell = 2\eta$ for the sake of convenience and

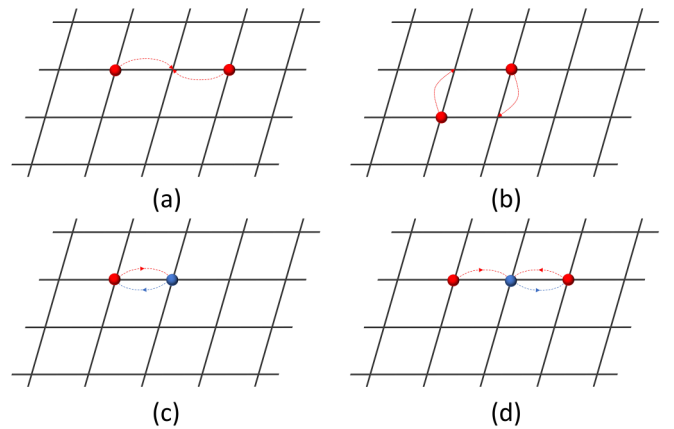


FIG. 1. Illustration of the hopping terms [Eqs. (78) and (81)] on a square lattice. The balls of red and blue colors represent bosons of two distinct species. The hopping shown in (a) corresponds to diagonal $t_a^{(ij)}$ terms in Eq. (78). The hopping shown in (b) corresponds to off-diagonal $t_a^{(ij)}$ terms in Eq. (78). The hopping shown in (c) corresponds to $t_{ab}^{(i)}$ terms in Eq. (78). In (d), the blue ball represents one boson of species 1, and the red balls represent bosons of species 2. The hopping shown in (d) corresponds to the $t_{12}^{(i)}$ terms in Eq. (81).

finally end up with the following lattice model that conserves charges of each species, the hybrid moments, and other lower moments [Eqs. (27)–(30)]:

$$H_{\text{lat B}} = H_{\text{hop}} + H_{\text{onsite}}, \quad (80)$$

where

$$H_{\text{hop}} = \left[\sum_{\mathbf{i}, \mathbf{j}} -t_1^{(ij)} \hat{b}_{1, \mathbf{i}+\hat{x}_i+\hat{x}_j}^\dagger \hat{b}_{1, \mathbf{i}}^\dagger \hat{b}_{1, \mathbf{i}+\hat{x}_i} \hat{b}_{1, \mathbf{i}+\hat{x}_j} \right. \\ \left. + \sum_{\mathbf{i}, \mathbf{j}} -t_{12}^{(ij)} \hat{b}_{1, \mathbf{i}+\hat{x}_i}^\dagger \hat{b}_{1, \mathbf{i}} \hat{b}_{2, \mathbf{i}+\hat{x}_i}^\dagger \hat{b}_{2, \mathbf{i}} \right] + \text{H.c.}, \quad (81)$$

$$H_{\text{onsite}} = \sum_a \left[-\mu \sum_{\mathbf{i}} \hat{n}_{a, \mathbf{i}} + \frac{U}{2} \sum_{\mathbf{i}} \hat{n}_{a, \mathbf{i}} (\hat{n}_{a, \mathbf{i}} - 1) \right]. \quad (82)$$

The $t_1^{(ij)}$ terms and the $t_{12}^{(ij)}$ terms correspond to the K_1^{ij} terms and the $\Gamma_{1,2}^i$ terms in Eq. (35), respectively. The hopping processes represented by the $t_{12}^{(ij)}$ terms are illustrated in Fig. 1(d) in the case of two-dimensional square lattice.

B. Mean-field phase diagram of Lattice Model Series A

We employ the mean-field approximation to sketch out the schematic phase diagram of Lattice Model Series A [Eq. (77)], as shown in Figs. 2 and 3. We set $t_a^{(ij)} = t_1$, $t_{ab}^{(i)} = t_2$, which are the coefficients of the hopping terms of the same species and different species, respectively.

At the point $t_2 = 0$, the Hamiltonian [Eq. (77)] reduces to two independent dipolar Bose-Hubbard models [74], and has a similar phase diagram [Fig. 2(a)]. At the limit $t_1 = 0$, all particles lose their mobility, and the system is at the Mott insulator (MI) phase with fixed average boson densities of species 1 and 2. At the small- U limit (i.e., large t_1/U , μ/U), the system is at the fractonic superfluid (FS) phase which is studied in Ref. [43]. Between the MI phase and the FS phase, the light green region in Fig. 2(a) shows the dipole condensate (DC) phase with nonzero $\langle \hat{b}_{a, \mathbf{i}+\hat{x}_i}^\dagger \hat{b}_{a, \mathbf{i}} \rangle$. At the mean-field level, the phase transition between MI and DC occurs at the point $\frac{1}{2dt_1} - \frac{2n(n+1)}{U} = 0$ [74].

When we take $t_2 > 0$, the hybridization of bosons of different species drives the emergence of another order parameter $\langle \hat{b}_{1, \mathbf{i}}^\dagger \hat{b}_{2, \mathbf{i}} \rangle$. Nonzero $\langle \hat{b}_{1, \mathbf{i}}^\dagger \hat{b}_{2, \mathbf{i}} \rangle$ implies the existence of exotic intermediate phases with condensates of dipolar bound states $\hat{b}_{1, \mathbf{i}}^\dagger \hat{b}_{2, \mathbf{i}}$. To distinguish it from the DC phase with condensates of dipoles formed by bosons of the same species, the phases are referred to as the *multidipole condensates* (MDC). The physics in the small- U limit of this lattice model is described by the HFS [Eq. (20)] in this paper. A summary of comparison between Bose-Hubbard-type models with different conservation laws is given in Table II in the Introduction.

By applying the dipolar and single-particle mean-field theory [74], we can determine the locations of the phase boundaries between different phases and complete the schematic phase diagram (Figs. 2 and 3). Detailed steps of the mean-field theory to determine the phase boundary between the MI phase and the MDC phase are given in Appendix E. To derive the phase diagram beyond mean-field theory, it will be meaningful to employ more advanced treatments, e.g., large-scale Monte Carlo and tensor network.

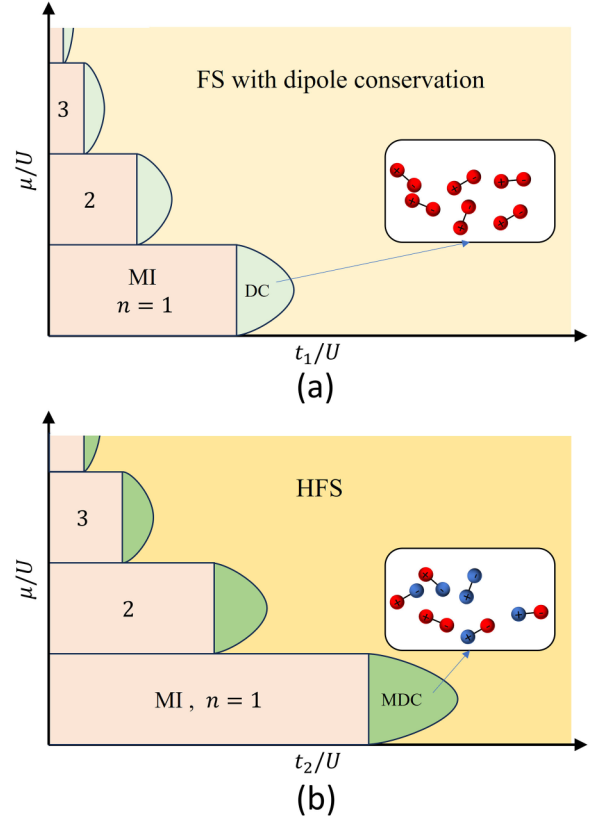


FIG. 2. Schematic phase diagrams from mean-field theory by setting (a) $t_2 = 0$, and (b) $t_1 = 0$. The pink regions are Mott insulators, and n represents the average boson filling per site. The light and dark green regions in the diagrams are dipole condensates (DC) and multidipole condensates (MDC), respectively. The light and dark yellow regions are fractonic superfluids with dipole conservation [43,74] and hybrid fractonic superfluids described by Eq. (20), respectively. In the dipole condensate phase, $\langle \hat{b}_{a, \mathbf{i}+\hat{x}_i}^\dagger \hat{b}_{a, \mathbf{i}} \rangle \neq 0$, $\langle \hat{b}_{1, \mathbf{i}}^\dagger \hat{b}_{2, \mathbf{i}} \rangle = 0$. In the multidipole condensate phase, $\langle \hat{b}_{a, \mathbf{i}+\hat{x}_i}^\dagger \hat{b}_{a, \mathbf{i}} \rangle \neq 0$, $\langle \hat{b}_{1, \mathbf{i}}^\dagger \hat{b}_{2, \mathbf{i}} \rangle \neq 0$.

C. Realizations of Lattice Model Series A in strongly tilted optical lattice

In the following, we discuss how to realize the Hamiltonian [Eq. (77)] of Lattice Model Series A in strongly tilted optical lattices. We start with a microscopic Hamiltonian with two species of bosons in a tilted optical lattice chain. Since there is only one direction, we can simplify the notation $\hat{x}_1 = 1$ and the lattice coordinate $\mathbf{i}$ becomes an integer $\mathbf{i} = 1, 2, \dots$. The Hamiltonian can be written as

$$H_{\text{tilted}} = -t \sum_{a, \mathbf{i}} (\hat{b}_{a, \mathbf{i}}^\dagger \hat{b}_{a, \mathbf{i}+1} + \text{H.c.}) - \mu \sum_{a, \mathbf{i}} \hat{n}_{a, \mathbf{i}} \\ + \frac{U}{2} \sum_{a, \mathbf{i}} \hat{n}_{a, \mathbf{i}} (\hat{n}_{a, \mathbf{i}} - 1) + U' \sum_{\mathbf{i}} \hat{n}_{1, \mathbf{i}} \hat{n}_{2, \mathbf{i}} + V \sum_{a, \mathbf{i}} \mathbf{i} \hat{n}_{a, \mathbf{i}}, \quad (83)$$

where t , μ , and U terms are just terms in the Bose-Hubbard model, and the U' term denotes the interaction between bosons of different species. V is the strength of the tilt potential. In the limit $V \gg t, U$, we perform a third-order

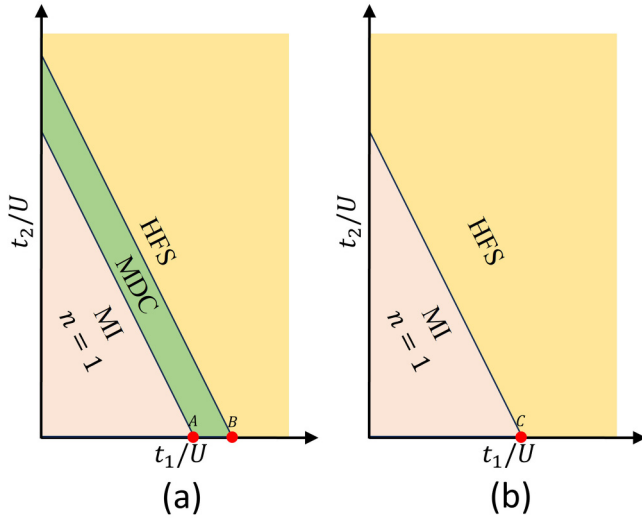


FIG. 3. Schematic phase diagrams from mean-field theory by setting (a) $\mu/U = \frac{1}{2}$, and (b) $\mu/U = 1$. In (a), along the horizontal axis ($t_2 = 0$), the region between the points A and B is the DC phase, and the region on the right of point B is fractonic superfluids with dipole conservation. In (b), along the horizontal axis ($t_2 = 0$), the region on the right of point C is fractonic superfluids with dipole conservation.

perturbation theory. The effective Hamiltonian becomes

$$\begin{aligned}
 H_{\text{eff}} = & \sum_{a,i} \left(-\frac{t^2 U}{V^2} \hat{b}_{a,i} (\hat{b}_{a,i+1}^\dagger)^2 \hat{b}_{a,i+2} + \text{H.c.} \right) \\
 & + \sum_{a \neq b, i} \left(\frac{t^2 U'}{V^2} \hat{b}_{a,i}^\dagger \hat{b}_{a,i+1} \hat{b}_{b,i+1} \hat{b}_{b,i+2}^\dagger \right. \\
 & \left. - \frac{t^2 U'}{V^2} \hat{b}_{a,i}^\dagger \hat{b}_{a,i+1} \hat{b}_{b,i} \hat{b}_{b,i+1}^\dagger + \text{H.c.} \right) \\
 & + \sum_{a,i} \left(-\mu - \frac{U}{2} + V\mathbf{i} \right) \hat{n}_{a,i} + \sum_{a,i} \left(\frac{U}{2} - \frac{2t_0^2 U}{V^2} \right) \hat{n}_{a,i}^2 \\
 & + \frac{4t_0^2 U}{V^2} \hat{n}_{a,i} \hat{n}_{a,i+1} + \sum_{a \neq b, i} \left[\left(\frac{U'}{2} - \frac{2t_0^2 U'}{V^2} \right) \hat{n}_{a,i} \hat{n}_{b,i} \right. \\
 & \left. + \frac{2t_0^2 U'}{V^2} \hat{n}_{a,i} \hat{n}_{b,i+1} \right]. \quad (84)
 \end{aligned}$$

The detailed derivation in Eq. (84) is given in Appendix F.

VI. SUMMARY AND OUTLOOK

In conclusion, in this paper we have completed the third part in the trilogy on fractonic superfluids [43,44], exploring the novel symmetries arising from higher-moment conservation and the associated symmetry-breaking phenomena. Just like the prior two papers about fractonic superfluids, the main goal of this paper is to lay the foundation for hybrid fractonic superfluids, leaving various promising discussions for future

studies. Therefore, we anticipate further investigations, such as those akin to Refs. [45–48], to deepen our understanding on new symmetries and symmetry-breaking phases in the presence of higher-moment conservation. Let us elaborate in more details.

By endowing algebraic structures [40] with conserved hybrid moments, a rich variety of effective field theories can be constructed. It is compelling to derive and numerically solve the hydrodynamic equations [45] of systems with conservation of hybrid moments. We can also employ renormalization group analysis to determine whether systems with conserved hybrid moments can emerge at low energies [46]. The finite-temperature phase transitions [47] in these hybrid-moment-conserving models are also worth investigating, where the configuration of symmetry defects (i.e., various generalized topological vortices) and hierarchical Kosterlitz-Thouless transitions potentially get enriched by moment hybridization. The results of the partial spontaneous breaking of charge symmetry will generalize the Hohenberg-Mermin-Wagner theorems [71,73]. As mobility restrictions lead to unconventional types of toric codes [95], it is interesting to explore further generalizations of discrete gauge theory and toric codes through additional possibilities of mobility constraints arising from conservations, such as the hybridization discussed in this work. Detecting such states in the context of stabilizer codes and cellular automata, e.g., in Refs. [32,96,97], will be an intriguing direction. In principle, these Bose-Hubbard models constructed in this paper can be extended to fermionic Hubbard models for realizing non-Fermi liquids [80]. Numerical simulations, such as quantum Monte Carlo, DMRG, and tensor network methods, can be applied to study these lattice models. These lattice models have the potential to be academically attractive in cold-atom communities, such as strongly tilted optical lattices discussed in Refs. [98–101] and general Josephson effects in Ref. [78]. We can construct some other lattice systems with hybrid moment conservation which are experimentally realizable in tilted optical lattices. By incorporating the concept of conserved hybridized moments, diffusion in spin models [42] can also be explored. The conservation of hybrid moments may induce different fragmentation behaviors in Hilbert space [98]. Finally, it will be interesting to investigate how the moment hybridization affects the dilaton spectrum in an unusual Ginzburg-Landau theory [69].

ACKNOWLEDGMENTS

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: DERIVATIONS OF GOLDSTONE MODES AND DISPERSION RELATIONS FOR MODEL SERIES A

1. Derivation of the Hamiltonian in the momentum space

Based on the Hamiltonian [Eq. (43)], we can derive the Hamiltonian of the momentum space [Eq. (44)] in the following steps. In the momentum space, we can perform a Fourier transform

$$H = \frac{1}{(2\pi)^{2d}} \int d^d \mathbf{x} d^d \mathbf{k} d^d \mathbf{k}' e^{i\mathbf{k} \cdot \mathbf{x} + i\mathbf{k}' \cdot \mathbf{x}} \sum_a \left[\frac{g}{2} \hat{\pi}_a(\mathbf{k}) \hat{\pi}_a(\mathbf{k}') + \frac{\kappa \rho_{a0}^2 |\mathbf{k}|^4}{2} \hat{\theta}_a(\mathbf{k}) \hat{\theta}_a(\mathbf{k}') \right. \\ \left. + \sum_{b \neq a} \frac{\Gamma \rho_{a0} \rho_{b0} |\mathbf{k}|^2}{2} [\hat{\theta}_a(\mathbf{k}) \hat{\theta}_a(\mathbf{k}') + \hat{\theta}_b(\mathbf{k}) \hat{\theta}_b(\mathbf{k}') - \hat{\theta}_a(\mathbf{k}) \hat{\theta}_b(\mathbf{k}') - \hat{\theta}_b(\mathbf{k}) \hat{\theta}_a(\mathbf{k}')] \right]. \quad (\text{A1})$$

Integrating over the real space, we can obtain a δ function so that only the integral of the momentum space is left in the Hamiltonian

$$H = \frac{1}{(2\pi)^d} \int d^d \mathbf{k} d^d \mathbf{k}' \delta(\mathbf{k} + \mathbf{k}') \sum_a \left[\frac{g}{2} \hat{\pi}_a(\mathbf{k}) \hat{\pi}_a(\mathbf{k}') + \frac{\kappa \rho_{a0}^2 |\mathbf{k}|^4}{2} \hat{\theta}_a(\mathbf{k}) \hat{\theta}_a(\mathbf{k}') \right. \\ \left. + \sum_{b \neq a} \frac{\Gamma \rho_{a0} \rho_{b0} |\mathbf{k}|^2}{2} [\hat{\theta}_a(\mathbf{k}) \hat{\theta}_a(\mathbf{k}') + \hat{\theta}_b(\mathbf{k}) \hat{\theta}_b(\mathbf{k}') - \hat{\theta}_a(\mathbf{k}) \hat{\theta}_b(\mathbf{k}') - \hat{\theta}_b(\mathbf{k}) \hat{\theta}_a(\mathbf{k}')] \right] \\ = \frac{1}{(2\pi)^d} \int d^d \mathbf{k} \sum_a \left[\frac{g}{2} \hat{\pi}_a(\mathbf{k}) \hat{\pi}_a(-\mathbf{k}) + \frac{\kappa \rho_{a0}^2 |\mathbf{k}|^4}{2} \hat{\theta}_a(\mathbf{k}) \hat{\theta}_a(-\mathbf{k}) \right. \\ \left. + \sum_{b \neq a} \frac{\Gamma \rho_{a0} \rho_{b0} |\mathbf{k}|^2}{2} [\hat{\theta}_a(\mathbf{k}) \hat{\theta}_a(-\mathbf{k}) + \hat{\theta}_b(\mathbf{k}) \hat{\theta}_b(-\mathbf{k}) - \hat{\theta}_a(\mathbf{k}) \hat{\theta}_b(-\mathbf{k}) - \hat{\theta}_b(\mathbf{k}) \hat{\theta}_a(-\mathbf{k})] \right]. \quad (\text{A2})$$

The Hamiltonian [Eq. (44)] can be obtained by writing it in the matrix form and using the notation in the main text

$$H = \frac{1}{2(2\pi)^d} \int d^d \mathbf{k} \sum_a \left[g \hat{\pi}_a(\mathbf{k}) \hat{\pi}_a(-\mathbf{k}) + \kappa \rho_{a0}^2 |\mathbf{k}|^4 \hat{\theta}_a(\mathbf{k}) \hat{\theta}_a(-\mathbf{k}) \right. \\ \left. + \sum_{b \neq a} \Gamma \rho_{a0} \rho_{b0} |\mathbf{k}|^2 [\hat{\theta}_a(\mathbf{k}) \hat{\theta}_a(-\mathbf{k}) + \hat{\theta}_b(\mathbf{k}) \hat{\theta}_b(-\mathbf{k}) - \hat{\theta}_a(\mathbf{k}) \hat{\theta}_b(-\mathbf{k}) - \hat{\theta}_b(\mathbf{k}) \hat{\theta}_a(-\mathbf{k})] \right] \\ = \frac{1}{2(2\pi)^d} \int d^d \mathbf{k} \psi^\dagger(\mathbf{k}) \begin{bmatrix} g & 0 & 0 & 0 \\ 0 & g & 0 & 0 \\ 0 & 0 & \kappa \rho_{10}^2 |\mathbf{k}|^4 + 2\Gamma \rho_{10} \rho_{20} |\mathbf{k}|^2 & -2\Gamma \rho_{10} \rho_{20} |\mathbf{k}|^2 \\ 0 & 0 & -2\Gamma \rho_{10} \rho_{20} |\mathbf{k}|^2 & \kappa \rho_{20}^2 |\mathbf{k}|^4 + 2\Gamma \rho_{10} \rho_{20} |\mathbf{k}|^2 \end{bmatrix} \psi(\mathbf{k}) \\ = \frac{1}{2(2\pi)^d} \int d^d \mathbf{k} \psi^\dagger(\mathbf{k}) M(\mathbf{k}) \psi(\mathbf{k}). \quad (\text{A3})$$

2. Matrix elements of the transformation matrix $T(\mathbf{k})$

In the main text, we give the commutation relations that the operators need to satisfy. To keep the commutation relations and diagonalize the matrix $M(\mathbf{k})$, the transformation matrix $T(\mathbf{k})$ needs to obey the equation below:

$$T(\mathbf{k}) \Lambda_1 T^\dagger(\mathbf{k}) = \Lambda_2, \quad (\text{A4})$$

where

$$\Lambda_1 = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix}, \quad \Lambda_2 = \text{diag}(1, 1, -1, -1). \quad (\text{A5})$$

The transformation matrix $T(\mathbf{k})$ is given by

$$T(\mathbf{k}) = \begin{bmatrix} T_{11}(\mathbf{k}) & T_{12}(\mathbf{k}) & T_{13}(\mathbf{k}) & T_{14}(\mathbf{k}) \\ T_{21}(\mathbf{k}) & T_{22}(\mathbf{k}) & T_{23}(\mathbf{k}) & T_{24}(\mathbf{k}) \\ T_{31}(\mathbf{k}) & T_{32}(\mathbf{k}) & T_{33}(\mathbf{k}) & T_{34}(\mathbf{k}) \\ T_{41}(\mathbf{k}) & T_{42}(\mathbf{k}) & T_{43}(\mathbf{k}) & T_{44}(\mathbf{k}) \end{bmatrix}, \quad (\text{A6})$$

where the matrix elements are

$$T_{11}(\mathbf{k}) = -T_{31}(\mathbf{k}) = (ig[\kappa|\mathbf{k}|^4(\rho_{20}^2 - \rho_{10}^2) + \sqrt{\kappa^2|\mathbf{k}|^8(\rho_{10}^2 - \rho_{20}^2)^2 + 16\Gamma^2k^4\rho_{10}^2\rho_{20}^2}]) / \{ \sqrt[4]{2}[g(16\Gamma^2|\mathbf{k}|^4\rho_{10}^2\rho_{20}^2 + (\kappa|\mathbf{k}|^4(\rho_{20}^2 - \rho_{10}^2) + \sqrt{\kappa^2k^8(\rho_{10}^2 - \rho_{20}^2)^2 + 16\Gamma^2|\mathbf{k}|^4\rho_{10}^2\rho_{20}^2})^2) \times \sqrt{g[\kappa|\mathbf{k}|^4(\rho_{10}^2 + \rho_{20}^2) + 4\Gamma|\mathbf{k}|^2\rho_{10}\rho_{20} - \sqrt{\kappa^2|\mathbf{k}|^8(\rho_{10}^2 - \rho_{20}^2)^2 + 16\Gamma^2|\mathbf{k}|^4\rho_{10}^2\rho_{20}^2}]]^{\frac{1}{2}} \}, \quad (\text{A7})$$

$$T_{12}(\mathbf{k}) = -T_{32}(\mathbf{k}) = (4ig\Gamma\rho_{10}\rho_{20}|\mathbf{k}|^2) / \{ \sqrt[4]{2}[g[16\Gamma^2|\mathbf{k}|^4\rho_{10}^2\rho_{20}^2 + (\kappa|\mathbf{k}|^4(\rho_{20}^2 - \rho_{10}^2) + \sqrt{\kappa^2k^8(\rho_{10}^2 - \rho_{20}^2)^2 + 16\Gamma^2|\mathbf{k}|^4\rho_{10}^2\rho_{20}^2})^2] \times \sqrt{g[\kappa|\mathbf{k}|^4(\rho_{10}^2 + \rho_{20}^2) + 4\Gamma|\mathbf{k}|^2\rho_{10}\rho_{20} - \sqrt{\kappa^2|\mathbf{k}|^8(\rho_{10}^2 - \rho_{20}^2)^2 + 16\Gamma^2|\mathbf{k}|^4\rho_{10}^2\rho_{20}^2}]]^{\frac{1}{2}} \}, \quad (\text{A8})$$

$$T_{13}(\mathbf{k}) = T_{33}(\mathbf{k}) = \left[\kappa|\mathbf{k}|^4(\rho_{20}^2 - \rho_{10}^2) + \sqrt{\kappa^2|\mathbf{k}|^8(\rho_{10}^2 - \rho_{20}^2)^2 + 16\Gamma^2|\mathbf{k}|^4\rho_{10}^2\rho_{20}^2} \right] \times \sqrt[4]{\frac{\kappa|\mathbf{k}|^4(\rho_{10}^2 + \rho_{20}^2) + 4\Gamma|\mathbf{k}|^2\rho_{10}\rho_{20} - \sqrt{\kappa^2|\mathbf{k}|^8(\rho_{10}^2 - \rho_{20}^2)^2 + 16\Gamma^2|\mathbf{k}|^4\rho_{10}^2\rho_{20}^2}}{g}} \Big/ (2^{3/4} \sqrt{16\Gamma^2|\mathbf{k}|^4\rho_{10}^2\rho_{20}^2 + [\kappa|\mathbf{k}|^4(\rho_{20}^2 - \rho_{10}^2) + \sqrt{\kappa^2|\mathbf{k}|^8(\rho_{10}^2 - \rho_{20}^2)^2 + 16\Gamma^2|\mathbf{k}|^4\rho_{10}^2\rho_{20}^2}]^2}), \quad (\text{A9})$$

$$T_{14}(\mathbf{k}) = T_{34}(\mathbf{k}) = \left(2\Gamma|\mathbf{k}|^2\rho_{10}\rho_{20} \sqrt[4]{\frac{2\kappa|\mathbf{k}|^4(\rho_{10}^2 + \rho_{20}^2) + 8\Gamma|\mathbf{k}|^2\rho_{10}\rho_{20} - 2\sqrt{\kappa^2|\mathbf{k}|^8(\rho_{10}^2 - \rho_{20}^2)^2 + 16\Gamma^2|\mathbf{k}|^4\rho_{10}^2\rho_{20}^2}}{g}} \right) / \left(\sqrt{16\Gamma^2|\mathbf{k}|^4\rho_{10}^2\rho_{20}^2 + (\kappa|\mathbf{k}|^4(\rho_{20}^2 - \rho_{10}^2) + \sqrt{\kappa^2|\mathbf{k}|^8(\rho_{10}^2 - \rho_{20}^2)^2 + 16\Gamma^2|\mathbf{k}|^4\rho_{10}^2\rho_{20}^2})^2} \right) \quad (\text{A10})$$

$$T_{21}(\mathbf{k}) = -T_{41}(\mathbf{k}) = -[ig(\kappa|\mathbf{k}|^4(\rho_{10}^2 - \rho_{20}^2) + \sqrt{\kappa^2|\mathbf{k}|^8(\rho_{10}^2 - \rho_{20}^2)^2 + 16\Gamma^2|\mathbf{k}|^4\rho_{10}^2\rho_{20}^2})] / \{ 2^{3/4}|\mathbf{k}|^2[g(16\Gamma^2\rho_{10}^2\rho_{20}^2 + \kappa^2|\mathbf{k}|^4(\rho_{10}^2 - \rho_{20}^2)^2 + \kappa(\rho_{10}^2 - \rho_{20}^2)\sqrt{\kappa^2|\mathbf{k}|^8(\rho_{10}^2 - \rho_{20}^2)^2 + 16\Gamma^2|\mathbf{k}|^4\rho_{10}^2\rho_{20}^2}) \times \sqrt{g[\kappa|\mathbf{k}|^4(\rho_{10}^2 + \rho_{20}^2) + 4\Gamma|\mathbf{k}|^2\rho_{10}\rho_{20} + \sqrt{\kappa^2|\mathbf{k}|^8(\rho_{10}^2 - \rho_{20}^2)^2 + 16\Gamma^2|\mathbf{k}|^4\rho_{10}^2\rho_{20}^2}]]^{\frac{1}{2}} \}, \quad (\text{A11})$$

$$T_{22}(\mathbf{k}) = -T_{42}(\mathbf{k}) = (2i\sqrt[4]{2}\Gamma g\rho_{10}\rho_{20}) / \left[\sqrt[4]{g(\kappa|\mathbf{k}|^4(\rho_{10}^2 + \rho_{20}^2) + 4\Gamma|\mathbf{k}|^2\rho_{10}\rho_{20} + \sqrt{\kappa^2k^8(\rho_{10}^2 - \rho_{20}^2)^2 + 16\Gamma^2|\mathbf{k}|^4\rho_{10}^2\rho_{20}^2})} \times \sqrt{g[16\Gamma^2\rho_{10}^2\rho_{20}^2 + \kappa^2|\mathbf{k}|^4(\rho_{10}^2 - \rho_{20}^2)^2 + \kappa(\rho_{10}^2 - \rho_{20}^2)\sqrt{\kappa^2|\mathbf{k}|^8(\rho_{10}^2 - \rho_{20}^2)^2 + 16\Gamma^2|\mathbf{k}|^4\rho_{10}^2\rho_{20}^2}]} \right], \quad (\text{A12})$$

$$T_{23}(\mathbf{k}) = T_{43}(\mathbf{k}) = \frac{\kappa|\mathbf{k}|^4(\rho_{20}^2 - \rho_{10}^2) - \sqrt{\kappa^2|\mathbf{k}|^8(\rho_{10}^2 - \rho_{20}^2)^2 + 16\Gamma^2|\mathbf{k}|^4\rho_{10}^2\rho_{20}^2}}{2\sqrt[4]{2}|\mathbf{k}|^2 \sqrt{\frac{g(16\Gamma^2\rho_{10}^2\rho_{20}^2 + \kappa^2|\mathbf{k}|^4(\rho_{10}^2 - \rho_{20}^2)^2 + \kappa\rho_{10}^2\sqrt{\kappa^2|\mathbf{k}|^8(\rho_{10}^2 - \rho_{20}^2)^2 + 16\Gamma^2|\mathbf{k}|^4\rho_{10}^2\rho_{20}^2} - \kappa\rho_{20}^2\sqrt{\kappa^2|\mathbf{k}|^8(\rho_{10}^2 - \rho_{20}^2)^2 + 16\Gamma^2|\mathbf{k}|^4\rho_{10}^2\rho_{20}^2})}{\sqrt{g(\kappa|\mathbf{k}|^4(\rho_{10}^2 + \rho_{20}^2) + 4\Gamma|\mathbf{k}|^2\rho_{10}\rho_{20} + \sqrt{\kappa^2|\mathbf{k}|^8(\rho_{10}^2 - \rho_{20}^2)^2 + 16\Gamma^2|\mathbf{k}|^4\rho_{10}^2\rho_{20}^2})}}}}, \quad (\text{A13})$$

$$T_{24}(\mathbf{k}) = T_{44}(\mathbf{k}) = \frac{2^{3/4}\Gamma\rho_{10}\rho_{20}}{\sqrt{\frac{g(16\Gamma^2\rho_{10}^2\rho_{20}^2 + \kappa^2|\mathbf{k}|^4(\rho_{10}^2 - \rho_{20}^2)^2 + \kappa\rho_{10}^2\sqrt{\kappa^2|\mathbf{k}|^8(\rho_{10}^2 - \rho_{20}^2)^2 + 16\Gamma^2|\mathbf{k}|^4\rho_{10}^2\rho_{20}^2} - \kappa\rho_{20}^2\sqrt{\kappa^2|\mathbf{k}|^8(\rho_{10}^2 - \rho_{20}^2)^2 + 16\Gamma^2|\mathbf{k}|^4\rho_{10}^2\rho_{20}^2})}{\sqrt{g(\kappa|\mathbf{k}|^4(\rho_{10}^2 + \rho_{20}^2) + 4\Gamma|\mathbf{k}|^2\rho_{10}\rho_{20} + \sqrt{\kappa^2|\mathbf{k}|^8(\rho_{10}^2 - \rho_{20}^2)^2 + 16\Gamma^2|\mathbf{k}|^4\rho_{10}^2\rho_{20}^2})}}}}}. \quad (\text{A14})$$

3. Derivation of the Hamiltonian represented by the creation and annihilation operators

After diagonalization, the Hamiltonian [Eq. (45)] becomes

$$H = \frac{1}{2(2\pi)^d} \phi^\dagger(\mathbf{k}) [T^{-1}(\mathbf{k})]^\dagger M(\mathbf{k}) T^{-1}(\mathbf{k}) \phi(\mathbf{k})$$

$$= \frac{1}{2(2\pi)^d} \int d^d \mathbf{k} D_1(\mathbf{k}) [\hat{\alpha}^\dagger(\mathbf{k}) \hat{\alpha}(\mathbf{k}) + \hat{\alpha}(-\mathbf{k}) \hat{\alpha}^\dagger(-\mathbf{k})] + D_2(\mathbf{k}) [\hat{\beta}^\dagger(\mathbf{k}) \hat{\beta}(\mathbf{k}) + \hat{\beta}(-\mathbf{k}) \hat{\beta}^\dagger(-\mathbf{k})]. \quad (\text{A15})$$

Because of $D_1(\mathbf{k}) = D_1(-\mathbf{k})$, $D_2(\mathbf{k}) = D_2(-\mathbf{k})$, we rewrite the Hamiltonian as

$$H = \frac{1}{2(2\pi)^d} \int d^d \mathbf{k} D_1(\mathbf{k}) [\hat{\alpha}^\dagger(\mathbf{k}) \hat{\alpha}(\mathbf{k}) + \hat{\alpha}(\mathbf{k}) \hat{\alpha}^\dagger(\mathbf{k})] + D_2(\mathbf{k}) [\hat{\beta}^\dagger(\mathbf{k}) \hat{\beta}(\mathbf{k}) + \hat{\beta}(\mathbf{k}) \hat{\beta}^\dagger(\mathbf{k})]$$

$$= \frac{1}{(2\pi)^d} \int d^d \mathbf{k} D_1(\mathbf{k}) \hat{\alpha}^\dagger(\mathbf{k}) \hat{\alpha}(\mathbf{k}) + D_2(\mathbf{k}) \hat{\beta}^\dagger(\mathbf{k}) \hat{\beta}(\mathbf{k}) + \int d^d \mathbf{k} \frac{D_1(\mathbf{k}) + D_2(\mathbf{k})}{2} \delta^{(d)}(\mathbf{0})$$

$$= \frac{1}{(2\pi)^d} \int d^d \mathbf{k} D_1(\mathbf{k}) \hat{\alpha}^\dagger(\mathbf{k}) \hat{\alpha}(\mathbf{k}) + D_2(\mathbf{k}) \hat{\beta}^\dagger(\mathbf{k}) \hat{\beta}(\mathbf{k}) + \frac{\tilde{V}}{(2\pi)^d} \int d^d \mathbf{k} \frac{D_1(\mathbf{k}) + D_2(\mathbf{k})}{2}, \quad (\text{A16})$$

where $\tilde{V}$ is the volume of the d -dimensional space, the last term is the vacuum zero-point energy. Then, we obtain the dispersion relations of Goldstone mode [Eq. (50)].

APPENDIX B: THE CALCULATION DETAILS OF CORRELATION FUNCTION OF MODEL SERIES A

In this Appendix, we will give the calculation details of correlation functions of Model Series A. From the main text and Appendix A 2, we have the Bogoliubov transformation from $\hat{\theta}$ and $\hat{\pi}$ to the independent modes $\hat{\alpha}$ and $\hat{\beta}$:

$$\psi(\mathbf{k}) = [\hat{\pi}_1(\mathbf{k}) \quad \hat{\pi}_2(\mathbf{k}) \quad \hat{\theta}_1(\mathbf{k}) \quad \hat{\theta}_2(\mathbf{k})]^T = T^{-1}(\mathbf{k}) \phi(\mathbf{k}) = \Lambda_1 T^\dagger(\mathbf{k}) \Lambda_2 \phi(\mathbf{k})$$

$$= \begin{bmatrix} -iT_{13}^*(\mathbf{k}) & -iT_{23}^*(\mathbf{k}) & iT_{33}^*(\mathbf{k}) & iT_{43}^*(\mathbf{k}) \\ -iT_{14}^*(\mathbf{k}) & -iT_{24}^*(\mathbf{k}) & iT_{34}^*(\mathbf{k}) & iT_{44}^*(\mathbf{k}) \\ iT_{11}^*(\mathbf{k}) & iT_{21}^*(\mathbf{k}) & -iT_{31}^*(\mathbf{k}) & -iT_{41}^*(\mathbf{k}) \\ iT_{12}^*(\mathbf{k}) & iT_{22}^*(\mathbf{k}) & -iT_{32}^*(\mathbf{k}) & -iT_{42}^*(\mathbf{k}) \end{bmatrix} \begin{bmatrix} \hat{\alpha}(\mathbf{k}) \\ \hat{\beta}(\mathbf{k}) \\ \hat{\alpha}^\dagger(-\mathbf{k}) \\ \hat{\beta}^\dagger(-\mathbf{k}) \end{bmatrix}. \quad (\text{B1})$$

In the main text, we express two-operator and four-operator correlation functions of the boson operator $\hat{\Phi}_a$ in terms of two-operator correlation functions of the phase field operator $\hat{\theta}_a$. Two-operator correlation functions of the phase field operator $\hat{\theta}_a$ can be given in terms of the matrix elements of the matrix $T(\mathbf{k})$ in the following procedure:

$$\langle \hat{\theta}_1(\mathbf{x}) \hat{\theta}_1(\mathbf{0}) \rangle = \left\langle \int \frac{d^d \mathbf{k} d^d \mathbf{k}'}{(2\pi)^{2d}} e^{i\mathbf{k} \cdot \mathbf{x}} \hat{\theta}_1(\mathbf{k}) \hat{\theta}_1(\mathbf{k}') \right\rangle = \left\langle \int \frac{d^d \mathbf{k} d^d \mathbf{k}'}{(2\pi)^{2d}} e^{i\mathbf{k} \cdot \mathbf{x}} \hat{\theta}_1(\mathbf{k}) \hat{\theta}_1^\dagger(-\mathbf{k}') \right\rangle$$

$$= \int \frac{d^d \mathbf{k} d^d \mathbf{k}'}{(2\pi)^{2d}} e^{i\mathbf{k} \cdot \mathbf{x}} \langle [iT_{11}^*(\mathbf{k}) \hat{\alpha}(\mathbf{k}) + iT_{21}^*(\mathbf{k}) \hat{\beta}(\mathbf{k}) - iT_{31}^*(\mathbf{k}) \hat{\alpha}^\dagger(-\mathbf{k}) - iT_{41}^*(\mathbf{k}) \hat{\beta}^\dagger(-\mathbf{k})]$$

$$\times [-iT_{11}(-\mathbf{k}') \hat{\alpha}^\dagger(-\mathbf{k}') - iT_{21}(-\mathbf{k}') \hat{\beta}^\dagger(-\mathbf{k}') + iT_{31}(-\mathbf{k}') \hat{\alpha}(\mathbf{k}') + iT_{41}(-\mathbf{k}') \hat{\beta}(\mathbf{k}')] \rangle. \quad (\text{B2})$$

For the expectation value of the creation and annihilation operators in the ground state, we have

$$\langle \hat{\alpha}(\mathbf{k}') \hat{\alpha}(\mathbf{k}) \rangle = \langle \hat{\beta}(\mathbf{k}') \hat{\beta}(\mathbf{k}) \rangle = \langle \hat{\alpha}^\dagger(-\mathbf{k}') \hat{\alpha}(\mathbf{k}) \rangle = \langle \hat{\beta}^\dagger(-\mathbf{k}') \hat{\beta}(\mathbf{k}) \rangle = 0, \quad (\text{B3})$$

$$\langle \hat{\alpha}(\mathbf{k}) \hat{\alpha}^\dagger(-\mathbf{k}') \rangle = \langle \hat{\beta}(\mathbf{k}) \hat{\beta}^\dagger(-\mathbf{k}') \rangle = (2\pi)^d \delta^{(d)}(\mathbf{k} + \mathbf{k}'). \quad (\text{B4})$$

Then we can obtain the correlation function

$$\langle \hat{\theta}_1(\mathbf{x}) \hat{\theta}_1(\mathbf{0}) \rangle = \int \frac{d^d \mathbf{k}}{(2\pi)^d} e^{i\mathbf{k} \cdot \mathbf{x}} [|T_{11}(\mathbf{k})|^2 + |T_{21}(\mathbf{k})|^2]. \quad (\text{B5})$$

Similarly, we can get

$$\langle \hat{\theta}_2(\mathbf{x}) \hat{\theta}_2(\mathbf{0}) \rangle = \int \frac{d^d \mathbf{k}}{(2\pi)^d} e^{i\mathbf{k} \cdot \mathbf{x}} [|T_{12}(\mathbf{k})|^2 + |T_{22}(\mathbf{k})|^2]. \quad (\text{B6})$$

The correlation function of different species of phase field operators is given by

$$\begin{aligned}
\langle \hat{\theta}_1(\mathbf{x}) \hat{\theta}_2(\mathbf{0}) \rangle &= \left\langle \int \frac{d^d \mathbf{k} d^d \mathbf{k}'}{(2\pi)^{2d}} e^{i\mathbf{k} \cdot \mathbf{x}} \hat{\theta}_1(\mathbf{k}) \hat{\theta}_2^\dagger(-\mathbf{k}) \right\rangle \\
&= \int \frac{d^d \mathbf{k} d^d \mathbf{k}'}{(2\pi)^{2d}} e^{i\mathbf{k} \cdot \mathbf{x}} \langle [iT_{11}^*(\mathbf{k}) \hat{\alpha}(\mathbf{k}) + iT_{21}^*(\mathbf{k}) \hat{\beta}(\mathbf{k}) - iT_{31}^*(\mathbf{k}) \hat{\alpha}^\dagger(-\mathbf{k}) - iT_{41}^*(\mathbf{k}) \hat{\beta}^\dagger(-\mathbf{k})] \\
&\quad \times [-iT_{12}(-\mathbf{k}') \hat{\alpha}^\dagger(-\mathbf{k}') - iT_{22}(-\mathbf{k}') \hat{\beta}^\dagger(-\mathbf{k}') + iT_{32}(-\mathbf{k}') \hat{\alpha}(\mathbf{k}') + iT_{42}(-\mathbf{k}') \hat{\beta}(\mathbf{k}')] \rangle \\
&= \int \frac{d^d \mathbf{k}}{(2\pi)^d} e^{i\mathbf{k} \cdot \mathbf{x}} [T_{11}^*(\mathbf{k}) T_{12}(\mathbf{k}) + T_{21}^*(\mathbf{k}) T_{22}(\mathbf{k})].
\end{aligned} \tag{B7}$$

For the $T(\mathbf{k})$ matrix of Model Series A in Appendix A 2, we can give some approximations:

$$\begin{aligned}
|T_{11}(\mathbf{k})|^2 &\approx \frac{g}{2\sqrt{2g\kappa(\rho_{10}^2 + \rho_{20}^2)}|\mathbf{k}|^2} = \frac{1}{c_1|\mathbf{k}|^2}, \\
|T_{12}(\mathbf{k})|^2 &\approx \frac{g}{2\sqrt{2g\kappa(\rho_{10}^2 + \rho_{20}^2)}|\mathbf{k}|^2} = \frac{1}{c_1|\mathbf{k}|^2}, \\
|T_{21}(\mathbf{k})|^2 &\approx \frac{g}{8\sqrt{g\Gamma\rho_{10}\rho_{20}}|\mathbf{k}|} = \frac{1}{c_2|\mathbf{k}|}, \\
|T_{22}(\mathbf{k})|^2 &\approx \frac{g}{8\sqrt{g\Gamma\rho_{10}\rho_{20}}|\mathbf{k}|} = \frac{1}{c_2|\mathbf{k}|}, \\
T_{11}^*(\mathbf{k})T_{12}(\mathbf{k}) &\approx \frac{g}{2\sqrt{2g\kappa(\rho_{10}^2 + \rho_{20}^2)}|\mathbf{k}|^2} = \frac{1}{c_1|\mathbf{k}|^2}, \\
T_{21}^*(\mathbf{k})T_{22}(\mathbf{k}) &\approx -\frac{g}{8\sqrt{g\Gamma\rho_{10}\rho_{20}}|\mathbf{k}|} = -\frac{1}{c_2|\mathbf{k}|}.
\end{aligned} \tag{B8}$$

Based on the above derivation, we can calculate correlation functions in different spatial dimensions.

1. One spatial dimension

In one spatial dimension, we can plug in the results of the above approximation. The correlation functions of the operator $\hat{\theta}$ are

$$\begin{aligned}
\langle \hat{\theta}_1(x) \hat{\theta}_1(0) \rangle &= \langle \hat{\theta}_2(x) \hat{\theta}_2(0) \rangle = \int \frac{dk}{2\pi} e^{ikx} \left(\frac{1}{c_1|k|^2} + \frac{1}{c_2|k|} \right) \\
&= \frac{1}{2\pi c_1} \int dk \frac{e^{ikx}}{|k|^2} + \frac{1}{2\pi c_2} \int dk \frac{e^{ikx}}{|k|} \\
&= \frac{1}{2\pi c_1} \left(\int_{-\infty}^{-\frac{2\pi}{L}} dk \frac{e^{ikx}}{k^2} + \int_{\frac{2\pi}{L}}^{\infty} dk \frac{e^{ikx}}{k^2} \right) + \frac{1}{2\pi c_2} \left(\int_{-\infty}^{-\frac{2\pi}{L}} dk \frac{e^{ikx}}{-k} + \int_{\frac{2\pi}{L}}^{\infty} dk \frac{e^{ikx}}{k} \right).
\end{aligned} \tag{B9}$$

We need to compute these integrals separately. Considering $L > 0$, $x > 0$, we have the first two integrals

$$\int_{-\infty}^{-\frac{2\pi}{L}} dk \frac{e^{ikx}}{k^2} + \int_{\frac{2\pi}{L}}^{\infty} dk \frac{e^{ikx}}{k^2} = \frac{1}{2\pi} \left[e^{-2i\pi x} L + e^{2i\pi x} L + 2i\pi x \Gamma\left(0, -\frac{2i\pi x}{L}\right) - 2i\pi x \Gamma\left(0, -\frac{2i\pi x}{L}\right) \right] \stackrel{L \rightarrow \infty}{=} \frac{L}{\pi} - \pi x, \tag{B10}$$

where $\Gamma(a, z) = \int_z^\infty t^{a-1} e^{-t} dt$ is the incomplete gamma function. The last two integrals are

$$\int_{-\infty}^{-\frac{2\pi}{L}} dk \frac{e^{ikx}}{-k} + \int_{\frac{2\pi}{L}}^{\infty} dk \frac{e^{ikx}}{k} = \Gamma\left(0, -\frac{2i\pi x}{L}\right) + \Gamma\left(0, \frac{2i\pi x}{L}\right) \stackrel{L \rightarrow \infty}{=} -2\left[\gamma + \ln\left(\frac{2\pi x}{L}\right)\right], \tag{B11}$$

where γ is the Euler-Mascheroni constant. Then, we can get the correlation functions.

$$\langle \hat{\theta}_1(x) \hat{\theta}_1(0) \rangle = \langle \hat{\theta}_2(x) \hat{\theta}_2(0) \rangle \stackrel{L \rightarrow \infty}{=} \frac{1}{2\pi c_1} \left(\frac{L}{\pi} - \pi|x| \right) + \frac{1}{\pi c_2} \left[-\gamma + \ln\left(\frac{L}{2\pi|x|}\right) \right] \tag{B12}$$

Similarly, we have

$$\langle \hat{\theta}_1(x) \hat{\theta}_2(0) \rangle \stackrel{L \rightarrow \infty}{=} \frac{1}{2\pi c_1} \left(\frac{L}{\pi} - \pi |x| \right) - \frac{1}{\pi c_2} \left[-\gamma + \ln \left(\frac{L}{2\pi |x|} \right) \right]. \quad (\text{B13})$$

To get $\langle \hat{\theta}_a^2(0) \rangle = \lim_{x \rightarrow 0} \langle \theta_a(x) \theta_a(0) \rangle$, we just need to replace $|x|$ by ξ_c . ξ_c is the coherence length,

$$\langle \hat{\theta}_1^2(0) \rangle = \langle \hat{\theta}_2^2(0) \rangle = \frac{1}{2\pi c_1} \left(\frac{L}{\pi} - \pi \xi_c \right) + \frac{1}{\pi c_2} \left[-\gamma + \ln \left(\frac{L}{2\pi \xi_c} \right) \right], \quad (\text{B14})$$

$$\langle \hat{\theta}_1(0) \hat{\theta}_2(0) \rangle = \frac{1}{2\pi c_1} \left(\frac{L}{\pi} - \pi \xi_c \right) - \frac{1}{\pi c_2} \left[-\gamma + \ln \left(\frac{L}{2\pi \xi_c} \right) \right]. \quad (\text{B15})$$

Then, we can calculate correlation functions of $\hat{\Phi}$. The two-operator correlation functions are

$$\langle \hat{\Phi}_1^\dagger(x) \hat{\Phi}_1(0) \rangle = \rho_{10} e^{\langle \hat{\theta}_1(x) \hat{\theta}_1(0) \rangle - \langle \hat{\theta}_1^2(0) \rangle} = \rho_{10} e^{-\frac{|x|}{2c_1} + \frac{\xi_c}{2c_1}} \left(\frac{|x|}{\xi_c} \right)^{-\frac{1}{\pi c_2}} \approx \rho_{10} e^{-\frac{|x|}{2c_1}} \left(\frac{|x|}{\xi_c} \right)^{-\frac{1}{\pi c_2}}, \quad (\text{B16})$$

$$\langle \hat{\Phi}_2^\dagger(x) \hat{\Phi}_2(0) \rangle = \rho_{20} e^{\langle \hat{\theta}_2(x) \hat{\theta}_2(0) \rangle - \langle \hat{\theta}_2^2(0) \rangle} = \rho_{20} e^{-\frac{|x|}{2c_1} + \frac{\xi_c}{2c_1}} \left(\frac{|x|}{\xi_c} \right)^{-\frac{1}{\pi c_2}} \approx \rho_{20} e^{-\frac{|x|}{2c_1}} \left(\frac{|x|}{\xi_c} \right)^{-\frac{1}{\pi c_2}}, \quad (\text{B17})$$

$$\langle \hat{\Phi}_1^\dagger(x) \hat{\Phi}_2(0) \rangle = \sqrt{\rho_{10} \rho_{20}} e^{\langle \hat{\theta}_1(x) \hat{\theta}_2(0) \rangle - \frac{1}{2} \langle \hat{\theta}_1^2(0) \rangle - \frac{1}{2} \langle \hat{\theta}_2^2(0) \rangle} = \sqrt{\rho_{10} \rho_{20}} e^{-\frac{|x|}{2c_1} + \frac{\xi_c}{2c_1} + \frac{2\gamma}{\pi c_2}} \left(\frac{L^2}{4\pi^2 \xi_c |x|} \right)^{-\frac{1}{\pi c_2}} \stackrel{L \rightarrow \infty}{=} 0. \quad (\text{B18})$$

The four-operator correlation functions are

$$\begin{aligned} \langle \hat{\Phi}_1^\dagger(\mathbf{x}) \hat{\Phi}_2(\mathbf{x}) \hat{\Phi}_1(0) \hat{\Phi}_2(0) \rangle &= \rho_{10} \rho_{20} \exp \left(\langle \hat{\theta}_1(\mathbf{x}) \hat{\theta}_1(0) \rangle + \langle \hat{\theta}_2(\mathbf{x}) \hat{\theta}_2(0) \rangle + 2 \langle \hat{\theta}_1(0) \hat{\theta}_2(0) \rangle - \langle \hat{\theta}_1^2(0) \rangle - \langle \hat{\theta}_2^2(0) \rangle - 2 \langle \hat{\theta}_1(\mathbf{x}) \hat{\theta}_2(0) \rangle \right) \\ &= \rho_{10} \rho_{20} \left(\frac{|x|}{\xi_c} \right)^{-\frac{4}{\pi c_2}}, \end{aligned} \quad (\text{B19})$$

$$\begin{aligned} \langle \hat{\Phi}_1^\dagger(\mathbf{x}) \hat{\Phi}_2^\dagger(\mathbf{x}) \hat{\Phi}_1(0) \hat{\Phi}_2(0) \rangle &= \rho_{10} \rho_{20} \exp \left(\langle \hat{\theta}_1(\mathbf{x}) \hat{\theta}_1(0) \rangle + \langle \hat{\theta}_2(\mathbf{x}) \hat{\theta}_2(0) \rangle - 2 \langle \hat{\theta}_1(0) \hat{\theta}_2(0) \rangle - \langle \hat{\theta}_1^2(0) \rangle - \langle \hat{\theta}_2^2(0) \rangle + 2 \langle \hat{\theta}_1(\mathbf{x}) \hat{\theta}_2(0) \rangle \right) \\ &= \rho_{10} \rho_{20} e^{\frac{2\xi_c}{c_1}} e^{-\frac{2}{c_1} |x|} \approx \rho_{10} \rho_{20} e^{-\frac{2}{c_1} |x|}. \end{aligned} \quad (\text{B20})$$

Since all these correlation functions do not saturate to constants, Model Series A does not break the charge symmetry in one spatial dimension. Since the correlation function $\langle \hat{\Phi}_1^\dagger(\mathbf{x}) \hat{\Phi}_2(\mathbf{x}) \hat{\Phi}_1(0) \hat{\Phi}_2(0) \rangle$ is power-law decay, the system is algebraically ordered in one spatial dimension.

2. Two spatial dimensions

In two spatial dimensions, we can plug in the results of the above approximation. The correlation functions of the operator $\hat{\theta}$ are

$$\begin{aligned} \langle \hat{\theta}_1(\mathbf{x}) \hat{\theta}_1(0) \rangle &= \langle \hat{\theta}_2(\mathbf{x}) \hat{\theta}_2(0) \rangle = \int \frac{d^2 \mathbf{k}}{(2\pi)^2} e^{i\mathbf{k} \cdot \mathbf{x}} \left(\frac{1}{c_1 |\mathbf{k}|^2} + \frac{1}{c_2 |\mathbf{k}|} \right) \\ &= \frac{1}{4\pi^2 c_1} \int dk d\theta \frac{e^{ik|x| \cos \theta}}{k} + \frac{1}{4\pi^2 c_2} \int dk d\theta e^{ik|x| \cos \theta} \\ &= \frac{1}{4\pi^2 c_1} \int_0^{2\pi} d\theta \int_{\frac{2\pi}{L}}^\infty dk \frac{e^{ik|x| \cos \theta}}{k} + \frac{1}{4\pi^2 c_2} \int_0^{2\pi} d\theta \int_0^\infty dk e^{ik|x| \cos \theta}. \end{aligned} \quad (\text{B21})$$

We need to compute these integrals separately,

$$\int_0^{2\pi} d\theta \int_{\frac{2\pi}{L}}^\infty dk \frac{e^{ik|x| \cos \theta}}{k} \stackrel{L \rightarrow \infty}{=} -2\pi \left[\gamma + \ln \left(\frac{\pi |x|}{L} \right) \right], \quad (\text{B22})$$

$$\int_0^{2\pi} d\theta \int_0^\infty dk e^{ik|x| \cos \theta} = 2\pi \int_0^\infty dk J_0(k|x|) = \frac{2\pi}{|x|}, \quad (\text{B23})$$

where $J_n(z)$ is the Bessel function of the first kind. Then, we can get the correlation functions.

$$\langle \hat{\theta}_1(\mathbf{x}) \hat{\theta}_1(0) \rangle = \langle \hat{\theta}_2(\mathbf{x}) \hat{\theta}_2(0) \rangle \stackrel{L \rightarrow \infty}{=} \frac{1}{2\pi c_1} \left[-\gamma + \ln \left(\frac{L}{\pi |x|} \right) \right] + \frac{1}{2\pi c_2 |x|} \quad (\text{B24})$$

Similarly, we have

$$\langle \hat{\theta}_1(\mathbf{x}) \hat{\theta}_2(\mathbf{0}) \rangle \stackrel{L \rightarrow \infty}{=} \frac{1}{2\pi c_1} \left[-\gamma + \ln \left(\frac{L}{\pi |\mathbf{x}|} \right) \right] - \frac{1}{2\pi c_2 |\mathbf{x}|}. \quad (\text{B25})$$

To get $\langle \hat{\theta}_a^2(0) \rangle = \lim_{x \rightarrow 0} \langle \theta_a(x) \theta_a(0) \rangle$, we just need to replace $|x|$ by ξ_c ,

$$\langle \hat{\theta}_1^2(\mathbf{0}) \rangle = \langle \hat{\theta}_2^2(\mathbf{0}) \rangle = \frac{1}{2\pi c_1} \left[-\gamma + \ln \left(\frac{L}{\pi \xi_c} \right) \right] + \frac{1}{2\pi c_2 \xi_c}, \quad (\text{B26})$$

$$\langle \hat{\theta}_1(\mathbf{0}) \hat{\theta}_2(\mathbf{0}) \rangle = \frac{1}{2\pi c_1} \left[-\gamma + \ln \left(\frac{L}{\pi \xi_c} \right) \right] - \frac{1}{2\pi c_2 \xi_c}. \quad (\text{B27})$$

Then, we can calculate correlation functions of $\hat{\Phi}$. The two-operator correlation functions are

$$\langle \hat{\Phi}_1^\dagger(\mathbf{x}) \hat{\Phi}_1(\mathbf{0}) \rangle = \rho_{10} e^{\langle \hat{\theta}_1(\mathbf{x}) \hat{\theta}_1(\mathbf{0}) \rangle - \langle \hat{\theta}_1^2(\mathbf{0}) \rangle} = \rho_{10} \left(\frac{|\mathbf{x}|}{\xi_c} \right)^{-\frac{1}{2\pi c_1}} e^{\frac{1}{2\pi c_2 |\mathbf{x}|} - \frac{1}{2\pi c_2 \xi_c}}, \quad (\text{B28})$$

$$\langle \hat{\Phi}_2^\dagger(\mathbf{x}) \hat{\Phi}_2(\mathbf{0}) \rangle = \rho_{20} e^{\langle \hat{\theta}_2(\mathbf{x}) \hat{\theta}_2(\mathbf{0}) \rangle - \langle \hat{\theta}_2^2(\mathbf{0}) \rangle} = \rho_{20} \left(\frac{|\mathbf{x}|}{\xi_c} \right)^{-\frac{1}{2\pi c_1}} e^{\frac{1}{2\pi c_2 |\mathbf{x}|} - \frac{1}{2\pi c_2 \xi_c}}, \quad (\text{B29})$$

$$\langle \hat{\Phi}_1^\dagger(\mathbf{x}) \hat{\Phi}_2(\mathbf{0}) \rangle = \sqrt{\rho_{10} \rho_{20}} e^{\langle \hat{\theta}_1(\mathbf{x}) \hat{\theta}_2(\mathbf{0}) \rangle - \frac{1}{2} \langle \hat{\theta}_1^2(\mathbf{0}) \rangle - \frac{1}{2} \langle \hat{\theta}_2^2(\mathbf{0}) \rangle} = \sqrt{\rho_{10} \rho_{20}} \left(\frac{|\mathbf{x}|}{\xi_c} \right)^{-\frac{1}{2\pi c_1}} e^{-\frac{1}{2\pi c_2 |\mathbf{x}|} - \frac{1}{2\pi c_2 \xi_c}}. \quad (\text{B30})$$

The four-operator correlation functions are

$$\begin{aligned} \langle \hat{\Phi}_1^\dagger(\mathbf{x}) \hat{\Phi}_2(\mathbf{x}) \hat{\Phi}_1(\mathbf{0}) \hat{\Phi}_2^\dagger(\mathbf{0}) \rangle &= \rho_{10} \rho_{20} \exp \left(\langle \hat{\theta}_1(\mathbf{x}) \hat{\theta}_1(\mathbf{0}) \rangle + \langle \hat{\theta}_2(\mathbf{x}) \hat{\theta}_2(\mathbf{0}) \rangle + 2 \langle \hat{\theta}_1(\mathbf{0}) \hat{\theta}_2(\mathbf{0}) \rangle - \langle \hat{\theta}_1^2(\mathbf{0}) \rangle - \langle \hat{\theta}_2^2(\mathbf{0}) \rangle - 2 \langle \hat{\theta}_1(\mathbf{x}) \hat{\theta}_2(\mathbf{0}) \rangle \right) \\ &= \rho_{10} \rho_{20} e^{\frac{2}{\pi c_1 |\mathbf{x}|} - \frac{2}{\pi c_1 \xi_c}}, \end{aligned} \quad (\text{B31})$$

$$\begin{aligned} \langle \hat{\Phi}_1^\dagger(\mathbf{x}) \hat{\Phi}_2^\dagger(\mathbf{x}) \hat{\Phi}_1(\mathbf{0}) \hat{\Phi}_2(\mathbf{0}) \rangle &= \rho_{10} \rho_{20} \exp \left(\langle \hat{\theta}_1(\mathbf{x}) \hat{\theta}_1(\mathbf{0}) \rangle + \langle \hat{\theta}_2(\mathbf{x}) \hat{\theta}_2(\mathbf{0}) \rangle - 2 \langle \hat{\theta}_1(\mathbf{0}) \hat{\theta}_2(\mathbf{0}) \rangle - \langle \hat{\theta}_1^2(\mathbf{0}) \rangle - \langle \hat{\theta}_2^2(\mathbf{0}) \rangle + 2 \langle \hat{\theta}_1(\mathbf{x}) \hat{\theta}_2(\mathbf{0}) \rangle \right) \\ &= \rho_{10} \rho_{20} \left(\frac{|\mathbf{x}|}{\xi_c} \right)^{-\frac{2}{\pi c_1}}. \end{aligned} \quad (\text{B32})$$

Since the correlation function $\langle \hat{\Phi}_1^\dagger(\mathbf{x}) \hat{\Phi}_2(\mathbf{x}) \hat{\Phi}_1(\mathbf{0}) \hat{\Phi}_2^\dagger(\mathbf{0}) \rangle$ saturates to a constant when $|\mathbf{x}| \rightarrow \infty$, Model Series A spontaneously breaks the relative charge symmetry $U(1)_{-C}$ and has a true ODLRO in two spatial dimensions.

3. Three spatial dimensions

In three spatial dimensions, we can plug in the results of the above approximation. The correlation functions of the operator $\hat{\theta}$ are

$$\begin{aligned} \langle \hat{\theta}_1(\mathbf{x}) \hat{\theta}_1(\mathbf{0}) \rangle &= \langle \hat{\theta}_2(\mathbf{x}) \hat{\theta}_2(\mathbf{0}) \rangle = \int \frac{d^3 \mathbf{k}}{(2\pi)^3} e^{i\mathbf{k} \cdot \mathbf{x}} \left(\frac{1}{c_1 |\mathbf{k}|^2} + \frac{1}{c_2 |\mathbf{k}|} \right) \\ &= \frac{1}{8\pi^3 c_1} \int_0^\pi d\psi \int_0^{2\pi} d\theta \int_0^\infty dk (\sin \psi e^{ik|\mathbf{x}| \cos \theta \sin \psi}) \\ &\quad + \frac{1}{8\pi^3 c_2} \int_0^\pi d\psi \int_0^{2\pi} d\theta \int_0^{\frac{2\pi}{\xi_c}} dk (k \sin \psi e^{ik|\mathbf{x}| \cos \theta \sin \psi}). \end{aligned} \quad (\text{B33})$$

We need to compute these integrals separately,

$$\int_0^\pi d\psi \int_0^{2\pi} d\theta \int_0^\infty dk (\sin \psi e^{ik|\mathbf{x}| \cos \theta \sin \psi}) = \int_0^\infty dk \frac{4\pi \sin(k|\mathbf{x}|)}{k|\mathbf{x}|} = \frac{2\pi^2}{|\mathbf{x}|}, \quad (\text{B34})$$

$$\int_0^\pi d\psi \int_0^{2\pi} d\theta \int_0^{\frac{2\pi}{\xi_c}} dk (k \sin \psi e^{ik|\mathbf{x}| \cos \theta \sin \psi}) = \int_0^{\frac{2\pi}{\xi_c}} dk \frac{4\pi k \sin(k|\mathbf{x}|)}{k|\mathbf{x}|} = \frac{8\pi \sin^2 \left(\frac{\pi |\mathbf{x}|}{\xi_c} \right)}{|\mathbf{x}|^2}. \quad (\text{B35})$$

Then, we can get the correlation functions

$$\langle \hat{\theta}_1(\mathbf{x}) \hat{\theta}_1(\mathbf{0}) \rangle = \langle \hat{\theta}_2(\mathbf{x}) \hat{\theta}_2(\mathbf{0}) \rangle = \frac{1}{4\pi c_1 |\mathbf{x}|} + \frac{\sin^2\left(\frac{\pi|\mathbf{x}|}{\xi_c}\right)}{\pi^2 c_2 |\mathbf{x}|^2}. \quad (\text{B36})$$

Similarly, we have

$$\langle \hat{\theta}_1(\mathbf{x}) \hat{\theta}_2(\mathbf{0}) \rangle = \frac{1}{4\pi c_1 |\mathbf{x}|} - \frac{\sin^2\left(\frac{\pi|\mathbf{x}|}{\xi_c}\right)}{\pi^2 c_2 |\mathbf{x}|^2}. \quad (\text{B37})$$

Since the coherence length is introduced by the previous integration, we cannot just set $|\mathbf{x}| = \xi_c$ to get $\langle \hat{\theta}_a^2(\mathbf{0}) \rangle$. We need to calculate the integral

$$\int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{c_2 |\mathbf{k}|} = \frac{1}{8\pi^3 c_2} \int_0^{\frac{2\pi}{\xi_c}} dk \left(4\pi k^2 \frac{1}{k} \right) = \frac{1}{2\pi^2 c_2} \int_0^{\frac{2\pi}{\xi_c}} k dk = \frac{1}{c_2 \xi_c^2}. \quad (\text{B38})$$

Then, we can get

$$\langle \hat{\theta}_1^2(\mathbf{0}) \rangle = \langle \hat{\theta}_2^2(\mathbf{0}) \rangle = \frac{1}{4\pi c_1 \xi_c} + \frac{1}{c_2 \xi_c^2}, \quad (\text{B39})$$

$$\langle \hat{\theta}_1(\mathbf{0}) \hat{\theta}_2(\mathbf{0}) \rangle = \frac{1}{4\pi c_1 \xi_c} - \frac{1}{c_2 \xi_c^2}. \quad (\text{B40})$$

Then, we can calculate correlation functions of $\hat{\Phi}$. The two-operator correlation functions are

$$\langle \hat{\Phi}_1^\dagger(\mathbf{x}) \hat{\Phi}_1(\mathbf{0}) \rangle = \rho_{10} e^{\langle \hat{\theta}_1(\mathbf{x}) \hat{\theta}_1(\mathbf{0}) \rangle - \langle \hat{\theta}_1^2(\mathbf{0}) \rangle} \xrightarrow{|\mathbf{x}| \rightarrow \infty} \rho_{10} e^{-\frac{1}{4\pi c_1 \xi_c} - \frac{1}{c_2 \xi_c^2}}, \quad (\text{B41})$$

$$\langle \hat{\Phi}_2^\dagger(\mathbf{x}) \hat{\Phi}_2(\mathbf{0}) \rangle = \rho_{20} e^{\langle \hat{\theta}_2(\mathbf{x}) \hat{\theta}_2(\mathbf{0}) \rangle - \langle \hat{\theta}_2^2(\mathbf{0}) \rangle} \xrightarrow{|\mathbf{x}| \rightarrow \infty} \rho_{20} e^{-\frac{1}{4\pi c_1 \xi_c} - \frac{1}{c_2 \xi_c^2}}, \quad (\text{B42})$$

$$\langle \hat{\Phi}_1^\dagger(\mathbf{x}) \hat{\Phi}_2(\mathbf{0}) \rangle = \sqrt{\rho_{10} \rho_{20}} e^{\langle \hat{\theta}_1(\mathbf{x}) \hat{\theta}_2(\mathbf{0}) \rangle - \frac{1}{2} \langle \hat{\theta}_1^2(\mathbf{0}) \rangle - \frac{1}{2} \langle \hat{\theta}_2^2(\mathbf{0}) \rangle} \xrightarrow{|\mathbf{x}| \rightarrow \infty} \sqrt{\rho_{10} \rho_{20}} e^{-\frac{1}{4\pi c_1 \xi_c} - \frac{1}{c_2 \xi_c^2}}. \quad (\text{B43})$$

The four-operator correlation functions are

$$\begin{aligned} \langle \hat{\Phi}_1^\dagger(\mathbf{x}) \hat{\Phi}_2(\mathbf{x}) \hat{\Phi}_1(\mathbf{0}) \hat{\Phi}_2^\dagger(\mathbf{0}) \rangle &= \rho_{10} \rho_{20} \exp \left(\langle \hat{\theta}_1(\mathbf{x}) \hat{\theta}_1(\mathbf{0}) \rangle + \langle \hat{\theta}_2(\mathbf{x}) \hat{\theta}_2(\mathbf{0}) \rangle + 2 \langle \hat{\theta}_1(\mathbf{0}) \hat{\theta}_2(\mathbf{0}) \rangle - \langle \hat{\theta}_1^2(\mathbf{0}) \rangle - \langle \hat{\theta}_2^2(\mathbf{0}) \rangle - 2 \langle \hat{\theta}_1(\mathbf{x}) \hat{\theta}_2(\mathbf{0}) \rangle \right) \\ &\xrightarrow{|\mathbf{x}| \rightarrow \infty} \rho_{10} \rho_{20} e^{-\frac{4}{c_2 \xi_c^2}}, \end{aligned} \quad (\text{B44})$$

$$\begin{aligned} \langle \hat{\Phi}_1^\dagger(\mathbf{x}) \hat{\Phi}_2^\dagger(\mathbf{x}) \hat{\Phi}_1(\mathbf{0}) \hat{\Phi}_2(\mathbf{0}) \rangle &= \rho_{10} \rho_{20} \exp \left(\langle \hat{\theta}_1(\mathbf{x}) \hat{\theta}_1(\mathbf{0}) \rangle + \langle \hat{\theta}_2(\mathbf{x}) \hat{\theta}_2(\mathbf{0}) \rangle - 2 \langle \hat{\theta}_1(\mathbf{0}) \hat{\theta}_2(\mathbf{0}) \rangle - \langle \hat{\theta}_1^2(\mathbf{0}) \rangle - \langle \hat{\theta}_2^2(\mathbf{0}) \rangle + 2 \langle \hat{\theta}_1(\mathbf{x}) \hat{\theta}_2(\mathbf{0}) \rangle \right) \\ &\xrightarrow{|\mathbf{x}| \rightarrow \infty} \rho_{10} \rho_{20} e^{-\frac{1}{\pi c_1 \xi_c}}. \end{aligned} \quad (\text{B45})$$

Since all the correlation functions above saturate to constants when $|\mathbf{x}| \rightarrow \infty$, Model Series A breaks the charge symmetry and has a true ODLRO in three spatial dimensions.

APPENDIX C: DERIVATIONS OF GOLDSTONE MODES AND DISPERSION RELATIONS FOR MODEL SERIES B

In this Appendix, we give the detailed derivation of Goldstone mode and dispersion relation for Model Series B in Sec. IV B.

1. Derivation of the effective Hamiltonian density

Considering quantum fluctuations, we have $\hat{\Phi}_a(\mathbf{x}) = \sqrt{\rho_{a0} + \hat{f}_a(\mathbf{x})} e^{i\hat{\theta}_a(\mathbf{x})}$, where $\hat{f}_a(\mathbf{x}) \ll \rho_{a0}$. The commutation relation between $\hat{\theta}_a$ and $\hat{f}_a$ is given by $[\hat{\theta}_a(\mathbf{x}), \hat{f}_b(\mathbf{y})] = -i\delta^{(d)}(\mathbf{x} - \mathbf{y})\delta_{ab}$. Therefore, the conjugate momentum operator of $\hat{\theta}_a$ is $\hat{\pi}_a = -\hat{f}_a$, and the operators $\hat{\theta}_a$ and $\hat{\pi}_a$ satisfy the commutation relations $[\hat{\theta}_a(\mathbf{x}), \hat{\pi}_b(\mathbf{y})] = i\delta_{ab}\delta^{(d)}(\mathbf{x} - \mathbf{y})$. We can derive the effective

Hamiltonian

$$\begin{aligned}
\mathcal{H} = & \frac{g}{2}\hat{\pi}_1^2 + \frac{g}{2}\hat{\pi}_2^2 + \sum_{i,j}^d K_1^{ij} \left[\rho_{10}^2 (\partial_i \partial_j \hat{\theta}_1)^2 + \frac{(\partial_i \partial_j \hat{\pi}_1)^2}{4} \right] \\
& + \sum_{i,j,k}^d K_2^{ijk} \left[\rho_{20}^3 (\partial_i \partial_j \partial_k \hat{\theta}_2)^2 + \frac{1}{4} \rho_{20} (\partial_i \partial_j \partial_k \hat{\pi}_2)^2 + \frac{1}{2} (\partial_i \hat{\pi}_2 \partial_j \partial_k \hat{\pi}_2 + \partial_j \hat{\pi}_2 \partial_i \partial_k \hat{\pi}_2 + \partial_k \hat{\pi}_2 \partial_i \partial_j \hat{\pi}_2) \partial_i \partial_j \partial_k \hat{\pi}_2 \right] \\
& + \sum_i^d \Gamma_{1,2}^i \left[4\rho_{10} \rho_{20}^2 (\partial_i \hat{\theta}_1)^2 - 4\ell \rho_{10} \rho_{20}^2 \partial_i \hat{\theta}_1 \partial_i^2 \hat{\theta}_2 + \ell^2 \rho_{10} \rho_{20}^2 (\partial_i^2 \hat{\theta}_2)^2 + \frac{\rho_{20}^2}{\rho_{10}} (\partial_i \hat{\rho}_1)^2 - \ell \partial_i \hat{\pi}_1 (\partial_i \hat{\rho}_2)^2 \right. \\
& \left. - \ell \rho_{20} \partial_i \hat{\pi}_1 \partial_i^2 \hat{\pi}_2 + \frac{\ell^2 \rho_{10}}{4\rho_{20}^2} (\partial_i \hat{\pi}_2)^4 + \frac{\ell^2 \rho_{10}}{2\rho_{20}} (\partial_i \hat{\pi}_2)^2 \partial_i^2 \hat{\pi}_2 + \frac{1}{4} \ell^2 \rho_{10} (\partial_i^2 \hat{\pi}_2)^2 \right]. \tag{C1}
\end{aligned}$$

We can obtain the time evolution of $\hat{\theta}_a$:

$$\dot{\hat{\theta}}_1 = g\hat{\pi}_1 + \sum_{i,j}^d \frac{1}{2} K_1^{ij} \partial_i^2 \partial_j^2 \hat{\pi}_1 + \sum_i^d -2\Gamma_{1,2}^i \frac{\rho_{20}^2}{\rho_{10}} \partial_i^2 \hat{\pi}_1 + 2\Gamma_{1,2}^i \ell \partial_i \hat{\pi}_2 \partial_i^2 \hat{\pi}_2 + \Gamma_{1,2}^i \ell \rho_{20} \partial_i^3 \hat{\pi}_2, \tag{C2}$$

$$\begin{aligned}
\dot{\hat{\theta}}_2 = & g\hat{\pi}_2 + \sum_{i,j,k}^d \frac{1}{2} K_2^{ijk} \left[-\rho_{20} \partial_i^2 \partial_j^2 \partial_k^2 \hat{\pi}_2 - 3(\partial_i \partial_j \partial_k \hat{\pi}_2)^2 - 2\partial_i \partial_j \hat{\pi}_2 \partial_i \partial_j \partial_k^2 \hat{\pi}_2 - 2\partial_i \partial_k^2 \hat{\pi}_2 \partial_i \partial_j^2 \hat{\pi}_2 - 2\partial_i \partial_k \hat{\pi}_2 \partial_i \partial_j^2 \partial_k \hat{\pi}_2 \right. \\
& \left. - 2\partial_j^2 \partial_k \hat{\pi}_2 \partial_i^2 \partial_k \hat{\pi}_2 - 2\partial_j \partial_k^2 \hat{\pi}_2 \partial_i^2 \partial_j \hat{\pi}_2 - 2\partial_j \partial_k \hat{\pi}_2 \partial_i^2 \partial_j \partial_k \hat{\pi}_2 - \partial_i^2 \hat{\pi}_2 \partial_j^2 \partial_k^2 \hat{\pi}_2 - \partial_j^2 \hat{\pi}_2 \partial_i^2 \partial_k^2 \hat{\pi}_2 - \partial_k^2 \hat{\pi}_2 \partial_i^2 \partial_j^2 \hat{\pi}_2 \right] \\
& + \sum_i^d \Gamma_{1,2}^i \left[2\ell \partial_i^2 \hat{\pi}_1 \partial_i \hat{\pi}_2 + 2\ell \partial_i \hat{\pi}_1 \partial_i^2 \hat{\pi}_2 - \ell \rho_{20} \partial_i^3 \hat{\pi}_1 - \frac{3\ell^2 \rho_{10}}{\rho_{20}^2} (\partial_i \hat{\pi}_2)^2 \partial_i^2 \hat{\pi}_2 - \frac{1}{2} \ell^2 \rho_{10} \partial_i^4 \hat{\pi}_2 \right]. \tag{C3}
\end{aligned}$$

In the long-wavelength limit, we have

$$g\hat{\pi}_1 \gg \sum_{i,j}^d \frac{1}{2} K_1^{ij} \partial_i^2 \partial_j^2 \hat{\pi}_1 + \sum_i^d -2\Gamma_{1,2}^i \frac{\rho_{20}^2}{\rho_{10}} \partial_i^2 \hat{\pi}_1 + 2\Gamma_{1,2}^i \ell \partial_i \hat{\pi}_2 \partial_i^2 \hat{\pi}_2 + \Gamma_{1,2}^i \ell \rho_{20} \partial_i^3 \hat{\pi}_2, \tag{C4}$$

$$\begin{aligned}
g\hat{\pi}_2 \gg & \sum_{i,j,k}^d \frac{1}{2} K_2^{ijk} \left[-\rho_{20} \partial_i^2 \partial_j^2 \partial_k^2 \hat{\pi}_2 - 3(\partial_i \partial_j \partial_k \hat{\pi}_2)^2 - 2\partial_i \partial_j \hat{\pi}_2 \partial_i \partial_j \partial_k^2 \hat{\pi}_2 - 2\partial_i \partial_k^2 \hat{\pi}_2 \partial_i \partial_j^2 \hat{\pi}_2 - 2\partial_i \partial_k \hat{\pi}_2 \partial_i \partial_j^2 \partial_k \hat{\pi}_2 \right. \\
& \left. - 2\partial_j^2 \partial_k \hat{\pi}_2 \partial_i^2 \partial_k \hat{\pi}_2 - 2\partial_j \partial_k^2 \hat{\pi}_2 \partial_i^2 \partial_j \hat{\pi}_2 - 2\partial_j \partial_k \hat{\pi}_2 \partial_i^2 \partial_j \partial_k \hat{\pi}_2 - \partial_i^2 \hat{\pi}_2 \partial_j^2 \partial_k^2 \hat{\pi}_2 - \partial_j^2 \hat{\pi}_2 \partial_i^2 \partial_k^2 \hat{\pi}_2 - \partial_k^2 \hat{\pi}_2 \partial_i^2 \partial_j^2 \hat{\pi}_2 \right] \\
& + \sum_i^d \Gamma_{1,2}^i \left[2\ell \partial_i^2 \hat{\pi}_1 \partial_i \hat{\pi}_2 + 2\ell \partial_i \hat{\pi}_1 \partial_i^2 \hat{\pi}_2 - \ell \rho_{20} \partial_i^3 \hat{\pi}_1 - \frac{3\ell^2 \rho_{10}}{\rho_{20}^2} (\partial_i \hat{\pi}_2)^2 \partial_i^2 \hat{\pi}_2 - \frac{1}{2} \ell^2 \rho_{10} \partial_i^4 \hat{\pi}_2 \right]. \tag{C5}
\end{aligned}$$

Here we just need to consider the lowest order and get

$$g \gg \sum_i^d 2\Gamma_{1,2}^i \frac{\rho_{20}^2}{\rho_{10}} k_i^2. \tag{C6}$$

If we set an isotropic $K_1^{ij} = \frac{\kappa_1}{2}$, $K_2^{ijk} = \frac{\kappa_2}{2}$, and $\Gamma_{1,2}^i = \frac{\Gamma}{2}$, we can get the coherence length $\xi_c = 2\pi \sqrt{\frac{\Gamma \rho_{20}^2}{g \rho_{10}}}$. Then, we can drop the higher-order terms in Eq. (C1), and get the effective Hamiltonian

$$\mathcal{H} = \frac{g}{2}\hat{\pi}_1^2 + \frac{g}{2}\hat{\pi}_2^2 + \frac{\kappa_1 \rho_{10}^2}{2} (\partial_i \partial_j \hat{\theta}_1)^2 + \frac{\kappa_2 \rho_{20}^3}{2} (\partial_i \partial_j \partial_k \hat{\theta}_2)^2 + \frac{\Gamma \rho_{10} \rho_{20}^2}{2} (\ell \partial_i^2 \hat{\theta}_2 - 2\partial_i \hat{\theta}_1)^2. \tag{C7}$$

2. Derivation of the Hamiltonian in the momentum space

In the momentum space, we can perform a Fourier transform,

$$\begin{aligned}
 H = & \frac{1}{(2\pi)^{2d}} \int d^d \mathbf{x} d^d \mathbf{k} d^d \mathbf{k}' e^{i\mathbf{k}\cdot\mathbf{x}+i\mathbf{k}'\cdot\mathbf{x}} \left[\frac{g}{2} \hat{\pi}_1(\mathbf{k}) \hat{\pi}_1(\mathbf{k}') + \frac{g}{2} \hat{\pi}_2(\mathbf{k}) \hat{\pi}_2(\mathbf{k}') + \frac{\kappa_1 \rho_{10}^2}{2} \sum_{i,j}^d k_i^2 k_j^2 \hat{\theta}_1(\mathbf{k}) \hat{\theta}_1(\mathbf{k}') \right. \\
 & + \frac{\kappa_2 \rho_{20}^3}{2} \sum_{i,j,k}^d k_i^2 k_j^2 k_k^2 \hat{\theta}_2(\mathbf{k}) \hat{\theta}_2(\mathbf{k}') + \frac{\Gamma \rho_{10} \rho_{20}^2}{2} \sum_i^d [\ell^2 k_i^4 \hat{\theta}_2(\mathbf{k}) \hat{\theta}_2(\mathbf{k}') + 4k_i^2 \hat{\theta}_1(\mathbf{k}) \hat{\theta}_1(\mathbf{k}') + 2i\ell k_i^3 \hat{\theta}_1(\mathbf{k}) \hat{\theta}_2(\mathbf{k}') \\
 & \left. - 2i\ell k_i^3 \hat{\theta}_2(\mathbf{k}) \hat{\theta}_1(\mathbf{k}') \right]. \quad (C8)
 \end{aligned}$$

Integrating over the real space, we can obtain a δ function so that only the integral of the momentum space is left in the Hamiltonian,

$$\begin{aligned}
 H = & \frac{1}{(2\pi)^d} \int d^d \mathbf{k} d^d \mathbf{k}' \delta(\mathbf{k} + \mathbf{k}') \left[\frac{g}{2} \hat{\pi}_1(\mathbf{k}) \hat{\pi}_1(\mathbf{k}') + \frac{g}{2} \hat{\pi}_2(\mathbf{k}) \hat{\pi}_2(\mathbf{k}') + \frac{\kappa_1 \rho_{10}^2 |\mathbf{k}|^4}{2} \hat{\theta}_1(\mathbf{k}) \hat{\theta}_1(\mathbf{k}') + \frac{\kappa_2 \rho_{20}^3 |\mathbf{k}|^6}{2} \hat{\theta}_2(\mathbf{k}) \hat{\theta}_2(\mathbf{k}') \right. \\
 & + \frac{\Gamma \rho_{10} \rho_{20}^2}{2} \sum_i^d [\ell^2 k_i^4 \hat{\theta}_2(\mathbf{k}) \hat{\theta}_2(\mathbf{k}') + 4k_i^2 \hat{\theta}_1(\mathbf{k}) \hat{\theta}_1(\mathbf{k}') + 2i\ell k_i^3 \hat{\theta}_1(\mathbf{k}) \hat{\theta}_2(\mathbf{k}') - 2i\ell k_i^3 \hat{\theta}_2(\mathbf{k}) \hat{\theta}_1(\mathbf{k}')] \Big] \\
 = & \frac{1}{(2\pi)^d} \int d^d \mathbf{k} \left[\frac{g}{2} \hat{\pi}_1(\mathbf{k}) \hat{\pi}_1(-\mathbf{k}) + \frac{g}{2} \hat{\pi}_2(\mathbf{k}) \hat{\pi}_2(-\mathbf{k}) + \frac{\kappa_1 \rho_{10}^2 |\mathbf{k}|^4}{2} \hat{\theta}_1(\mathbf{k}) \hat{\theta}_1(-\mathbf{k}) + \frac{\kappa_2 \rho_{20}^3 |\mathbf{k}|^6}{2} \hat{\theta}_2(\mathbf{k}) \hat{\theta}_2(-\mathbf{k}) \right. \\
 & + \frac{\Gamma \rho_{10} \rho_{20}^2}{2} \sum_i^d [\ell^2 k_i^4 \hat{\theta}_2(\mathbf{k}) \hat{\theta}_2(-\mathbf{k}) + 4k_i^2 \hat{\theta}_1(\mathbf{k}) \hat{\theta}_1(-\mathbf{k}) + 2i\ell k_i^3 \hat{\theta}_1(\mathbf{k}) \hat{\theta}_2(-\mathbf{k}) - 2i\ell k_i^3 \hat{\theta}_2(\mathbf{k}) \hat{\theta}_1(-\mathbf{k})] \Big]. \quad (C9)
 \end{aligned}$$

Using the same method we used to compute Model 1, we can write the Hamiltonian in matrix form

$$\begin{aligned}
 H = & \frac{1}{2(2\pi)^d} \int d^d \mathbf{k} \left[g \hat{\pi}_1(\mathbf{k}) \hat{\pi}_1(-\mathbf{k}) + g \hat{\pi}_2(\mathbf{k}) \hat{\pi}_2(-\mathbf{k}) \right. \\
 & + \begin{bmatrix} \hat{\theta}_1(-\mathbf{k}) & \hat{\theta}_2(-\mathbf{k}) \end{bmatrix} \begin{bmatrix} \kappa_1 \rho_{10}^2 |\mathbf{k}|^4 + 4\Gamma \rho_{10} \rho_{20}^2 |\mathbf{k}|^2 & -2i\Gamma \rho_{10} \rho_{20}^2 \ell \sum_i^d k_i^3 \\ 2i\Gamma \rho_{10} \rho_{20}^2 \ell \sum_i^d k_i^3 & \kappa_2 \rho_{20}^3 |\mathbf{k}|^6 + \Gamma \rho_{10} \rho_{20}^2 \ell^2 \sum_i^d k_i^4 \end{bmatrix} \begin{bmatrix} \hat{\theta}_1(\mathbf{k}) \\ \hat{\theta}_2(\mathbf{k}) \end{bmatrix} \Big] \\
 = & \frac{1}{2(2\pi)^d} \int d^d \mathbf{k} \psi^\dagger(\mathbf{k}) \begin{bmatrix} g & 0 & 0 & 0 \\ 0 & g & 0 & 0 \\ 0 & 0 & \kappa_1 \rho_{10}^2 |\mathbf{k}|^4 + 4\Gamma \rho_{10} \rho_{20}^2 |\mathbf{k}|^2 & -2i\Gamma \rho_{10} \rho_{20}^2 \ell \sum_i^d k_i^3 \\ 0 & 0 & 2i\Gamma \rho_{10} \rho_{20}^2 \ell \sum_i^d k_i^3 & \kappa_2 \rho_{20}^3 |\mathbf{k}|^6 + \Gamma \rho_{10} \rho_{20}^2 \ell^2 \sum_i^d k_i^4 \end{bmatrix} \psi(\mathbf{k}) \\
 = & \frac{1}{2(2\pi)^d} \int d^d \mathbf{k} \psi^\dagger(\mathbf{k}) M(\mathbf{k}) \psi(\mathbf{k}). \quad (C10)
 \end{aligned}$$

In the momentum space, the operators $\hat{\theta}_a(\mathbf{k})$ and the conjugate momentum densities $\hat{\pi}_b(\mathbf{k}')$ satisfy the commutation relations $[\hat{\theta}_a(\mathbf{k}), \hat{\pi}_b(\mathbf{k}')] = i(2\pi)^d \delta_{ab} \delta^{(d)}(\mathbf{k} + \mathbf{k}')$.

3. Bogoliubov transformation and the transformation matrix $T(\mathbf{k})$

In this Hamiltonian, two species of bosons couple to each other. By the Bogoliubov transformation, we can convert the coupling bosons into two independent modes. We introduce a transformation matrix $T(\mathbf{k})$ which changes the basis from the basis $\psi(\mathbf{k})$ into the basis $\phi(\mathbf{k})$:

$$\phi(\mathbf{k}) = [\hat{\alpha}(\mathbf{k}) \quad \hat{\beta}(\mathbf{k}) \quad \hat{\alpha}^\dagger(-\mathbf{k}) \quad \hat{\beta}^\dagger(-\mathbf{k})]^T = T(\mathbf{k}) \psi(\mathbf{k}), \quad (C11)$$

where $\hat{\alpha}(\mathbf{k})$ and $\hat{\beta}(\mathbf{k})$ are annihilation operators of two independent modes, and they satisfy the commutation relations

$$[\hat{\alpha}(\mathbf{k}), \hat{\alpha}^\dagger(\mathbf{k}')] = [\hat{\beta}(\mathbf{k}), \hat{\beta}^\dagger(\mathbf{k}')] = (2\pi)^d \delta^{(d)}(\mathbf{k} - \mathbf{k}'), \quad (C12)$$

$$[\hat{\alpha}(\mathbf{k}), \hat{\alpha}(\mathbf{k}')] = [\hat{\beta}(\mathbf{k}), \hat{\beta}(\mathbf{k}')] = [\hat{\alpha}(\mathbf{k}), \hat{\beta}(\mathbf{k}')] = [\hat{\alpha}(\mathbf{k}), \hat{\beta}^\dagger(\mathbf{k}')] = [\hat{\beta}(\mathbf{k}), \hat{\alpha}^\dagger(\mathbf{k}')] = 0. \quad (C13)$$

To keep the commutation relations and diagonalize the matrix $M(\mathbf{k})$, the transformation matrix $T(\mathbf{k})$ needs to obey the equation below:

$$T(\mathbf{k})\Lambda_1 T^\dagger(\mathbf{k}) = \Lambda_2, \quad (\text{C14})$$

where

$$\Lambda_1 = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix}, \quad \Lambda_2 = \text{diag}(1, 1, -1, -1). \quad (\text{C16})$$

The diagonalized Hamiltonian becomes

$$H = \frac{1}{2(2\pi)^d} \phi^\dagger(\mathbf{k}) [T^{-1}(\mathbf{k})]^\dagger M(\mathbf{k}) T^{-1}(\mathbf{k}) \phi(\mathbf{k}), \quad (\text{C17})$$

where the matrix $[T^{-1}(\mathbf{k})]^\dagger M(\mathbf{k}) T^{-1}(\mathbf{k}) = \text{diag}(D_1(\mathbf{k}), D_2(\mathbf{k}), D_3(\mathbf{k}), D_4(\mathbf{k}))$ is a diagonal matrix with the diagonal elements

$$\begin{aligned} D_1(\mathbf{k}) = D_3(\mathbf{k}) &= \left\{ \frac{g}{2} \left(\kappa_1 |\mathbf{k}|^4 \rho_{10}^2 + \kappa_2 |\mathbf{k}|^6 \rho_{20}^3 + 4\Gamma |\mathbf{k}|^2 \rho_{10} \rho_{20}^2 + \Gamma \ell^2 \rho_{10} \rho_{20}^2 \sum_i^d k_i^4 \right. \right. \\ &\quad \left. \left. + \sqrt{\left(\kappa_1 |\mathbf{k}|^4 \rho_{10}^2 - \rho_{20}^2 \left(\kappa_2 |\mathbf{k}|^6 \rho_{20} + \Gamma \rho_{10} \left(\ell^2 \sum_i^d k_i^4 - 4|\mathbf{k}|^2 \right) \right) \right)^2 + 16\Gamma^2 \ell^2 \rho_{10}^2 \rho_{20}^4 \left(\sum_i^d k_i^3 \right)^2} \right) \right\}^{\frac{1}{2}}, \\ D_2(\mathbf{k}) = D_4(\mathbf{k}) &= \left\{ \frac{g}{2} \left(\kappa_1 |\mathbf{k}|^4 \rho_{10}^2 + \kappa_2 |\mathbf{k}|^6 \rho_{20}^3 + 4\Gamma |\mathbf{k}|^2 \rho_{10} \rho_{20}^2 + \Gamma \ell^2 \rho_{10} \rho_{20}^2 \sum_i^d k_i^4 \right. \right. \\ &\quad \left. \left. - \sqrt{\left(\kappa_1 |\mathbf{k}|^4 \rho_{10}^2 - \rho_{20}^2 \left(\kappa_2 |\mathbf{k}|^6 \rho_{20} + \Gamma \rho_{10} \left(\ell^2 \sum_i^d k_i^4 - 4|\mathbf{k}|^2 \right) \right) \right)^2 + 16\Gamma^2 \ell^2 \rho_{10}^2 \rho_{20}^4 \left(\sum_i^d k_i^3 \right)^2} \right) \right\}^{\frac{1}{2}}. \end{aligned} \quad (\text{C18})$$

The transformation matrix $T(\mathbf{k})$ is given by

$$T(\mathbf{k}) = \begin{bmatrix} T_{11}(\mathbf{k}) & T_{12}(\mathbf{k}) & T_{13}(\mathbf{k}) & T_{14}(\mathbf{k}) \\ T_{21}(\mathbf{k}) & T_{22}(\mathbf{k}) & T_{23}(\mathbf{k}) & T_{24}(\mathbf{k}) \\ T_{31}(\mathbf{k}) & T_{32}(\mathbf{k}) & T_{33}(\mathbf{k}) & T_{34}(\mathbf{k}) \\ T_{41}(\mathbf{k}) & T_{42}(\mathbf{k}) & T_{43}(\mathbf{k}) & T_{44}(\mathbf{k}) \end{bmatrix}, \quad (\text{C19})$$

where the matrix elements are

$$\begin{aligned} T_{11}(\mathbf{k}) = -T_{31}(\mathbf{k}) &= - \left\{ g \left[-\kappa_2 |\mathbf{k}|^6 \rho_{20}^3 + \kappa_1 |\mathbf{k}|^4 \rho_{10}^2 + 4\Gamma |\mathbf{k}|^2 \rho_{10} \rho_{20}^2 - \Gamma \ell^2 \rho_{10} \rho_{20}^2 \sum_i^d k_i^4 \right. \right. \\ &\quad \left. \left. + \sqrt{\left(\kappa_1 |\mathbf{k}|^4 \rho_{10}^2 - \rho_{20}^2 \left(\kappa_2 |\mathbf{k}|^6 \rho_{20} + \Gamma \rho_{10} \left(\ell^2 \sum_i^d k_i^4 - 4|\mathbf{k}|^2 \right) \right) \right)^2 + 16\Gamma^2 \ell^2 \rho_{10}^2 \rho_{20}^4 \left(\sum_i^d k_i^3 \right)^2} \right] \right\} / \\ &\quad \left\{ 2\sqrt{2}\Gamma \ell \rho_{10} \rho_{20}^2 \sum_i^d k_i^3 \left[g \left(4 + \left(\left(-\kappa_2 |\mathbf{k}|^6 \rho_{20}^3 + \kappa_1 |\mathbf{k}|^4 \rho_{10}^2 + 4\Gamma |\mathbf{k}|^2 \rho_{10} \rho_{20}^2 - \Gamma \ell^2 \rho_{10} \rho_{20}^2 \sum_i^d k_i^4 \right. \right. \right. \right. \right. \\ &\quad \left. \left. \left. + \sqrt{\left(\kappa_1 |\mathbf{k}|^4 \rho_{10}^2 - \rho_{20}^2 \left(\kappa_2 |\mathbf{k}|^6 \rho_{20} + \Gamma \rho_{10} \left(\ell^2 \sum_i^d k_i^4 - 4|\mathbf{k}|^2 \right) \right) \right)^2 + 16\Gamma^2 \ell^2 \rho_{10}^2 \rho_{20}^4 \left(\sum_i^d k_i^3 \right)^2} \right) \right) \right. \right. \\ &\quad \left. \left. \left(4\Gamma^2 \ell^2 \rho_{10}^2 \rho_{20}^4 \left(\sum_i^d k_i^3 \right)^2 \right) \right) \left(g \left(\kappa_2 |\mathbf{k}|^6 \rho_{20}^3 + \kappa_1 |\mathbf{k}|^4 \rho_{10}^2 + 4\Gamma |\mathbf{k}|^2 \rho_{10} \rho_{20}^2 + \Gamma \ell^2 \rho_{10} \rho_{20}^2 \sum_i^d k_i^4 \right. \right. \right. \right. \\ &\quad \left. \left. \left. + \sqrt{\left(\kappa_1 |\mathbf{k}|^4 \rho_{10}^2 - \rho_{20}^2 \left(\kappa_2 |\mathbf{k}|^6 \rho_{20} + \Gamma \rho_{10} \left(\ell^2 \sum_i^d k_i^4 - 4|\mathbf{k}|^2 \right) \right) \right)^2 + 16\Gamma^2 \ell^2 \rho_{10}^2 \rho_{20}^4 \left(\sum_i^d k_i^3 \right)^2} \right) \right) \right] \right\}^{\frac{1}{2}} \right\}^{\frac{1}{2}}, \end{aligned} \quad (\text{C20})$$

$$\begin{aligned}
T_{12}(\mathbf{k}) = -T_{32}(\mathbf{k}) = & (i2^{3/4}g) \Bigg/ \left\{ \left[g \left(4 + \left(\left(-\kappa_2 |\mathbf{k}|^6 \rho_{20}^3 + \kappa_1 |\mathbf{k}|^4 \rho_{10}^2 + 4\Gamma |\mathbf{k}|^2 \rho_{10} \rho_{20}^2 - \Gamma \ell^2 \rho_{10} \rho_{20}^2 \sum_i^d k_i^4 \right. \right. \right. \right. \right. \\
& + \sqrt{\left(\kappa_1 |\mathbf{k}|^4 \rho_{10}^2 - \rho_{20}^2 \left(\kappa_2 |\mathbf{k}|^6 \rho_{20} + \rho_{10} \left(\Gamma \ell^2 \sum_i^d k_i^4 - 4\Gamma |\mathbf{k}|^2 \right) \right) \right)^2 + 16\Gamma^2 \ell^2 \rho_{10}^2 \rho_{20}^4 \left(\sum_i^d k_i^3 \right)^2} \right)^2 \Bigg/ \\
& 4\Gamma^2 \ell^2 \rho_{10}^2 \rho_{20}^4 \left(\sum_i^d k_i^3 \right)^2 \Bigg] \left[g \left(\kappa_2 |\mathbf{k}|^6 \rho_{20}^3 + \kappa_1 |\mathbf{k}|^4 \rho_{10}^2 + 4\Gamma |\mathbf{k}|^2 \rho_{10} \rho_{20}^2 + \Gamma \ell^2 \rho_{10} \rho_{20}^2 \sum_i^d k_i^4 \right. \right. \\
& + \sqrt{\left(\kappa_1 |\mathbf{k}|^4 \rho_{10}^2 - \rho_{20}^2 \left(\kappa_2 |\mathbf{k}|^6 \rho_{20} + \rho_{10} \left(\Gamma \ell^2 \sum_i^d k_i^4 - 4\Gamma |\mathbf{k}|^2 \right) \right) \right)^2 + 16\Gamma^2 \ell^2 \rho_{10}^2 \rho_{20}^4 \left(\sum_i^d k_i^3 \right)^2} \right)^{\frac{1}{2}} \Bigg] \Bigg]^{\frac{1}{2}} \Bigg\}, \quad (C21)
\end{aligned}$$

$$\begin{aligned}
T_{13}(\mathbf{k}) = T_{33}(\mathbf{k}) = & \left\{ i \left[-\kappa_2 |\mathbf{k}|^6 \rho_{20}^3 + \kappa_1 |\mathbf{k}|^4 \rho_{10}^2 + 4\Gamma |\mathbf{k}|^2 \rho_{10} \rho_{20}^2 - \Gamma \ell^2 \rho_{10} \rho_{20}^2 \sum_i^d k_i^4 \right. \right. \\
& + \sqrt{\left(\kappa_1 |\mathbf{k}|^4 \rho_{10}^2 - \rho_{20}^2 \left(\kappa_2 |\mathbf{k}|^6 \rho_{20} + \rho_{10} \left(\Gamma \ell^2 \sum_i^d k_i^4 - 4\Gamma |\mathbf{k}|^2 \right) \right) \right)^2 + 16\Gamma^2 \ell^2 \rho_{10}^2 \rho_{20}^4 \left(\sum_i^d k_i^3 \right)^2} \Bigg] \\
& \times \left(g \left(\kappa_2 |\mathbf{k}|^6 \rho_{20}^3 + \kappa_1 |\mathbf{k}|^4 \rho_{10}^2 + 4\Gamma |\mathbf{k}|^2 \rho_{10} \rho_{20}^2 + \Gamma \ell^2 \rho_{10} \rho_{20}^2 \sum_i^d k_i^4 \right. \right. \\
& + \sqrt{\left(\kappa_1 |\mathbf{k}|^4 \rho_{10}^2 - \rho_{20}^2 \left(\kappa_2 |\mathbf{k}|^6 \rho_{20} + \rho_{10} \left(\Gamma \ell^2 \sum_i^d k_i^4 - 4\Gamma |\mathbf{k}|^2 \right) \right) \right)^2 + 16\Gamma^2 \ell^2 \rho_{10}^2 \rho_{20}^4 \left(\sum_i^d k_i^3 \right)^2} \Bigg) \Bigg]^{\frac{1}{4}} \Bigg\} / \\
& \left\{ 2^{3/4} \Gamma \ell \rho_{10} \rho_{20}^2 \sum_i^d k_i^3 \left[g \left(\left(-\kappa_2 |\mathbf{k}|^6 \rho_{20}^3 + \kappa_1 |\mathbf{k}|^4 \rho_{10}^2 + 4\Gamma |\mathbf{k}|^2 \rho_{10} \rho_{20}^2 - \Gamma \ell^2 \rho_{10} \rho_{20}^2 \sum_i^d k_i^4 \right. \right. \right. \right. \right. \\
& + \sqrt{\left(\kappa_1 |\mathbf{k}|^4 \rho_{10}^2 - \rho_{20}^2 \left(\kappa_2 |\mathbf{k}|^6 \rho_{20} + \rho_{10} \left(\Gamma \ell^2 \sum_i^d k_i^4 - 4\Gamma |\mathbf{k}|^2 \right) \right) \right)^2 + 16\Gamma^2 \ell^2 \rho_{10}^2 \rho_{20}^4 \left(\sum_i^d k_i^3 \right)^2} \Bigg) \Bigg]^2 \Bigg/ \\
& \left(4\Gamma^2 \ell^2 \rho_{10}^2 \rho_{20}^4 \left(\sum_i^d k_i^3 \right)^2 + 4 \right) \Bigg]^{\frac{1}{2}} \Bigg\}, \quad (C22)
\end{aligned}$$

$$\begin{aligned}
T_{14}(\mathbf{k}) = T_{34}(\mathbf{k}) = & \left\{ \sqrt[4]{2} \left[g \left(\kappa_2 |\mathbf{k}|^6 \rho_{20}^3 + \kappa_1 |\mathbf{k}|^4 \rho_{10}^2 + 4\Gamma |\mathbf{k}|^2 \rho_{10} \rho_{20}^2 + \Gamma \ell^2 \rho_{10} \rho_{20}^2 \sum_i^d k_i^4 \right. \right. \right. \\
& + \sqrt{\left(\kappa_1 |\mathbf{k}|^4 \rho_{10}^2 - \rho_{20}^2 \left(\kappa_2 |\mathbf{k}|^6 \rho_{20} + \rho_{10} \left(\Gamma \ell^2 \sum_i^d k_i^4 - 4\Gamma |\mathbf{k}|^2 \right) \right) \right)^2 + 16\Gamma^2 \ell^2 \rho_{10}^2 \rho_{20}^4 \left(\sum_i^d k_i^3 \right)^2} \Bigg] \Bigg]^{\frac{1}{4}} \Bigg\} / \\
& \left\{ \left[g \left(\left(-\kappa_2 |\mathbf{k}|^6 \rho_{20}^3 + \kappa_1 |\mathbf{k}|^4 \rho_{10}^2 + 4\Gamma |\mathbf{k}|^2 \rho_{10} \rho_{20}^2 - \Gamma \ell^2 \rho_{10} \rho_{20}^2 \sum_i^d k_i^4 \right. \right. \right. \right. \right. \\
& + \sqrt{\left(\kappa_1 |\mathbf{k}|^4 \rho_{10}^2 - \rho_{20}^2 \left(\kappa_2 |\mathbf{k}|^6 \rho_{20} + \rho_{10} \left(\Gamma \ell^2 \sum_i^d k_i^4 - 4\Gamma |\mathbf{k}|^2 \right) \right) \right)^2 + 16\Gamma^2 \ell^2 \rho_{10}^2 \rho_{20}^4 \left(\sum_i^d k_i^3 \right)^2} \Bigg) \Bigg]^2 \Bigg/ \\
& \left(4\Gamma^2 \ell^2 \rho_{10}^2 \rho_{20}^4 \left(\sum_i^d k_i^3 \right)^2 + 4 \right) \Bigg]^{\frac{1}{2}} \Bigg\}, \quad (C23)
\end{aligned}$$

$$\begin{aligned}
T_{21}(\mathbf{k}) = -T_{41}(\mathbf{k}) = & 2^{-\frac{3}{4}} \text{sgn}\left(\sum_i^d k_i^3\right) \left\{ \left[g \left(\left(\kappa_2 |\mathbf{k}|^6 \rho_{20}^3 - \kappa_1 |\mathbf{k}|^4 \rho_{10}^2 - 4\Gamma |\mathbf{k}|^2 \rho_{10} \rho_{20}^2 + \Gamma \ell^2 \rho_{10} \rho_{20}^2 \sum_i^d k_i^4 \right. \right. \right. \\
& + \sqrt{\left(\kappa_1 k^4 \rho_{10}^2 - \rho_{20}^2 \left(\kappa_2 |\mathbf{k}|^6 \rho_{20} + \rho_{10} \left(\Gamma \ell^2 \sum_i^d k_i^4 - 4\Gamma |\mathbf{k}|^2 \right) \right) \right)^2 + 16\Gamma^2 \ell^2 \rho_{10}^2 \rho_{20}^4 \left(\sum_i^d k_i^3 \right)^2} \right) \\
& \times \left(\rho_{20}^2 \left(\kappa_2 |\mathbf{k}|^6 \rho_{20} + \rho_{10} \left(\Gamma \ell^2 \sum_i^d k_i^4 - 4\Gamma |\mathbf{k}|^2 \right) \right) - \kappa_1 |\mathbf{k}|^4 \rho_{10}^2 \right) + 16\Gamma^2 \ell^2 \rho_{10}^2 \rho_{20}^4 \left(\sum_i^d k_i^3 \right)^2 \Big] / \\
& \left[\left(\left(\kappa_1 |\mathbf{k}|^4 \rho_{10}^2 - \rho_{20}^2 \left(\kappa_2 |\mathbf{k}|^6 \rho_{20} + \rho_{10} \left(\Gamma \ell^2 \sum_i^d k_i^4 - 4\Gamma |\mathbf{k}|^2 \right) \right) \right)^2 + 16\Gamma^2 \ell^2 \rho_{10}^2 \rho_{20}^4 \left(\sum_i^d k_i^3 \right)^2 \right) \right. \\
& \times \left(g \left(\kappa_2 |\mathbf{k}|^6 \rho_{20}^3 + \kappa_1 |\mathbf{k}|^4 \rho_{10}^2 + 4\Gamma |\mathbf{k}|^2 \rho_{10} \rho_{20}^2 + \Gamma \ell^2 \rho_{10} \rho_{20}^2 \sum_i^d k_i^4 \right. \right. \\
& \left. \left. - \sqrt{\left(\kappa_1 |\mathbf{k}|^4 \rho_{10}^2 - \rho_{20}^2 \left(\kappa_2 |\mathbf{k}|^6 \rho_{20} + \rho_{10} \left(\Gamma \ell^2 \sum_i^d k_i^4 - 4\Gamma |\mathbf{k}|^2 \right) \right) \right)^2 + 16\Gamma^2 \ell^2 \rho_{10}^2 \rho_{20}^4 \left(\sum_i^d k_i^3 \right)^2} \right)^{\frac{1}{2}} \right] \right\}^{\frac{1}{2}}, \quad (\text{C24})
\end{aligned}$$

$$\begin{aligned}
T_{22}(\mathbf{k}) = -T_{42}(\mathbf{k}) = & \left(2i\sqrt{2}\Gamma\ell\rho_{10}\rho_{20}^2 \left| \sum_i^d k_i^3 \right| \right) / \\
& \left\{ \left[\left(\kappa_2 |\mathbf{k}|^6 \rho_{20}^3 + \kappa_1 |\mathbf{k}|^4 \rho_{10}^2 + 4\Gamma |\mathbf{k}|^2 \rho_{10} \rho_{20}^2 + \Gamma \ell^2 \rho_{10} \rho_{20}^2 \sum_i^d k_i^4 \right. \right. \right. \\
& \left. \left. - \sqrt{\left(\kappa_1 |\mathbf{k}|^4 \rho_{10}^2 - \rho_{20}^2 \left(\kappa_2 |\mathbf{k}|^6 \rho_{20} + \rho_{10} \left(\Gamma \ell^2 \sum_i^d k_i^4 - 4\Gamma |\mathbf{k}|^2 \right) \right) \right)^2 + 16\Gamma^2 \ell^2 \rho_{10}^2 \rho_{20}^4 \left(\sum_i^d k_i^3 \right)^2} \right)^{\frac{1}{2}} \right. \\
& \times \left[\left(16\Gamma^2 \ell^2 \rho_{10}^2 \rho_{20}^4 \left(\sum_i^d k_i^3 \right)^2 - \left(\kappa_1 |\mathbf{k}|^4 \rho_{10}^2 - \rho_{20}^2 \left(\kappa_2 |\mathbf{k}|^6 \rho_{20} + \rho_{10} \left(\Gamma \ell^2 \sum_i^d k_i^4 - 4\Gamma |\mathbf{k}|^2 \right) \right) \right) \right) \right. \\
& \times \left(\kappa_2 |\mathbf{k}|^6 \rho_{20}^3 - \kappa_1 |\mathbf{k}|^4 \rho_{10}^2 - 4\Gamma |\mathbf{k}|^2 \rho_{10} \rho_{20}^2 + \Gamma \ell^2 \rho_{10} \rho_{20}^2 \sum_i^d k_i^4 \right. \\
& \left. \left. + \sqrt{\left(\kappa_1 |\mathbf{k}|^4 \rho_{10}^2 - \rho_{20}^2 \left(\kappa_2 |\mathbf{k}|^6 \rho_{20} + \rho_{10} \left(\Gamma \ell^2 \sum_i^d k_i^4 - 4\Gamma |\mathbf{k}|^2 \right) \right) \right)^2 + 16\Gamma^2 \ell^2 \rho_{10}^2 \rho_{20}^4 \left(\sum_i^d k_i^3 \right)^2} \right) \right] / \\
& \left(\left(g \left(\kappa_2 |\mathbf{k}|^6 \rho_{20}^3 + \kappa_1 |\mathbf{k}|^4 \rho_{10}^2 + 4\Gamma |\mathbf{k}|^2 \rho_{10} \rho_{20}^2 + \Gamma \ell^2 \rho_{10} \rho_{20}^2 \sum_i^d k_i^4 \right. \right. \right. \\
& \left. \left. - \sqrt{\left(\kappa_1 |\mathbf{k}|^4 \rho_{10}^2 - \rho_{20}^2 \left(\kappa_2 |\mathbf{k}|^6 \rho_{20} + \rho_{10} \left(\Gamma \ell^2 \sum_i^d k_i^4 - 4\Gamma |\mathbf{k}|^2 \right) \right) \right)^2 + 16\Gamma^2 \ell^2 \rho_{10}^2 \rho_{20}^4 \left(\sum_i^d k_i^3 \right)^2} \right)^{\frac{1}{2}} \right) \right] \right\}, \quad (\text{C25})
\end{aligned}$$

$$\begin{aligned}
T_{23}(\mathbf{k}) = T_{43}(\mathbf{k}) = & - \left\{ i \text{sgn}\left(\sum_i^d k_i^3\right) \left[\kappa_2 |\mathbf{k}|^6 \rho_{20}^3 - \kappa_1 |\mathbf{k}|^4 \rho_{10}^2 - 4\Gamma |\mathbf{k}|^2 \rho_{10} \rho_{20}^2 + \Gamma \ell^2 \rho_{10} \rho_{20}^2 \sum_i^d k_i^4 \right. \right. \\
& \left. \left. + \sqrt{\left(\kappa_1 |\mathbf{k}|^4 \rho_{10}^2 - \rho_{20}^2 \left(\kappa_2 |\mathbf{k}|^6 \rho_{20} + \rho_{10} \left(\Gamma \ell^2 \sum_i^d k_i^4 - 4\Gamma |\mathbf{k}|^2 \right) \right) \right)^2 + 16\Gamma^2 \ell^2 \rho_{10}^2 \rho_{20}^4 \left(\sum_i^d k_i^3 \right)^2} \right] \right\} / \\
& \left\{ 2\sqrt{2} \left[- \left(g \left(\left(\kappa_1 |\mathbf{k}|^4 \rho_{10}^2 - \rho_{20}^2 \left(\kappa_2 |\mathbf{k}|^6 \rho_{20} + \rho_{10} \left(\Gamma \ell^2 \sum_i^d k_i^4 - 4\Gamma |\mathbf{k}|^2 \right) \right) \right) \right) \right) \right. \right.
\end{aligned}$$

$$\begin{aligned}
& \times \left(\kappa_2 |\mathbf{k}|^6 \rho_{20}^3 - \kappa_1 |\mathbf{k}|^4 \rho_{10}^2 - 4\Gamma |\mathbf{k}|^2 \rho_{10} \rho_{20}^2 + \Gamma \ell^2 \rho_{10} \rho_{20}^2 \sum_i^d k_i^4 \right. \\
& + \sqrt{\left(\kappa_1 |\mathbf{k}|^4 \rho_{10}^2 - \rho_{20}^2 \left(\kappa_2 |\mathbf{k}|^6 \rho_{20} + \rho_{10} \left(\Gamma \ell^2 \sum_i^d k_i^4 - 4\Gamma |\mathbf{k}|^2 \right) \right) \right)^2 + 16\Gamma^2 \ell^2 \rho_{10}^2 \rho_{20}^4 \left(\sum_i^d k_i^3 \right)^2} \\
& \left. - 16\Gamma^2 \ell^2 \rho_{10}^2 \rho_{20}^4 \left(\sum_i^d k_i^3 \right)^2 \right) \Bigg/ \left(\left(g \left(\kappa_2 |\mathbf{k}|^6 \rho_{20}^3 + \kappa_1 |\mathbf{k}|^4 \rho_{10}^2 + 4\Gamma |\mathbf{k}|^2 \rho_{10} \rho_{20}^2 + \Gamma \ell^2 \rho_{10} \rho_{20}^2 \sum_i^d k_i^4 \right. \right. \right. \\
& \left. \left. \left. - \sqrt{\left(\kappa_1 |\mathbf{k}|^4 \rho_{10}^2 - \rho_{20}^2 \left(\kappa_2 |\mathbf{k}|^6 \rho_{20} + \rho_{10} \left(\Gamma \ell^2 \sum_i^d k_i^4 - 4\Gamma |\mathbf{k}|^2 \right) \right) \right)^2 + 16\Gamma^2 \ell^2 \rho_{10}^2 \rho_{20}^4 \left(\sum_i^d k_i^3 \right)^2} \right) \right)^{\frac{1}{2}} \right) \Bigg\}, \quad (\text{C26}) \\
T_{24}(\mathbf{k}) = T_{44}(\mathbf{k}) = & \left(2^{3/4} \Gamma \ell \rho_{10} \rho_{20}^2 \left| \sum_i^d k_i^3 \right| \right) \Bigg/ \\
& \left\{ \left[- \left(g \left(\kappa_1 |\mathbf{k}|^4 \rho_{10}^2 - \rho_{20}^2 \left(\kappa_2 |\mathbf{k}|^6 \rho_{20} + \rho_{10} \left(\Gamma \ell^2 \sum_i^d k_i^4 - 4\Gamma |\mathbf{k}|^2 \right) \right) \right) \right) \right. \right. \\
& \times \left(\kappa_2 |\mathbf{k}|^6 \rho_{20}^3 - \kappa_1 |\mathbf{k}|^4 \rho_{10}^2 - 4\Gamma |\mathbf{k}|^2 \rho_{10} \rho_{20}^2 + \Gamma \ell^2 \rho_{10} \rho_{20}^2 \sum_i^d k_i^4 \right. \\
& + \sqrt{\left(\kappa_1 |\mathbf{k}|^4 \rho_{10}^2 - \rho_{20}^2 \left(\kappa_2 |\mathbf{k}|^6 \rho_{20} + \rho_{10} \left(\Gamma \ell^2 \sum_i^d k_i^4 - 4\Gamma |\mathbf{k}|^2 \right) \right) \right)^2 + 16\Gamma^2 \ell^2 \rho_{10}^2 \rho_{20}^4 \left(\sum_i^d k_i^3 \right)^2} \\
& \left. \left. - 16\Gamma^2 \ell^2 \rho_{10}^2 \rho_{20}^4 \left(\sum_i^d k_i^3 \right)^2 \right) \right] \Bigg/ \left(\left(g \left(\kappa_2 |\mathbf{k}|^6 \rho_{20}^3 + \kappa_1 |\mathbf{k}|^4 \rho_{10}^2 + 4\Gamma |\mathbf{k}|^2 \rho_{10} \rho_{20}^2 + \Gamma \ell^2 \rho_{10} \rho_{20}^2 \sum_i^d k_i^4 \right. \right. \right. \\
& \left. \left. \left. - \sqrt{\left(\kappa_1 |\mathbf{k}|^4 \rho_{10}^2 - \rho_{20}^2 \left(\kappa_2 |\mathbf{k}|^6 \rho_{20} + \rho_{10} \left(\Gamma \ell^2 \sum_i^d k_i^4 - 4\Gamma |\mathbf{k}|^2 \right) \right) \right)^2 + 16\Gamma^2 \ell^2 \rho_{10}^2 \rho_{20}^4 \left(\sum_i^d k_i^3 \right)^2} \right) \right)^{\frac{1}{2}} \right) \Bigg\}. \quad (\text{C27})
\end{aligned}$$

4. Derivation of the Hamiltonian represented by the creation and annihilation operators

After diagonalized, the Hamiltonian becomes

$$H = \frac{1}{2(2\pi)^d} \int d^d \mathbf{k} D_1(\mathbf{k}) [\hat{\alpha}^\dagger(\mathbf{k}) \hat{\alpha}(\mathbf{k}) + \hat{\alpha}(-\mathbf{k}) \hat{\alpha}^\dagger(-\mathbf{k})] + D_2(\mathbf{k}) [\hat{\beta}^\dagger(\mathbf{k}) \hat{\beta}(\mathbf{k}) + \hat{\beta}(-\mathbf{k}) \hat{\beta}^\dagger(-\mathbf{k})]. \quad (\text{C28})$$

Because of $D_1(\mathbf{k}) = D_1(-\mathbf{k})$, $D_2(\mathbf{k}) = D_2(-\mathbf{k})$, we rewrite the Hamiltonian as

$$\begin{aligned}
H &= \frac{1}{2(2\pi)^d} \int d^d \mathbf{k} D_1(\mathbf{k}) [\hat{\alpha}^\dagger(\mathbf{k}) \hat{\alpha}(\mathbf{k}) + \hat{\alpha}(\mathbf{k}) \hat{\alpha}^\dagger(\mathbf{k})] + D_2(\mathbf{k}) [\hat{\beta}^\dagger(\mathbf{k}) \hat{\beta}(\mathbf{k}) + \hat{\beta}(\mathbf{k}) \hat{\beta}^\dagger(\mathbf{k})] \\
&= \frac{1}{(2\pi)^d} \int d^d \mathbf{k} D_1(\mathbf{k}) \hat{\alpha}^\dagger(\mathbf{k}) \hat{\alpha}(\mathbf{k}) + D_2(\mathbf{k}) \hat{\beta}^\dagger(\mathbf{k}) \hat{\beta}(\mathbf{k}) + \int d^d \mathbf{k} \frac{D_1(\mathbf{k}) + D_2(\mathbf{k})}{2} \delta^{(d)}(\mathbf{0}) \\
&= \frac{1}{(2\pi)^d} \int d^d \mathbf{k} D_1(\mathbf{k}) \hat{\alpha}^\dagger(\mathbf{k}) \hat{\alpha}(\mathbf{k}) + D_2(\mathbf{k}) \hat{\beta}^\dagger(\mathbf{k}) \hat{\beta}(\mathbf{k}) + \frac{\tilde{V}}{(2\pi)^d} \int d^d \mathbf{k} \frac{D_1(\mathbf{k}) + D_2(\mathbf{k})}{2}, \quad (\text{C29})
\end{aligned}$$

where $\tilde{V}$ is the volume of the d -dimensional space, the last term is the vacuum zero-point energy. The dispersion relations are

$$\omega_1 = D_1(\mathbf{k}), \quad \omega_2 = D_2(\mathbf{k}). \quad (\text{C30})$$

APPENDIX D: THE CALCULATION DETAILS OF CORRELATION FUNCTIONS OF MODEL SERIES B

From the main text, we have the Bogoliubov transformation from $\hat{\theta}$ and $\hat{\pi}$ to the independent modes $\hat{\alpha}$ and $\hat{\beta}$:

$$\begin{aligned} \psi(\mathbf{k}) &= [\hat{\pi}_1(\mathbf{k}) \quad \hat{\pi}_2(\mathbf{k}) \quad \hat{\theta}_1(\mathbf{k}) \quad \hat{\theta}_2(\mathbf{k})]^T = T^{-1}(\mathbf{k})\phi(\mathbf{k}) = \Lambda_1 T^\dagger(\mathbf{k})\Lambda_2\phi(\mathbf{k}) \\ &= \begin{bmatrix} -iT_{13}^*(\mathbf{k}) & -iT_{23}^*(\mathbf{k}) & iT_{33}^*(\mathbf{k}) & iT_{43}^*(\mathbf{k}) \\ -iT_{14}^*(\mathbf{k}) & -iT_{24}^*(\mathbf{k}) & iT_{34}^*(\mathbf{k}) & iT_{44}^*(\mathbf{k}) \\ iT_{11}^*(\mathbf{k}) & iT_{21}^*(\mathbf{k}) & -iT_{31}^*(\mathbf{k}) & -iT_{41}^*(\mathbf{k}) \\ iT_{12}^*(\mathbf{k}) & iT_{22}^*(\mathbf{k}) & -iT_{32}^*(\mathbf{k}) & -iT_{42}^*(\mathbf{k}) \end{bmatrix} \begin{bmatrix} \hat{\alpha}(\mathbf{k}) \\ \hat{\beta}(\mathbf{k}) \\ \hat{\alpha}^\dagger(-\mathbf{k}) \\ \hat{\beta}^\dagger(-\mathbf{k}) \end{bmatrix}. \end{aligned} \quad (\text{D1})$$

The two-operator correlation function can be written as

$$\langle \hat{\Phi}_1(\mathbf{x})\hat{\Phi}_1(\mathbf{0}) \rangle = \rho_{10}e^{-\frac{1}{2}(\|\hat{\theta}_1(\mathbf{x})-\hat{\theta}_1(\mathbf{0})\|^2)} = \rho_{10}e^{(\hat{\theta}_1(\mathbf{x})\hat{\theta}_1(\mathbf{0})-\langle\hat{\theta}_1^2(\mathbf{0})\rangle)}, \quad (\text{D2})$$

where

$$\begin{aligned} \langle \hat{\theta}_1(\mathbf{x})\hat{\theta}_1(\mathbf{0}) \rangle &= \left\langle \int \frac{d^d\mathbf{k} d^d\mathbf{k}'}{(2\pi)^{2d}} e^{i\mathbf{k}\cdot\mathbf{x}} \hat{\theta}_1(\mathbf{k})\hat{\theta}_1(\mathbf{k}') \right\rangle = \left\langle \int \frac{d^d\mathbf{k} d^d\mathbf{k}'}{(2\pi)^{2d}} e^{i\mathbf{k}\cdot\mathbf{x}} \hat{\theta}_1(\mathbf{k})\hat{\theta}_1^\dagger(-\mathbf{k}') \right\rangle \\ &= \int \frac{d^d\mathbf{k} d^d\mathbf{k}'}{(2\pi)^{2d}} e^{i\mathbf{k}\cdot\mathbf{x}} \langle [iT_{11}^*(\mathbf{k})\hat{\alpha}(\mathbf{k}) + iT_{21}^*(\mathbf{k})\hat{\beta}(\mathbf{k}) - iT_{31}^*(\mathbf{k})\hat{\alpha}^\dagger(-\mathbf{k}) - iT_{41}^*(\mathbf{k})\hat{\beta}^\dagger(-\mathbf{k})] \\ &\quad \times [-iT_{11}(-\mathbf{k}')\hat{\alpha}^\dagger(-\mathbf{k}') - iT_{21}(-\mathbf{k}')\hat{\beta}^\dagger(-\mathbf{k}') + iT_{31}(-\mathbf{k}')\hat{\alpha}(\mathbf{k}') + iT_{41}(-\mathbf{k}')\hat{\beta}(\mathbf{k}')] \rangle. \end{aligned} \quad (\text{D3})$$

For the expectation value of the creation and annihilation operators in the ground state, we have

$$\langle \hat{\alpha}(\mathbf{k}')\hat{\alpha}(\mathbf{k}) \rangle = \langle \hat{\beta}(\mathbf{k}')\hat{\beta}(\mathbf{k}) \rangle = \langle \hat{\alpha}^\dagger(-\mathbf{k}')\hat{\alpha}(\mathbf{k}) \rangle = \langle \hat{\beta}^\dagger(-\mathbf{k}')\hat{\beta}(\mathbf{k}) \rangle = 0, \quad (\text{D4})$$

$$\langle \hat{\alpha}(\mathbf{k})\hat{\alpha}^\dagger(-\mathbf{k}') \rangle = \langle \hat{\beta}(\mathbf{k})\hat{\beta}^\dagger(-\mathbf{k}') \rangle = (2\pi)^d \delta^{(d)}(\mathbf{k} + \mathbf{k}'). \quad (\text{D5})$$

Then we can obtain the correlation function

$$\langle \hat{\theta}_1(\mathbf{x})\hat{\theta}_1(\mathbf{0}) \rangle = \int \frac{d^d\mathbf{k}}{(2\pi)^d} e^{i\mathbf{k}\cdot\mathbf{x}} [|T_{11}(\mathbf{k})|^2 + |T_{21}(\mathbf{k})|^2]. \quad (\text{D6})$$

Similarly, we can get

$$\langle \hat{\theta}_2(\mathbf{x})\hat{\theta}_2(\mathbf{0}) \rangle = \int \frac{d^d\mathbf{k}}{(2\pi)^d} e^{i\mathbf{k}\cdot\mathbf{x}} [|T_{12}(\mathbf{k})|^2 + |T_{22}(\mathbf{k})|^2]. \quad (\text{D7})$$

Therefore, to calculate correlation functions, we need to calculate $|T_{11}(\mathbf{k})|^2$, $|T_{21}(\mathbf{k})|^2$, $|T_{12}(\mathbf{k})|^2$, and $|T_{22}(\mathbf{k})|^2$ in different spatial dimensions.

1. One spatial dimension

In one spatial dimension, the dispersion relations are

$$\omega_1 \approx \sqrt{4g\Gamma\rho_{10}\rho_{20}^2}|k|}, \quad \omega_2 \approx \sqrt{\kappa_2 g \rho_{20}^3 + \frac{\kappa_1 g \rho_{10}^2 \ell^2}{4}|k|^3}. \quad (\text{D8})$$

We can approximate these four terms:

$$\begin{aligned} |T_{11}(\mathbf{k})|^2 &\approx \frac{\sqrt{g\Gamma\rho_{10}}}{4\Gamma\rho_{10}\rho_{20}} \frac{1}{|k|} = \frac{1}{c_{11}|k|}, \quad |T_{12}(\mathbf{k})|^2 \approx \frac{\sqrt{g\Gamma\rho_{10}\ell^2}}{16\Gamma\rho_{10}\rho_{20}} |k| = \frac{|k|}{c_{12}}, \\ |T_{21}(\mathbf{k})|^2 &\approx \frac{\sqrt{g\ell^2}}{4\sqrt{\kappa_1\rho_{10}^2\ell^2 + 4\kappa_2\rho_{20}^3}} \frac{1}{|k|} = \frac{1}{c_{21}|k|}, \quad |T_{22}(\mathbf{k})|^2 \approx \frac{\sqrt{g}}{\sqrt{\kappa_1\rho_{10}^2\ell^2 + 4\kappa_2\rho_{20}^3}} \frac{1}{|k|^3} = \frac{1}{c_{22}|k|^3}. \end{aligned} \quad (\text{D9})$$

The correlation function of $\hat{\theta}_1$ is

$$\begin{aligned} \langle \hat{\theta}_1(x)\hat{\theta}_1(0) \rangle &= \int \frac{dk}{2\pi} e^{ikx} \left(\frac{1}{c_{11}|k|} + \frac{1}{c_{21}|k|} \right) = \frac{1}{2\pi} \left(\frac{1}{c_{11}} + \frac{1}{c_{21}} \right) \int dk \frac{e^{ikx}}{|k|} \\ &= \frac{1}{2\pi} \left(\frac{1}{c_{11}} + \frac{1}{c_{21}} \right) \left(\int_{-\infty}^{-\frac{2\pi}{L}} dk \frac{e^{ikx}}{-k} + \int_{\frac{2\pi}{L}}^{\infty} dk \frac{e^{ikx}}{k} \right). \end{aligned} \quad (\text{D10})$$

Considering $L > 0, x > 0$, we have the two integrals

$$\int_{-\infty}^{-\frac{2\pi}{L}} dk \frac{e^{ikx}}{-k} + \int_{\frac{2\pi}{L}}^{\infty} \frac{e^{ikx}}{k} = \Gamma\left(0, -\frac{2i\pi x}{L}\right) + \Gamma\left(0, \frac{2i\pi x}{L}\right) \stackrel{L \rightarrow \infty}{=} -2\left[\gamma + \ln\left(\frac{2\pi x}{L}\right)\right], \quad (\text{D11})$$

where γ is the Euler-Mascheroni constant. Then, we can get the correlation functions,

$$\langle \hat{\theta}_1(x) \hat{\theta}_1(0) \rangle \stackrel{L \rightarrow \infty}{=} \frac{1}{2\pi} \left(\frac{1}{c_{11}} + \frac{1}{c_{21}} \right) \left[-2 \left(\gamma + \ln \left(\frac{2\pi |x|}{L} \right) \right) \right] = \frac{c_{11} + c_{21}}{\pi c_{11} c_{21}} \left[-\gamma + \ln \left(\frac{L}{2\pi |x|} \right) \right]. \quad (\text{D12})$$

To get $\langle \hat{\theta}_1^2(0) \rangle = \lim_{x \rightarrow 0} \langle \hat{\theta}_1(x) \hat{\theta}_1(0) \rangle$, we just need to replace $|x|$ by ξ_c . ξ_c is the coherence length,

$$\langle \hat{\theta}_1^2(0) \rangle = \langle \hat{\theta}_1(\xi_c) \hat{\theta}_1(0) \rangle = \frac{c_{11} + c_{21}}{\pi c_{11} c_{21}} \left[-\gamma + \ln \left(\frac{L}{2\pi \xi_c} \right) \right]. \quad (\text{D13})$$

Then, we obtain the correlation function $\langle \hat{\Phi}_1^\dagger(x) \hat{\Phi}_1(0) \rangle$,

$$\langle \hat{\Phi}_1^\dagger(x) \hat{\Phi}_1(0) \rangle = \rho_{10} e^{\langle \hat{\theta}_1(x) \hat{\theta}_1(0) \rangle - \langle \hat{\theta}_1^2(0) \rangle} = \rho_{10} \left(\frac{|x|}{\xi_c} \right)^{-\frac{c_{11} + c_{21}}{\pi c_{11} c_{21}}}. \quad (\text{D14})$$

The correlation function $\langle \hat{\Phi}_1^\dagger(\mathbf{x}) \hat{\Phi}_1(\mathbf{0}) \rangle$ is a power-law decay. The correlation function of $\hat{\theta}_2$ is

$$\langle \hat{\theta}_2(x) \hat{\theta}_2(0) \rangle = \int \frac{dk}{2\pi} e^{ikx} \left(\frac{|k|}{c_{12}} + \frac{1}{c_{22}|k|^3} \right) = \frac{1}{2\pi c_{12}} \int dk e^{ikx} |k| + \frac{1}{2\pi c_{22}} \int dk \frac{e^{ikx}}{|k|^3}. \quad (\text{D15})$$

Here we calculate the these integrals separately:

$$\int_{-\frac{2\pi}{\xi_c}}^{\frac{2\pi}{\xi_c}} dk e^{ikx} |k| = \frac{4\pi |x| \sin\left(\frac{2\pi |x|}{\xi_c}\right) - 4\xi_c \sin^2\left(\frac{\pi |x|}{\xi_c}\right)}{|x|^2 \xi_c}, \quad (\text{D16})$$

$$\int_{\frac{2\pi}{L}}^{\infty} dk \frac{e^{ikx}}{|k|^3} + \int_{-\infty}^{-\frac{2\pi}{L}} dk \frac{e^{ikx}}{|k|^3} \stackrel{L \rightarrow \infty}{=} \frac{L^2}{4\pi^2} + \frac{|x|^2}{2} \left[-3 + 2\gamma + 2 \ln \left(\frac{2\pi |x|}{L} \right) \right]. \quad (\text{D17})$$

Then, we obtain the correlation functions $\langle \hat{\theta}_2(x) \hat{\theta}_2(0) \rangle$ and $\langle \hat{\theta}_2^2(0) \rangle$,

$$\begin{aligned} \langle \hat{\theta}_2(x) \hat{\theta}_2(0) \rangle &= \int \frac{dk}{2\pi} e^{ikx} \left(\frac{|k|}{c_{12}} + \frac{1}{c_{22}|k|^3} \right) = \frac{1}{2\pi c_{12}} \int dk e^{ikx} |k| + \frac{1}{2\pi c_{22}} \int dk \frac{e^{ikx}}{|k|^3} \\ &= \frac{1}{2\pi c_{12}} \left[\frac{4\pi |x| \sin\left(\frac{2\pi |x|}{\xi_c}\right) - 4\xi_c \sin^2\left(\frac{\pi |x|}{\xi_c}\right)}{|x|^2 \xi_c} \right] + \frac{1}{2\pi c_{22}} \left[\frac{L^2}{4\pi^2} + \frac{|x|^2}{2} \left(-3 + 2\gamma + 2 \ln \left(\frac{2\pi |x|}{L} \right) \right) \right], \end{aligned} \quad (\text{D18})$$

$$\langle \hat{\theta}_2^2(0) \rangle = \int \frac{dk}{2\pi} \left(\frac{|k|}{c_{12}} + \frac{1}{c_{22}|k|^3} \right) = \int_{-\frac{2\pi}{\xi_c}}^{\frac{2\pi}{\xi_c}} dk |k| + \int_{\frac{2\pi}{L}}^{\infty} dk \frac{1}{|k|^3} + \int_{-\infty}^{-\frac{2\pi}{L}} dk \frac{1}{|k|^3} = \frac{1}{2\pi c_{12}} \frac{4\pi^2}{\xi_c^2} + \frac{1}{2\pi c_{22}} \frac{L^2}{4\pi^2}. \quad (\text{D19})$$

Then, we obtain the correlation function $\langle \hat{\Phi}_2^\dagger(\mathbf{x}) \hat{\Phi}_2(\mathbf{0}) \rangle$,

$$\langle \hat{\Phi}_2^\dagger(x) \hat{\Phi}_2(0) \rangle = \rho_{20} e^{\langle \hat{\theta}_2(x) \hat{\theta}_2(0) \rangle - \langle \hat{\theta}_2^2(0) \rangle} \rightarrow 0. \quad (\text{D20})$$

The correlation function $\langle \hat{\Phi}_2^\dagger(x) \hat{\Phi}_2(0) \rangle$ decays exponentially with $|x|^2$. The correlation function $\langle \hat{\Phi}_1^\dagger(\mathbf{x}) \hat{\Phi}_1(\mathbf{0}) \rangle$ is a power-law decay. Model Series B does not break the charge symmetry and is algebraically ordered in one spatial dimension.

2. Two spatial dimensions

In two spatial dimensions, we set $k_1 = |\mathbf{k}| \cos \theta$, and $k_2 = |\mathbf{k}| \sin \theta$, for $|\mathbf{k}| \geq 0, 0 \leq \theta < 2\pi$. Then, we can get the dispersion relations,

$$\omega_1 \approx \sqrt{4g\Gamma\rho_{10}\rho_{20}^2} |\mathbf{k}|, \quad (\text{D21})$$

$$\omega_2 \approx \begin{cases} \frac{1}{2} \rho_{20} \ell |\sin 2\theta| \sqrt{g\rho_{10}\Gamma(1 - \sin 2\theta)} |\mathbf{k}|^2 & (\theta \neq 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}), \\ \sqrt{\kappa_2 g \rho_{20}^3 + \frac{\kappa_1 g \rho_{10}^2 \ell^2}{4}} |\mathbf{k}|^3 & (\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}), \\ \sqrt{\kappa_2 g \rho_{20}^3 + \frac{\kappa_1 g \rho_{10}^2 \ell^2}{8}} |\mathbf{k}|^3 & (\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}). \end{cases} \quad (\text{D22})$$

In two spatial dimensions, the dispersion relations are anisotropic. The same is true for the two terms $|T_{21}(\mathbf{k})|^2$ and $|T_{22}(\mathbf{k})|^2$:

$$|T_{11}(\mathbf{k})|^2 \approx \frac{\sqrt{g\Gamma\rho_{10}}}{4\Gamma\rho_{10}\rho_{20}} \frac{1}{|\mathbf{k}|}, \quad |T_{12}(\mathbf{k})|^2 \approx \frac{\sqrt{g\Gamma\rho_{10}}\ell^2(\cos^3\theta + \sin^3\theta)^2}{16\Gamma\rho_{10}\rho_{20}} |\mathbf{k}|, \quad (\text{D23})$$

$$|T_{21}(\mathbf{k})|^2 \begin{cases} \approx \frac{\sqrt{g\Gamma\rho_{10}}\ell(3\cos\theta + \cos 3\theta + 4\sin^3\theta)^2}{64\Gamma\rho_{10}\rho_{20}|\sin 2\theta|\sqrt{1-\sin 2\theta}} & (\theta \neq 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}), \\ \approx \frac{\sqrt{g}\ell^2}{4\sqrt{\kappa_1\rho_{10}^2\ell^2 + 4\kappa_2\rho_{20}^3}} \frac{1}{|\mathbf{k}|} & (\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}), \\ \approx \frac{\sqrt{g}\ell^2}{4\sqrt{2}\sqrt{\kappa_1\rho_{10}^2\ell^2 + 8\kappa_2\rho_{20}^3}} \frac{1}{|\mathbf{k}|} & (\theta = \frac{\pi}{4}, \frac{5\pi}{4}), \\ = 0 & (\theta = \frac{3\pi}{4}, \frac{7\pi}{4}). \end{cases}, \quad (\text{D24})$$

$$|T_{22}(\mathbf{k})|^2 \begin{cases} \approx \frac{\sqrt{g\Gamma\rho_{10}}}{\Gamma\ell\rho_{10}\rho_{20}\sqrt{1-\sin 2\theta}|\sin 2\theta||\mathbf{k}|^2} & (\theta \neq 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}), \\ \approx \frac{\sqrt{g}}{\sqrt{\kappa_1\rho_{10}^2\ell^2 + 4\kappa_2\rho_{20}^3}} \frac{1}{|\mathbf{k}|^3} & (\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}), \\ \approx \frac{\sqrt{2g}}{\sqrt{\kappa_1\rho_{10}^2\ell^2 + 8\kappa_2\rho_{20}^3}} \frac{1}{|\mathbf{k}|^3} & (\theta = \frac{\pi}{4}, \frac{5\pi}{4}), \\ = 0 & (\theta = \frac{3\pi}{4}, \frac{7\pi}{4}). \end{cases}. \quad (\text{D25})$$

In the long-wavelength limit, to simplify the calculation, we can always choose the lowest order to give a rough approximation:

$$|T_{11}(\mathbf{k})|^2 \approx \frac{1}{c_{11}|\mathbf{k}|}, \quad |T_{12}(\mathbf{k})|^2 \lesssim \frac{|\mathbf{k}|}{c_{12}}, \quad |T_{21}(\mathbf{k})|^2 \lesssim \frac{1}{c_{21}|\mathbf{k}|}, \quad |T_{22}(\mathbf{k})|^2 \lesssim \frac{1}{c_{22}|\mathbf{k}|^3}, \quad (\text{D26})$$

where $c_{11} = \frac{4\Gamma\rho_{10}\rho_{20}}{\sqrt{g\Gamma\rho_{10}}}$, $c_{12} = \frac{16\Gamma\rho_{10}\rho_{20}}{\sqrt{g\Gamma\rho_{10}}\ell^2}$, $c_{21} = \frac{4\sqrt{\kappa_1\rho_{10}^2\ell^2 + 4\kappa_2\rho_{20}^3}}{\sqrt{g}\ell^2}$, $c_{22} = \frac{\sqrt{\kappa_1\rho_{10}^2\ell^2 + 8\kappa_2\rho_{20}^3}}{\sqrt{2g}}$.

Following this rough approximation, if the calculation results show that there is a spontaneous breaking of charge symmetry, there must be, but not the opposite. With the above approximation, we can compute correlation functions,

$$\langle \hat{\theta}_1(\mathbf{x})\hat{\theta}_1(\mathbf{0}) \rangle = \int \frac{k dk d\theta}{(2\pi)^2} e^{ik|\mathbf{x}|\cos\theta} \left(\frac{1}{c_{11}k} + \frac{1}{c_{21}k} \right) = \frac{c_{11} + c_{21}}{4\pi^2 c_{11}c_{21}} \int dk d\theta e^{ik|\mathbf{x}|\cos\theta} = \frac{c_{11} + c_{21}}{2\pi c_{11}c_{21}|\mathbf{x}|}. \quad (\text{D27})$$

To get $\langle \hat{\theta}_1^2(\mathbf{0}) \rangle = \lim_{|\mathbf{x}| \rightarrow 0} \langle \hat{\theta}_1(\mathbf{x})\hat{\theta}_1(\mathbf{0}) \rangle$, we just need to replace $|\mathbf{x}|$ by ξ_c . ξ_c is the coherence length

$$\langle \hat{\theta}_1^2(\mathbf{0}) \rangle = \langle \hat{\theta}_1(\xi_c)\hat{\theta}_1(\mathbf{0}) \rangle = \frac{c_{11} + c_{21}}{2\pi c_{11}c_{21}\xi_c}. \quad (\text{D28})$$

Then, we can calculate the correlation function $\langle \hat{\Phi}_1^\dagger(\mathbf{x})\hat{\Phi}_1(\mathbf{0}) \rangle$:

$$\langle \hat{\Phi}_1^\dagger(\mathbf{x})\hat{\Phi}_1(\mathbf{0}) \rangle = \rho_{10} e^{\langle \hat{\theta}_1(\mathbf{x})\hat{\theta}_1(\mathbf{0}) \rangle - \langle \hat{\theta}_1^2(\mathbf{0}) \rangle} \stackrel{|\mathbf{x}| \rightarrow \infty}{=} \rho_{10} e^{-\frac{c_{11}+c_{21}}{2\pi c_{11}c_{21}\xi_c}}. \quad (\text{D29})$$

The correlation function of $\hat{\theta}_2$ is

$$\langle \hat{\theta}_2(\mathbf{x})\hat{\theta}_2(\mathbf{0}) \rangle = \int \frac{k dk d\theta}{(2\pi)^2} e^{ik|\mathbf{x}|\cos\theta} \left(\frac{k}{c_{12}} + \frac{1}{c_{22}k^3} \right) = \frac{1}{4\pi^2 c_{12}} \int dk d\theta e^{ik|\mathbf{x}|\cos\theta} k^2 + \frac{1}{4\pi^2 c_{22}} \int dk d\theta e^{ik|\mathbf{x}|\cos\theta} \frac{1}{k^2}. \quad (\text{D30})$$

Here we compute the two integrals separately:

$$\begin{aligned} \frac{1}{4\pi^2 c_{12}} \int dk d\theta e^{ik|\mathbf{x}|\cos\theta} k^2 &= \frac{1}{4\pi^2 c_{12}} \int_0^{\frac{2\pi}{\xi_c}} 2\pi J_0(k|\mathbf{x}|) k^2 \\ &= \frac{1}{4\pi^2 c_{12}} \frac{2\pi^3 [J_1(\frac{2\pi|\mathbf{x}|}{\xi_c})(4|\mathbf{x}| - \xi_c H_0(\frac{2\pi|\mathbf{x}|}{\xi_c})) + \xi_c J_0(\frac{2\pi|\mathbf{x}|}{\xi_c}) H_1(\frac{2\pi|\mathbf{x}|}{\xi_c})]}{|\mathbf{x}|^2 \xi_c^2} \stackrel{|\mathbf{x}| \rightarrow \infty}{=} 0, \end{aligned} \quad (\text{D31})$$

where $J_a(z)$ is the Bessel function of the first kind, and $H_n(z)$ is the Struve function

$$\frac{1}{4\pi^2 c_{22}} \int dk d\theta e^{ik|\mathbf{x}|\cos\theta} \frac{1}{k^2} = \frac{1}{4\pi^2 c_{22}} \int_{\frac{2\pi}{L}}^{\infty} \frac{2\pi J_0(k|\mathbf{k}|)}{k^2} \stackrel{L \rightarrow \infty}{=} \frac{1}{4\pi^2 c_{22}} (L - 2\pi|\mathbf{x}|). \quad (\text{D32})$$

Then, we can get

$$\langle \hat{\theta}_2(\mathbf{x})\hat{\theta}_2(\mathbf{0}) \rangle \stackrel{L \rightarrow \infty}{=} \frac{1}{4\pi^2 c_{22}} (L - 2\pi|\mathbf{x}|). \quad (\text{D33})$$

We can also calculate $\langle \hat{\theta}_2^2(\mathbf{0}) \rangle$,

$$\langle \hat{\theta}_2^2(\mathbf{0}) \rangle = \frac{1}{4\pi^2 c_{12}} \int_0^{\frac{2\pi}{\xi_c}} dk \int_0^{2\pi} d\theta k^2 + \frac{1}{4\pi^2 c_{22}} \int_{\frac{2\pi}{L}}^{\infty} dk \int_0^{2\pi} d\theta \frac{1}{k^2} = \frac{4\pi^2}{3c_{12}\xi_c^3} + \frac{L}{4\pi^2 c_{22}}. \quad (\text{D34})$$

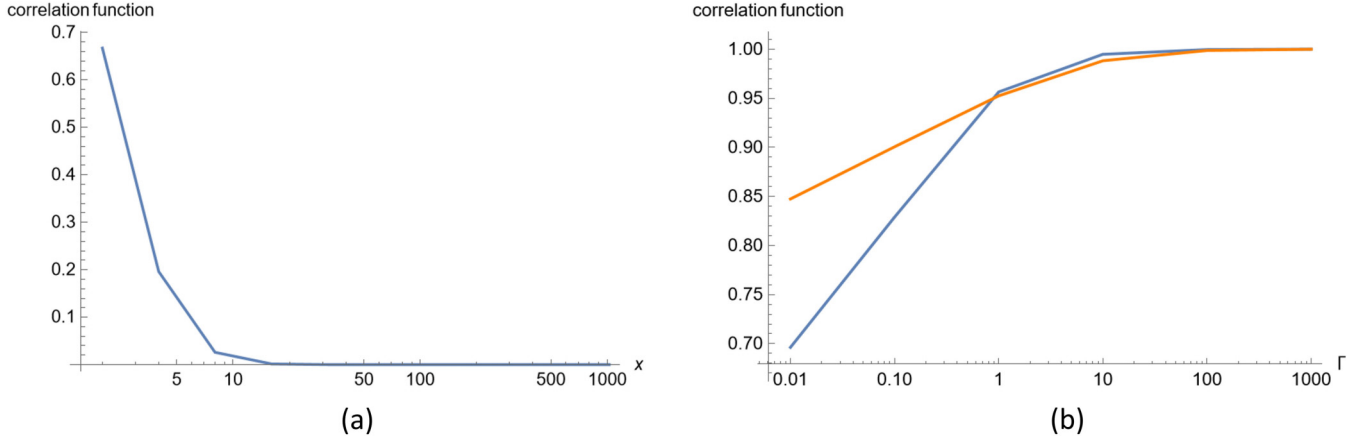


FIG. 4. Results obtained by numerical calculations. (a) Shows the graph of the correlation function $\langle \hat{\Phi}_2^\dagger(x, 0) \hat{\Phi}_2(\mathbf{0}) \rangle$ decaying with increasing x in two spatial dimensions. In (b), the blue line shows the variation of correlation function $\langle \hat{\Phi}_1^\dagger(10^{10}, 0) \hat{\Phi}_1(\mathbf{0}) \rangle$ with increasing Γ in two spatial dimensions, and the orange line shows the variation of correlation function $\langle \hat{\Phi}_2^\dagger(10^{10}, 0, 0) \hat{\Phi}_2(\mathbf{0}) \rangle$ with increasing Γ in three spatial dimensions.

Then, we can calculate the correlation function $\langle \hat{\Phi}_2^\dagger(\mathbf{x}) \hat{\Phi}_2(\mathbf{0}) \rangle$:

$$\langle \hat{\Phi}_2^\dagger(\mathbf{x}) \hat{\Phi}_2(\mathbf{0}) \rangle = \rho_{20} e^{\langle \hat{\theta}_2(\mathbf{x}) \hat{\theta}_2(\mathbf{0}) \rangle - \langle \hat{\theta}_2^2(\mathbf{0}) \rangle} \stackrel{|\mathbf{x}| \rightarrow \infty}{=} \rho_{20} e^{-\frac{4\pi^2}{3c_{12}\xi_c^2} - \frac{|\mathbf{x}|}{2\pi c_{22}}} \stackrel{|\mathbf{x}| \rightarrow \infty}{=} 0. \quad (\text{D35})$$

According to the results of the above analytical calculation, the correlation function $\langle \hat{\Phi}_1^\dagger(\mathbf{x}) \hat{\Phi}_1(\mathbf{0}) \rangle$ saturates to a constant and $\langle \hat{\Phi}_2^\dagger(\mathbf{x}) \hat{\Phi}_2(\mathbf{0}) \rangle$ is a power-law decay when $|\mathbf{x}| \rightarrow \infty$. We can test these results numerically. By setting $\kappa_1 = 1$, $\kappa_2 = 1$, $\Gamma = 1$, $g = 1$, $\rho_{10} = 1$, $\rho_{20} = 1$, $\ell = 1$, $\xi_c = 2\pi$, $\frac{2\pi}{\xi_c} = 1$, $|\mathbf{x}| \sim 10^{10}$, $L \sim 10^{20}$, we can calculate the correlation function by using the numerical methods. Firstly, we can calculate $\langle \hat{\Phi}_1^\dagger(10^{10}, 0) \hat{\Phi}_1(\mathbf{0}) \rangle$ numerically. For the integrals, we have

$$\int_0^1 dk \int_0^{2\pi} d\theta |T_{11}(\mathbf{k})|^2 k e^{i10^{10}k \cos \theta} - \int_0^1 dk \int_0^{2\pi} d\theta |T_{11}(\mathbf{k})|^2 k \approx (5.90 \times 10^{-13} + 8.41 \times 10^{-16}i) - 1.42 \approx -1.42, \quad (\text{D36})$$

$$\int_0^1 dk \int_0^{2\pi} d\theta |T_{21}(\mathbf{k})|^2 k e^{i10^{10}k \cos \theta} - \int_0^1 dk \int_0^{2\pi} d\theta |T_{21}(\mathbf{k})|^2 k \approx (-5.23 \times 10^{-4} - 6.08 \times 10^{-4}i) - 0.351 \approx -0.351. \quad (\text{D37})$$

Then, we obtain the result

$$\langle \hat{\Phi}_1^\dagger(10^{10}, 0) \hat{\Phi}_1(\mathbf{0}) \rangle \approx e^{\frac{1}{(2\pi)^2}(-1.42-0.351)} \approx 0.956. \quad (\text{D38})$$

From this we can believe that $\langle \hat{\Phi}_1^\dagger(\mathbf{x}) \hat{\Phi}_1(\mathbf{0}) \rangle$ saturates to a constant when $|\mathbf{x}| \rightarrow \infty$. We can also calculate the correlation function $\langle \hat{\Phi}_2^\dagger(10^{10}, 0) \hat{\Phi}_2(\mathbf{0}) \rangle$. For the integrals, we have

$$\int_0^1 dk \int_0^{2\pi} d\theta |T_{12}(\mathbf{k})|^2 k e^{i10^{10}k \cos \theta} - \int_0^1 dk \int_0^{2\pi} d\theta |T_{12}(\mathbf{k})|^2 k \approx (2.34 \times 10^{-4} + 3.56 \times 10^{-4}i) - 0.0623 \approx -0.0623, \quad (\text{D39})$$

$$\int_{10^{-20}}^1 dk \int_0^{2\pi} d\theta |T_{22}(\mathbf{k})|^2 k (e^{i10^{10}k \cos \theta} - 1) \approx -874 - 10.9i \approx -874, \quad (\text{D40})$$

$$\langle \hat{\Phi}_2^\dagger(10^{10}, 0) \hat{\Phi}_2(\mathbf{0}) \rangle \approx e^{\frac{1}{(2\pi)^2}(-874)} \approx 2.43 \times 10^{-10}. \quad (\text{D41})$$

From this we can know that $\langle \hat{\Phi}_2^\dagger(\mathbf{x}) \hat{\Phi}_2(\mathbf{0}) \rangle$ do not saturate to a constant when $|\mathbf{x}| \rightarrow \infty$. The correlation function $\langle \hat{\Phi}_2^\dagger(\mathbf{x}) \hat{\Phi}_2(\mathbf{0}) \rangle$ is shown in Fig. 4(a) below. The figure shows that the correlation function $\langle \hat{\Phi}_2^\dagger(\mathbf{x}) \hat{\Phi}_2(\mathbf{0}) \rangle$ decays power law. Since the correlation function $\langle \hat{\Phi}_1^\dagger(\mathbf{x}) \hat{\Phi}_1(\mathbf{0}) \rangle$ decays in a power law when Γ is 0, the value of Γ affects the value of the correlation function. We can also give the dependence of the correlation function $\langle \hat{\Phi}_1^\dagger(\mathbf{x}) \hat{\Phi}_1(\mathbf{0}) \rangle$ on Γ [see the blue line of Fig. 4(b)]. The constant behavior of the correlation function $\langle \hat{\Phi}_1^\dagger(\mathbf{x}) \hat{\Phi}_1(\mathbf{0}) \rangle$ shows the spontaneous breaking of the charge symmetry $U(1)_{1,C}$ of bosons of species 1. Model Series B has a true ODLRO in two spatial dimensions.

3. Three spatial dimensions

In three spatial dimensions, we set $k_1 = |\mathbf{k}| \cos \theta \sin \psi$, $k_2 = |\mathbf{k}| \sin \theta \sin \psi$, $k_3 = |\mathbf{k}| \cos \psi$. Due to the large error in the analytical approximation of the 3D results, we directly use numerical methods to calculate the correlation functions

$$\int_0^1 dk \int_0^{2\pi} d\theta \int_0^\pi d\psi |T_{11}(\mathbf{k})|^2 k^2 \sin \psi e^{i10^{10}k \cos \theta \sin \psi} - \int_0^1 dk \int_0^{2\pi} d\theta \int_0^\pi d\psi |T_{11}(\mathbf{k})|^2 k^2 \sin \psi \approx (-0.0239 + 0.0113i) - 1.39 \approx -1.41, \quad (\text{D42})$$

$$\int_0^1 dk \int_0^{2\pi} d\theta \int_0^\pi d\psi |T_{21}(\mathbf{k})|^2 k^2 \sin \psi e^{i10^{10}k \cos \theta \sin \psi} - \int_0^1 dk \int_0^{2\pi} d\theta \int_0^\pi d\psi |T_{21}(\mathbf{k})|^2 k^2 \sin \psi \approx (-9.34 \times 10^{-4} - 9.71 \times 10^{-4}i) - 0.258 \approx -0.258. \quad (\text{D43})$$

From the above results, we can obtain the correlation function $\langle \hat{\Phi}_1^\dagger(\mathbf{x}) \hat{\Phi}_1(\mathbf{0}) \rangle$:

$$\langle \hat{\Phi}_1^\dagger(10^{10}, 0, 0) \hat{\Phi}_1(\mathbf{0}) \rangle \approx e^{\frac{1}{(2\pi)^3}(-1.41-0.258i)} \approx 0.993. \quad (\text{D44})$$

From this we can believe that $\langle \hat{\Phi}_1^\dagger(\mathbf{x}) \hat{\Phi}_1(\mathbf{0}) \rangle$ saturates to a constant when $|\mathbf{x}| \rightarrow \infty$:

$$\int_0^1 dk \int_0^{2\pi} d\theta \int_0^\pi d\psi |T_{12}(\mathbf{k})|^2 k^2 \sin \psi e^{i10^{10}k \cos \theta \sin \psi} - \int_0^1 dk \int_0^{2\pi} d\theta \int_0^\pi d\psi |T_{12}(\mathbf{k})|^2 k^2 \sin \psi \approx (1.05 \times 10^{-3} + 6.24 \times 10^{-5}i) - 0.0654 \approx -0.064, \quad (\text{D45})$$

$$\int_{10^{-20}}^1 dk \int_0^{2\pi} d\theta \int_0^\pi d\psi |T_{22}(\mathbf{k})|^2 k^2 \sin \psi (e^{i10^{10}k \cos \theta \sin \psi} - 1) \approx -11.6, \quad (\text{D46})$$

$$\langle \hat{\Phi}_2^\dagger(10^{10}, 0, 0) \hat{\Phi}_2(\mathbf{0}) \rangle \approx e^{\frac{1}{(2\pi)^3}(-0.064-11.6i)} \approx 0.954. \quad (\text{D47})$$

From this we can believe that $\langle \hat{\Phi}_2^\dagger(\mathbf{x}) \hat{\Phi}_2(\mathbf{0}) \rangle$ saturates to a constant when $|\mathbf{x}| \rightarrow \infty$. We can also give the dependence of the correlation function $\langle \hat{\Phi}_2^\dagger(\mathbf{x}) \hat{\Phi}_2(\mathbf{0}) \rangle$ on Γ [see the orange line of Fig. 4(b)]. The constant behaviors of the correlation functions $\langle \hat{\Phi}_1^\dagger(\mathbf{x}) \hat{\Phi}_1(\mathbf{0}) \rangle$ and $\langle \hat{\Phi}_2^\dagger(\mathbf{x}) \hat{\Phi}_2(\mathbf{0}) \rangle$ show the spontaneous breaking of charge symmetries $U(1)_{1,C}$ and $U(1)_{2,C}$. This system has a true ODLRO in three spatial dimensions.

APPENDIX E: MEAN-FIELD THEORY OF LATTICE MODEL A

To determine the phase boundary between the MI phase and the MDC phase in Lattice Model Series A, we present a simple mean-field approximation.

We define the local operators of dipoles: $\hat{d}_{a,j}^i = \hat{b}_{a,j+\hat{x}_i}^\dagger \hat{b}_{a,j}$, and set $t_a^{(ij)} = t_1$, $t_{ab}^{(i)} = t_2$, which mean the coefficients of the hopping terms for the same species and different species, respectively. Then, the hopping terms in the Hamiltonian [Eq. (77)] can be written as

$$H_{\text{hop}} = \sum_a \left[\sum_{i,j} -t_1 (\hat{d}_{a,i}^i)^\dagger \hat{d}_{a,i+\hat{x}_j}^i + \sum_{b \neq a} \sum_{i,j} -t_2 (\hat{d}_{a,i}^i)^\dagger \hat{d}_{b,i}^i \right] + \text{H.c.} \quad (\text{E1})$$

We introduce the mean dipole fields to decompose the hopping terms. The mean dipole fields $\Psi_{a,i}$ are given by

$$\Psi_{a,i} = \sum_j t_1 \langle \hat{d}_{a,i-\hat{x}_j}^i + \hat{d}_{a,i+\hat{x}_j}^i \rangle + \sum_{b \neq a} t_2 \langle \hat{d}_{b,i}^i \rangle. \quad (\text{E2})$$

By the mean-field approximation, the hopping terms become

$$H_{\text{hop}} \approx - \sum_a \sum_{i,j} \left[\Psi_{a,i} (\hat{d}_{a,i}^i)^\dagger + \Psi_{a,i}^* \hat{d}_{a,i}^i - \frac{|\Psi_{a,i}|^2}{2dt_1 + t_2} \right], \quad (\text{E3})$$

and the total Hamiltonian is written as a local piece $h_i^{(0)}$ plus a perturbation:

$$H = \sum_i \left[h_i^{(0)} - \sum_a \sum_i (\Psi_{a,i} (\hat{d}_{a,i}^i)^\dagger + \Psi_{a,i}^* \hat{d}_{a,i}^i) \right], \quad (\text{E4})$$

where

$$h_i^{(0)} = \sum_a \left[-\mu \hat{n}_{a,i} + \frac{U}{2} \hat{n}_{a,i} (\hat{n}_{a,i} - 1) + \sum_i \frac{|\Psi_{a,i}|^2}{2dt_1 + t_2} \right]. \quad (\text{E5})$$

The terms $-\sum_a \sum_i (\Psi_{a,i} (\hat{d}_{a,i}^i)^\dagger + \Psi_{a,i}^* \hat{d}_{a,i}^i)$ are treated as the perturbation. The energy density can be expressed as a series expansion in even powers of the parameter $\Psi_{a,i}$: $E[\Psi_{a,i}] = \sum_a \sum_i [\text{constant} + R |\Psi_{a,i}|^2 + W |\Psi_{a,i}|^4 + \dots]$. We assume the coefficient $W > 0$, and the phase transition between the MI phase and the MDC phase occurs at the point $R = 0$. We calculate the coefficient R by using second-order perturbation theory. The zeroth-order energy from $h_i^{(0)}$ is

given by

$$E_{a,n}^{(0)} = \langle n | h_{a,i}^{(0)} | n \rangle = -\mu n + \frac{U}{2} n(n-1) + \sum_i \frac{|\Psi_{a,i}|^2}{2dt_1 + t_2}, \quad (\text{E6})$$

where n is the filling numbers of bosons of species 1 and 2. To calculate second-order energy contribution, we consider a site and one of its nearest neighbors in the direction labeled by i :

$$\begin{aligned} E_{a,(n,n)}^{(2)} &= \sum_i |\Psi_{a,i}|^2 \sum_{(m,m') \neq (n,n)} \frac{|\langle n, n | (\hat{d}_{a,i}^i)^\dagger + \hat{d}_{a,i}^i | m, m' \rangle|^2}{E_{a,(n,n)}^{(0)} - E_{a,(m,m')}^{(0)}} \\ &= \sum_i |\Psi_{a,i}|^2 \left[\frac{|\langle n, n | (\hat{d}_{a,i}^i)^\dagger + \hat{d}_{a,i}^i | n-1, n+1 \rangle|^2}{E_{a,(n,n)}^{(0)} - E_{a,(n-1,n+1)}^{(0)}} \right. \\ &\quad \left. + \frac{|\langle n, n | (\hat{d}_{a,i}^i)^\dagger + \hat{d}_{a,i}^i | n+1, n-1 \rangle|^2}{E_{a,(n,n)}^{(0)} - E_{a,(n+1,n-1)}^{(0)}} \right], \quad (\text{E7}) \end{aligned}$$

where $E_{a,(n-1,n+1)}^{(0)} = E_{a,(n+1,n-1)}^{(0)} = E_{a,(n,n)}^{(0)} + U$. Then, we obtain the second-order energy

$$E_{a,(n,n)}^{(2)} = - \sum_i |\Psi_{a,i}|^2 \frac{2n(n+1)}{U}. \quad (\text{E8})$$

The energy $E_{a,(n,n)}^{(2)}$ is shared by the two sites, and each site has two nearest neighbors in the direction labeled by i . Thus, the energy in Eq. (E8) is exactly the second-order energy of a single site. With the contribution of the zeroth-order and second-order energy, the quadratic coefficient R is

$$R = \frac{1}{2dt_1 + t_2} - \frac{2n(n+1)}{U}. \quad (\text{E9})$$

At $R > 0$, this system is under the Mott insulator phase, while at $R < 0$, the dipole condensation occurs. In the coefficient R , the coupling coefficient t_2 plays a similar role as t_1 . It is worth mentioning that at $t_1 > 0$, $t_2 > 0$, we have $\langle \hat{b}_{a,i+\hat{x}_i}^\dagger \hat{b}_{a,i} \rangle \neq 0$, $\langle \hat{b}_{1,i}^\dagger \hat{b}_{2,i} \rangle \neq 0$, which implies condensates with a variety of particle-hole bound states in this system as long as $t_2 > 0$. This phase is called multidipole condensate phase (MDC).

According to the result of mean-field theory, we can draw schematic phase diagrams of this model, as shown in Figs. 2 and 3.

APPENDIX F: DERIVATION OF LATTICE MODEL A IN A TILTED LATTICE

In a microscopic Hamiltonian with two species of bosons in a tilted optical lattice chain, a U' term Hubbard term between different species is included:

$$\begin{aligned} H_{\text{tilted}} &= -t \sum_{a,i} (\hat{b}_{a,i}^\dagger \hat{b}_{a,i+1} + \text{H.c.}) - \mu \sum_{a,i} \hat{n}_{a,i} + \frac{U}{2} \sum_{a,i} \hat{n}_{a,i} (\hat{n}_{a,i} - 1) + U' \sum_i \hat{n}_{1,i} \hat{n}_{2,i} + V \sum_{a,i} \mathbf{i} \hat{n}_{a,i} \\ &\equiv H_t + H_U + H_V. \end{aligned} \quad (\text{F1})$$

In the limit $V \gg t, U, U'$, we use a Nakajima transformation $e^S H e^{-S} = H + [S, H] + \frac{1}{2}[S, [S, H]] + \dots$, where S is anti-Hermitian. We take $S = \sum_{n=1}^{\infty} S_n$, where S_n is order n in t/V , U/V , and U'/V . In this derivation, we need an effective Hamiltonian to the third order, and thus we can set S_n to 0 for $n \geq 4$. Then, the effective Hamiltonian becomes

$$\begin{aligned} H_{\text{eff}} &= H_V + H_t + H_U + [S_1, H_V] + [S_1, H_U] + [S_1, H_t] + [S_2, H_V] + [S_2, H_U] + [S_2, H_t] + [S_3, H_V] + \frac{1}{2}[S_1, [S_1, H_V]] \\ &\quad + \frac{1}{2}[S_1, [S_1, H_U]] + \frac{1}{2}[S_1, [S_1, H_t]] + \frac{1}{2}[S_1, [S_2, H_V]] + \frac{1}{2}[S_2, [S_1, H_V]] + \frac{1}{6}[S_1, [S_1, [S_1, H_V]]]. \end{aligned} \quad (\text{F2})$$

Firstly, we need $[S_1, H_V] + H_t = 0$, and we can get $S_1 = \frac{t}{V} \sum_{a,i} (\hat{b}_{a,i}^\dagger \hat{b}_{a,i+1} - \text{H.c.})$, and we have

$$[S_1, H_V] = -H_t, \quad [S_1, H_t] = 0, \quad (\text{F3})$$

$$[S_1, H_U] = \frac{t_0 U}{2V} \sum_{a,i} \{\hat{n}_{a,i}, (\hat{b}_{a,i-1}^\dagger \hat{b}_{a,i} - \hat{b}_{a,i}^\dagger \hat{b}_{a,i+1} + \text{H.c.})\} + \frac{t_0 U'}{V} \sum_{a \neq b, i} (\hat{b}_{a,i-1}^\dagger \hat{b}_{a,i} - \hat{b}_{a,i}^\dagger \hat{b}_{a,i+1} + \text{H.c.}) \hat{n}_{b,i}. \quad (\text{F4})$$

Plugging Eq. (F3) into the effective Hamiltonian, we obtain

$$H_{\text{eff}} = H_V + H_U + [S_1, H_U] + [S_2, H_V] + [S_2, H_U] + \frac{1}{2}[S_2, H_t] + [S_3, H_V] + \frac{1}{2}[S_1, [S_1, H_U]] + \frac{1}{2}[S_1, [S_2, H_V]]. \quad (\text{F5})$$

Then we require S_2 to satisfy $[S_2, H_V] + [S_1, H_U] = 0$. S_2 is given by

$$S_2 = -\frac{t_0 U}{2V^2} \sum_{a,i} \{\hat{n}_{a,i}, (\hat{b}_{a,i-1}^\dagger \hat{b}_{a,i} - \hat{b}_{a,i}^\dagger \hat{b}_{a,i+1} - \text{H.c.})\} - \frac{t_0 U'}{V^2} \sum_{a \neq b, i} (\hat{b}_{a,i-1}^\dagger \hat{b}_{a,i} - \hat{b}_{a,i}^\dagger \hat{b}_{a,i+1} - \text{H.c.}) \hat{n}_{b,i}. \quad (\text{F6})$$

And thus, the effective Hamiltonian becomes $H_{\text{eff}} = H_V + H_U + [S_2, H_U] + \frac{1}{2}[S_2, H_t] + [S_3, H_V]$. To the third order, we need $[S_3, H_V]$ to cancel the terms in $[S_2, H_U] + \frac{1}{2}[S_2, H_t]$ which do not conserve the total dipole moment. Here we define a projection

operator P which projects onto total dipole-conserving terms, and we have

$$\begin{aligned}
 [S_2, H_U] + \frac{1}{2}[S_2, H_t] + [S_3, H_V] &= P([S_2, H_U] + \frac{1}{2}[S_2, H_t])P = \frac{1}{2}P[S_2, H_t]P \\
 &= \sum_{a,i} \left(-\frac{2t_0^2 U}{V^2} \hat{n}_{a,i}^2 + \frac{4t_0^2 U}{V^2} \hat{n}_{a,i} \hat{n}_{a,i+1} \right) + \left(-\frac{t^2 U}{V^2} \hat{b}_{a,i} (\hat{b}_{a,i+1}^\dagger)^2 \hat{b}_{a,i+2} + \text{H.c.} \right) \\
 &\quad + \sum_{a \neq b, i} \left(-\frac{2t_0^2 U'}{V^2} \hat{n}_{a,i} \hat{n}_{b,i} + \frac{2t_0^2 U'}{V^2} \hat{n}_{a,i} \hat{n}_{b,i+1} \right) \\
 &\quad + \left(\frac{t^2 U'}{V^2} \hat{b}_{a,i}^\dagger \hat{b}_{a,i+1} \hat{b}_{b,i+1} \hat{b}_{b,i+2}^\dagger - \frac{t^2 U'}{V^2} \hat{b}_{a,i}^\dagger \hat{b}_{a,i+1} \hat{b}_{b,i} \hat{b}_{b,i+1}^\dagger + \text{H.c.} \right). \tag{F7}
 \end{aligned}$$

At last, the effective Hamiltonian is

$$\begin{aligned}
 H_{\text{eff}} &= \sum_{a,i} \left(-\frac{t^2 U}{V^2} \hat{b}_{a,i} (\hat{b}_{a,i+1}^\dagger)^2 \hat{b}_{a,i+2} + \text{H.c.} \right) + \sum_{a \neq b, i} \left(\frac{t^2 U'}{V^2} \hat{b}_{a,i}^\dagger \hat{b}_{a,i+1} \hat{b}_{b,i+1} \hat{b}_{b,i+2}^\dagger - \frac{t^2 U'}{V^2} \hat{b}_{a,i}^\dagger \hat{b}_{a,i+1} \hat{b}_{b,i} \hat{b}_{b,i+1}^\dagger + \text{H.c.} \right) \\
 &\quad + \sum_{a,i} \left(-\mu - \frac{U}{2} + V\mathbf{i} \right) \hat{n}_{a,i} + \sum_{a,i} \left(\frac{U}{2} - \frac{2t_0^2 U}{V^2} \right) \hat{n}_{a,i}^2 + \frac{4t_0^2 U}{V^2} \hat{n}_{a,i} \hat{n}_{a,i+1} \\
 &\quad + \sum_{a \neq b, i} \left[\left(\frac{U'}{2} - \frac{2t_0^2 U'}{V^2} \right) \hat{n}_{a,i} \hat{n}_{b,i} + \frac{2t_0^2 U'}{V^2} \hat{n}_{a,i} \hat{n}_{b,i+1} \right]. \tag{F8}
 \end{aligned}$$

After the above derivation, we obtain effective hopping terms with the conservation of the total dipole moments on the tilted optical lattice.

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- [1] C. Chamon, Quantum glassiness in strongly correlated clean systems: An example of topological overprotection, *Phys. Rev. Lett.* **94**, 040402 (2005).
 - [2] S. Vijay, J. Haah, and L. Fu, A new kind of topological quantum order: A dimensional hierarchy of quasiparticles built from stationary excitations, *Phys. Rev. B* **92**, 235136 (2015).
 - [3] S. Vijay, J. Haah, and L. Fu, Fracton topological order, generalized lattice gauge theory, and duality, *Phys. Rev. B* **94**, 235157 (2016).
 - [4] W. Shirley, K. Slagle, Z. Wang, and X. Chen, Fracton models on general three-dimensional manifolds, *Phys. Rev. X* **8**, 031051 (2018).
 - [5] H. Song, J. Schönmeier-Kromer, K. Liu, O. Viyuela, L. Pollet, and M. A. Martin-Delgado, Optimal thresholds for fracton codes and random spin models with subsystem symmetry, *Phys. Rev. Lett.* **129**, 230502 (2022).
 - [6] H. Ma, E. Lake, X. Chen, and M. Hermele, Fracton topological order via coupled layers, *Phys. Rev. B* **95**, 245126 (2017).
 - [7] W. Shirley, K. Slagle, and X. Chen, Foliated fracton order from gauging subsystem symmetries, *SciPost Phys.* **6**, 041 (2019).
 - [8] A. Prem, J. Haah, and R. Nandkishore, Glassy quantum dynamics in translation invariant fracton models, *Phys. Rev. B* **95**, 155133 (2017).
 - [9] A. Dua, I. H. Kim, M. Cheng, and D. J. Williamson, Sorting topological stabilizer models in three dimensions, *Phys. Rev. B* **100**, 155137 (2019).
 - [10] R. M. Nandkishore and M. Hermele, Fractons, *Annu. Rev. Condens. Matter Phys.* **10**, 295 (2019).
 - [11] D. Bulmash and M. Barkeshli, Gauging fractons: Immobile non-Abelian quasiparticles, fractals, and position-dependent degeneracies, *Phys. Rev. B* **100**, 155146 (2019).
 - [12] A. Prem, S.-J. Huang, H. Song, and M. Hermele, Cage-net fracton models, *Phys. Rev. X* **9**, 021010 (2019).
 - [13] M.-Y. Li and P. Ye, Fracton physics of spatially extended excitations, *Phys. Rev. B* **101**, 245134 (2020).
 - [14] M.-Y. Li and P. Ye, Fracton physics of spatially extended excitations. II. Polynomial ground state degeneracy of exactly solvable models, *Phys. Rev. B* **104**, 235127 (2021).
 - [15] M.-Y. Li and P. Ye, Hierarchy of entanglement renormalization and long-range entangled states, *Phys. Rev. B* **107**, 115169 (2023).
 - [16] K. Slagle, Foliated quantum field theory of fracton order, *Phys. Rev. Lett.* **126**, 101603 (2021).
 - [17] C. Zhou, M.-Y. Li, Z. Yan, P. Ye, and Z. Y. Meng, Evolution of dynamical signature in the x-cube fracton topological order, *Phys. Rev. Res.* **4**, 033111 (2022).
 - [18] G.-Y. Zhu, J.-Y. Chen, P. Ye, and S. Trebst, Topological fracton quantum phase transitions by tuning exact tensor network states, *Phys. Rev. Lett.* **130**, 216704 (2023).
 - [19] G. Canossa, L. Pollet, M. A. Martin-Delgado, H. Song, and K. Liu, Exotic symmetry breaking properties of self-dual fracton spin models, *Phys. Rev. Res.* **6**, 013304 (2024).
 - [20] B.-X. Li, Y. Zhou, and P. Ye, Three-dimensional fracton topological orders with boundary toeplitz braiding, *Phys. Rev. B* **110**, 205108 (2024).
 - [21] T. Devakul, S. A. Parameswaran, and S. L. Sondhi, Correlation function diagnostics for type-i fracton phases, *Phys. Rev. B* **97**, 041110 (2018).
 - [22] T. Devakul and D. J. Williamson, Universal quantum computation using fractal symmetry-protected cluster phases, *Phys. Rev. A* **98**, 022332 (2018).

- [23] Y. You, T. Devakul, F. J. Burnell, and S. L. Sondhi, Subsystem symmetry protected topological order, *Phys. Rev. B* **98**, 035112 (2018).
- [24] T. Devakul, Y. You, F. J. Burnell, and S. L. Sondhi, Fractal symmetric phases of matter, *SciPost Phys.* **6**, 007 (2019).
- [25] T. Devakul, Classifying local fractal subsystem symmetry-protected topological phases, *Phys. Rev. B* **99**, 235131 (2019).
- [26] D. T. Stephen, H. P. Nautrup, J. Bermejo-Vega, J. Eisert, and R. Raussendorf, Subsystem symmetries, quantum cellular automata, and computational phases of quantum matter, *Quantum* **3**, 142 (2019).
- [27] A. T. Schmitz, S.-J. Huang, and A. Prem, Entanglement spectra of stabilizer codes: A window into gapped quantum phases of matter, *Phys. Rev. B* **99**, 205109 (2019).
- [28] Y. You, T. Devakul, S. L. Sondhi, and F. J. Burnell, Fractonic Chern-Simons and Bf theories, *Phys. Rev. Res.* **2**, 023249 (2020).
- [29] J. F. San Miguel, A. Dua, and D. J. Williamson, Bifurcating subsystem symmetric entanglement renormalization in two dimensions, *Phys. Rev. B* **103**, 035148 (2021).
- [30] F. J. Burnell, T. Devakul, P. Gorantla, H. T. Lam, and S.-H. Shao, Anomaly inflow for subsystem symmetries, *Phys. Rev. B* **106**, 085113 (2022).
- [31] C. Zhou, M.-Y. Li, Z. Yan, P. Ye, and Z. Y. Meng, Detecting subsystem symmetry protected topological order through strange correlators, *Phys. Rev. B* **106**, 214428 (2022).
- [32] J.-Y. Zhang, M.-Y. Li, and P. Ye, Higher-order cellular automata generated symmetry-protected topological phases and detection through multi point strange correlators, *PRX Quantum* **5**, 030342 (2024).
- [33] M. Pretko, Generalized electromagnetism of subdimensional particles: A spin liquid story, *Phys. Rev. B* **96**, 035119 (2017).
- [34] M. Pretko, Subdimensional particle structure of higher rank $u(1)$ spin liquids, *Phys. Rev. B* **95**, 115139 (2017).
- [35] M. Pretko, The fracton gauge principle, *Phys. Rev. B* **98**, 115134 (2018).
- [36] S. Pai and M. Pretko, Dynamical scar states in driven fracton systems, *Phys. Rev. Lett.* **123**, 136401 (2019).
- [37] H. Ma and M. Pretko, Higher-rank deconfined quantum criticality at the Lifshitz transition and the exciton Bose condensate, *Phys. Rev. B* **98**, 125105 (2018).
- [38] S. Pai and M. Pretko, Fractonic line excitations: An inroad from three-dimensional elasticity theory, *Phys. Rev. B* **97**, 235102 (2018).
- [39] S. Pai, M. Pretko, and R. M. Nandkishore, Localization in fractonic random circuits, *Phys. Rev. X* **9**, 021003 (2019).
- [40] A. Gromov, Towards classification of fracton phases: The multipole algebra, *Phys. Rev. X* **9**, 031035 (2019).
- [41] J. Wang and S.-T. Yau, Non-Abelian gauged fracton matter field theory: Sigma models, superfluids, and vortices, *Phys. Rev. Res.* **2**, 043219 (2020).
- [42] J. Feldmeier, P. Sala, G. De Tomasi, F. Pollmann, and M. Knap, Anomalous diffusion in dipole- and higher-moment-conserving systems, *Phys. Rev. Lett.* **125**, 245303 (2020).
- [43] J.-K. Yuan, S. A. Chen, and P. Ye, Fractonic superfluids, *Phys. Rev. Res.* **2**, 023267 (2020).
- [44] S. A. Chen, J.-K. Yuan, and P. Ye, Fractonic superfluids. II. Condensing subdimensional particles, *Phys. Rev. Res.* **3**, 013226 (2021).
- [45] J.-K. Yuan, S. A. Chen, and P. Ye, Quantum hydrodynamics of fractonic superfluids with lineon condensate: From Navier-Stokes-like equations to Landau-like criterion, *Chin. Phys. Lett.* **39**, 057101 (2022).
- [46] H. Li and P. Ye, Renormalization group analysis on emergence of higher rank symmetry and higher moment conservation, *Phys. Rev. Res.* **3**, 043176 (2021).
- [47] J.-K. Yuan, S. A. Chen, and P. Ye, Hierarchical proliferation of higher-rank symmetry defects in fractonic superfluids, *Phys. Rev. B* **107**, 205134 (2023).
- [48] S. A. Chen and P. Ye, Many-body physics of spontaneously broken higher-rank symmetry: From fractonic superfluids to dipolar Hubbard model, [arXiv:2305.00941](https://arxiv.org/abs/2305.00941).
- [49] A. Gromov, A. Lucas, and R. M. Nandkishore, Fracton hydrodynamics, *Phys. Rev. Res.* **2**, 033124 (2020).
- [50] A. Jain, K. Jensen, R. Liu, and E. Mefford, Dipole superfluid hydrodynamics, *J. High Energy Phys.* **09** (2023) 184.
- [51] P. Glorioso, X. Huang, J. Guo, J. F. Rodriguez-Nieva, and A. Lucas, Goldstone bosons and fluctuating hydrodynamics with dipole and momentum conservation, *J. High Energy Phys.* **05** (2023) 022.
- [52] A. Głodkowski, F. Peña-Benítez, and P. Surówka, Dissipative fracton superfluids, *J. High Energy Phys.* **07** (2024) 285.
- [53] J. Armas and E. Have, Ideal fracton superfluids, *SciPost Phys.* **16**, 039 (2024).
- [54] A. Jain, K. Jensen, R. Liu, and E. Mefford, Dipole superfluid hydrodynamics. Part II, *J. High Energy Phys.* **07** (2024) 197.
- [55] A. Morningstar, N. O'Dea, and J. Richter, Hydrodynamics in long-range interacting systems with center-of-mass conservation, *Phys. Rev. B* **108**, L020304 (2023).
- [56] C. Stahl, M. Qi, P. Glorioso, A. Lucas, and R. Nandkishore, Fracton superfluid hydrodynamics, *Phys. Rev. B* **108**, 144509 (2023).
- [57] A. Osborne and A. Lucas, Infinite families of fracton fluids with momentum conservation, *Phys. Rev. B* **105**, 024311 (2022).
- [58] J. Boesl, P. Zechmann, J. Feldmeier, and M. Knap, Deconfinement dynamics of fractons in tilted Bose-Hubbard chains, *Phys. Rev. Lett.* **132**, 143401 (2024).
- [59] D. Doshi and A. Gromov, Vortices as fractons, *Commun. Phys.* **4**, 44 (2021).
- [60] K. T. Grosvenor, C. Hoyos, F. Peña Benítez, and P. Surówka, Hydrodynamics of ideal fracton fluids, *Phys. Rev. Res.* **3**, 043186 (2021).
- [61] K. T. Grosvenor, C. Hoyos, F. Peña-Benítez, and P. Surówka, Space-dependent symmetries and fractons, *Front. Phys.* **9**, 792621 (2022).
- [62] P. Zechmann, J. Boesl, J. Feldmeier, and M. Knap, Dynamical spectral response of fractonic quantum matter, *Phys. Rev. B* **109**, 125137 (2024).
- [63] S. Angus, M. Kim, and J.-H. Park, Fractons, non-Riemannian geometry, and double field theory, *Phys. Rev. Res.* **4**, 033186 (2022).
- [64] F. Peña-Benítez, Fractons, symmetric gauge fields and geometry, *Phys. Rev. Res.* **5**, 013101 (2023).
- [65] E. Bertolini, N. Maggiore, and G. Palumbo, Covariant fracton gauge theory with boundary, *Phys. Rev. D* **108**, 025009 (2023).
- [66] E. Bertolini and H. Kim, Covariant interacting fractons, *Phys. Rev. D* **111**, 025006 (2025).

- [67] E. Bertolini and H. Kim, Strings as hyper-fractons, [arXiv:2410.11678](#).
- [68] J. Wang, K. Xu, and S.-T. Yau, Higher-rank tensor non-Abelian field theory: Higher-moment or subdimensional polynomial global symmetry, algebraic variety, Noether's theorem, and gauging, *Phys. Rev. Res.* **3**, 013185 (2021).
- [69] R. Argurio, C. Hoyos, D. Musso, and D. Naegels, Fractons in effective field theories for spontaneously broken translations, *Phys. Rev. D* **104**, 105001 (2021).
- [70] J. Iaconis, A. Lucas, and R. Nandkishore, Multipole conservation laws and subdiffusion in any dimension, *Phys. Rev. E* **103**, 022142 (2021).
- [71] C. Stahl, E. Lake, and R. Nandkishore, Spontaneous breaking of multipole symmetries, *Phys. Rev. B* **105**, 155107 (2022).
- [72] L. Radzihovsky, Lifshitz gauge duality, *Phys. Rev. B* **106**, 224510 (2022).
- [73] A. Kapustin and L. Spodyneiko, Hohenberg-Mermin-Wagner-type theorems and dipole symmetry, *Phys. Rev. B* **106**, 245125 (2022).
- [74] E. Lake, M. Hermele, and T. Senthil, Dipolar Bose-Hubbard model, *Phys. Rev. B* **106**, 064511 (2022).
- [75] E. Lake, H.-Y. Lee, J. H. Han, and T. Senthil, Dipole condensates in tilted Bose-Hubbard chains, *Phys. Rev. B* **107**, 195132 (2023).
- [76] K. Giergiel, R. Lier, P. Surówka, and A. Kosior, Bose-Hubbard realization of fracton defects, *Phys. Rev. Res.* **4**, 023151 (2022).
- [77] P. Zechmann, E. Altman, M. Knap, and J. Feldmeier, Fractonic Luttinger liquids and supersolids in a constrained Bose-Hubbard model, *Phys. Rev. B* **107**, 195131 (2023).
- [78] W. Xu, C. Lv, and Q. Zhou, Multipolar condensates and multipolar Josephson effects, *Nat. Commun.* **15**, 4786 (2024).
- [79] P. Gorantla, H. T. Lam, N. Seiberg, and S.-H. Shao, Global dipole symmetry, compact Lifshitz theory, tensor gauge theory, and fractons, *Phys. Rev. B* **106**, 045112 (2022).
- [80] E. Lake and T. Senthil, Non-fermi liquids from kinetic constraints in tilted optical lattices, *Phys. Rev. Lett.* **131**, 043403 (2023).
- [81] J. Molina-Vilaplana, A post-Gaussian approach to dipole symmetries and interacting fractons, *J. High Energy Phys.* **08** (2023) 065.
- [82] P. Gorantla, H. T. Lam, N. Seiberg, and S.-H. Shao, (2+1)-dimensional compact Lifshitz theory, tensor gauge theory, and fractons, *Phys. Rev. B* **108**, 075106 (2023).
- [83] A. Anakru and Z. Bi, Non-Fermi liquids from dipolar symmetry breaking, *Phys. Rev. B* **108**, 165112 (2023).
- [84] X. Huang, A Chern-Simons theory for dipole symmetry, *SciPost Phys.* **15**, 153 (2023).
- [85] E. Afxonidis, A. Caddeo, C. Hoyos, and D. Musso, Dipole symmetry breaking and fractonic Nambu-Goldstone mode, *SciPost Phys. Core* **6**, 082 (2023).
- [86] H. T. Lam, Classification of dipolar symmetry-protected topological phases: Matrix product states, stabilizer Hamiltonians, and finite tensor gauge theories, *Phys. Rev. B* **109**, 115142 (2024).
- [87] J. H. Han, E. Lake, and S. Ro, Scaling and localization in multipole-conserving diffusion, *Phys. Rev. Lett.* **132**, 137102 (2024).
- [88] F. J. Burnell, S. Moudgalya, and A. Prem, Filling constraints on translation invariant dipole conserving systems, *Phys. Rev. B* **110**, L121113 (2024).
- [89] A. Chavda, D. Naegels, and J. Staunton, Fractonic coset construction for spontaneously broken translations, *Phys. Rev. D* **111**, 085018 (2025).
- [90] Y.-M. Hu and B. Lian, The bosonic quantum breakdown Hubbard model, [arXiv:2401.04309](#).
- [91] K. Jensen and A. Raz, Large n fractons, *Phys. Rev. Lett.* **132**, 071603 (2024).
- [92] D. Gaiotto, A. Kapustin, N. Seiberg, and B. Willett, Generalized global symmetries, *J. High Energy Phys.* **02** (2015) 172.
- [93] J. McGreevy, Generalized symmetries in condensed matter, *Annu. Rev. Condens. Matter Phys.* **14**, 57 (2023).
- [94] H. Zhai, *Ultracold Atomic Physics* (Cambridge University Press, Cambridge, 2021), pp. 247–261.
- [95] Y.-T. Oh, J. Kim, E.-G. Moon, and J. H. Han, Rank-2 toric code in two dimensions, *Phys. Rev. B* **105**, 045128 (2022).
- [96] Z. Liang, Y. Xu, J. T. Iosue, and Y.-A. Chen, Extracting topological orders of generalized Pauli stabilizer codes in two dimensions, *PRX Quantum* **5**, 030328 (2024).
- [97] Z. Liang, B. Yang, J. T. Iosue, and Y.-A. Chen, Operator algebra and algorithmic construction of boundaries and defects in (2+1)D topological Pauli stabilizer codes, [arXiv:2410.11942](#).
- [98] V. Khemani, M. Hermele, and R. Nandkishore, Localization from Hilbert space shattering: From theory to physical realizations, *Phys. Rev. B* **101**, 174204 (2020).
- [99] E. Guardado-Sanchez, A. Morningstar, B. M. Spar, P. T. Brown, D. A. Huse, and W. S. Bakr, Subdiffusion and heat transport in a tilted two-dimensional Fermi-Hubbard system, *Phys. Rev. X* **10**, 011042 (2020).
- [100] S. Scherg, T. Kohlert, P. Sala, F. Pollmann, B. Hebbe Madhusudhana, I. Bloch, and M. Aidelsburger, Observing non-ergodicity due to kinetic constraints in tilted Fermi-Hubbard chains, *Nat. Commun.* **12**, 4490 (2021).
- [101] T. Kohlert, S. Scherg, P. Sala, F. Pollmann, B. Hebbe Madhusudhana, I. Bloch, and M. Aidelsburger, Exploring the regime of fragmentation in strongly tilted Fermi-Hubbard chains, *Phys. Rev. Lett.* **130**, 010201 (2023).