

Asymptotically independent fluctuations of stabilizer Rényi entropy and entanglement in random unitary circuits

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We investigate numerically the joint distribution of magic (as described by the stabilizer 2-Rényi entropy M_2) and entanglement (as described by the von Neumann entropy S) in N -qubit Haar-random quantum states. The distribution $P_N(M_2, S)$ and the marginals become exponentially localized, and centered around the typical values, $\tilde{M}_2 \rightarrow N - 2$ and $\tilde{S} \rightarrow N/2$ as $N \rightarrow \infty$. Magic and entanglement fluctuations are, however, found to become exponentially uncorrelated. While the set of stabilizer states grows exponentially with N , they nevertheless form a measure-zero subset of Haar-random states and thus occur with vanishing probability. Typical quantum states are characterized by large magic and entanglement entropy, and uncorrelated magic and entanglement fluctuations. The mutual information of the joint distribution vanishes exponentially as $N \rightarrow \infty$, implying that fluctuations of these two resources are not only uncorrelated but asymptotically independent.

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I. INTRODUCTION

Although it may still be considered somewhat elusive, quantum complexity is a central theme in quantum computing. Certain “simple” quantum states are in some sense close to classical, while others are “complicated” as they are strongly nonlocal and possess quantum correlations between remote components or faraway states of a system. To distinguish “simple” and “complicated” states, several tools and concepts, such as entanglement, nonlocality, quantum coherence, or quantum correlations, have been proposed to quantify quantum complexity in the framework of quantum resource theory [1–4].

In this context, although difficult to measure [5–7], *entanglement* is of utmost significance and has long been identified as one of the major quantum resources. Entanglement entropy helps to characterize the amount of classical resources needed to represent a state, and entanglement is the key ingredient of quantum teleportation [8–11], quantum repeaters [12–14], and many quantum algorithms [15,16].

Entanglement and nonlocality, however, do not reflect the entire complexity of a state. Another way to characterize the complexity of a state $|\psi\rangle$ is to investigate the complexity of the quantum circuit that produces this state. While generic quantum dynamics typically turns simple product states into states that are impossible to represent on a classical computer [17], yet highly entangled states can be noncomplex in this sense. A striking example is given by entangled quantum states produced by classically efficiently simulable Clifford circuits [18–23].

The lack of complexity of these states and the corresponding circuits can be captured through the *spectral property* of the Clifford circuits [24] or by the notion of *magic* or *nonstabilizing power*. The concept of nonstabilizing power, in particular, is based on the notion of *stabilizer Rényi entropy* (SRE) [25]. The latter quantifies the hardness to simulate classically a quantum state in terms of simple stabilizer states [26] and has raised significant attention recently [27–46].

In a recent work, a distinction between entanglement-dominated and magic-dominated states was identified, suggesting a division of the Hilbert space into two distinct regimes [47]. The interplay between magic and entanglement has been further explored in the context of matrix product states [48,49], particularly in a spin-1 model, by examining the relationship between magic and the bond dimension required to approximate the ground state [48]. An even more ambitious study investigated the comparison between many-body entanglement and magic in nuclear structure [50]. The relationship between these two quantities remains ambiguous:

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It is possible to find states with negligible magic but high entanglement, as well as states with low entanglement yet substantial magic. Ground states, in this context, stand out as atypical, as they tend to exhibit reduced entropy compared to more typical states. However, entanglement and magic are related [49,51], as entanglement is *instrumental* to generate high-magic states. Indeed, the N -qubit state with the highest magic that can be generated *without* entanglement is given by $|\psi\rangle = \bigotimes_{j=1}^N T_j H_j T_j H_j |0\rangle_j$, where T is the single-qubit $\pi/4$ phase gate and H is the Hadamard gate. The corresponding magic, measured by the SRE, is therefore $M_2(|\psi\rangle) = N \times 0.585\dots$ [24], which is substantially lower than the theoretical upper bound of the SRE, $\log_2((2^N + 1)/2)$ [25].

In this work, we address the question whether magic and entanglement are related in generic, Haar-random quantum states. First, we study numerically the distribution of magic $P_N(M_2)$ of few-qubit quantum systems. We find that for Haar-random states of an N -qubit circuit, $P_N(M_2)$ is an exponentially sharp distribution centered at around $\tilde{M}_2 \approx N - 2$. This follows also from the concentration property of the linear magic M_{lin} , which is a consequence of Levy's lemma [25]. The entanglement entropy of Haar-random N -qubit states displays a similar, exponentially sharp distribution around $\tilde{S}$, approaching $N/2$ in the large N limit. The typical values of entropy and magic are therefore strongly correlated. However, their joint distribution $P_N(M_2, S)$ becomes uncorrelated for large N 's, and the covariance of the entropy and magic fluctuations vanishes exponentially fast.

This paper is structured as follows: In Sec. II, we examine the stabilizer Rényi entropy (magic) and present results for its distribution for various system sizes. In Sec. III, we focus on entanglement, while Sec. IV explores the relationship between magic and entanglement, along with results for their joint distribution. Finally, we conclude in Sec. V with a summary of the findings.

II. STABILIZER RÉNYI ENTROPY DISTRIBUTION OF HAAR-RANDOM STATES

Stabilizer Rényi entropy (SRE), also referred to as magic, represents the hardness to classically simulate a quantum state [25,31,52] within the stabilizer formalism. This entropy is closely related to the structure of Clifford circuits, i.e., unitary operations mapping Pauli N -strings, $\sigma \in \mathcal{P}_N \equiv \{1, X, Y, Z\}^N$, to N -strings—apart from an overall sign. Each state $|\psi\rangle\langle\psi|$ can be decomposed in terms of Pauli strings, and one can associate with it a corresponding string probability distribution

$$\Xi_\psi(\sigma) = \frac{1}{d} \langle \psi | \sigma | \psi \rangle^2, \quad (1)$$

where $d = 2^N$ is the dimension of the Hilbert space of N qubits. The corresponding stabilizer α -Rényi entropy is defined as

$$M_\alpha(|\psi\rangle) \equiv (1 - \alpha)^{-1} \log_2 \sum_{\sigma \in \mathcal{P}_N} \Xi_\psi^\alpha(\sigma) - \log_2(d), \quad (2)$$

with $\mathcal{P}_N$ the set of unsigned Pauli strings. α -SRE fulfills the following important properties [25]: (1) M_α remains unaltered under Clifford operations; (2) $M_\alpha(|\psi\rangle) = 0$ if $|\psi\rangle$ is a stabilizer state; (3) similar to entropy, M_α is additive for product states; and (4) it is upper bounded by $M_\alpha \leq \log_2(d)$.

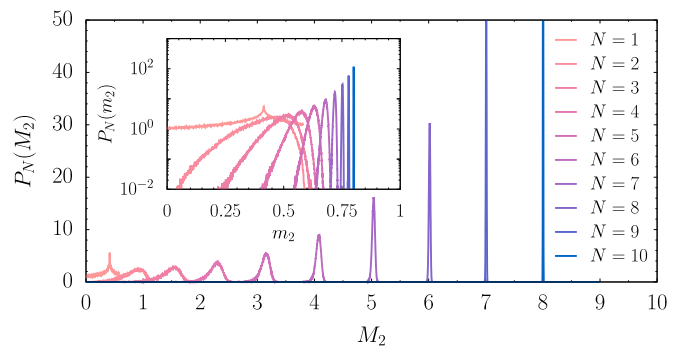


FIG. 1. Distribution of magic $P_N(M_2)$ generated by random unitary operators $U \in \mathcal{U}(2^N)$. Increasing N , the distribution becomes concentrated around the value $\tilde{M}_2 \approx N - 2$. Inset: Distribution of magic density $m_2 = M_2/N$.

lizer state; (3) similar to entropy, M_α is additive for product states; and (4) it is upper bounded by $M_\alpha \leq \log_2(d)$.

We obtain the distribution of magic by sampling Haar-random unitary $U \in \mathcal{U}(2^N)$ acting on the reference state $|0\rangle \equiv \bigotimes_{j=1}^N |0\rangle_j$ of an N -qubit register, i.e., $|\psi\rangle = U|0\rangle$. However, in Appendixes A 1 and A 2 we show that the results we obtain in the following naturally extend to other classes of random circuits that can be used to approximate Haar unitary operators, such as *sufficiently deep* brickwall circuits.

Moreover, in the following, we restrict ourselves to the stabilizer Rényi entropy of Rényi index $\alpha = 2$, for which a tighter upper bound is given by $M_2(|\psi\rangle) \leq \log_2((2^N + 1)/2)$ [25]. Henceforth, we refer to the stabilizer 2-Rényi entropy as SRE or magic, for simplicity. α -SREs of different Rényi indices α have also been investigated in the literature, and it has been shown that they follow a hierarchy and display qualitatively similar behavior [25,31,52]. In this sense, the choice to focus on M_2 is not restrictive, and indeed, in Appendix A 2 we show explicitly in some cases that the properties we discuss in the following also generically apply to stabilizer Rényi entropies of Rényi index α .

Figure 1 displays the evolution of $P_N(M_2)$ with increasing number of qubits. The case $N = 1$ is somewhat peculiar [24]; the saddle point states on the Bloch sphere give rise to logarithmic van Hove singularities. For larger values of N , however, the distribution gradually shifts to higher values and is concentrated around $\tilde{M}_2 \approx N - 2$. The numerically observed maximum is extremely close to the lower bound, $\mathbb{E}[M_2(|\psi_{\text{Haar}}\rangle)] \geq N - 2 + \log_2(1 + 3/2^N)$ [25].

The logarithmic inset of Fig. 1 shows the rescaled magic, or magic density $m_2 = M_2/N$, and demonstrates that the distribution becomes exponentially narrow with increasing N , and the magic density of typical Haar-random states converges to $m_2 \approx 1 - 2/N \rightarrow 1$ in the large N limit, as dictated by the bounds reported earlier.

The width of the distribution scales as $\delta M_2 = \sqrt{\text{var}(M_2)} \approx 2^{-N}$ (see also Fig. 6). This behavior is in contrast to standard thermodynamic behavior, which would correspond to a variance scaling as $\sim N$. The ultimate reason behind this behavior is that, apart from an overall phase and the normalization, Haar-random states are characterized by exponentially many ($\sim 2^N$) complex numbers, and the sum in Eq. (2) can therefore be considered as a sum of $N_\sigma = 4^N$ weakly correlated

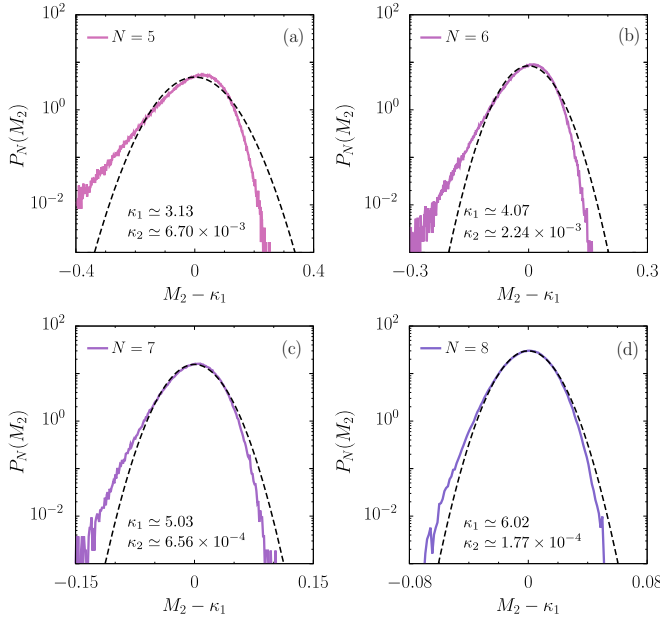


FIG. 2. Distribution of magic $P_N(M_2)$ centered around the first cumulant $\kappa_1 = \langle M_2 \rangle$ on a semilog scale. Black dashed lines correspond to a normal distribution with mean κ_1 and variance $\kappa_2 = \langle (M_2 - \langle M_2 \rangle)^2 \rangle$ extracted from $P_N(M_2)$.

terms. A simple calculation yields $\delta M_2 \sim 1/N_{\sigma}^{1/2} \sim 2^{-N}$, as we indeed observe it numerically. A similar result holds for the linear magic [25].

Although the magic distribution $P_N(M_2)$ is strongly skewed for small N 's (see Fig. 2), we find that higher-order cumulants vanish faster than the variance, i.e., $\kappa_{n>2}/\kappa_2^{n/2} \rightarrow 0$, and in the thermodynamic limit the distribution converges toward a Gaussian with a vanishing variance $P_{N \rightarrow \infty}(M_2) \rightarrow \mathcal{N}(\kappa_1 \rightarrow \tilde{M}_2, \kappa_2 \rightarrow 0)$.

III. ENTANGLEMENT ENTROPY DISTRIBUTION

Entanglement entropy displays a behavior very similar to that of magic. We have computed the von Neumann entropy of Haar-random states by cutting the system into two parts of equal size, $N/2$ for even N and, of almost equal size, $(N \pm 1)/2$ for odd N . The evolution of entropy distributions $P_N(S)$ is presented in Fig. 3. Also for the entanglement entropy, we observe an exponentially narrow distribution, concentrated around $\tilde{S} \approx N/2 - \log_2(e)/2$ for even N and around $\tilde{S} \approx (N+1)/2 - \log_2(e)$ or $\tilde{S} \approx (N-1)/2 - \log_2(e)/4$ for odd N , for the subsystem with $(N \pm 1)/2$ qubits, consistent with the Page formula [53,54]. In contrast, the probability of maximally entangled states is very small [55]. The distribution of the rescaled entropy, $s = S/N \approx 1/2 - \log_2(e)/(2N)$, shown in the inset of Fig. 3, gradually shifts toward $\tilde{s} = 1/2$, which corresponds to the entropy density of a state at infinite temperature.

The observed trend can be explained in terms of the eigenstate thermalization hypothesis [56]. Generic quantum states of a large system produce a thermal state when projected to a subsystem. Infinite temperature states of a subsystem have the largest entropy due to the fact that they are exponentially

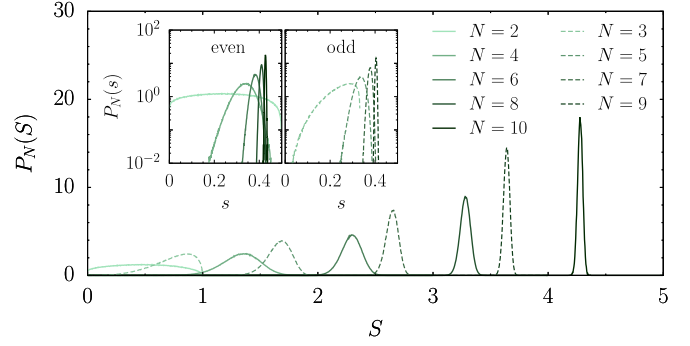


FIG. 3. Distribution of entanglement entropy $P_N(S)$ generated by random unitary operators $U \in \mathcal{U}(2^N)$. Increasing N , the distribution becomes concentrated around the value $\tilde{S}_2 \approx N/2$. Insets: Distribution of entropy density $s = S/N$.

abundant when sampling states randomly in the Hilbert space [57,58]. For generic quantum states, the entropy of a subsystem approaches $N/2 - \log_2(e)/2$ with increasing N , though finite-size corrections introduce deviations.

Similarly to magic, entanglement distributions become exponentially narrow with increasing N (see Fig. 4). This can be understood similarly to the sharpness of magic distributions. Entanglement is an average over all 2^M density matrix eigenvalues of an M -qubit subsystem. Assuming weak correlations between these, one therefore arrives at entropy fluctuations of size $\delta S = \sqrt{\text{var}(S)} \approx 2^{-N/2}$, as indeed verified by the quantitative analysis presented in the next section and in agreement with previous results [58].

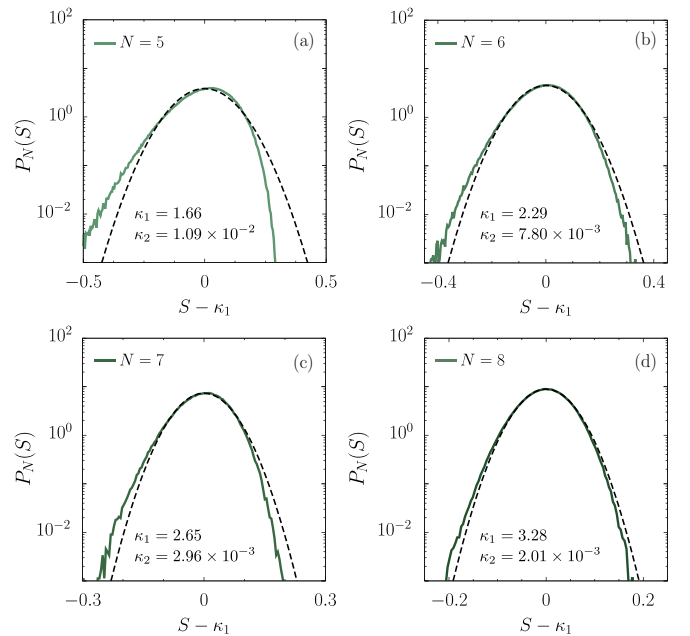


FIG. 4. Distribution of magic $P_N(S)$ centered around the first cumulant $\kappa_1 = \langle S \rangle$ on a semilog scale. Black dashed lines correspond to a normal distribution with mean κ_1 and variance $\kappa_2 = \langle (S - \langle S \rangle)^2 \rangle$ extracted from $P_N(S)$.

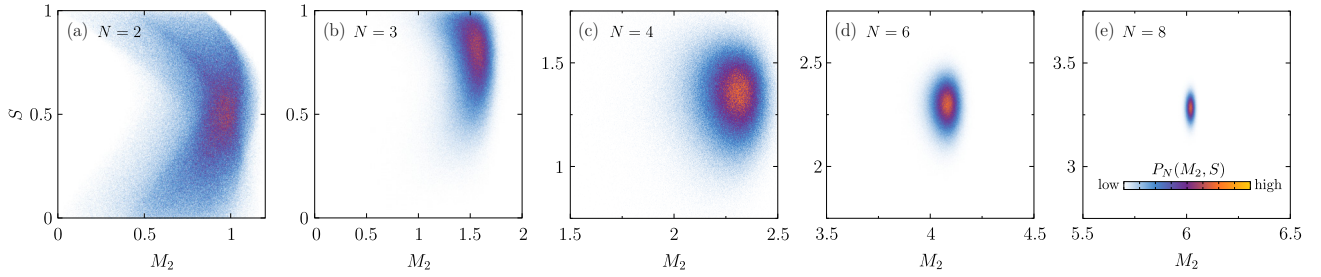


FIG. 5. Joint distribution $P_N(M_2, S)$ of magic and entanglement entropy. Since the variance of magic and entropy vanish at a different rate, the joint distribution assumes a characteristic ellipsoidal shape as it concentrates around $\bar{M}_2$ and $\bar{S}$.

IV. JOINT DISTRIBUTION AND CORRELATIONS

Let us now turn to the main subject of this work, the analysis of joint magic and entanglement entropy distributions of Haar-random states. Figure 5 displays the evolution of the joint distribution with increasing qubit number. Clearly, for small values of $N = \{2, 3\}$, $P_N(M_2, S)$ displays a particular structure, as shown in Figs. 5(a) and 5(b). In particular, for $N = 3$, the distribution is skewed toward the higher end of the entanglement spectrum. This asymmetry arises due to the upper bound on entanglement imposed by the small system size, i.e., $S \leq \min_M (\log_2(d_M), \log_2(d_{N-M}))$. As the system size N increases, the averages of the rescaled magic and entanglement remain fixed at $\bar{M}_2 \approx N - 2$ and $\bar{S} \approx N/2$, respectively, while their variances exhibit rapid exponential decay: $\text{var}(M_2) \propto 4^{-N}$ and $\text{var}(S) \propto 2^{-N}$.

These findings are in excellent agreement with the simple estimates derived in previous sections, as corroborated by the numerical data for the variance of the distributions presented in Fig. 6(a). Notably, the numerical results are best described by exponential decay, confirming the robustness of the predictions.

In Fig. 6(b), we also present the covariance between the two quantities, M_2 and S , which follows an exponential decay with a slightly larger exponent: $\text{cov}(M_2, S) \propto 2^{-3N}$. Importantly, the covariance decays faster than the geometric mean

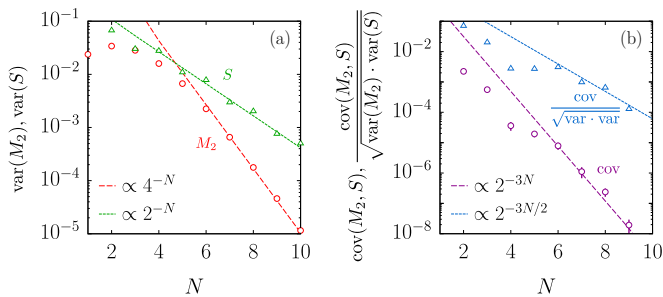


FIG. 6. (a) Scaling of the variance of the magic (red circles) and entanglement (green triangles) distributions. Both are exponentially vanishing in the limit $N \rightarrow \infty$. (b) Scaling of the covariance of the joint distribution $P_N(M_2, S)$ (magenta circles), which is exponentially suppressed with system size, also faster than the geometric mean of the individual variances (blue triangles), demonstrating that magic and entanglement fluctuations are asymptotically uncorrelated. The color dashed lines highlight the corresponding exponential scaling for each quantity.

of the variance of M_2 and S . This indicates that the correlation between magic and entanglement becomes negligible as $N \rightarrow \infty$. This behavior strongly supports the conclusion that, in the thermodynamic limit, the two quantities become effectively uncorrelated. Recent results from Hamma and co-workers [59] show analytically that for the linear magic and entanglement entropy, the correlation vanishes *identically* for any N . Our numerical results extend and generalize this for M_2 and the von Neumann entropy, showing that the correlation vanishes exponentially with system size.

We also computed the mutual information (MI) of the joint distribution, defined as

$$I(M_2, S) = \int_{M_2} \int_S P_N(M_2, S) \log_2 \left(\frac{P_N(M_2, S)}{P_N(M_2)P_N(S)} \right). \quad (3)$$

The MI measures the dependence of the two resources in the joint distribution relative to the marginal distributions $P_N(M_2)$ and $P_N(S)$.

Since the joint distribution $P_N(M_2, S)$ is sampled statistically, the value of the MI calculated from Eq. (3) has a distribution and an average $\bar{I}$ that both depend on the number of samples, n_s : $\bar{I} = \bar{I}(n_s)$. We find that $\bar{I}(n_s)$ converges exponentially to its $n_s \rightarrow \infty$ value, $\bar{I}(n_s) \approx \bar{I}_\infty + a_1 e^{-|a_2|n_s}$. The extracted results are displayed in Fig. 7. The MI vanishes exponentially in the thermodynamic limit, implying that magic and entanglement entropy are not only asymptotically uncorrelated but also *asymptotically independent* as $N \rightarrow \infty$, $P_{N \rightarrow \infty}(M_2, S) \rightarrow P_\infty(M_2)P_\infty(S)$.

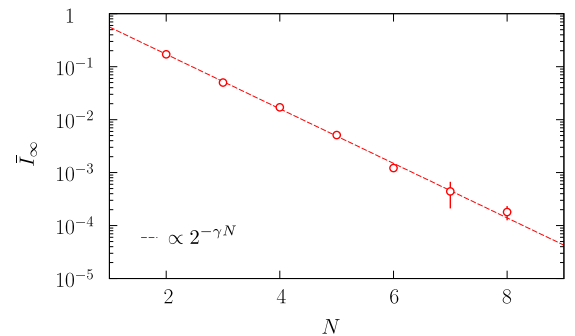


FIG. 7. The mutual information of magic and entanglement $\bar{I}_\infty$ vanishes exponentially in the limit $N \rightarrow \infty$, showing that, asymptotically, M_2 and S become independent, $P_{N \rightarrow \infty}(M_2, S) \rightarrow P_\infty(M_2)P_\infty(S)$.

V. CONCLUSIONS

In this work, we investigated the joint distribution of stabilizer Rényi entropy (magic) and von Neumann entanglement entropy in Haar-random quantum states of N qubits, uncovering key statistical properties of these quantum resources. Our numerical analysis revealed that both magic and entanglement exhibit exponentially sharp distributions as the system size increases, with magic centering around $\tilde{M}_2 \approx N - 2$ and entanglement entropy peaking near $\tilde{S} \approx N/2$. While the typical values of the two resources are dependent, their fluctuations become exponentially uncorrelated in the thermodynamic limit, as demonstrated by the rapid decay of their covariance. For the linear magic and the von Neumann entropy, it can be shown analytically that the covariance identically vanishes for any finite N [59]. Remarkably we find that the mutual information of the joint distribution of M_2 and S also vanishes in the thermodynamic limit $N \rightarrow \infty$, thereby implying that the fluctuations of these two resources become asymptotically independent, even though their averages are related. This result emphasizes the fundamental difference of these two measures of quantum complexity in large systems, which account for different aspects of quantumness.

Additionally, our work demonstrates that quantum states with vanishing magic but high entanglement entropy, or states with high magic but low entanglement, are exponentially rare compared to the abundance of states characterized by both high magic and high entanglement, indicating the predominance of highly complex quantum states in the Hilbert space, as most generic states are characterized by large values of both resources.

Our findings highlight the utility of random unitary circuits as a framework for exploring quantum resource theories, particularly for understanding the interplay and statistical properties of key measures such as magic and entanglement.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

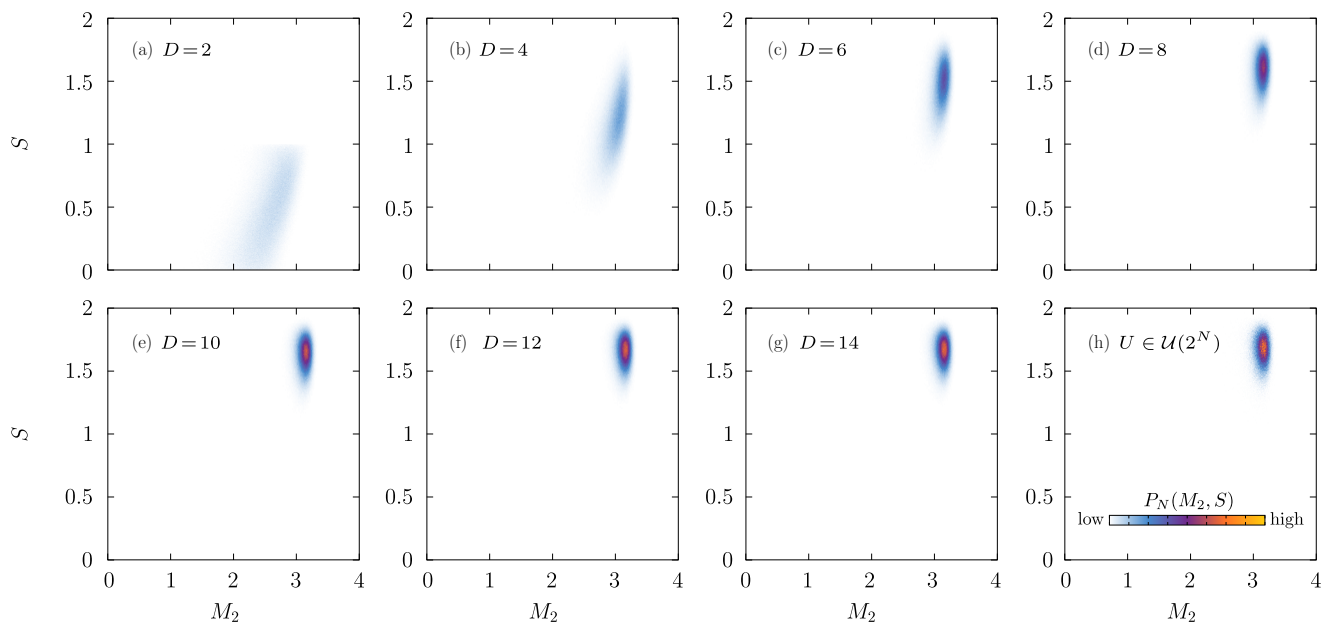


FIG. 8. Joint distribution $P_N(M_2, S)$ of magic and entanglement entropy generated by sampling brickwall circuits of different depths D , compared to the distribution generated by sampling Haar-random unitaries $U \in \mathcal{U}(2^N)$. As the depth increases, the distributions become statistically indistinguishable.

APPENDIX: CONVERGENCE OF BRICKWALL CIRCUIT OF UNITARY GATES TO HAAR UNITARY OPERATOR

1. Joint distribution

In quantum systems with a large number of qubits, *exact* methods for computing properties of generic unitary operators become intractable due to the exponential growth of corresponding Hilbert space. Consequently, it is important to assess to what extent approximate constructions—such as brickwall quantum circuits composed of two-qubit unitaries—are representative of the behavior of Haar-random unitaries. In particular, a brickwall construction is a natural candidate for quantum circuit simulations with tensor network techniques, which give access to efficient ways to calculate entanglement entropies and stabilizer Rényi entropies [37,41,48,60].

To this end, we consider a decomposition of a general $U \in \mathcal{U}(2^N)$ into a layered brickwall structure of local gates $U_2 \in \mathcal{U}(4)$, each acting on pairs of adjacent qubits. We expect that, as the circuit depth D increases, such construction approximates the statistical properties of Haar-random unitaries. In particular, we demonstrate that the joint distribution of magic and entanglement entropy, as well as the individual distributions, obtained with the brickwall construction, converges to the Haar unitary counterpart.

In Fig. 8, we show the evolution of the joint distribution of stabilizer 2-Rényi entropy (magic) and entanglement (von Neumann) entropy $P_{N;D}^{\text{BW}}(M_2, S)$ obtained from sampling brickwall circuits, as a function of circuit depth D for a representative case with $N = 5$ qubits. For shallow circuits (e.g., for $D = 2$), the entanglement is strongly limited due to both the restricted connectivity and the finite light cone associated with local gate propagation. In this regime, only nearby qubits can become entangled, and the resulting joint distribution reflects these constraints. As the depth increases to $D \approx N$, the brickwall circuit becomes capable of entangling all qubits in principle, thus allowing access to the full range of the entanglement and magic spectra. At this point, the joint distribution closely resembles that of random unitaries.

Notably, we also observe that, for depths exceeding the system size ($D > N$), the *individual* distributions of both entanglement and magic become statistically indistinguishable from those obtained using Haar-random unitaries. This validates the use of sufficiently deep brickwall circuits as proxies for generic unitary dynamics in large systems, particularly in settings where efficient numerical simulation is essential.

2. α -SRE distribution

Concentrating on the distribution of magic, we can quantify the differences between the brickwall circuits and Haar unitaries. For each circuit depth D , as well as for Haar unitaries, for $N = 5$ qubits, we collect $n_s = 2^{20}$ samples and construct the distribution of α -SRE for a few different values of the Rényi index α . Figure 9 shows the evolution of the distribution $P_{N;D}^{\text{BW}}(M_\alpha)$ as a function of brickwall circuit depth D .

While α -SREs differ quantitatively, they exhibit qualitatively similar behavior across different values of the Rényi indices α . As D increases, the distribution becomes increasingly narrow and concentrated around an α -dependent value

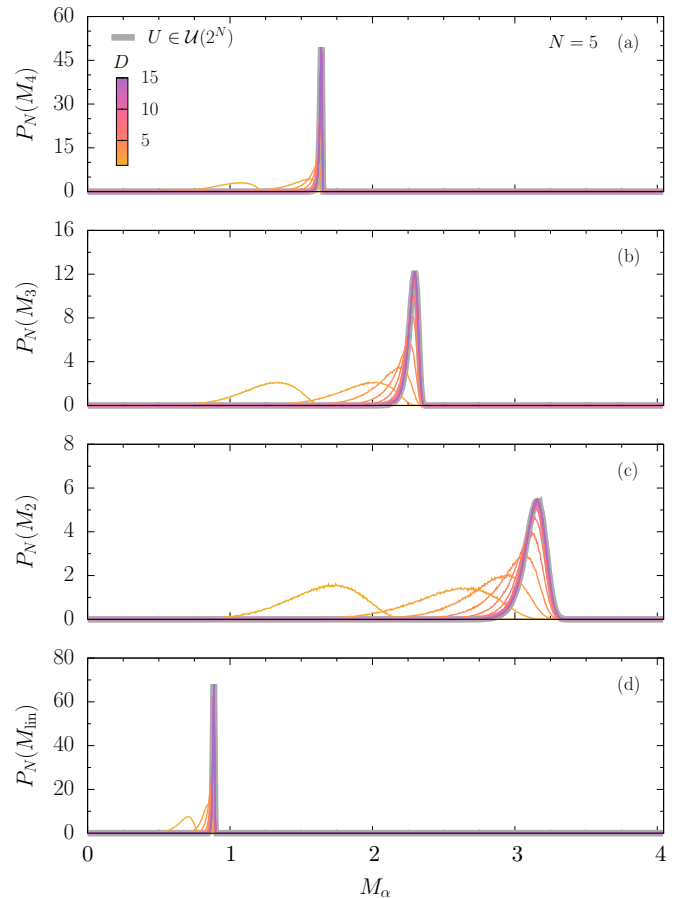


FIG. 9. Convergence with circuit depth. Comparison between the distribution of stabilizer α -Rényi entropy $P_N(M_\alpha)$ generated by random unitary operators $U \in \mathcal{U}(2^N)$ and in a brickwall construction with only two-qubit gates, for a representative case with $N = 5$ qubits. For any α , the brickwall distribution converges to the one generated by a single unitary operator as the circuit depth increases. The sample size for each depth is $n_s = 2^{20}$.

$\tilde{M}_\alpha$, which decreases with increasing α , indicating a stronger bias toward the dominant components of the Pauli spectrum. When computing SRE for large systems, particularly for highly nonstabilizer states, the individual coefficients in the Pauli spectrum typically scale as $\mathcal{O}(d^{-2})$. Therefore, the choice of α also governs numerical sensitivity, as these small coefficients must be raised to the power 2α , thus amplifying the contribution of small numerical differences. Note that the *linear* magic M_{lin} , shown in Fig. 9(d), escapes the above quantitative trend, because its upper bound is $M_{\text{lin}}(|\psi\rangle) < 1 - 2(d+1)^{-1} \leq 1$ [25].

For any value of α and for $D \gtrsim N$, the distribution $P_{N;D}^{\text{BW}}(M_\alpha)$, obtained by sampling brickwall circuits, becomes statistically indistinguishable from the corresponding distribution $P_N(M_\alpha)$, obtained by sampling Haar-random unitaries.

A more quantitative analysis of the distribution convergence can be obtained considering an estimator that measures the distance between the distributions, as a function of the brickwall circuit depth D and sample size n_s . We choose

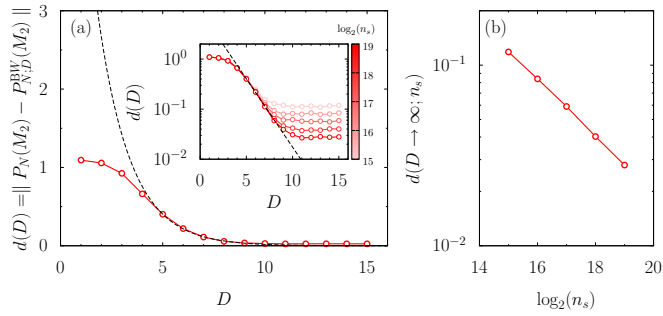


FIG. 10. (a) Distance $d(D)$ between the stabilizer Rényi entropy (M_2) distributions obtained from Haar-random unitaries, $P_N(M_2)$, and from brickwall circuits, $P_{N,D}^{BW}(M_2)$, for $N = 5$ qubits, as a function of the brickwall circuit depth D . Inset: Convergence of $d(D)$ as a function of depth, for different sample sizes n_s . The dashed line shows the exponential convergence for depths $D \gtrsim N$, which is cut off by statistical noise. (b) The asymptotic distance $d(D \rightarrow \infty)$ converges exponentially to zero as a function of sample size.

as estimator the root-mean-square distance, defined as

$$d(D) = \|P_N(M_\alpha) - P_{N,D}^{BW}(M_\alpha)\|. \quad (\text{A1})$$

As reported in Fig. 10(a) and the corresponding inset, we observe a regime starting around $D \approx N$, where the distance between the distributions decreases *exponentially* as a function of sample size. The exponential convergence is cut off at a certain circuit depth due to the statistical noise originating from the operators' sampling. In Fig. 10(b), we show that the asymptotic value of the distance $d(D \rightarrow \infty)$ decreases strictly exponentially as a function of sample size; i.e., it decreases linearly on a semilogarithmic scale against $\log_2(n_s)$.

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