

Propagation of the space-time entanglement of twin photons and twin beams

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We investigate how the space-time entanglement of photon pairs generated by multimode parametric down-conversion (PDC) propagates away from the source. We show that the entangled two-photon probability, tightly confined in space and time at the source, quickly broadens and decays out of a focal plane, behaving as a focused coherent beam of approximately twice the wavelength of the twin photons. This behavior is in strong contrast with the much slower decay rate of the classical mutual coherence of the PDC radiation, which follows the prescription of the generalized Van Cittert-Zernicke theorem. Besides their fundamental interest, our results could be relevant for the design of experiments, such as quantum-enhanced two-photon processes, that exploit the spatiotemporal entanglement of the pairs.

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I. INTRODUCTION

When an intense laser beam propagates through certain nonlinear materials, some of its photons can occasionally split into a pair of lower-energy photons, conventionally called signal and idler, through the process of optical parametric down-conversion (PDC). Due to the multiple possibilities for energy and momentum conservation, photon pairs born in this way are emitted in a wide range of transverse and frequency modes, resulting in a high degree of entanglement in their spatiotemporal degrees of freedom. The temporal and transverse bunching of the PDC pairs has been known since early investigations [1,2] and found several applications, ranging from historical experiments of two-photon quantum interference [3] to, e.g., quantum-enhanced clock synchronization [4] or quantum imaging (see the reviews [5–8] and references therein). In particular, over the last 30 years there has been a growing interest in using the photonic entanglement to increase the efficiency of multiphoton interactions, such as two-photon absorption (TPA) (see Refs. [9–14] and references therein) and second harmonic generation (SHG) [15–20], with potential applications to nonlinear microscopy and spectroscopy. The main idea is that the entangled signal and idler photons, generated by splitting a single parent photon of frequency ω_p , may remain confined within an exceptionally small interaction area and time, still maintaining the strict anticorrelation of their frequencies $\omega_s + \omega_i = \omega_p$.

Because of this, they can behave in two-photon processes as a single particle of twice the energy, greatly increasing the probability of interaction. In the literature, the emphasis is often placed on the time-frequency entanglement, due to the purely temporal models adopted. However, since early studies [9], it was clear that crucial for the quantum enhancement of these processes is the spatiotemporal entanglement volume in which the pair is confined, product of an entanglement area and an entanglement [21] time, with the interaction efficiency scaling as the inverse of this volume [9,19].

However, little is known about the question whether the spatiotemporal confinement of PDC pairs can be maintained from the source to the medium in which the interaction occurs, i.e., on how biphoton entanglement propagates. In contrast, there is an extended literature on the propagation of the classical light speckles generated from incoherent sources, which can be considered the closer classical counterpart, since they describe the bunching of classically correlated photons. In this context, a milestone is the Van Cittert-Zernike theorem [23,24], which describes the transverse properties of the radiation emitted from an incoherent source in terms of the two-dimensional Fourier transform of the source intensity distribution. Its generalizations [25,26] and extensions [27,28] provide a thorough view of the propagation of the classical transverse coherence.

On the quantum side, several authors [29–33] investigated the transition from the position correlation of PDC pairs in the near field of the source to their momentum correlation in the far field, focusing on the transverse correlation width and exploring fundamental aspects such as the variation of the Fedorov ratio and the related migration of entanglement during propagation [29–31]. However, there was—to our knowledge—no attempt to provide a general model of the propagation of the entangled two-photon probability comparable to those elaborated for the classical coherence [25–28].

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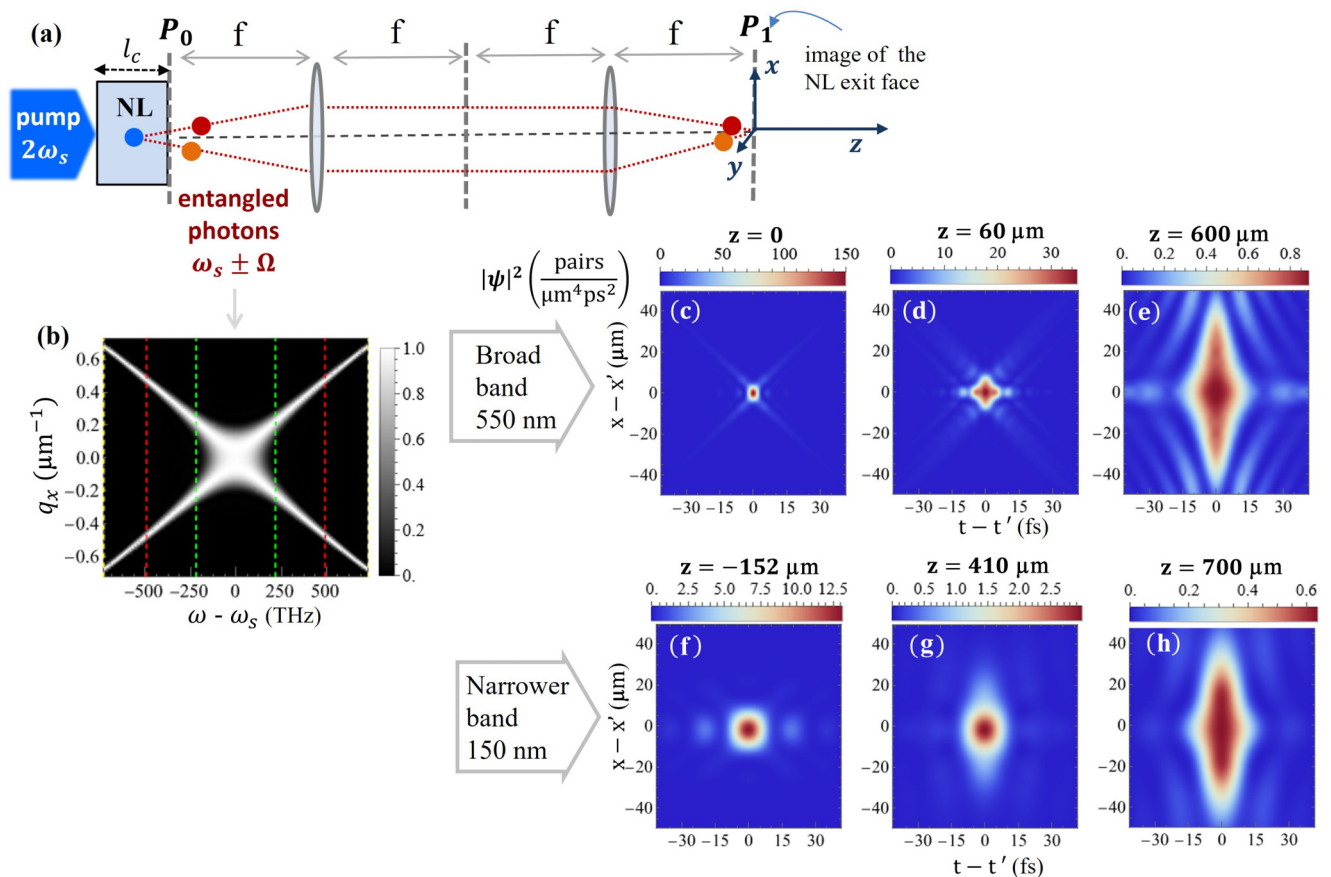


FIG. 1. (a) Entangled photon pairs, generated by PDC in the nonlinear crystal NL, are recollected at the plane P_1 by an achromatic $4f$ imaging scheme. We study the probability $|\Psi|^2$ of finding an entangled pair with a temporal and a transverse separation $t - t'$ and $x - x'$, as a function of the distance z from P_1 . (b) Fourier spectrum of light generated by PDC in a 2-mm β -barium borate (BBO) crystal pumped at 400 nm in quasicollinear conditions, calculated from Eq. (38). The dashed lines enclose the bandwidths considered. Panels (c)–(e), obtained by numerical integration of Eq. (24b), show $|\Psi|^2$ at increasing distances from plane P_1 , for the 550 nm bandwidth. Panels (f)–(h) are similar plots, but for the 150 nm bandwidth, showing a slower decay with propagation distance.

The aim of this work is to provide a picture of the propagation of the entangled two-photon probability, with a quantitative model valid in any gain regime and under very general conditions. Unlike previous studies [29–33] limited to the transverse spatial properties, within a narrow-band approximation, our description of the propagation will be fully spatiotemporal. The reason is that, in view of applications to entangled multiphoton interactions, a broad bandwidth is highly desirable to ensure a tight temporal confinement of the pair. Under these conditions, the spatial and temporal properties of PDC emission are not independent [34,35] due to the angular dispersion relations ruling the process. Our study focuses on the near-field region close to the PDC source and propagation in free space. In a related work [36], the analysis will be extended to propagation in dense media.

The key result that will emerge is that quantum entanglement propagates in a radically different way than its classical counterparts, and the probability of two entangled photons decays and broadens more rapidly than the classical probability of two photons under equivalent conditions. Figures 1(c)–1(e) anticipate some results that will be derived in the following [in particular, Eq. (24b)] and show how the joint probability of finding a pair of entangled photons propagates with distance

from the source: At 60 μm from the crystal exit face, the joint distribution has already spread out both in time and in the transverse direction, and the probability of detecting two entangled photons together has decreased by a factor of 4, while after 600 μm it has reduced 180 times. On the other hand, we will find that at these distances, which fall within the so-called deep Fresnel region [27,37], the classical coherence speckles of the PDC radiation remain practically unchanged (see Fig. 7). We will instead show that the biphoton probability behaves in propagation similarly to a focused coherent pulse having twice the wavelength of the PDC light. Hence, also in this case, it diffracts faster than the classical reference (a coherent field at the same wavelength). Besides their fundamental interest, these findings can be of great importance when designing experiments aimed at exploiting the temporal and spatial quantum correlation of the entangled pairs.

In order not to burden the formalism, we focus on the widely used type I PDC process, close to collinear and degenerate conditions, in which the Fourier spectrum has the characteristic X-shape shown in Fig. 1(b), but our results can be straightforwardly generalized to other configurations.

The paper is organized as follows. Section II introduces the scheme studied and the formalism used. Section III reviews

the properties of the space-time entanglement generated at the source, using models for PDC recently developed [38,39]. Section IV is the core of the paper and presents our results for the propagation of the biphoton entanglement. Sections IV A and IV B highlight the fast decay and broadening of the space-time entanglement in the zone near to the source, while Sec. IV C sketches some results about propagation in the far zone where the position entanglement transforms into momentum entanglement. To complete the picture, Sec. V studies for comparison how the classical coherence properties of the PDC radiation change in propagation.

II. GENERAL FORMALISM

The scheme is shown in Fig. 1(a): An intense coherent pump beam, centered at the frequency $2\omega_s$, propagates in the nonlinear crystal NL and generates by parametric down-conversion entangled photon pairs in a broad band around frequency ω_s [Fig. 1(b)]. We assume paraxial propagation around a mean axis z and a type I process in which PDC photons have the same polarization. We introduce the quantum operator associated with the down-converted field at the NL output plane P_0 (the source plane), as

$$\hat{A}_0(\vec{r}, t) = \int \frac{d^2q}{2\pi} \int \frac{d\Omega}{\sqrt{2\pi}} e^{i\vec{q}\cdot\vec{r}} e^{-i\Omega t} \hat{A}_0(\vec{q}, \Omega), \quad (1)$$

where $\vec{r} = x\vec{e}_x + y\vec{e}_y$ is the position in the transverse plane, $\vec{q} = q_x\vec{e}_x + q_y\vec{e}_y$ is the transverse wave vector, and a frame of reference oscillating at the central frequency ω_s has been adopted, so that $\Omega = \omega - \omega_s$. Dimensions are such that $\hat{A}_0^\dagger(\vec{r}, t)\hat{A}_0(\vec{r}, t)$ is a photon number per unit area and time. For brevity of notation, when convenient the space-time coordinates in $\mathbb{R}^3$ will be indicated as

$$\begin{aligned} \vec{\xi} &:= (x, y, t), \\ \vec{w} &:= (q_x, q_y, \Omega), \\ \text{with } \vec{w} \cdot \vec{\xi} &= xq_x + yq_y - \Omega t. \end{aligned} \quad (2)$$

We are interested in the properties of the field that has propagated in free space a distance z from the source. Precisely, z will be the longitudinal coordinate from the image plane P_1 of the crystal output (hence can be positive or negative), formed by an achromatic $4f$ imaging scheme. In concrete implementations that involve a broad frequency band, imaging lenses are replaced by dispersion-free mirrors [20,40,41], or the dispersion from the optics is compensated [17]. Thus, it is reasonable to neglect any dispersion effects from optical elements, and to assume that the only significant losses come from the presence of a frequency filter that limits the transmitted bandwidth. In these conditions, the field at the plane located at a distance z from P_1 reads

$$\begin{aligned} \hat{A}(\vec{w}, z) &= \chi_f(\Omega) h_z(\vec{w}) \hat{A}_0(\vec{w}) + [1 - \chi_f(\Omega)]^{\frac{1}{2}} \hat{v}(\vec{w}), \\ h_z(\vec{q}, \Omega) &= \exp \left[-i \frac{q^2 z}{4\pi} \lambda(\Omega) \right], \end{aligned} \quad (3)$$

where $\chi_f(\Omega)$ is the rectangle function of the interval $[-\Omega_{\max}, \Omega_{\max}]$, simulating the bandpass filter; $h_z(\vec{q}, \Omega)$ is the standard paraxial propagator in free space for a

monochromatic wave at frequency $\omega_s + \Omega$, where

$$\lambda(\Omega) = \frac{2\pi c}{\omega_s + \Omega} \quad (4)$$

is the corresponding wavelength; and $\hat{v}$ is the vacuum contribution from the frequency regions that are filtered out, whose form is not relevant for our calculations.

The main object of our investigation is the so-called biphoton amplitude, which in the space-time domain reads

$$\Psi(\vec{\xi}, \vec{\xi}'; z) = \langle \hat{A}(\vec{\xi}, z) \hat{A}(\vec{\xi}', z) \rangle. \quad (5)$$

This particular field-field correlation is characteristic of processes, where particles are created in pairs, and is at the heart of the entanglement of the PDC state. The other quantity of interest is the coherence function,

$$G^{(1)}(\vec{\xi}, \vec{\xi}'; z) = \langle \hat{A}^\dagger(\vec{\xi}', z) \hat{A}(\vec{\xi}, z) \rangle, \quad (6)$$

characterizing the classical spatiotemporal statistics of the radiation and, in particular, for $\vec{\xi}' = \vec{\xi}$, giving the photon flux distribution. The biphoton amplitude (5) and the coherence function (6) are sufficient to calculate all the physical observables, because the state generated by PDC from vacuum is a Gaussian state (i.e., a state whose characteristic function that generates the moments is a Gaussian), with zero mean $\langle \hat{A}_0(\vec{\xi}) \rangle = 0$. Then, the quantum version [42] of the Isserlis theorem [43] for the factorization of the Gaussian moments ensures that for Gaussian states higher-order moments can be factorized into products of the second-order moments [42]. This can be also seen as the generalization of the Wick's theorem [44], ubiquitous in quantum field theory, to processes in which the anomalous propagator $\langle \hat{A}(\vec{\xi}) \hat{A}(\vec{\xi}') \rangle$ does not vanish [45]. In particular, the joint probability of detecting two photons [46] at the space-time positions $\vec{\xi}$ and $\vec{\xi}'$ at a plane z can be expanded as

$$\begin{aligned} \langle \hat{A}^\dagger(\vec{\xi}, z) \hat{A}^\dagger(\vec{\xi}', z) \hat{A}(\vec{\xi}', z) \hat{A}(\vec{\xi}, z) \rangle \\ = \mathcal{I}(\vec{\xi}, z) \mathcal{I}(\vec{\xi}', z) + |G^{(1)}(\vec{\xi}, \vec{\xi}'; z)|^2 + |\Psi(\vec{\xi}, \vec{\xi}'; z)|^2, \end{aligned} \quad (7)$$

where $\mathcal{I}(\vec{\xi}, z) = G^{(1)}(\vec{\xi}, \vec{\xi}, z)$ is the PDC photon flux. The first two terms are typical of classical thermal statistics and represent the contribution of uncorrelated and classically correlated photons, respectively. The third term is the additional contribution arising from pairing of quantum entangled photons, with $|\Psi(\vec{\xi}, \vec{\xi}'; z)|^2$ being the joint probability of finding two entangled photons at points $\vec{\xi}$ and $\vec{\xi}'$. Altogether, the three terms in Eq. (7) give rise to what is known as the extrabunching [47,48] of the PDC state [49].

By performing the Fourier transform of Eq. (5), and inserting the field propagation described by Eq. (3), one has

$$\begin{aligned} \Psi(\vec{\xi}, \vec{\xi}'; z) &= \int \frac{d^3w}{(2\pi)^{\frac{3}{2}}} e^{i\vec{\xi} \cdot \vec{w}} \int \frac{d^3w'}{(2\pi)^{\frac{3}{2}}} e^{i\vec{\xi}' \cdot \vec{w}'} \langle \hat{A}(\vec{w}, z) \hat{A}(\vec{w}', z) \rangle \\ &= \int \frac{d^3w}{(2\pi)^{\frac{3}{2}}} \int \frac{d^3w'}{(2\pi)^{\frac{3}{2}}} [\Psi_0(\vec{w}, \vec{w}') e^{i(\vec{\xi} \cdot \vec{w} + \vec{\xi}' \cdot \vec{w}')} \\ &\quad \times \chi_f(\Omega) \chi_f(\Omega') h_z(\vec{w}) h_z(\vec{w}')], \end{aligned} \quad (8)$$

where $\Psi_0(\vec{w}, \vec{w}') = \langle \hat{A}_0(\vec{w}) \hat{A}_0(\vec{w}') \rangle$ is calculated at the source plane P_0 . In a similar way, the propagation of the coherence

function is described by

$$G^{(1)}(\vec{\xi}, \vec{\xi}'; z) = \int \frac{d^3 w}{(2\pi)^{\frac{3}{2}}} \int \frac{d^3 w'}{(2\pi)^{\frac{3}{2}}} [G_0^{(1)}(\vec{w}, \vec{w}') e^{i(\vec{\xi} \cdot \vec{w} - \vec{\xi}' \cdot \vec{w}')} \times \chi_f(\Omega) \chi_f(\Omega') h_z(\vec{w}) h_z^*(\vec{w}')] , \quad (9)$$

where $G_0^{(1)}(\vec{w}, \vec{w}') = \langle \hat{A}_0^\dagger(\vec{w}') \hat{A}_0(\vec{w}) \rangle$ is again calculated at the source plane P_0 .

The next sections will focus on the propagation of the quantum correlation (8), while the propagation of the classical coherence will be postponed to Sec. V.

III. SPACE-TIME ENTANGLEMENT AT THE SOURCE

The properties of the down-converted light at the source plane P_0 (or at its image plane P_1) are described using a model for PDC introduced in Refs. [38,39], which we called quasistationary (QS). This model is based on the existence of two well-separated spatiotemporal scales in the radiation field: a fast scale associated with the decay of the correlation and coherence peaks and a slow scale associated with the decay of the field envelope. It provides a useful extension of the well-known plane-wave and cw description (see, e.g., Refs. [50–53]; for brevity in the following plane wave pump (PWP) model) to the more realistic case of a pump pulse with finite cross section and duration. It is valid in any gain regime, ranging from the spontaneous regime of coincidence counts to the stimulated one in which bright twin beams are generated. At low gain, it converges to the well-known perturbative solution widely used in the literature to describe spontaneous parametric down-conversion (SPDC) (see, e.g., Ref. [54]). Physically, it corresponds to a highly multimode state, which can be depicted as a collection of approximately independent two-mode squeezed vacua generated in symmetric Fourier modes.

Its validity conditions are close to many experimental realizations [20,41,55–57] and require that the cross section and the duration of the pump pulse driving PDC are much larger than the characteristic temporal and spatial broadening due to dispersion and diffraction in the medium [39]. A further simplifying assumption is that the temporal and transverse walk-off effects can be substantially ignored. Under these assumptions, the equation of propagation for the quantum field operator associated with the PDC field can be approximately solved in terms of generalized Bogoljubov relations linking the fields at the input and output of the nonlinear crystal in a nonlocal way [39]. A simple, semianalytical form of the biphoton amplitude (5) and the coherence function (6) can hence be derived. Note that in this limit the two functions factorize into the product of a slowly varying envelope, which substitutes the unphysical plane wave of infinite energy of the PWP theory, and a localized correlation peak, reproducing the results [34,35,58] calculated within the PWP model. Thus, the QS approximation for PDC generalizes the concept of spatially quasihomogeneous sources [25] widely used for the study of classical spatial coherence, extending it to space-time and to the quantum domain.

In particular, the biphoton correlation at the medium output reads

$$\Psi_0(\vec{\xi}, \vec{\xi}') = \langle \hat{A}_0(\vec{\xi}) \hat{A}_0(\vec{\xi}') \rangle = F_\psi \left(\frac{\vec{\xi} + \vec{\xi}'}{2} \right) \psi_0(\vec{\xi}' - \vec{\xi}), \quad (10)$$

where F_ψ is a slowly varying envelope, decaying on the scale of the pump pulse, and defined [38] as

$$F_\psi(\vec{\xi}) = \frac{\sinh[2g\mathcal{A}_p(\vec{\xi})]}{\sinh(2g)}, \quad (11)$$

where $\mathcal{A}_p(\vec{\xi})$ is the space-time profile of the pump field, which can be described as, e.g., a Gaussian $\mathcal{A}_p(\vec{\xi}) = \exp[-\frac{x^2+y^2}{w_p^2} - \frac{t^2}{\tau_p^2}]$. Notice that at low gain, $F_\psi(\vec{\xi})$ converges to the pump amplitude $\mathcal{A}_p(\vec{\xi})$, while at high gain, as discussed in detail in Refs. [38,39], it shrinks by a factor $\sqrt{\frac{\tanh(2g)}{2g}}$ and as soon as $g \gtrsim 1$ it coincides with the intensity distribution $\mathcal{I}(\vec{\xi})$ of the PDC beam, given by Eq. (37). Notice that our results do not depend too much on the exact shape of the PDC envelopes, provided they are smooth peaks that decay slowly with respect to the correlation functions. For simplicity, in the following we will not hence distinguish the size of the biphoton envelope $F_\psi(\vec{\xi})$ from that of the beam envelope $\mathcal{I}(\vec{\xi})$, and we will indicate them as D_s (transverse size, FWHM) and T_s (duration, FWHM).

In Eq. (10), $\psi_0(\vec{\xi}' - \vec{\xi})$ is a well-localized peak, describing the quantum correlation of biphotons. This was extensively studied in the PWP approximation in Refs. [35,58], where it was defined as the Fourier transform of the corresponding spectral two-photon amplitude:

$$\psi_0(\vec{\xi}' - \vec{\xi}) = \int \frac{d^3 w}{(2\pi)^3} \Phi_{\text{PDC}}(\vec{w}) e^{i(\vec{\xi}' - \vec{\xi}) \cdot \vec{w}}, \quad (12)$$

where

$$\Phi_{\text{PDC}}(\vec{w}) = \frac{g \sinh \Gamma(\vec{w})}{\Gamma(\vec{w})} \left[\cosh \Gamma(\vec{w}) + i \frac{D(\vec{w}) l_c}{2\Gamma(\vec{w})} \sinh \Gamma(\vec{w}) \right] \quad (13)$$

provides, up to a constant factor, the probability amplitude of generating an entangled pair in the conjugate Fourier modes $\vec{w}$ and $-\vec{w}$. Equation (13) is the result of the PWP theory (see e.g. Eq. (7) in Ref. [37]), where $g = \chi l_c |\alpha_p|$ is the parametric gain, with α_p being the peak amplitude of the pump field and l_c the crystal length, $\Gamma(\vec{w}) = \sqrt{g^2 - \frac{|D(\vec{w}) l_c|^2}{4}}$, and

$$D(\vec{q}, \Omega) = \sqrt{k_s^2(\Omega) - q^2} + \sqrt{k_s^2(-\Omega) - q^2} - k_p \quad (14)$$

is the phase-mismatch function, accounting for the energy and momentum conservation, where $k_s(\pm\Omega)$ and k_p are the wave numbers of the signal and pump waves at the frequencies $\omega_s \pm \Omega$ and ω_p , respectively. Examples of the spectral amplitude (13) are shown in Figs. 2(a) and 2(b) (they differ only in the selected bandwidth). Like the spectrum in Fig. 1(b), $\Phi_{\text{PDC}}(\vec{w})$ is sharply peaked around the surface where $D(\vec{w}) = 0$, which in the collinear condition of this work has an hourglass shape (an “X” in two dimensions) [34,60]. This geometry can be understood by making the second-order series expansion

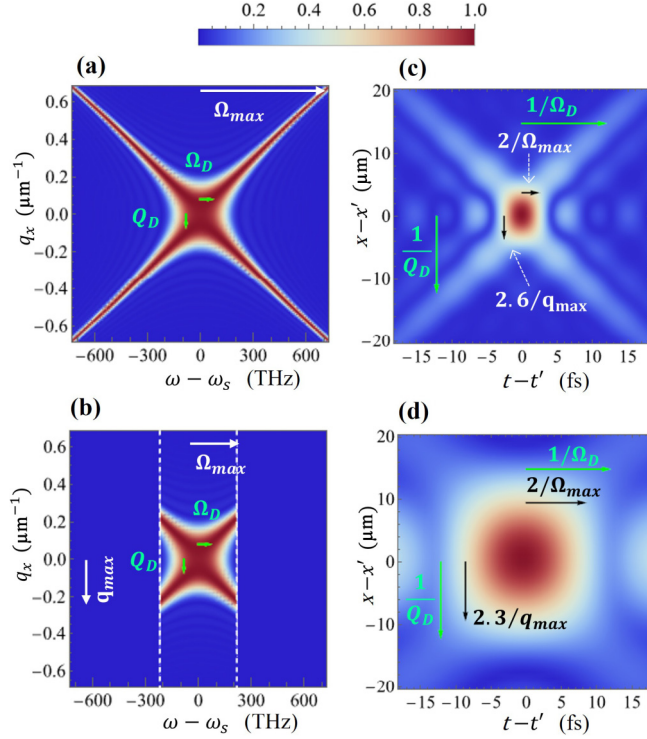


FIG. 2. First column: Two-photon spectral amplitude $|\Phi_{\text{PDC}}|$ in the Fourier plane (Ω, q_x) , for a filtered bandwidth: (a) 550 nm and (b) 150 nm. The green arrows show its scales of variation [Eq. (15)], $\Omega_D=82$ THz and $Q_D=0.08 \mu\text{m}^{-1}$. Second column: Corresponding biphoton correlation $|\psi_0|$ at the source, as a function of $t-t'$ and $x-x'$. The black arrows show here the size of the correlation peak, inversely proportional to the selected bandwidths Ω_{max} and q_{max} . Two-millimeter BBO crystal, cut for type I collinear conversion $400 \rightarrow 800$ nm, pumped by a 600 fs, 600 μm pulse. Gain $g = 2.2$.

of the phase mismatch in Eq. (14), $D(\vec{q}, \Omega) \simeq (2k_s - k_p) + k_s''\Omega^2 - \frac{q^2}{k_s} + \dots$ (here and in the following, $k_s = k_s|_{\vec{w}=0}$, $k_s' = dk_s/d\Omega|_{\vec{w}=0}$, etc.). Hence,

$$\Omega_D = \tau_D^{-1} = \frac{1}{\sqrt{k_s''}l_c}, \quad Q_D = l_D^{-1} = \sqrt{\frac{k_s}{l_c}} \quad (15)$$

provide the scales of variation of Φ_{PDC} , indicated by the green arrows in Figs. 2(a) and 2(b). Their inverse, τ_D and l_D , are the characteristic temporal and spatial broadening of the light wave packet due to group velocity dispersion and diffraction in the medium.

The second column of Fig. 2 shows instead the space-time correlation as a function of the two-photon separation. Because of the Fourier relation (12), the correlation function has also an X-shape [35,56,58], whose size is related to the inverse size of the Fourier amplitude. In particular, as depicted by the arrows in Fig. 2, the width of the central correlation peak is the inverse of the width of the whole Fourier spectrum, as selected by the scheme, with the half widths at half maximum of the peak being

$$\tau_{\text{ent}} \simeq \frac{1.9}{\Omega_{\text{max}}}, \quad \rho_{\text{ent}} \simeq \frac{2.6}{q_{\text{max}}} \quad (16)$$

in the temporal and transverse domains, respectively (the numbers 1.9–2.6 come from fits of the numerically calculated correlation; see later). These values set the size of the entanglement volume in which the photon pairs are confined, crucial for the quantum enhancement of multiphoton processes [9,19]. As we shall see, they also set the rate at which the biphoton probability decays with propagation distance. Conversely, the inverse sizes $1/\Omega_D$ and $1/Q_D$ rule the distances over which the arm of the X-shaped correlation decays [35,58,59].

IV. PROPAGATION OF ENTANGLEMENT

We are now in condition of calculating the biphoton amplitude propagated to the plane z , by inserting these results into Eq. (8). The quasistationary correlation (10) in the Fourier domain becomes [39]

$$\Psi_0(\vec{w}, \vec{w}') = \mu_\psi(\vec{w} + \vec{w}')\Phi_{\text{PDC}}\left(\frac{\vec{w} - \vec{w}'}{2}\right), \quad (17)$$

where

$$\mu_\psi(\vec{w}) = \int \frac{d^3\xi}{(2\pi)^3} e^{-i\vec{w}\cdot\vec{\xi}} F_\psi(\vec{\xi}) \quad (18)$$

is now a localized peak, which forces $\vec{w}' + \vec{w} \simeq 0$ within the narrow Fourier spectrum of the pump beam, as required by the energy and transverse momentum conservation. In the quasistationary approximation, its widths (e.g., the widths $1/e$) in the transverse wave-vector domain $\delta q_0 \simeq 3/D_s$ and in the frequency domain $\delta\Omega_0 \simeq 3/T_s$ are much narrower than the scales of variation Q_D and Ω_D of the overall two-photon distribution [see Eq. (15)].

Inserting Eq. (17) inside the integral (8), and making the change of variables $(\vec{w}, \vec{w}') \rightarrow (\vec{w}_0 = \vec{w} + \vec{w}', \vec{w} = \frac{\vec{w}_0}{2}(\vec{w} - \vec{w}'))$, one has

$$\begin{aligned} \Psi(\vec{\xi}, \vec{\xi}'; z) &= \int d^3w_0 \int \frac{d^3w}{(2\pi)^3} \mu_\psi(\vec{w}_0) \Phi_{\text{PDC}}(\vec{w}) e^{i\vec{w}_0 \cdot (\frac{\vec{\xi} + \vec{\xi}'}{2})} \\ &\times e^{i\vec{w} \cdot (\vec{\xi} - \vec{\xi}')} \chi_f\left(\frac{\Omega_0}{2} + \Omega\right) \chi_f\left(\frac{\Omega_0}{2} - \Omega\right) h_z \\ &\times \left(\frac{\vec{w}_0}{2} + \vec{w}\right) h_z \left(\frac{\vec{w}_0}{2} - \vec{w}\right), \end{aligned} \quad (19)$$

where the dummy integration variable was renamed as $\vec{W} \rightarrow \vec{w}$. Some simplifications, based on the QS approximation, can be made: First, we assume that the selected bandwidth Ω_{max} is much larger than the fast decay scale $\delta\Omega_0$ of μ_ψ . This condition not only ensures a high temporal localization of biphotons, but is also necessary to preserve, after filtering, the multimodal nature of the field and the validity of the QS approximation. This allows us to set $\chi_f(\frac{\Omega_0}{2} + \Omega)\chi_f(\frac{\Omega_0}{2} - \Omega) \rightarrow \chi_f(\Omega)\chi_f(-\Omega) = \chi_f(\Omega)$. By expanding the product of propagators inside the integral, and after neglecting terms on the order of $\frac{\delta\Omega_0}{2\omega_s} \approx \frac{1}{T_s\omega_s} \ll 1$ (needs only a pulse duration much longer than the optical cycle; see Appendix A for

details:

$$\begin{aligned}
 & h_z \left(\frac{\vec{w}_0}{2} + \vec{w} \right) h_z \left(\frac{\vec{w}_0}{2} - \vec{w} \right) \\
 &= \exp \left[-i \frac{q_0^2 z}{4\pi} \frac{\lambda(\Omega) + \lambda(-\Omega)}{4} \right] \\
 &\quad \times \exp \left\{ -i \frac{\vec{q} \cdot \vec{q}_0 z}{4\pi} [\lambda(\Omega) - \lambda(-\Omega)] \right\} \\
 &\quad \times \exp \left\{ -i \frac{q^2 z}{4\pi} [\lambda(\Omega) + \lambda(-\Omega)] \right\}. \quad (20)
 \end{aligned}$$

The first factor on the right-hand side of Eq. (20) contains only the fast variable q_0 , which is restricted to the small domain δq_0 , the second one couples slow and fast variables, while the third one, as we shall see, provides a much faster decay of the two-photon probability. Having in mind applications to multiphoton processes, we concentrate on small propagation distances from the plane where the two-photon probability has its maximum. Let us see then the conditions under which the first two terms are negligible.

The first factor $\sim \exp(-i \frac{q_0^2 z}{4\pi} \lambda_p)$ contributes only for distances on the order of the Rayleigh range of the pump beam. For example, for a 600 μm waist pump and taking into account the shrinking of the signal beam, one needs $z \gtrsim \frac{\pi w_p^2}{\lambda_p} \frac{\tanh(2g)}{2g} = 642 \text{ mm}$ for $g = 2.2$.

The second phase factor,

$$\frac{\vec{q} \cdot \vec{q}_0 z}{4\pi} [\lambda(\Omega) - \lambda(-\Omega)], \quad (21)$$

couples slow and fast variables. It is reminiscent of the phase factor [see Eq. (40)] that for the classical speckles is responsible for the transition from a deep Fresnel region, where the transverse coherence does not change upon propagation [27,28,37], to the VCZ zone, where the Van Cittert-Zernike theorem holds, and the transverse speckle size increases $\propto z$. The transition takes place at a distance

$$z_{\text{DF}} = \frac{\rho D_s}{\lambda_s}, \quad (22)$$

where ρ is the transverse speckle size and D_s is the beam diameter. A heuristic explanation for this behavior was given in Ref. [37]: The light speckles have at the source a finite divergence angle $\sim \frac{\lambda}{\rho}$; hence, at small distances a speckle interferes only with a limited number of surrounding speckles, namely, those contained in a region of diameter $D^* \simeq z \frac{\lambda}{\rho}$ and the VCZ theorem applied to this region provides a transverse coherence size $\frac{\lambda z}{D^*} = \rho$, which does not depend on z . At increasing distances z , the process continues till $D^* = D_s$, which exactly sets the boundary $z = z_{\text{DF}}$ of the deep Fresnel zone. However, for entangled biphotons, the quantity in Eq. (21) remains negligible over much longer distances, due to the presence of the wavelength difference in place of the wavelength $\lambda(\Omega)$ of its classical counterpart [see Eq. (40)]. In the narrow-band limit $\Delta\lambda/\lambda_s \rightarrow 0$, the factor (21) vanishes and there is nothing like the VCZ theorem—at least in its standard form—for entangled biphotons.

The main reason for this behavior, so different from that of the classical coherence, is that the phase of entangled photons

sum up to the fixed phase of the pump beam. Hence, biphotons originating from different transverse positions constructively interfere and, as we shall see in the following, their joint propagation has close analogies with the propagation of coherent light. Moreover, under quasistationarity the conservation of transverse momentum requires that twin photons are emitted with $\vec{q}_i \simeq -\vec{q}_s$, so that, close to degeneracy, they propagate out of the crystal at symmetric angles $\vec{\theta}_s + \vec{\theta}_i = 0$. Although the signal and idler wave packets exhibit a large angular divergence, the “biphoton” propagates forward. Clearly, far away from degeneracy this is no longer true, and for a photon pair emitted at frequencies $\omega_s \pm \Omega$, one has $|\vec{\theta}_s + \vec{\theta}_i| \simeq |\frac{\vec{q}_s \lambda(\Omega)}{2\pi} - \frac{\vec{q}_s \lambda(-\Omega)}{2\pi}|$, which shows exactly the Ω dependence of the phase factor in Eq. (21). However, only photon pairs well outside degeneracy contribute, and only for distances such that $z \gtrsim \frac{2\lambda_s}{\lambda(-\Omega) - \lambda(\Omega)} z_{\text{DF}} \gg z_{\text{DF}}$, so that this factor starts to give some contribution at distances larger than $\frac{2\lambda_s}{\Delta\lambda} z_{\text{DF}}$, which in the worst case of the 550 nm bandwidth corresponds for our parameters to $\sim 6.6 \text{ mm}$. The validity of this approximation is further investigated in Appendixes B and C, where it is shown that the contribution of this phase factor actually remains small even for z in the centimeter range [see, in particular, Fig. 9(c)].

We are now ready to state the key result of this section. In a zone close enough to the source,

$$|z| < \frac{D_s \rho_{\text{ent}}}{\lambda_s} \frac{2\lambda_s}{\Delta\lambda} = z_{\text{DF}} \frac{2\lambda_s}{\Delta\lambda}, \quad (23)$$

the biphoton amplitude (19) propagated at a distance z maintains a factorized form:

$$\Psi(\vec{\xi}, \vec{\xi}'; z) = F_\psi \left(\frac{\vec{\xi} + \vec{\xi}'}{2} \right) \psi(\vec{\xi} - \vec{\xi}'; z), \quad (24a)$$

with the correlation peak being given by

$$\begin{aligned}
 \psi(\vec{\xi} - \vec{\xi}'; z) &= \int \frac{d^3 w}{(2\pi)^3} \chi_f(\Omega) \Phi_{\text{PDC}}(\vec{w}) e^{i\vec{w} \cdot (\vec{\xi} - \vec{\xi}')} \\
 &\quad \times \exp \left[-i \frac{q^2 z}{4\pi} [\lambda(\Omega) + \lambda(-\Omega)] \right]. \quad (24b)
 \end{aligned}$$

Thus, as a function of the midpoint $\frac{\vec{\xi} + \vec{\xi}'}{2}$, the biphoton amplitude keeps the same profile it had at the source plane. Conversely, as a function of the difference $\vec{\xi} - \vec{\xi}'$ between the photon coordinates, the correlation undergoes a broadening and decay with propagation distance, as shown by the examples of Figs. 1(c)–1(h). In particular, the evolution along z of the biphoton correlation is governed by the propagator

$$H_z^{2\text{-ph}}(\vec{q}, \Omega) = \exp \left[-i \frac{q^2 z}{4\pi} [\lambda(\Omega) + \lambda(-\Omega)] \right]. \quad (25)$$

Comparing with Eq. (3), this is just the Fresnel propagator describing standard diffraction of a monochromatic beam at the wavelength $\lambda(\Omega) + \lambda(-\Omega)$. This implies that each entangled pair with frequencies $\omega_s \pm \Omega$ follows the diffraction laws of a coherent light beam having the wavelength sum $\lambda(\Omega) + \lambda(-\Omega) = \frac{2\lambda_s}{1 - \Omega^2/\omega_s^2} \simeq 2\lambda_s$. This finding may seem counterintuitive, since twin photons are often associated with a single particle with double the frequency, reflecting the conservation of energy. However, propagation rather involves the sum of

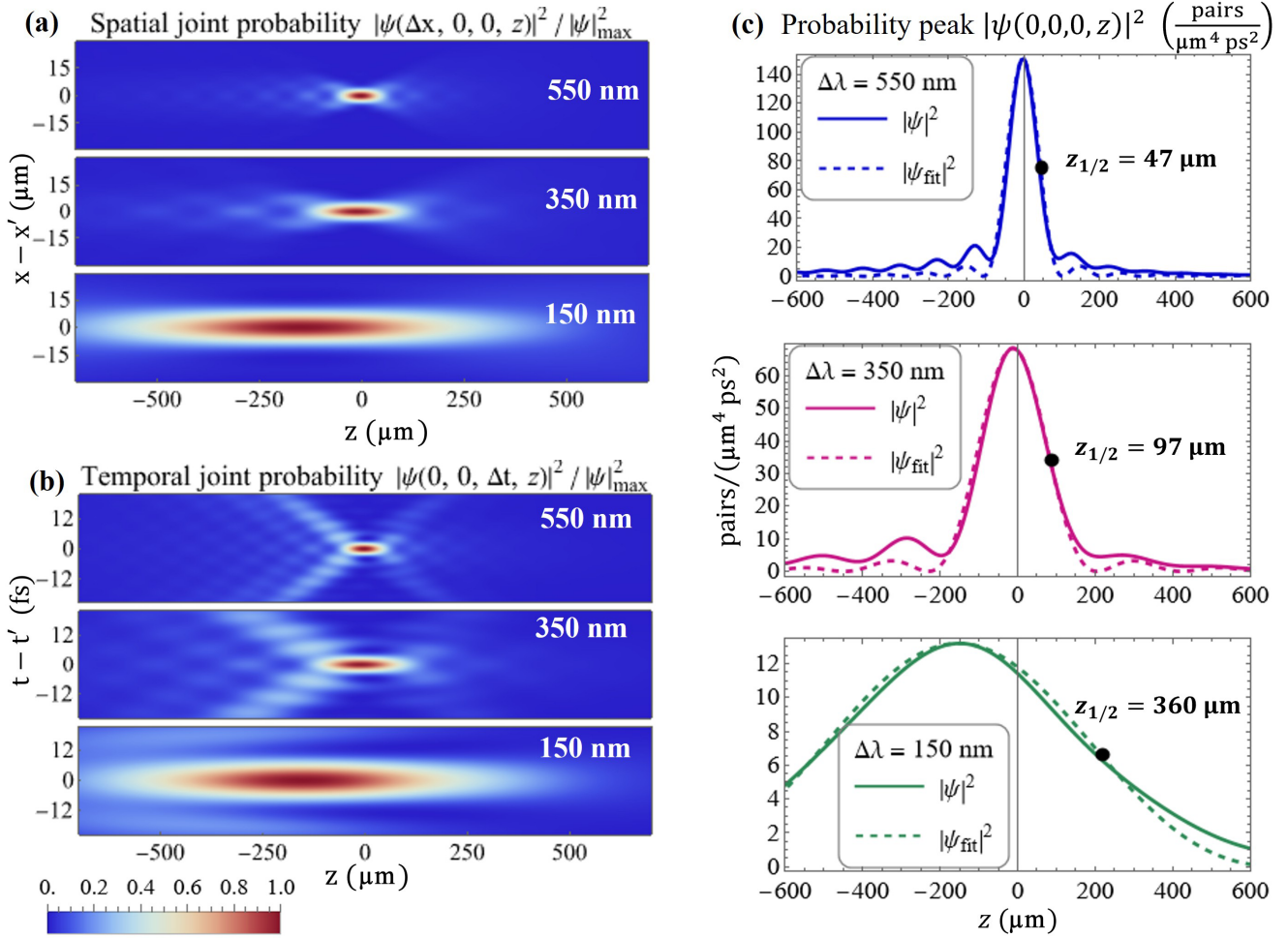


FIG. 3. Evolution of the entangled two-photon probability $|\psi(\vec{\xi} - \vec{\xi}', z)|^2$ away from the source for the three bandwidths indicated in the plots, from numerical integration of Eq. (24b). (a) Spatial sections at equal times $t = t'$. (b) Temporal sections at equal transverse coordinates $(x, y) = (x', y')$. (c) Peak at $\vec{\xi}' = \vec{\xi}$, giving the probability of detecting an entangled pair at the same location. The solid lines plot Eq. (24), the dashed lines are fits with the functions in Eq. (27), and the black dots mark the half width at half maximum of the peaks. Parameters as in Fig. 2.

the paraxial phase factors $\frac{q^2}{4\pi}\lambda(\Omega)$, where momentum conservation requires that the momenta of the pairs are anticorrelated $\vec{q}_s = -\vec{q}_i$. For this reason, the biphoton behaves in propagation as a single particle with double the wavelength.

Considering the structure of Eq. (24b), we can therefore conclude that the two-photon amplitude, as a function of the difference of coordinates, propagates in free space as a coherent light field of twice the wavelength, having the same transverse profile as the biphoton amplitude at the source plane.

A. Short distance decay $|z| \ll z_{DF}$

Based on the analogy with the propagation of a coherent beam developed in the previous section, we expect the two-photon amplitude in Eq. (24), and hence the joint probability of detecting two entangled photons together in space and time, to decay over a characteristic distance

$$z_{\text{decay}} \sim \frac{\pi \rho_{\text{ent}}^2}{2\lambda_s}, \quad (26)$$

which can be quite small when the photon pair is localized within a narrow entanglement area. Figure 3 reports some results obtained by numerical integration of Eq. (24), considering the case of type I collinear PDC in a 2-mm BBO crystal, pumped at 400 nm, and using the Sellmeier relations for BBO in Ref. [61] in order to calculate the Fourier amplitude Φ_{PDC} .

Figures 3(a) and 3(b) describe how the joint probability $|\psi(\vec{\xi} - \vec{\xi}', z)|^2$ of detecting an entangled pair separated by $\vec{\xi}' - \vec{\xi}$ evolves with the distance z from the source, for the three bandwidths considered, showing its sections along the transverse and temporal coordinates. Figure 3(c) details the evolution along z of the peak $|\psi(\vec{\xi}' = \vec{\xi}, z)|^2$, giving the joint probability of detecting an entangled pair together in space and time. Note that the maximum of the two-photon probability does not necessarily occur at the $4f$ plane, but at a focal plane z_{foc} slightly before it (we will come back to this point). Notice also the very different decay rates away from z_{foc} for the different bandwidths. Because of the hyperbolic structure of the Fourier spectrum, selecting frequencies up to Ω_{max} implies also selecting a transverse

TABLE I. Summary of useful parameters for the three bandwidths considered. $q_{\max}$ are evaluated from the truncated plots of the spectral amplitude Φ_{PDC} , z_{decay} is the longitudinal decay rate of the biphoton amplitude (HWHM of $|\psi(\xi' = \xi, z)|$), and ρ_{ent} and τ_{ent} are its transverse and temporal HWHM widths at the focal planes z_{foc} .

$\Delta\lambda$ (nm)	$\Omega_{\max}$ (THz)	$q_{\max}$ (μm^{-1})	z_{decay} (μm)	z_{foc} (μm)	ρ_{ent} (μm)	τ_{ent} (fs)
550	731	0.68	64	−0.0	3.8	2.59
350	492	0.47	134	−10.0	5.5	3.85
150	219	0.24	530	−150	10.2	8.66

bandwidth up to $q_{\max} \simeq Q_D/\Omega_D \Omega_{\max}$, thus performing an angular cut on the emitted radiation. Let us approximate the probability amplitude of detecting two simultaneous photons in the transverse q plane as a disk of radius $q_{\max}$ (with the addition of a quadratic phase variation). By recalling Eq. (24) for the entanglement propagation, the decay along z of the biphoton amplitude can then be approximated by $\simeq \int_0^{q_{\max}} dq q e^{-i\frac{q^2 \lambda_s}{2\pi}(z-z_{\text{foc}})} \propto \text{sinc}[\frac{q_{\max}^2 \lambda_s (z-z_{\text{foc}})}{4\pi}]$. Indeed, the dashed curves in Fig. 3(c) plot the function

$$\psi_{\text{fit}}(z) = \psi_{\max} \text{sinc}\left[(z - z_{\text{foc}}) \frac{q_{\max}^2 \lambda_s}{4\pi}\right], \quad (27)$$

where $\psi_{\max}$ and z_{foc} are fit parameters, but $q_{\max}$ are evaluated from the spectral amplitude Φ_{PDC} truncated at $\Omega_{\max}$, from plots like the ones in Fig. 2. Their values are reported, together with other parameters, in Table I. We see that this approximation, although rather crude, provides a very reasonable fit of the short-distance decay with z .

The same approximation can be used to estimate the transverse size of the biphoton correlation. Taking the probability amplitude of detecting two simultaneous photons at $\pm \vec{q}$ as a disk of radius $q_{\max}$, the probability of detecting them simultaneously at the transverse positions $\vec{r}$ and $\vec{r}'$ in the focal plane $z = z_{\text{foc}}$ is the Airy pattern:

$$\left| \frac{\psi(\vec{r} - \vec{r}', 0, z_{\text{foc}})}{\psi_{\max}} \right|^2 \simeq \left| \frac{2J_1(\eta|\vec{r} - \vec{r}'|q_{\max})}{\eta|\vec{r} - \vec{r}'|q_{\max}} \right|^2, \quad (28)$$

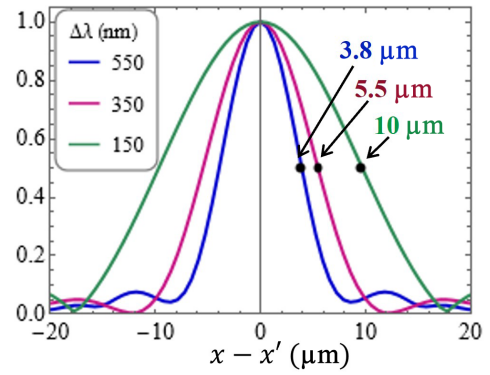
where J_1 is the Bessel function of first order and $\eta \simeq 0.85\text{--}0.95$ is used to fit the profile of the numerically calculated peak (see Appendix D). According to this formula, the radial size of the correlation can be calculated as the HWHM of the peak, which provides $\rho_{\text{ent}} = \frac{2.215}{\eta q_{\max}} \simeq \frac{2.6}{q_{\max}}$ for $\eta = 0.85$. These entanglement radii are shown by the black dots in Fig. 4(a), superimposed on the transverse biphoton amplitude, numerically calculated from Eq. (24) at the focal planes.

These results provide a quantitative support of our initial ansatz (26) for the decay rate of the biphoton correlation, based on the analogy with the propagation of a coherent beam. Using the fit function (27), the HWHM of $|\psi(\xi' = \xi, z)|$ can be evaluated as

$$z_{\text{decay}} = 1.9 \frac{4\pi}{q_{\max}^2 \lambda_s} \simeq 2.2 \frac{\pi \rho_{\text{ent}}^2}{2\lambda_s}. \quad (29)$$

The numerical value 2.2 should not be taken too strictly, since it depends on the shape of the PDC spectrum, and may differ for, e.g., different phase matching conditions.

(a) Spatial correlation $|\psi(\Delta x, 0, 0, z_{\text{foc}})|/\psi_{\max}$



(b) Temporal correlation $|\psi(0, 0, \Delta t, z_{\text{foc}})|/\psi_{\max}$

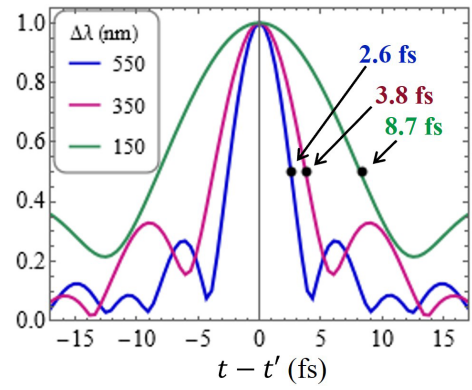


FIG. 4. (a) Transverse correlation $|\psi(x - x', 0, 0, z_{\text{foc}})|$ at the focal planes. The black dots are the HWHM of the function in Eq. (28), i.e., $\frac{2.6}{q_{\max}}$ for the two larger bandwidths ($\eta = 0.85$) and $\frac{2.4}{q_{\max}}$ for the small bandwidth ($\eta = 0.92$). (b) Temporal correlation $|\psi(0, 0, t - t', z_{\text{foc}})|$ at the focal planes. The dots are the HWHM of $\text{sinc}(\Omega_{\max} t)$, located at $t - t' = \frac{1.9}{\Omega_{\max}}$. Parameters as in Fig. 2.

We conclude this section with a remark on the propagation of the temporal entanglement of biphotons. Figure 3(b) describes the evolution along z of the probability of detecting an entangled pair at the same transverse position delayed by $t - t'$, which is a quantity of primary importance for entangled two-photon processes, and shows that it undergoes a quick decay and broadening away from the focal plane. This behavior, which was clearly observed in experiments [41], is naturally explained by our three-dimensional (3D) model, since diffraction of the biphoton amplitude quickly deteriorates the probability of finding two photons at the same transverse location. However, this would be not obvious at all in the purely temporal one-dimensional (1D) models widely employed in the literature: In free space, there is no chromatic dispersion, and there is apparently no reason why twin photons that are generated at the source almost simultaneously (or within a short entanglement time) should separate in time during propagation, nor any other clear reason why their temporal entanglement should sharply decrease. In other words, if one looks at Fig. 3(b) alone, there would be no sensible explanation of the behavior reported there, which stresses the need of using a complete space-time description of the biphoton entanglement.

Incidentally, we note that for the two larger bandwidths $\psi(\vec{\xi} - \vec{\xi}'; z)$ broadens steeply in time during propagation [this is perhaps clearer from panels (c)–(h) of Fig. 1], while for the smaller bandwidth it only decays without changing its width too much. The temporal broadening is indeed caused by the hyperbolic nonfactorable geometry of the PDC spectrum [41], not captured when only the central part of the spectrum is selected.

B. Compensation of dispersion

Figure 3 shows that, especially for the small bandwidth, the two-photon probability focuses before the image plane P_1 of the crystal output, which in practice means that a higher rate of entangled coincident photons is obtained by imaging some plane inside the medium. As was already recognized [19,60,63], this is due to the diffraction broadening of the two-photon wave packet while propagating in the non-linear medium, which can be partially compensated by a backward propagation in free space. An analogous broadening occurs in the time domain, because of chromatic dispersion in the medium: In the microscopic PDC process, twin photons are generated exactly at the same space-time position, but then propagation to the crystal end face causes their wave packet to spread. Mathematically, this reflects into the phase of the Fourier two-photon amplitude. By keeping in Eq. (13) the dominant term in the phase mismatch $\mathcal{D}(\vec{w})l_c$, and making its second-order Taylor expansion, one gets

$$\frac{\Phi_{\text{PDC}}(\vec{w})}{|\Phi_{\text{PDC}}(\vec{w})|} \simeq e^{i\frac{\mathcal{D}(\vec{w})l_c}{2} \frac{\tanh(g)}{g}} \simeq e^{i(k_s''\Omega^2 - \frac{q^2}{k_s}) \frac{l_c \tanh(g)}{2g}}. \quad (30)$$

The effect of the quadratic phase modulation of Φ_{PDC} is that the biphoton amplitude at the medium output, which is its Fourier transform [see Eq. (12)], does not reach its maximum space-time localization, just as it would happen for a non-transform-limited coherent beam. The average diffraction broadening along the medium is described by $-\frac{q^2 l_c}{k_s} \frac{\tanh(g)}{2g}$, while $k_s''\Omega^2 l_c \frac{\tanh(g)}{2g}$ accounts for the average dispersion. Notice the dependence on the gain g , due to stimulated PDC processes, which at higher gain tend to promote pair creation closer to the crystal end face, lowering the average distance propagated till the crystal output. Two limit situations can be identified:

(1) $\Omega_{\text{max}} \ll \Omega_D$. For a small detected bandwidth, dispersion is negligible. The detrimental effect of the q^2 modulation in Eq. (30) can be partially compensated by backpropagation in free space. Reminding the form of the two-photon propagator in Eq. (25), one expects the focal plane to be located at

$$z_{\text{foc}}^{\text{lim}} = -\frac{l_c}{2n_s} \frac{\tanh(g)}{g} \quad (31)$$

(n_s being the refractive index at ω_s). Note again the dependence on the gain g : Indeed, in the experiment [20], where SHG from PDC photons was monitored ranging from the spontaneous to the stimulated regime, the position of the SHG crystal had to be carefully optimized each time the gain was varied.

(2) $\Omega_{\text{max}} \gg \Omega_D$. For a broad bandwidth, which captures the hyperbolic geometry of the spectrum, diffraction effects

are naturally compensated by dispersion, which is the very mechanism at the basis of the X-shape of the PDC emission [34,35]. This case roughly corresponds to the $\Delta\lambda = 550$ nm example, in which $z_{\text{foc}} = 0$. The $\Delta\lambda = 150$ nm example, in which $z_{\text{foc}} = -150 \mu\text{m}$ (to be compared with $z_{\text{foc}}^{\text{lim}} = -267 \mu\text{m}$), is intermediate between the two limits.

One may at this point wonder to what extent the mechanism of compensation of diffraction and dispersion above described is fully effective in optimizing the two-photon probability. To this end, we have considered the possibility of impressing a frequency chirp on the PDC radiation, to approximately compensate the Ω variation of the phase in Eq. (30). Namely, we set

$$\hat{A}_0(\vec{q}, \Omega) \rightarrow \hat{A}_0(\vec{q}, \Omega) e^{-ik_s''\Omega^2 \frac{l_c \tanh(g)}{4g}}, \quad (32)$$

which for our parameters at $g = 2.2$ corresponds to a chirp of -16.3 fs². With this, the propagation of the biphoton correlation in Eq. (24) becomes

$$\psi(\vec{\xi} - \vec{\xi}', z) = \int \frac{d^3 w}{(2\pi)^3} \chi_f(\Omega) [\Phi_{\text{PDC}}(\vec{w}) e^{i\vec{w} \cdot (\vec{\xi} - \vec{\xi}')} \times e^{-ik_s''\Omega^2 \frac{l_c \tanh(g)}{2g}} e^{-iq^2 z \frac{\lambda(\omega) + \lambda(-\Omega)}{4\pi}}]. \quad (33)$$

We have then repeated the calculations of Sec. IV A, but in the presence of the chirp. In general, the results look qualitatively similar, but with the relevant difference that the two-photon probability peak is enhanced. Figure 5 reports the two-photon probability evolved at a distance z from the source in the presence of the frequency chirp (32), and has to be compared with the analogous plots of Fig. 3(c) in the absence of the chirp. We notice first of all that the focal plane is the same for all the bandwidths considered, and it is basically what is predicted by the small-bandwidth limit in Eq. (31), which confirms that the chirp we have chosen effectively matches the phase variation with Ω of the spectral amplitude Φ_{PDC} . More importantly, we notice that the joint probability peak is enhanced by a 40%–50% factor in the case of the two larger bandwidths, while the improvement is 17% for the smaller bandwidth. In this figure, $g = 2.2$. At smaller gains, we obtained even sharper results, because pairs are generated more uniformly along the crystal, and on average travel a longer distance in the dispersive medium. For example, at $g = 1.5$ dispersion compensation provides up to 70%–80% enhancement of the joint probability. These improvements of the coincidence rate may seem not particularly impressive, but they could prove crucial for the quantum enhancement of two-photon processes.

C. Beyond z_{decay} : far-field entanglement

The previous sections highlighted the rapid decay and broadening of the two-photon probability in the region near the source, which is a crucial issue for two-photon processes. A natural question is what happens when the distance increases further, and the entanglement area eventually approaches the beam cross section.

To answer this question, we have studied the propagation at distances $z \gg z_{\text{decay}}$ from the source. Our results are reported in Appendix B, while in the following we summarize the main points without demonstration. For simplicity, we also restrict

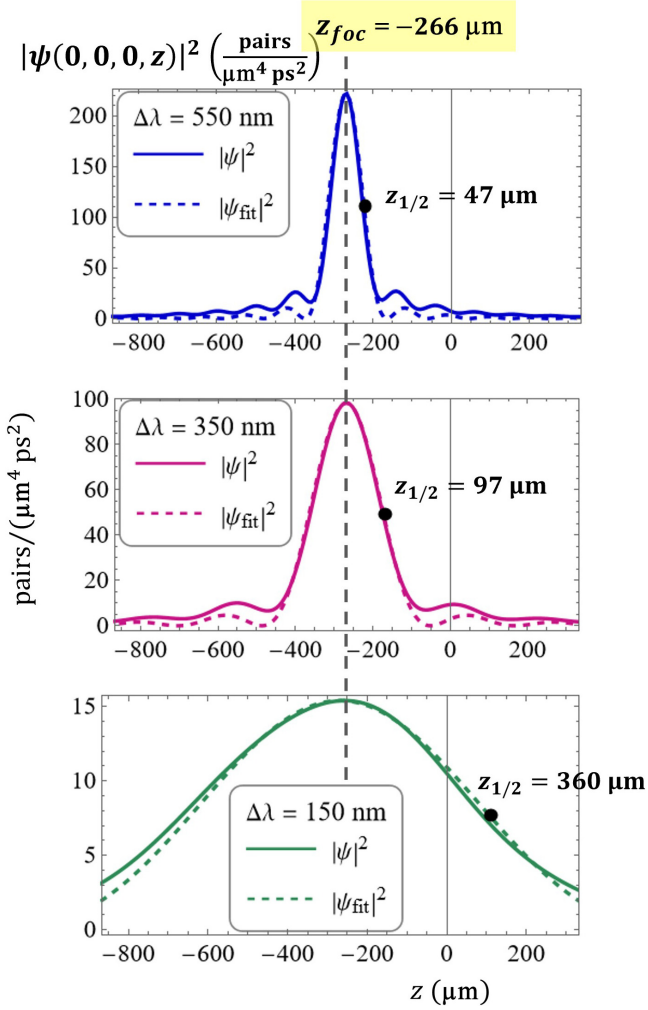


FIG. 5. Effect of a frequency chirp to compensate the dispersion broadening inside the PDC crystal. The solid lines plot $|\psi(\vec{\xi}' = \vec{\xi}, z)|^2$ from numerical integration of Eq. (33), in the presence of the optimal chirp, and must be compared with the analogous plots of Fig. 3(c) in the absence of the chirp. The dashed lines are fits with the functions in Eq. (27). Parameters as in Fig. 2.

to a bandwidth small enough that condition (23) is satisfied for any z of interest, and the biphoton amplitude maintains the factorized form (24). The general case can be found in the Appendix B. Due to the analogy with the propagation of a coherent beam, we expect a “Fraunhofer-like” behavior when z overcomes a distance related to the transverse extension of the X-correlation. Indeed, as shown in Appendix B, in the limit $z \gg \frac{\pi \rho_X^2}{2\lambda_s} \gtrsim z_{\text{decay}}$, where $\rho_X \geq \rho_{\text{ent}}$ is the global spatial extension of the X-correlation, the biphoton amplitude takes the form

$$\Psi(\vec{\xi}, \vec{\xi}'; z) \rightarrow F_\psi\left(\frac{\vec{\xi} + \vec{\xi}'}{2}\right) \psi_{\text{far}}(\vec{\xi} - \vec{\xi}'; z), \quad (34)$$

$$\begin{aligned} \psi_{\text{far}}(\vec{\xi} - \vec{\xi}'; z) &= \int \frac{d\Omega}{2\pi} \chi_f(\Omega) e^{-i\Omega(t-t')} \\ &\times \left\{ \frac{\bar{k}(\Omega)}{4\pi i z} \Phi_{\text{PDC}}\left[\left(\vec{r} - \vec{r}'\right) \frac{\bar{k}(\Omega)}{2z}, \Omega\right] e^{i|\vec{r} - \vec{r}'|^2 \frac{\bar{k}(\Omega)}{4z}} \right\}, \quad (35) \end{aligned}$$

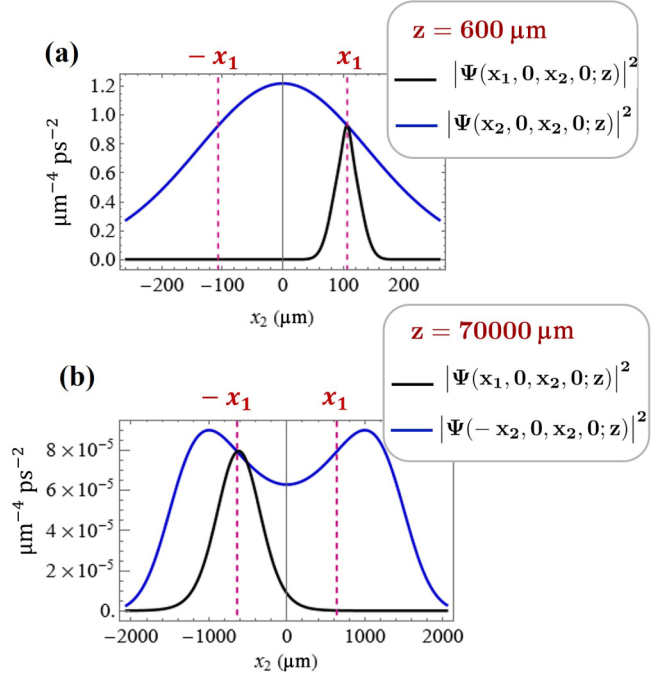


FIG. 6. Transition from position to momentum entanglement in the far zone. The black lines plot $|\Psi(\vec{\xi}_1, \vec{\xi}_2; z)|^2$ by varying the transverse position x_2 in $\vec{\xi}_2 = (x_2, 0, 0)$ for a fixed $\vec{\xi}_1 = (x_1, 0, 0)$. (a) $z = 600 \mu\text{m}$. $|\Psi|^2$, from Eq. (24), is peaked at $x_2 = x_1$, and the envelope (blue) follows the profile of F_ψ . (b) $z = 70000 \mu\text{m}$. The peak of $|\Psi|^2$, from Eq. (B9), has moved to $x_2 = -x_1$, and the envelope (blue) follows the profile of the Fourier amplitude Φ_{PDC} . $\Delta\lambda = 150 \text{ nm}$, other parameters as in Fig. 2.

where, for brevity of notation, we introduced $\bar{k}(\Omega) = \frac{1}{2} \left[\frac{\lambda(\Omega)}{2\pi} + \frac{\lambda(-\Omega)}{2\pi} \right]^{-1} = \frac{2\pi}{\lambda_s} \left(1 - \frac{\Omega^2}{\omega_s^2} \right)$, representing the harmonic mean of the wave numbers of each conjugate pair. As a function of the spatial separation $\vec{r} - \vec{r}'$, the biphoton amplitude in Eq. (35) follows now the profile of the Fourier amplitude Φ_{PDC} along $\vec{q}$, integrated over the frequencies. For example, panels (e) and (h) of Fig. 1 already show this behavior, although they are taken just at the boundary of the zone where Eq. (35) holds, since $\frac{\pi \rho_X^2}{2\lambda_s} \simeq \frac{l_c}{2n_s} = 602 \mu\text{m}$. In these examples, the transverse width of $|\psi|^2$ is on the order of 30–40 μm , hence smaller than the beam size, and the entangled pair is still localized inside the beam cross section. However, the radial size of ψ_{far} grows with z , with the separation of twin photons extending up to $|\vec{r} - \vec{r}'|_{\text{max}} \sim q_{\text{max}} \frac{z\lambda_s}{\pi}$. When the region over which the pair is delocalized approaches the beam size, the spatial correlation is basically lost, because if a photon is found at position $\vec{r}$, its twin can be anywhere in the beam cross section. Noticeably, by setting $|\vec{r} - \vec{r}'|_{\text{max}} = D_s$, we get that this distance roughly corresponds to the boundary of the deep Fresnel zone for the classical coherence:

$$z = \frac{\pi D_s}{\lambda_s q_{\text{max}}} \sim \frac{D_s \rho_{\text{ent}}}{\lambda_s} = z_{\text{DF}}. \quad (36)$$

As illustrated in Fig. 6, at distances larger than z_{DF} the spatial correlation is recovered, but it moves to the opposite sides of the cross section, because in the biphoton amplitude of Eq. (34), $F_\psi\left(\frac{\vec{r} + \vec{r}'}{2}, \dots\right)$ becomes narrower than

$\Phi_{\text{PDC}}[(\vec{r} - \vec{r}', \dots)_{\frac{\vec{k}(\Omega)}{2z}}]$, and forces $\vec{r}' = -\vec{r}$. This is nothing else but the conservation of transverse momentum, which dominates the spatial entanglement in the far-field zone: Entangled photons are generated at correlated wave vectors $\vec{q} + \vec{q}' \simeq 0$ (within the pump transverse bandwidth), and close to degeneracy propagate out of the crystal at symmetric angles. In the far-field zone, angles are mapped into positions, and entangled photons are thus found at symmetric positions. The complete expression in Eq. (B9) reflects more correctly the conservation of the transverse momentum in the general case: As could be expected, the difference is that since out of degeneracy the propagation angles are not exactly symmetric, the correlation is no more located exactly at opposite sides of the far-field plane.

The transverse momentum entanglement of twin photons is indeed well known [2] and is typically (and more conveniently) observed in the focal plane of a lens [31,32,55,64]. We have here shown how the near-field position entanglement transforms into a momentum entanglement with free-space propagation. Remarkably, the transition from the position to momentum entanglement takes place at a distance roughly corresponding to the boundary of the deep Fresnel zone for the classical coherence (see next section). Thus, we recover a sort of VCZ behavior for the quantum spatial correlation, although in a completely different form.

V. PROPAGATION OF THE CLASSICAL COHERENCE

This section considers the propagation of the classical coherence function of Eq. (6), providing an overview of the results relevant to this work. The treatment is similar to that in Refs. [27,28], with some additional complications due to the broad bandwidth and the X-shape of the spectrum.

When single photons, instead of pairs, are detected, the light intensity appears speckled, because of the large phase fluctuations of each individual signal or idler photon. In the quasistationary approximation, the coherence function at the source plane has the form [38,39]

$$G_0^{(1)}(\vec{\xi}, \vec{\xi}') = F_{\text{coh}}\left(\frac{\vec{\xi} + \vec{\xi}'}{2}\right) \int \frac{d^3w}{(2\pi)^3} e^{i(\vec{\xi} - \vec{\xi}') \cdot \vec{w}} |V(\vec{w})|^2, \quad (37)$$

where $F_{\text{coh}}(\vec{\xi}) = \frac{\sinh^2[gA_p(\vec{\xi})]}{\sinh^2(g)}$ describes the slowly varying profile of the intensity envelope, and

$$|V(\vec{w})|^2 = \left| g \frac{\sinh \Gamma(\vec{w})}{\Gamma(\vec{w})} \right|^2 \quad (38)$$

is the Fourier spectrum of light [see Fig. 1(b)], which has similar geometrical properties as the two-photon probability amplitude, as can be readily appreciated from the identity $|\Phi_{\text{PDC}}(\vec{w}, -\vec{w})|^2 = |V(\vec{w})|^2(1 + |V(\vec{w})|^2)$. In turn, the coherence peak as a function of $\vec{\xi} - \vec{\xi}'$ has similar geometrical properties as the biphoton correlation, in particular, the X-shape, for which it was named X-coherence in Ref. [34], and the narrow localization in space-time [see Fig. 7(a)]. However, since it describes an incoherent superposition, it propagates in a very different way with distance from the source.

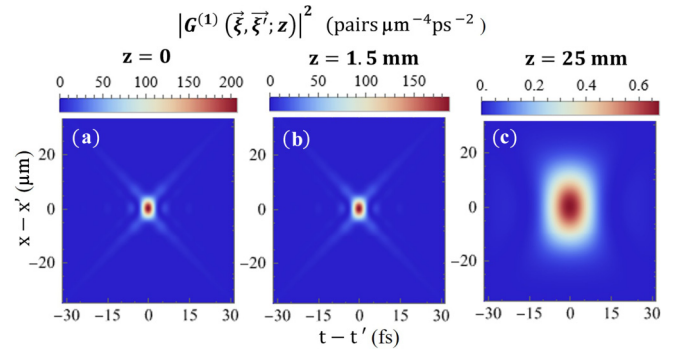


FIG. 7. Mutual coherence $|G^{(1)}(\frac{\vec{\xi} - \vec{\xi}'}{2}, -\frac{\vec{\xi} - \vec{\xi}'}{2}, z)|^2$ calculated from Eq. (39), at (a) $z = 0$, (b) $z = 1.5$ mm (deep Fresnel zone), and (c) $z = 25$ mm (VCZ zone). $\Delta\lambda = 550$ nm; other parameters as in Fig. 2.

Equation (8) describing the propagation of the biphoton amplitude, in the case of the classical coherence, is substituted by

$$G^{(1)}(\vec{\xi}, \vec{\xi}', z) = \int d^3w_0 \int \frac{d^3w}{(2\pi)^3} \chi_f(\Omega) \left[\mu_{\text{coh}}(\vec{w}_0) |V(\vec{w})|^2 \times e^{i\vec{w}_0 \cdot (\frac{\vec{\xi} + \vec{\xi}'}{2})} e^{i\vec{w} \cdot (\vec{\xi} - \vec{\xi}')} h_z^* \left(\vec{w} + \frac{\vec{w}_0}{2} \right) h_z \times \left(\vec{w} - \frac{\vec{w}_0}{2} \right) \right], \quad (39)$$

where we already assumed a large enough detected bandwidth, and $\mu_{\text{coh}}(\vec{w}_0) = \int \frac{d^3w}{(2\pi)^3} F_{\text{coh}}(\vec{\xi}) e^{i\vec{w}_0 \cdot \vec{\xi}}$ is the coherence peak in Fourier space, which decays on the scales $\delta q_0, \delta \Omega_0$ much smaller than the scales of variation of the spectrum. The main difference with Eq. (8) is that now, as usual, the propagation phases subtract, instead of summing as in the case of biphotons. The propagator ruling the evolution with z of the classical coherence is hence

$$h_z^* \left(\vec{w} - \frac{\vec{w}_0}{2} \right) h_z \left(\vec{w} + \frac{\vec{w}_0}{2} \right) \simeq e^{-i\vec{q} \cdot \vec{w}_0 z \frac{\lambda(\Omega)}{2\pi}}, \quad (40)$$

where the only approximation made was to neglect terms on the order of $\frac{\delta \Omega_0}{2\omega_s} \approx \frac{1}{T_s \omega_s} \ll 1$ (full calculation in Appendix A). As anticipated, this propagator couples slow and fast variables, and for each frequency Ω it has the same form as in the monochromatic case [28], since $\lambda(\Omega) = \lambda_s / (1 + \frac{\Omega}{\omega_s})$ is the wavelength at $\omega_s + \Omega$. Inserting this result into Eq. (39), and backtransforming with respect to $\vec{w}_0$, one easily gets

$$G^{(1)}(\vec{\xi}, \vec{\xi}', z) = \int \frac{d^3w}{(2\pi)^3} \chi_f(\Omega) |V(\vec{w})|^2 e^{i\vec{w} \cdot (\vec{\xi} - \vec{\xi}')} \times F_{\text{coh}} \left(\frac{\vec{r} + \vec{r}'}{2} - \frac{\vec{q} z \lambda(\Omega)}{2\pi}, \frac{t + t'}{2} \right). \quad (41)$$

From this formula, it is clear that the mutual coherence at the source plane of Eq. (37) is recovered in the deep Fresnel zone, in which

$$z \ll \frac{D_s \rho}{\lambda_{\text{max}}} \leq z_{\text{DF}},$$

where $\rho \propto \frac{\pi}{q_{\max}}$ is the transverse size of the narrow coherence peak. Conversely, when

$$z \gg \frac{D_s \rho_X}{\lambda_{\min}} \geq z_{\text{DF}},$$

where $\rho_X \propto \frac{\pi}{Q_D}$ is the overall transverse extension of the X-coherence, one finds a VCZ-like behavior. It can be shown that in these conditions, by setting $k_v(\Omega) = \frac{2\pi}{\lambda(\Omega)}$,

$$\begin{aligned} G^{(1)}(\vec{\xi}, \vec{\xi}', z) &\rightarrow \int \frac{d\Omega}{2\pi} e^{-i\Omega(t-t')} \left| V\left(\frac{\vec{r} + \vec{r}'}{2}, \frac{k_v(\Omega)}{z}, \Omega\right) \right|^2 \\ &\times \frac{k_v^2(\Omega)}{z^2} \int \frac{d^2s}{(2\pi)^2} F_{\text{coh}}\left(\vec{s}, \frac{t+t'}{2}\right) e^{-i\vec{s} \cdot (\vec{r} - \vec{r}') \frac{k_v(\Omega)}{z}} \\ &\times e^{-i(r^2 - r'^2) \frac{k_v(\Omega)}{2z}}. \end{aligned} \quad (42)$$

A detailed treatment of these results is beyond the scope of this paper: We only remark that in the VCZ zone, the transverse mutual coherence becomes the Fourier transform of the intensity profile $F_{\text{coh}}(\vec{r})$ at the source, while the intensity distribution as a function of the midpoint follows the profile of the spectrum $|V(\frac{\vec{r} + \vec{r}'}{2}, \frac{k_v(\Omega)}{z}, \Omega)|^2$, as prescribed by the generalized VCZ theorem [25]. The size of the far-field speckles becomes independent of their properties in the deep Fresnel zone (and hence of the selected bandwidth) and grows $\sim \frac{\lambda_s z}{D_s}$. This behavior is shown by panel (c) of Fig. 7, which plots $|G^{(1)}|^2$ in the plane $(t - t', x - x')$ at $z = 25$ mm from the source. Conversely, panels (a) and (b) show that after 1.5 mm of propagation, the function is practically unchanged ($z = 1.54$ mm is the boundary of the deep Fresnel zone), a behavior that should be compared with the rapid decay and broadening of the two-photon correlation in Fig. 1.

The slow evolution of the classical coherence is clearly exhibited in Fig. 8, which plots the peak value $G^{(1)}(0, 0, z) = \mathcal{I}(0, z)$ as a function of the distance from the source. Panel (b) in this figure, in particular, is a close-up in the deep Fresnel zone, comparing with the quick decay of the two-photon amplitude. Incidentally, the red symbols here are obtained by integrating the complete propagation equation (19) for Ψ , while the solid black line plots its approximated form (24), showing the validity of the approximation.

Such a difference in the decay rates of the quantum and classical correlation should be taken into account when dealing with quantum-enhanced two-photon processes: The probability of detecting two entangled photons rapidly vanishes out of a focal plane, whereas the probability of detecting nonentangled photons remains unchanged for much longer distances. Thus, if there is no means of selecting entangled from nonentangled processes, out of a focal plane one may be rapidly left only with classical two-photon processes.

VI. CONCLUSIONS

This work has provided a comprehensive formal analysis of the propagation of the space-time quantum correlation of entangled photons generated by the PDC. Previous experimental observations based on the up-conversion of PDC radiation [20,41] had indeed shown an unusually rapid decay of the

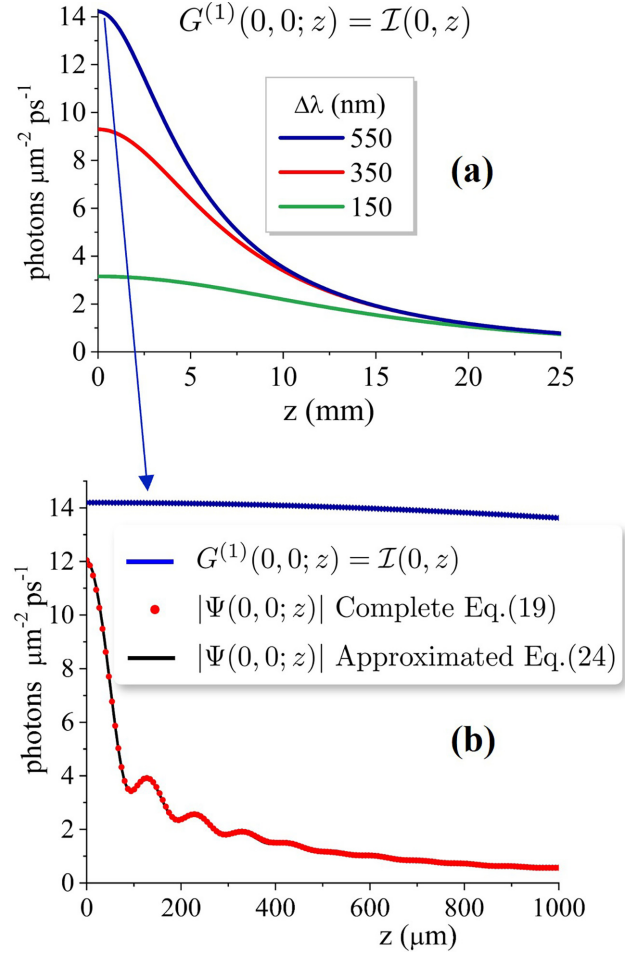


FIG. 8. (a) Decay of $G^{(1)}(0, 0; z) = \mathcal{I}(0, z)$ with the distance from the source, numerically calculated from Eq. (39). Panel (b) shows a close-up of the 550 nm blue curve in the deep Fresnel zone, where the coherence properties are invariant for propagation. The red symbols superimposed on the black line plot the much faster decay of the biphoton amplitude, numerically calculated from Eq. (19), and the reduced model in Eq. (24), respectively. Parameters as in Fig. 2.

two-photon probability out of a focal plane, but a systematic explanation was lacking. Our analysis revealed some results that were probably not entirely obvious:

(1) Bunches of entangled photons do not propagate as classical bunches of photons from an incoherent source, as one might expect, but rather as coherent pulses. This coherence derives from the phase conjugation, whereby the phases of the entangled photons add to the original pump phase, and the biphoton, viewed as a whole, behaves like a coherent optical field in propagation. As a result, if the entangled photons are confined at the source within a small correlation volume, they immediately spread out beyond a focal plane.

(2) Actually, the biphoton amplitude does propagate as a coherent field, but at approximately the wavelength $2\lambda_s$, so it diffracts even faster than a classical coherent field with the same space-time localization. Notice that in the scheme there is no optical field at the wavelength $2\lambda_s$.

(3) In contrast, the classical coherence function characterizing the nonentangled photon bunches of the PDC emission decays at a much slower rate. This must be taken into account when designing multiphoton interaction experiments with bright PDC light: Outside the focal plane, only processes involving nonentangled photons contribute (those processes that give rise to the so-called incoherent contribution in entangled SHG or TPA [13,16,19,63]).

The fast decay of the two-photon amplitude, on the other hand, might be an advantage in some situations, as it could, for example, establish a longitudinal resolution in two-photon microscopy with entangled photons, even under unfocused illumination and even in the regime where the TPA rate is linear with the incident photon flux. However, further investigations are needed to clarify this point, especially concerning propagation in dispersive media, such as biological tissues, which will be the subject of a separate study [36].

Additional interesting aspects that emerged from our analysis:

(1) The entangled two-photon probability can be enhanced by applying a proper frequency chirp to the emitted radiation, able to compensate the average dispersion broadening occurring inside the PDC crystal. It is beyond the scope of this work to discuss the experimental feasibility, but we indicate an opportunity.

(2) We have shown that the correlation at the transverse position is gradually lost during propagation, to be recovered at opposite areas of the cross-section, as a manifestation of the transverse momentum correlation. This behavior has

been observed in various experimental settings [29–32] and in numerical simulations [33], but formally demonstrating it in free propagation was nontrivial. Furthermore, our study provides an estimate of the distance at which the transition occurs [Eq. (36)], which remarkably matches the boundary of the classical deep Fresnel zone, $z_{\text{DF}} = \frac{D\rho}{\lambda_s}$. Our results agree well with the results shown, for example, in Refs. [31,32], where the Fedorov ratio decreases to a minimum and then begins to increase again.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: EXPANSION OF THE PROPAGATORS IN EQS. (19) AND (39)

Let us start from the propagation of the quantum entanglement. By reminding that the wavelength at frequency $\omega_s + \Omega$ is $\lambda(\Omega) = \frac{2\pi c}{\omega_s + \Omega}$, the product of Fresnel propagators appearing under the integral in Eq. (19) can be written as

$$\begin{aligned} h_z\left(\frac{\vec{w}_0}{2} + \vec{w}\right) h_z\left(\frac{\vec{w}_0}{2} - \vec{w}\right) &= \exp \left\{ -i \frac{z}{2} \left[\frac{|\frac{\vec{q}_0}{2} + \vec{q}|^2 c}{\omega_s + \Omega + \frac{\Omega_0}{2}} + \frac{|\frac{\vec{q}_0}{2} - \vec{q}|^2 c}{\omega_s - \Omega + \frac{\Omega_0}{2}} \right] \right\} \\ &\simeq \exp \left\{ -i \frac{z}{2} \left[\frac{|\frac{\vec{q}_0}{2} + \vec{q}|^2 c}{\omega_s + \Omega} + \frac{|\frac{\vec{q}_0}{2} - \vec{q}|^2 c}{\omega_s - \Omega} \right] \right\} \\ &= \exp \left\{ -i \frac{z}{4\pi} \left[\left(\frac{q_0^2}{4} + q^2 \right) [\lambda(\Omega) + \lambda(-\Omega)] + \vec{q} \cdot \vec{q}_0 [\lambda(\Omega) - \lambda(-\Omega)] \right] \right\}, \end{aligned} \quad (\text{A1})$$

where in the first line we neglected the terms $\frac{\Omega_0}{2}$ with respect to the optical frequency ω_s , because $\mu_\psi(\vec{w}_0)$ forces them to be restricted to the narrow bandwidth of the pump beam, $\delta\Omega_0 \ll \omega_s$.

We next consider the propagation of the classical coherence described by Eq. (39). The product of propagators thereby appearing reads

$$\begin{aligned} h_z^*\left(\vec{w} - \frac{\vec{w}_0}{2}\right) h_z\left(\vec{w} + \frac{\vec{w}_0}{2}\right) &= \exp \left\{ -i \frac{z c}{2} \left[\frac{|\vec{q} + \frac{\vec{q}_0}{2}|^2}{\omega_s + \Omega + \frac{\Omega_0}{2}} - \frac{|\vec{q} - \frac{\vec{q}_0}{2}|^2}{\omega_s + \Omega - \frac{\Omega_0}{2}} \right] \right\} \\ &\simeq \exp \left\{ -i \frac{z c}{2} \left[\frac{|\vec{q} + \frac{\vec{q}_0}{2}|^2 (\omega_s + \Omega - \frac{\Omega_0}{2})}{(\omega_s + \Omega)^2} - \frac{|\vec{q} - \frac{\vec{q}_0}{2}|^2 (\omega_s + \Omega + \frac{\Omega_0}{2})}{(\omega_s + \Omega)^2} \right] \right\} \\ &\simeq \exp \left\{ -i \vec{q} \cdot \vec{q}_0 z \frac{\lambda(\Omega)}{2\pi} \right\} \times \exp \left\{ +i \frac{q^2 \Omega_0 c}{(\omega_s + \Omega)^2} z \right\}, \end{aligned} \quad (\text{A2})$$

where in passing from the first line to the second line, we neglected terms on the order of $\frac{\delta\Omega_0^2}{4\omega_s^2} \ll 1$. In the third line, the zeroth-order term in $\frac{\delta\Omega_0}{2\omega_s} \ll 1$ gives rise to the first factor on the right-hand side of Eq. (A2), which is the dominant one. The second factor on the right-hand side of Eq. (A2) is a correction $\propto \frac{\Omega_0}{\omega_s}$, which can be in first approximation neglected (this is the approximation made in the main text), but this may lead to an unprecise evaluation in the VCZ zone, where z is large, especially for large bandwidths. Inserting this result into Eq. (39), and backtransforming with respect to $\bar{w}_0$, we get

$$\begin{aligned} G^{(1)}(\vec{\xi}, \vec{\xi}'; z) &= \int \frac{d^3w}{(2\pi)^3} \chi_f(\Omega) |V(\bar{w})|^2 e^{i\bar{w} \cdot (\vec{\xi} - \vec{\xi}')} F_{\text{coh}} \\ &\times \left(\frac{\vec{r} + \vec{r}'}{2} - \frac{\vec{q} z \lambda(\Omega)}{2\pi}, \frac{t + t'}{2} - \frac{c q^2 z}{(\omega_s + \Omega)^2} \right). \end{aligned} \quad (\text{A3})$$

Therefore, we see that, in general, a z dependence arises not only because the beam envelope has a finite spatial cross section, as it occurs in the standard description of the propagation of the transverse coherence, but also because it has a finite duration (the beam is not monochromatic). The spatial z dependence is negligible as long as $\frac{q_{\text{max}} z \lambda(\Omega)}{2\pi} \ll \frac{D_s}{2}$, which leads to

$$z \ll z_{\text{DF}} = \frac{D_s \rho}{\lambda_{\text{max}}},$$

where $\rho \simeq \frac{\pi}{q_{\text{max}}}$ is the transverse speckle size. This is the standard result of the generalized Van Cittert-Zernike theorem (see e.g., Ref. [26]). The z dependence in the temporal term can in turn be neglected as long as $\frac{c q_{\text{max}}^2 z}{(\omega_s + \Omega)^2} \ll \frac{T_s}{2}$, which leads to $z \ll z_{\text{DF}} \frac{\rho}{D_s} \frac{T_s \omega_s}{\pi}$. In the quasistationary approximation, the pulse duration needs to be on the order of hundreds of femtoseconds or longer, and is obviously much longer than the optical cycle: Therefore, this term starts to contribute at distances, in general, longer than z_{DF} . However, some caution should be taken to neglect it, because, for example, for the broader bandwidth, for our parameters this term is completely negligible only for $z \ll 3.4 z_{\text{DF}}$.

APPENDIX B: ENTANGLEMENT PROPAGATION AT LARGE DISTANCES

This Appendix studies the propagation of the quantum entanglement in the “far-field” zone. To simplify the treatment, we restrict to a range of distances in which the diffraction of the spatial envelope F_ψ alone would be negligible (we remind that for a pump waist of 600 μm , this needs $z > 640$ mm).

Let us start again from Eq. (20), and let us now retain in the two-photon propagator both the second and third terms. For brevity of notation, let us introduce

$$\bar{k}(\Omega) = \frac{1}{2} \left[\frac{2\pi}{\lambda(\omega)} + \frac{2\pi}{\lambda(-\omega)} \right] = \frac{2\pi}{\lambda_s} \left(1 - \frac{\Omega^2}{\omega_s^2} \right)$$

representing the harmonic mean of the wave numbers of each entangled pair. The two-photon propagator becomes in this

way $H_z^{2\text{-ph}}(\vec{q}, \Omega) \rightarrow \exp[-i \frac{q^2 z}{k(\Omega)} + i \frac{\vec{q} \cdot \vec{q}_0 z}{k(\Omega) \omega_s} \frac{\Omega}{\omega_s}]$, and the evolution along z of the biphoton amplitude in Eq. (19) reads

$$\Psi(\vec{\xi}, \vec{\xi}'; z) = \int d^3w_0 \mu_\psi(\bar{w}_0) e^{i\bar{w}_0 \cdot \frac{\vec{\xi} + \vec{\xi}'}{2}} S(\vec{\xi} - \vec{\xi}', \vec{q}_0, z), \quad (\text{B1})$$

$$\begin{aligned} S(\vec{\xi}_-, \vec{q}_0, z) &= \int \frac{d^3w}{(2\pi)^3} \chi_f(\Omega) \left\{ \Phi_{\text{PDC}}(\bar{w}) e^{i\bar{w} \cdot \vec{\xi}_-} \right. \\ &\times \exp \left[-i \frac{q^2 z}{k(\Omega)} + i \frac{\vec{q} \cdot \vec{q}_0 z}{k(\Omega) \omega_s} \frac{\Omega}{\omega_s} \right] \Big\}. \end{aligned} \quad (\text{B2})$$

Note that as long as the dependence on $\vec{q}_0$ in Eq. (B2) can be neglected, the above expression reduces to the factorized form of Eq. (24) explored in the main text. This is always true for z small enough, $z < z_{\text{DF}} \frac{2\lambda_s}{\Delta\lambda}$ [see Eq. (23)], or for a very narrow bandwidth, $\Delta\lambda/\lambda_s \rightarrow 0$. The following study will hence also explore validity of the approximation for Eq. (24) at large distances and/or large bandwidths.

For large distances, it is more convenient to use the Fresnel propagator in the direct transverse domain, rather than in the q space. To this end, by inverting Eq. (12) in the spatial domain, we use the identity

$$\Phi_{\text{PDC}}(\vec{q}, \Omega) = \int d^2r_- \psi_0(\vec{r}_-, \Omega) e^{-i\vec{q} \cdot \vec{r}_-}. \quad (\text{B3})$$

By substituting in Eq. (B2), one obtains

$$\begin{aligned} S(\vec{\xi}_-, \vec{q}_0, z) &= \int \frac{d\Omega}{2\pi} \chi_f(\Omega) e^{-i\Omega t_-} \int d^2r'_- \psi(\vec{r}'_-, \Omega) \\ &\times \int \frac{d^2q}{(2\pi)^2} e^{i\vec{q} \cdot [\vec{r}_- - \vec{r}'_- + \frac{\vec{q}_0 z}{k(\Omega) \omega_s} \frac{\Omega}{\omega_s}]} e^{-i \frac{q^2 z}{k(\Omega)}} \\ &= \int \frac{d\Omega}{2\pi} \chi_f(\Omega) e^{-i\Omega t_-} \int d^2r'_- \psi_0(\vec{r}'_-, \Omega) \\ &\times \frac{\bar{k}(\Omega)}{4\pi i z} e^{i|\vec{r}_- - \vec{r}'_- + \frac{\vec{q}_0 z}{k(\Omega) \omega_s} \frac{\Omega}{\omega_s}|^2 \frac{\bar{k}(\Omega)}{4z}}, \end{aligned} \quad (\text{B4})$$

where in the second line the Gaussian integral in d^2q has been performed, using $\int \frac{d^2q}{(2\pi)^2} \exp[-iq^2 A + i\vec{q} \cdot \vec{r}] = \frac{1}{4\pi i A} \exp[i \frac{\vec{r}^2}{4A}]$. Next, we follow a similar approach as for the propagation of a classical beam in the Fraunhofer zone. When z is large enough that $r'^2 \frac{\bar{k}(\Omega)}{4z} \ll 1$ for all $\vec{r}'_-$ for which $\psi_0(\vec{r}'_-, \Omega) \neq 0$, we can approximate

$$\begin{aligned} &\left| \vec{r}_- - \vec{r}'_- + \frac{\vec{q}_0 z}{k(\Omega) \omega_s} \frac{\Omega}{\omega_s} \right|^2 \frac{\bar{k}(\Omega)}{4z} \\ &\approx \left| \vec{r}_- + \frac{\vec{q}_0 z}{k(\Omega) \omega_s} \frac{\Omega}{\omega_s} \right|^2 \frac{\bar{k}(\Omega)}{4z} - 2\vec{r}'_- \cdot \left(\vec{r}_- + \frac{\vec{q}_0 z}{k(\Omega) \omega_s} \frac{\Omega}{\omega_s} \right) \frac{\bar{k}(\Omega)}{4z}. \end{aligned} \quad (\text{B5})$$

The validity of this approximation, however, does not depend on the beam size, as it would be for a coherent beam, but on the overall extension of the X-correlation in the transverse direction, i.e., on the spatial localization of the photon pairs at the source. Using the results in Sec. III (see, in particular, Fig. 2), we can take the transverse global size of the X-correlation as $\rho_X^2 \sim \frac{2}{Q_D^2} = \frac{2l_c}{n_s k_y}$. It follows then that the

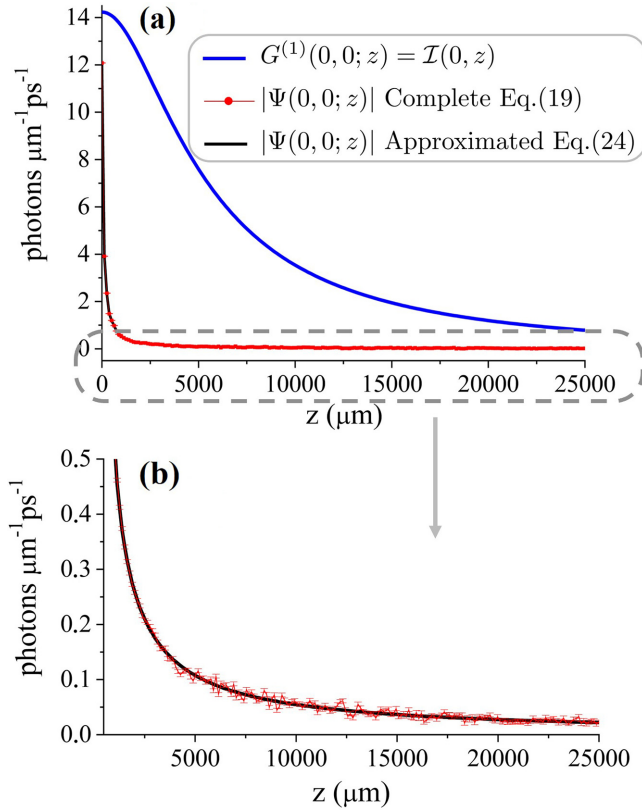


FIG. 9. (a) The red symbols and the black solid line plot the decay of the two-photon amplitude probability $|\Psi(\xi, \xi, z)|$, calculated from the complete equation (19), with a Monte Carlo sampling over 6×10^9 points, and the approximated equation (24), respectively. The blue curve shows for comparison the slow decay of the intensity envelope. Panel (b) is a close-up of panel (a). $\Delta\lambda = 550$ nm. Other parameters as in Fig. 2.

approximation (B5) is valid provided that

$$z \gg \frac{\rho_X^2 k_v}{4} = \frac{l_c}{2n_s}. \quad (\text{B6})$$

In this limit, by substituting Eq. (B5) into Eq. (B4) and using again Eq. (B3), one has

$$\begin{aligned} S(\vec{\xi}_-, \vec{q}_0, z) &\rightarrow \int \frac{d\Omega}{2\pi} \chi_f(\Omega) e^{-i\Omega t_-} \left\{ e^{i|\vec{r}_- + \frac{\vec{q}_0 z}{k(\Omega) \omega_s}|^2 \frac{k(\Omega)}{4c}} \right. \\ &\quad \times \frac{\bar{k}(\Omega)}{4\pi i z} \Phi_{\text{PDC}}\left(\vec{r}_- - \frac{\bar{k}(\Omega)}{2z}, \vec{q}_0 - \frac{\Omega}{2\omega_s}, \Omega\right) \Big\} \quad (\text{B7}) \\ &\simeq \int \frac{d\Omega}{2\pi} \chi_f(\Omega) e^{-i\Omega t_-} \left\{ e^{i\vec{r}_- \cdot \vec{q}_0 \frac{\Omega}{2\omega_s}} \right. \\ &\quad \times e^{i|\vec{r}_-|^2 \frac{k(\Omega)}{4c}} \frac{\bar{k}(\Omega)}{4\pi i z} \Phi_{\text{PDC}}\left(\vec{r}_- - \frac{\bar{k}(\Omega)}{2z}, \Omega\right) \Big\}, \quad (\text{B8}) \end{aligned}$$

where in Eq. (B8) the main dependence on $\vec{q}_0$ has been isolated in the factor $e^{i\vec{r}_- \cdot \vec{q}_0 \frac{\Omega}{2\omega_s}}$, by making the following approximations: (1) In the first line of Eq. (B7), we neglected a term $\frac{q_0^2 z}{4k(\Omega)} \left(\frac{\Omega}{\omega_s}\right)^2 \ll 1$, which would contribute at distances much larger than the Rayleigh range of the beam. (2) In the second line, we neglected the term $\vec{q}_0 \frac{\Omega}{2\omega_s}$ in the argument of Φ_{PDC} , because in the QS approximation $q_0 \ll Q_D$, where Q_D

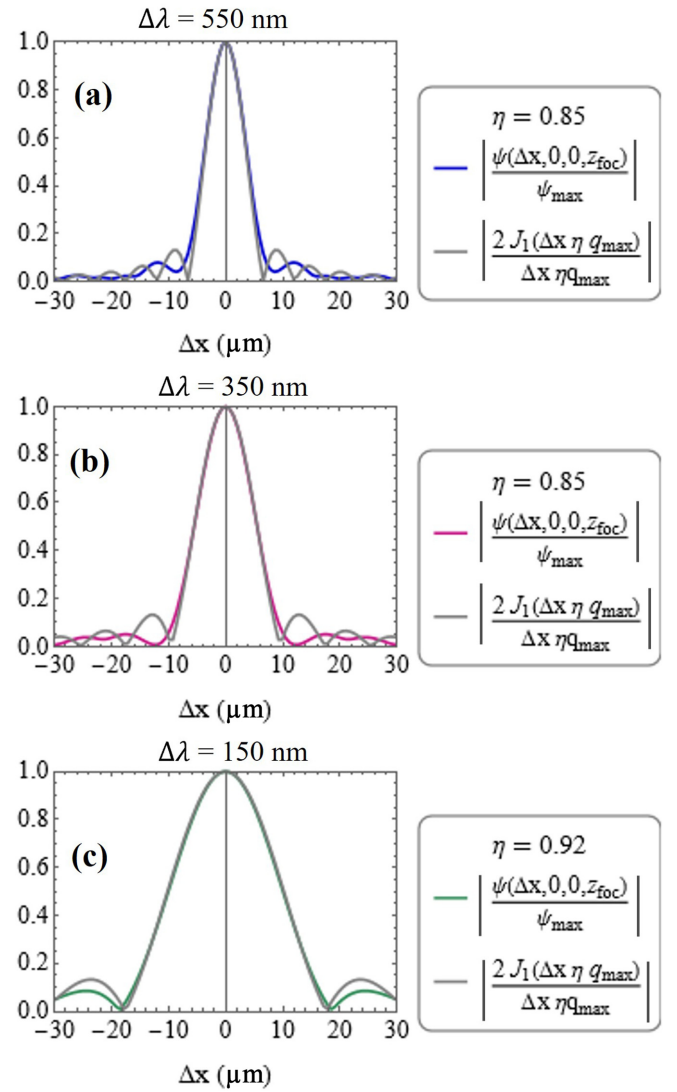


FIG. 10. Fits of the spatial biphoton correlation, calculated from Eq. (24) at the focal planes, with the Airy function in Eq. (28). (a) 550 nm bandwidth, $\eta = 0.85$; (b) 350 nm bandwidth, $\eta = 0.85$; and (c) 150 nm bandwidth, $\eta = 0.92$. The q_{max} are given in Table I.

represents the scale of variation of Φ_{PDC} as a function of q , and obviously $q_0 \frac{\Omega}{2\omega_s} \ll Q_D \frac{\Omega}{2\omega_s} \ll Q_D$.

Finally, the result (B8) can be inserted into Eq. (B1) for the biphoton amplitude. By performing first the integral over the fast variables $\vec{w}_0 = (\vec{q}_0, \Omega_0)$ and by using Eq. (18), one obtains

$$\begin{aligned} \Psi(\vec{\xi}, \vec{\xi}'; z) &= \int \frac{d\Omega}{2\pi} \chi_f(\Omega) e^{-i\Omega(t-t')} F_\psi \\ &\quad \times \left(\frac{\vec{r} + \vec{r}'}{2} + \frac{\vec{r} - \vec{r}'}{2} \frac{\Omega}{\omega_s}, \frac{t + t'}{2} \right) \\ &\quad \times \left\{ \frac{\bar{k}(\Omega)}{4\pi i z} \Phi_{\text{PDC}}\left[\left(\vec{r} - \vec{r}'\right) \frac{\bar{k}(\Omega)}{2z}, \Omega\right] e^{i|\vec{r} - \vec{r}'|^2 \frac{k(\Omega)}{4c}} \right\}. \quad (\text{B9}) \end{aligned}$$

This rather formidable expression has a simpler interpretation than it looks. Let us assume first that the dependence on $\vec{r} - \vec{r}'$

in the argument of F_ψ can be neglected: This always holds for a narrow enough bandwidth $\Omega/\omega_s \ll 1$, but also, in general, if $\vec{r} - \vec{r}'$ is restricted to a small enough region. Then, Eq. (B9) takes the factorized form

$$\Psi(\vec{\xi}, \vec{\xi}'; z) = F_\psi\left(\frac{\vec{r} + \vec{r}'}{2}, \frac{t + t'}{2}\right) \psi_{\text{far}}(\vec{r} - \vec{r}', t - t'; z), \quad (\text{B10})$$

$$\psi_{\text{far}}(\vec{r}_-, t_-; z) = \int \frac{d\Omega}{2\pi} \chi_f(\Omega) e^{-i\Omega t_-} \times \left\{ \frac{\bar{k}(\Omega)}{4\pi i z} \Phi_{\text{PDC}}\left[\vec{r}_- \frac{\bar{k}(\Omega)}{2z}, \Omega\right] e^{i|\vec{r}_-|^2 \frac{\bar{k}(\Omega)}{4z}} \right\}. \quad (\text{B11})$$

The correlation, as a function of the biphoton separation $\vec{r}_-$, follows the profile of the spectral amplitude Φ_{PDC} along $\vec{q}$, integrated over the frequencies, and its width grows with z , with the transverse separation of twin photons extending to distances up to $|\vec{r}_-|_{\text{max}} \sim q_{\text{max}} \frac{z\lambda_s}{\pi}$. As discussed extensively in the main text, the entanglement area approaches the beam cross section at $z = \frac{\pi D_s}{\lambda_s q_{\text{max}}} \sim \frac{D_s \rho_{\text{ent}}}{\lambda_s}$, comparable to the boundary of the deep Fresnel zone for the classical coherence. At larger distances, the spatial correlation moves to the opposite side of the cross section, because in these conditions $F_\psi(\frac{\vec{r} + \vec{r}'}{2})$, in Eq. (B10) becomes the narrower correlation peak, which forces $\vec{r}' = -\vec{r}$. This is a consequence of the conservation of transverse momentum, which dominates the spatial entanglement in the far-field zone: For a narrow-band, entangled photons generated with correlated wave vectors $\vec{q} + \vec{q}' \simeq 0$, propagate out of the crystal at angles $\vec{\theta} + \vec{\theta}' \simeq 0$, where $\vec{\theta} \simeq (\frac{q_x \lambda_s}{2\pi}, \frac{q_y \lambda_s}{2\pi})$. In the far field that maps angles into positions, entangled photons are therefore found at symmetric positions.

The complete expression in Eq. (B9) expresses more correctly the conservation of transverse momentum for a generic bandwidth. Actually, $\vec{q} + \vec{q}' \simeq 0$ must be considered together with the conservation of energy $\Omega + \Omega' \simeq 0$. For a

generic bandwidth, the angles are related by $\vec{\theta}'(1 - \Omega/\omega_s) \simeq \vec{\theta}(1 + \Omega/\omega_s)$, which implies $\vec{\theta} + \vec{\theta}' + (\vec{\theta}' - \vec{\theta}) \frac{\Omega}{\omega_s} \simeq 0$.

APPENDIX C: NUMERICAL INVESTIGATION OF THE APPROXIMATION (24)

We investigated the validity of the approximated evolution equation (24) numerically also. The six-dimensional integral involved in the complete propagation equation (19) was performed with a Monte Carlo method and compared with the results of the approximated equation. (24), which are also numerical, but involve an easier integration in two dimensions. Figure 9 plots the decay of the amplitude probability $|\Psi(\vec{\xi}, \vec{\xi}, z)|$ of finding two entangled photons at $\vec{\xi}' = \vec{\xi}$, obtained via Eq. (19) (red symbols) and Eq. (24) (solid black line), for the largest bandwidth of 550 nm. The results of the Monte Carlo integration are obtained by sampling over 6×10^9 points, and the error bars are quite small. According to Eq. (23), for these parameters ($D_s \simeq 415 \mu\text{m}$, $\rho_{\text{ent}} \sim 3.8 \mu\text{m}$, $\Delta\lambda = 550 \text{ nm}$), we expect the Monte Carlo result to stay close to the reduced model of Eq. (24) at least up to $z = 5700 \mu\text{m}$, but indeed the red curve shows no significant deviation from the reduced model up to the investigated distance $z = 25000 \mu\text{m}$. The blue curve in panel (a) shows for comparison the much slower decay of the classical coherence peak.

APPENDIX D: FITS OF THE TRANSVERSE CORRELATION

This Appendix investigates the validity of the approximation in Eq. (28) for the spatial biphoton correlation at the source plane. In Fig. 10 the spatial biphoton correlation at the focal plane, calculated from the complete Eq. (24), is fitted with the Airy function in Eq. (28), for the different bandwidth considered. For the fits we used a free parameter $\eta \simeq 1$ that slightly adjusts the effective transverse bandwidth.

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