

Rotation parameter modulation of dynamics in coupled fractional-order Van der Pol oscillators

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(Received 14 April 2026; accepted 20 July 2026; published 10 August 2026)

Traditional fixed-sign coupling schemes impose rigid directional constraints on interaction pathways, while integer-order differential models inherently lack the capacity to describe memory effects and long-range temporal correlations. To overcome these limitations, here we establish a fractional-order coupled Van der Pol oscillator system with a rotation parameter to investigate how the rotation parameter and fractional-order derivative control the dynamics. The rotation parameter distributes the coupling to the x and y variables via trigonometric functions, precisely regulating the sign and weight of coupling in each direction; this governs bifurcations and qualitatively determines the steady-state mode. Moreover, the memory effect of the fractional-order derivative exhibits quadrant dependence: As the fractional-order derivative decreases, the rotation parameter delays amplitude death in the second and third quadrants but accelerates incoherent oscillation death in the fourth quadrant. Finally, employing phase reduction theory, we derive the phase model and numerically verify the system's transition path. Our findings break away from traditional frameworks, deepen the understanding of fractional-order coupled systems, elucidate the regulatory mechanisms of rotation parameters, and provide a theoretical foundation for controlling synchronization, chimera states, and death states in real-world systems.

DOI: [10.1103/jh2w-x92t](https://doi.org/10.1103/jh2w-x92t)

I. INTRODUCTION

Coupled oscillator systems, as fundamental frameworks for describing nonlinear collective dynamics, are widely found in physics [1,2], biology [3], chemistry [4], engineering [5], and social sciences [6]. The collective behaviors they exhibit, including synchronization [7], amplitude death [8,9], oscillation death [10], and chimera states [11,12], represent frontier hot spots in contemporary nonlinear dynamics research. Synchronization, one of the most fundamental collective behaviors, refers to the coordinated evolution of phases or amplitudes among multiple interacting oscillators through mutual coupling. Typical examples include the rhythmic synchronization of cardiac pacemaker cells, collective flashing of fireflies, frequency synchronization in power grids, and the synchronized firing of neurons. This phenomenon is of great significance for understanding biological activities and ensuring the stable operation of engineering systems [13,14]. In contrast, amplitude death and oscillation death represent typical oscillation-suppressed behaviors, both reflecting the inhibitory effect of coupling on the self-sustained oscillations of the system. They can cause oscillatory behavior to dampen,

lose stability, or even cease entirely. Such behaviors hold significant theoretical and practical value for understanding the mechanisms of system instability and achieving oscillation control and suppression. In addition, the chimera state refers to the phenomenon in which coherent and incoherent oscillators coexist spatially in a system. It has been verified in numerous representative coupled systems and is closely associated with the unihemispheric sleep patterns [15], the treatment of certain neurological disorders [16,17], and the operation of power grids [18].

At present, most research on the collective behaviors of coupled oscillators is conducted within the framework of integer-order differential equations, with relevant findings now relatively systematic and well established [19,20]. However, integer-order models can only describe the evolution of the system's instantaneous states and cannot capture the memory effects, historical dependence, and nonlocal properties that are prevalent in real-world systems. In recent years, fractional calculus has emerged as a prominent research topic in interdisciplinary fields owing to its unique properties and has found wide applications in engineering [21,22], epidemiology [23,24], and biology [25]. For classical limit-cycle oscillators, Ngueuteu Mbouna *et al.* [26,27] examined the impact of fractional-order derivatives on dynamical behaviors in coupled Stuart-Landau oscillator networks and further extended the work to amplitude chimera states and symmetry-breaking death states in limit-cycle oscillator networks, reporting the first observation of amplitude chimera states in Van der Pol oscillators. Sun *et al.* [28–30] systematically analyzed chimera state transition pathways and

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parameter-dependent dynamical effects in fractional-order Rayleigh and Stuart-Landau oscillators with nonisochronicity parameters. In the field of neuronal network modeling, Ramadoss *et al.* [31] conducted an early study on the dynamical behaviors of coupled fractional-order FitzHugh-Nagumo neurons. Building on this line of research, Liang *et al.* [32] investigated fractional-order FitzHugh-Nagumo neuronal networks with modular fractal connectivity, elucidating the coregulatory effect of fractional-order derivatives and hierarchical connectivity on seizure activities and establishing a modeling framework for abnormal brain rhythms. Nevertheless, many questions remain in the study of dynamical behaviors of fractional-order coupled oscillator systems, and numerous underlying mechanisms in such systems under different coupling schemes and network structures have yet to be elucidated.

In addition, in the study of coupled oscillators, coupling schemes such as attractive-repulsive coupling [33,34], conjugate coupling [35], power-law coupling [26,27], and mean-field coupling [36,37] typically feature fixed signs or component structures, making it difficult to flexibly control the direction and strength of coupling. To address this issue, we introduce rotation parameters that enable the continuous distribution of coupling strength across distinct state variables, allowing flexible adjustment of the sign and weighting of each component, thereby accurately capturing asymmetric coupling behavior. This design concept originates from the classic paradigm in asymmetric coupling research, namely, the core idea of the Kuramoto-Sakaguchi model [38] that uses a phase shift to break coupling symmetry and control the synchronization transition of the system. Unlike the shift design in the phase space, the rotation parameter directly realizes the rotational manipulation of the coupling vector in the state space. In addition, the fractional-order Van der Pol oscillator model has a wide range of applications in neuroscience, physics, and engineering. For example, fractional-order Van der Pol oscillators can be used to simulate the nonlinear and chaotic electrical activities of spinal central pattern generators, as well as the sinoatrial node and atrioventricular node of the heart, providing a theoretical foundation for the study of biological locomotion mechanisms and bionic control [39,40]. Fractional-order Duffing-Van der Pol oscillators can also model viscoelastic vibration absorbers and nonlinear circuit systems, offering a reference paradigm for the controller design and synchronization strategy development of fractional-order chaotic systems [41,42].

Motivated by these considerations, our work investigates fractional-order coupled Van der Pol oscillators with rotation parameters, focusing on analyzing the mechanisms by which rotation parameters and fractional-order derivatives modulate the system's dynamical behaviors. This work aims to reveal the transition laws of system behaviors in different parameter regions and further refine the theory of collective behaviors in fractional-order coupled oscillator systems. The remainder of this paper is organized as follows: Sec. II introduces our model, which consists of a ring of N identical fractional-order Van der Pol oscillators with rotation parameters. In Sec. III, we analyze the effects of rotation parameters in different quadrants and the fractional-order derivative on the dynamical behaviors of the system, and verify the transition paths

using phase reduction theory. Finally, Sec. IV summarizes the research findings of this work.

II. MODEL

We consider a ring of N identical fractional-order coupled Van der Pol oscillators. The model is described as

$$\begin{aligned} D_t^\alpha x_j &= y_j + \frac{\sigma \cos \beta}{2P} \sum_{k=j-P}^{j+P} (x_k - x_j), \\ D_t^\alpha y_j &= \mu(1 - x_j^2)y_j - x_j + \frac{\sigma \sin \beta}{2P} \sum_{k=j-P}^{j+P} (y_k - y_j), \end{aligned} \quad (1)$$

where $j = 1, 2, \dots, N$, N represents the number of oscillators, and μ is the damping parameter. The coupling strength of the system is governed by the parameter σ and the coupling range is defined as P , meaning that each oscillator is coupled only with its P nearest-neighboring oscillators on both the left and right sides. Here, $P = 1$ corresponds to local coupling and $P = N/2$ (for even N) or $P = (N - 1)/2$ (for odd N) corresponds to global coupling. β ($-\pi \leq \beta < \pi$) denotes the rotation parameter that controls the sign and weight of coupling. When β lies in the first quadrant, both $\cos \beta$ and $\sin \beta$ are positive, and the system is termed positive-positive coupling in this case. When β falls in the second quadrant, $\cos \beta$ is negative and $\sin \beta$ is positive, corresponding to negative-positive coupling. Likewise, when β is located in the third and fourth quadrants, the system exhibits negative-negative coupling and positive-negative coupling, respectively. The fractional order is denoted by α ($0 < \alpha \leq 1$), and the Caputo fractional derivative operator D_t^α is defined as [43]

$$D_t^\alpha X = \frac{1}{\Gamma(1 - \alpha)} \int_0^t \frac{X'(\tau) d\tau}{(t - \tau)^\alpha}, \quad (2)$$

where $\Gamma(\cdot)$ denotes the Gamma function.

Under the modulation of different parameters, the system exhibits diverse collective dynamical states. Desynchronization (DS) refers to independent disordered oscillations of individual oscillators. Synchronization (SY) describes uniform in-phase oscillation with identical phase and amplitude across all oscillators. Traveling wave (TW) features a unidirectionally propagating wave structure. In the oscillation cluster (OC) state, uniform and nonuniform oscillations coexist. Transient state (TS) is the state of transition between two states. In the amplitude death (AD), all oscillators are drawn to the same fixed point. Chimera death (CD) is the coexistence of coherent oscillatory death (OD) and incoherent oscillatory death (IOD).

To quantitatively characterize the dynamical behaviors of the system, we adopt the strength of incoherence SI as a quantitative metric. We introduce the transformations $u_{1,j} = x_j - x_{j+1}$ and $u_{2,j} = y_j - y_{j+1}$ for $j = 1, 2, \dots, N$. All oscillators are divided into $K = \frac{N}{n}$ groups, with each group consisting of n oscillators. Subsequently, the standard deviation is introduced by

$$\sigma_l(m) = \left\langle \sqrt{\frac{1}{n} \sum_{k=n(m-1)+1}^{mn} [u_{l,k} - \langle u_l \rangle]^2} \right\rangle_t, \quad (3)$$

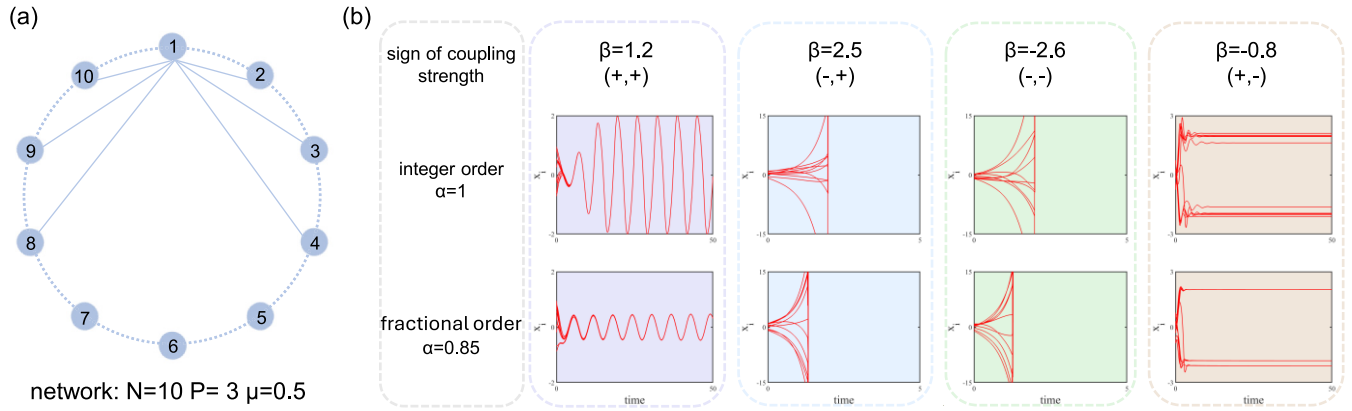


FIG. 1. (a) Network diagram. (b) Time-series plots of the system for different rotation parameters β ($\beta = 1.2, 2.5, -2.6, -0.8$) and fractional derivatives α ($\alpha = 1.0, 0.85$). Other parameters: $N = 10$, $P = 3$, $\mu = 0.5$, and $\sigma = 2.0$.

where $m = 1, 2, \dots, K$, $l = 1, 2$, $\langle u_l \rangle = \frac{1}{N} \sum_{j=1}^N u_{l,j}(t)$, and $\langle \cdot \rangle$ denotes the time average. The strength of incoherence SI can be expressed as [44]

$$SI = 1 - \frac{\sum_{m=1}^K s_m}{K}, \quad s_m = \Theta[\delta - \sigma_l(m)], \quad (4)$$

where $\Theta(\cdot)$ denotes the Heaviside step function and δ is a small threshold parameter. When SI equals 0 or 1, it corresponds to the coherent or incoherent state, respectively. When the value of SI falls between 0 and 1, it indicates the existence of the chimera state.

III. RESULTS

We employ the predictor-corrector algorithm for numerical simulation, with an integration time step of 0.01. The initial conditions for x and y are set to random numbers uniformly distributed between -1.0 and 1.0 . In addition, we fixed the damping parameter $\mu = 0.5$. To determine whether the rotation parameter β and the fractional-order derivative α affect the system dynamics, we consider a small-scale network with $N = 10$. The coupling range is set to $P = 3$, meaning each oscillator interacts only with its three nearest-neighboring oscillators on its left and right sides. A schematic of this network is presented in Fig. 1(a).

The rotation parameter β is set to 1.2 (first quadrant), 2.5 (second quadrant), -2.6 (third quadrant), and -0.8 (fourth quadrant), which are selected from four different quadrants and correspond to four distinct coupling types, respectively. The parameter α is set to 1 and 0.85, representing the integer-order and fractional-order derivatives, respectively. Figure 1(b) shows the time-series plots for ten oscillators under these eight conditions. It can be seen that the oscillatory behaviors of the oscillators differ under different values of β and α , confirming that the rotation parameter β and the fractional-order derivative α indeed exert a significant influence on the system dynamics.

However, in practical engineering and physical scenarios, coupled oscillator networks usually possess a large system size, and the coupling of multiple oscillators can give rise to rich collective dynamical behaviors. Therefore, in the subsequent part of this paper, we extend the system size to $N = 100$ with the coupling range set to $P = 30$. We conduct an

in-depth investigation into the mechanisms underlying the rotation parameter β and the fractional-order derivative α , with the aim of comprehensively revealing the dynamical control mechanisms of the coupled Van der Pol oscillator system. In addition, it should be noted that for all spatiotemporal plots of the variables x_i , the transient behavior before $t = 180$ s has been truncated, and only the segment from $t = 180$ s to $t = 210$ s is presented. The snapshots capture the system states of the 100 oscillators at $t = 210$ s.

A. Effects of the rotation parameter on the dynamical behaviors of the system

In this subsection, we focus on analyzing the influence of the rotation parameter β (i.e., four distinct coupling types) on the dynamical states of the system.

We first investigate the effect of the rotation parameter β on the transition paths of the coupled Van der Pol oscillator system with a fractional-order derivative of $\alpha = 0.9$. Figure 2 presents a two-parameter diagram of the coupling strength σ versus the rotation parameter β , where different colors indicate distinct dynamical states of the system. It can be observed that when the rotation parameter takes values in different quadrants, the dynamical transition paths of the system differ as the coupling strength σ increases, and the steady states that the system finally approaches are also different. For instance, the system remains in synchronization when β is in the first quadrant. The system eventually reaches the amplitude death state when β is in the second or third quadrant, while it

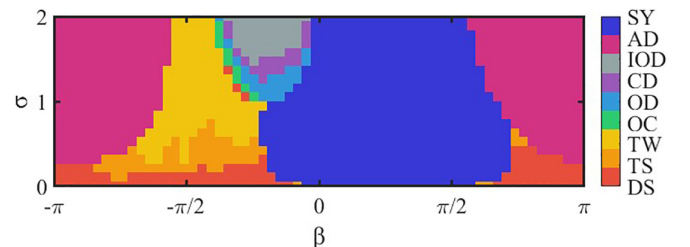


FIG. 2. The phase diagram for the coupling strength σ and the rotation parameter β . Here, the different dynamical behaviors are, respectively, SY, AD, IOD, CD, OD, OC, TW, TS, and DS. Other parameters: $N = 100$, $P = 30$, $\mu = 0.5$, and $\alpha = 0.9$.

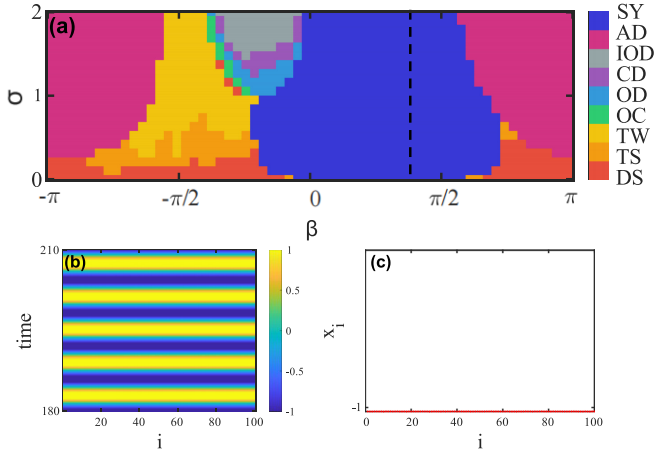


FIG. 3. Dynamical transition in the first quadrant: (a) The phase diagram for the coupling strength σ and the rotation parameter β at $\alpha = 0.9$. The black dashed line represents the transition path of the system at $\beta = 1.2$; (b) spatiotemporal plot and (c) snapshot of the oscillators at $\beta = 1.2$ and $\sigma = 1.0$. Other parameters: $N = 100$, $P = 30$, and $\mu = 0.5$.

ultimately evolves into the incoherent oscillation death state when β lies in the fourth quadrant.

To more clearly investigate the dynamical transition paths of the system in different quadrants, we select four values of the rotation parameter β from different quadrants based on Fig. 2, namely, 1.2 (first quadrant), 2.5 (second quadrant), -2.6 (third quadrant), and -0.8 (fourth quadrant), and observe the transition paths of the system with increasing coupling strength σ under these four values of the rotation parameter β .

For the dynamical transition in the first quadrant, the rotation parameter $\beta = 1.2$, and the system remains in synchronization for all values of the coupling strength σ . Specifically, the region corresponding to the black dashed line in Fig. 3(a) is uniformly dark blue. Figures 3(b) and 3(c) present the spatiotemporal plot and snapshot of the system at $\sigma = 1.0$ under this rotation parameter, where the phases and amplitudes of all oscillators are identical. In this case, β lies in the first quadrant, corresponding to positive-positive coupling for the system. That is, both the x and y components exhibit attractive coupling. Attractive coupling enhances the cooperativity between oscillators and suppresses individual differences. As the coupling strength σ increases, this attractive effect is amplified, and the states of all oscillators eventually converge to uniformity, forming synchronization.

For the dynamical transition in the second and third quadrants, the rotation parameter $\beta = 2.5$ or $\beta = -2.6$, and the system evolves from desynchronization to amplitude death as the coupling strength increases. The two black dashed lines in Fig. 4(a) represent the transition paths of the system for these two values of β , respectively. Figures 4(b1) and 4(b2) show the spatiotemporal plot and the snapshot of the system at $\beta = 2.5$ and $\sigma = 0.1$, where the system state is desynchronized. The oscillators exhibit distinct trajectories and their amplitudes differ significantly. Figure 4(c) shows the spatiotemporal plot for $\beta = 2.5$ and $\sigma = 2.0$, where the system is in the amplitude death state, and all oscillators are

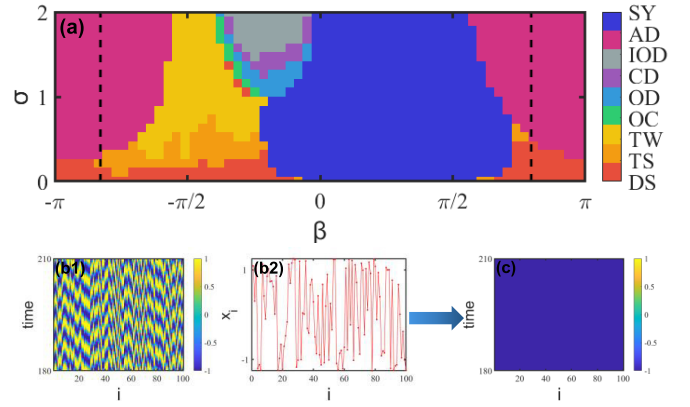


FIG. 4. Dynamical transition in the second and third quadrants: (a) The phase diagram for the coupling strength σ and the rotation parameter β at $\alpha = 0.9$. The black dashed lines represent the transition paths of the system at $\beta = 2.5$ and $\beta = -2.6$, respectively. (b1) Spatiotemporal plot and (b2) snapshot of the oscillators at $\beta = 2.5$ and $\sigma = 0.1$. (c) Spatiotemporal plot of the oscillators at $\beta = 2.5$ and $\sigma = 2.0$. Other parameters: $N = 100$, $P = 30$, and $\mu = 0.5$.

attracted to the same fixed point. When β is set to 2.5 in the second quadrant, the system exhibits negative-positive coupling. The x component is repulsive, while the y component is attractive. The repulsive coupling in the x direction directly breaks the nonlinear feedback loop that sustains oscillation, inducing a bifurcation from the limit cycle to the equilibrium point and ultimately leading to amplitude death. When β takes the value of -2.6 in the third quadrant, the system exhibits negative-negative coupling, with repulsive coupling in both the x and y directions, introducing an inhibitory effect. As the coupling strength σ increases, the damping effect intensifies, causing the oscillators' amplitudes to gradually decay until it eventually converges to the stationary equilibrium point, thereby reaching the amplitude death state.

For the dynamical transition in the fourth quadrant, the rotation parameter $\beta = -0.8$, and the general transition path of the system as the coupling strength increases is desynchronization $\rightarrow$ traveling wave $\rightarrow$ desynchronization $\rightarrow$ oscillatory cluster $\rightarrow$ coherent oscillation death $\rightarrow$ chimera death $\rightarrow$ incoherent oscillation death. The black dashed line in Fig. 5(a) denotes the transition path of the system in this case. It can be seen that the system finally evolves into the incoherent oscillation death, which is different from the cases where β takes values in other quadrants. When β takes the value of -0.8 in the fourth quadrant, the system exhibits positive-negative coupling, with attractive coupling in the x direction and repulsive coupling in the y direction. This coupling configuration enables the system to retain the oscillatory capability of individual oscillators while disrupting the synchronous correlation between them, ultimately leading to incoherent oscillation death.

To validate the above four transition paths, we plot the relationship between SI and σ for the four values of the rotation parameter, as shown in Fig. 6. When SI is 0, the system is in the coherent state; when SI is 1, the system is in the incoherent state; and when it lies between 0 and 1, the system is in the

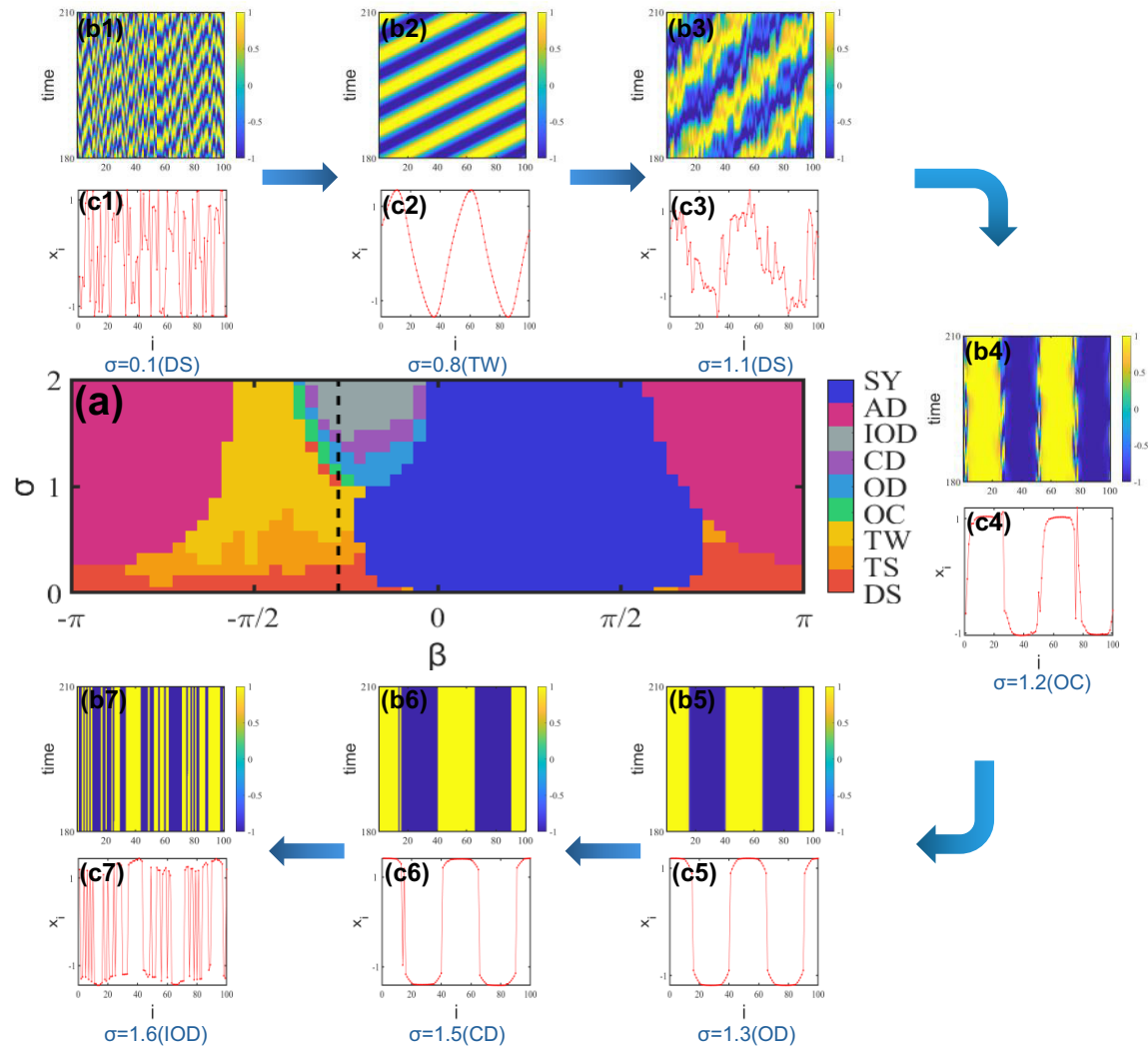


FIG. 5. Dynamical transition in the fourth quadrant: (a) The phase diagram for the coupling strength σ and the rotation parameter β at $\alpha = 0.9$. The black dashed line represents the transition path of the system at $\beta = -0.8$; (bi) spatiotemporal plots and (ci) snapshots of the oscillators at $\beta = -0.8$ under different coupling strengths σ . Other parameters: $N = 100$, $P = 30$, and $\mu = 0.5$.

chimera state. This relationship diagram further confirms the accuracy of the four transition paths described above.

Overall, β serves as a rotation parameter that uses trigonometric functions to distribute the coupling effect between the variables x and y , thereby precisely controlling the attractive and repulsive properties of coupling in each direction, as well as their relative weights. From Eq. (1) of the Van der Pol oscillator system, it can be seen that if repulsive coupling acts only on the y component, it only destroys the synchronization between oscillators without affecting the core structure of oscillation, leaving the system finally in the steady state of incoherent oscillation death. In contrast, if repulsive coupling acts on the x component, it directly breaks the core feedback loop sustaining oscillation, which not only disrupts synchronization but also induces a bifurcation from the limit cycle to an equilibrium point, driving the system into the completely quiescent steady state of amplitude death. This explains why the system reaches different steady states even though β lies in both the second and fourth quadrants, where the two components exhibit one attractive and one repulsive coupling in

both cases. This is because when the rotation parameter β is in the second quadrant, the x component exhibits repulsive coupling, which directly leads the system to amplitude death. In contrast, when β is in the fourth quadrant, the x component is attractive and the y component is repulsive, leaving the core oscillatory structure intact, so the system finally converges to incoherent oscillation death. Therefore, the rotation parameter β qualitatively determines the type of steady state, while the coupling strength σ quantitatively amplifies the coupling effect. The two parameters cooperatively regulate the synchronization, chimera states, and death behaviors of the fractional-order oscillatory network.

B. Effects of the fractional-order derivative on the dynamical behaviors of the system

Next, we investigate the influence of the fractional-order derivative α on the dynamical transition paths of the system. Similarly, we use the same four values of the rotation parameter from different quadrants as above and vary the values

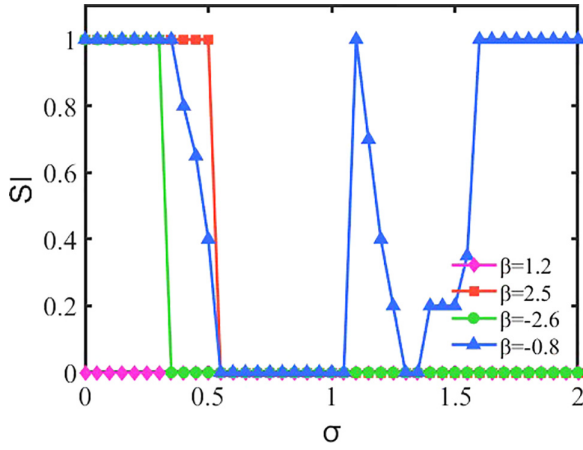


FIG. 6. The dependency of SI on the coupling strength σ for different values of the rotation parameter β . Other parameters: $N = 100$, $P = 30$, $\mu = 0.5$, and $\alpha = 0.9$.

of the fractional-order derivative to observe changes in the system's dynamical state. Here, four values of the fractional-order derivative are selected: 1, 0.95, 0.9, and 0.85.

When the rotation parameter takes the value $\beta = 1.2$ in the first quadrant, the system remains in synchronization regardless of the value of the fractional-order derivative, indicating that the synchronization force is absolutely dominant. When β takes the values 2.5 in the second quadrant and -2.6 in the third quadrant, the system still follows the transition path from desynchronization to amplitude death. However, the critical coupling strength for the system to reach amplitude death increases as the fractional order decreases. To verify this, the relationship between the strength of incoherence SI and the coupling strength σ is plotted for different fractional-order derivatives, as shown in Figs. 7(a) and 7(b). It can be seen that as α decreases from 1 to 0.85, the coupling strength corresponding to an SI value of 0 increases. In other words, the coupling strength required for the system to reach amplitude death becomes increasingly larger. This is because the formation of amplitude death depends on the instantaneous balance of self-sustained oscillations through coupled inhibition. The memory effect of the fractional-order derivative weakens the regulation efficiency of instantaneous coupling. Under the same coupling strength, the inhibitory effect cannot

accumulate rapidly enough to reach the bifurcation threshold of amplitude death, making it more difficult for the system to enter the amplitude death state.

When $\beta = -0.8$, changing the value of the fractional-order derivative α does not significantly alter the system's transition paths. To more clearly elucidate the influence of the fractional-order derivative, the two-parameter diagram of coupling strength σ versus fractional-order derivative α is plotted in Fig. 7(c). It can be seen that as α decreases, the coupling strength range for the initial occurrence of desynchronization becomes narrower, while the range corresponding to incoherent oscillation death becomes wider. The swing of desynchronization appears in the system when α exceeds 0.9, but this phenomenon vanishes once α drops below a certain threshold. Furthermore, in contrast to the second and third quadrants, the critical coupling strength for the system to reach incoherent oscillation death decreases as α decreases. This is because the bifurcation threshold of incoherent oscillation death is determined by the degree of mismatch accumulation rather than the instantaneous inhibition strength. The strong memory effect introduced by the fractional-order derivative accelerates the mismatch amplification effect, enabling the system to trigger coherence collapse at a lower coupling strength and thus enter the incoherent oscillation death state more rapidly. It can therefore be concluded that the memory effect of the fractional-order derivative can either delay or promote the coupling strength required for the system to reach its steady state.

C. Theoretical validation

To verify the accuracy of the system's transition paths in Fig. 2, we perform verification from an alternative perspective by fixing the coupling strength σ and varying the rotation parameter β , while applying the phase reduction theory. This subsection is based on the phase reduction theory proposed by Hiroya Nakao in 2016 [45] and draws on the analytical phase reduction method for weakly nonlinear oscillators, which he introduced in 2023 [46].

We fix $\sigma = 0.1$ and apply the phase reduction theory to verify the transition paths of the system as β varies from $-\pi$ to π . We apply this theory to the integer-order derivative case where $\alpha = 1$ (the transition paths for $\alpha = 0.9$ and $\alpha = 1$ are identical) and derive the phase model. Numerical simulations

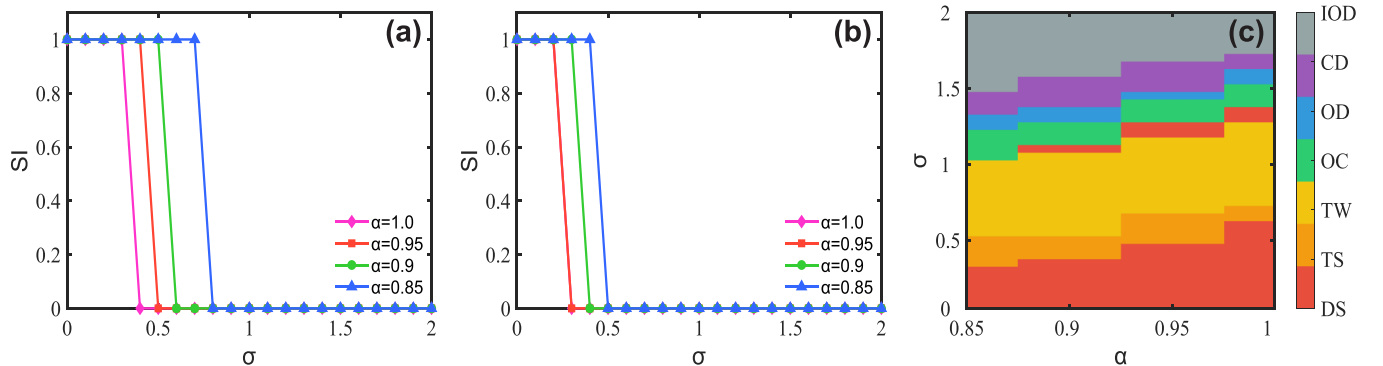


FIG. 7. The dependency of SI on the coupling strength σ for different values of α at (a) $\beta = 2.5$ and (b) $\beta = -2.6$; (c) $\beta = -0.8$: The phase diagram for the coupling strength σ and the fractional-order derivatives α . Other parameters: $N = 100$, $P = 30$, and $\mu = 0.5$.

of the phase model show that the transition process is consistent with the results obtained before applying the phase reduction theory. Therefore, considering $\alpha = 1$ and $\sigma = 0.1$, the system reduces to integer-order differential equations and the expression is

$$\begin{aligned}\frac{dx_j}{dt} &= y_j + \frac{\sigma \cos \beta}{2P} \sum_{k=j-P}^{j+P} (x_k - x_j), \\ \frac{dy_j}{dt} &= \mu(1 - x_j^2)y_j - x_j + \frac{\sigma \sin \beta}{2P} \sum_{k=j-P}^{j+P} (y_k - y_j).\end{aligned}\quad (5)$$

Define the Kuramoto-Daido order parameters

$$\begin{aligned}R_{1,j}e^{i\phi_{1,j}} &= \frac{1}{2P} \sum_{k=j-P}^{j+P} e^{i\theta_k}, \\ R_{3,j}e^{i\phi_{3,j}} &= \frac{1}{2P} \sum_{k=j-P}^{j+P} e^{i3\theta_k}.\end{aligned}\quad (6)$$

Consider a single Van der Pol oscillator, whose expressions are

$$\begin{aligned}\frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= \mu(1 - x^2)y - x.\end{aligned}\quad (7)$$

We use Poincaré-Lindstedt method [47–49] to derive analytical expressions for the frequency ω , the limit cycle $X_c = (x_c, y_c)$, and the phase sensitivity function $Z(\theta) = (Z_x(\theta), Z_y(\theta))$. Let $\theta = \omega t$ with ω unknown, and perform a power series expansion of μ in terms of ω , x_c , and y_c :

$$\begin{aligned}\omega &= \omega_0 + \mu\omega_1 + \mu^2\omega_2 + \dots, \\ x_c &= x_0 + \mu x_1 + \mu^2 x_2 + \dots, \\ y_c &= y_0 + \mu y_1 + \mu^2 y_2 + \dots.\end{aligned}\quad (8)$$

Substituting Eq. (8) into the system of Eq. (7) yields a set of ordinary differential equations at each order of μ .

The O(1) order ODE is given by

$$\begin{aligned}\omega_0 \frac{dx_0}{d\theta} &= y_0, \\ \omega_0 \frac{dy_0}{d\theta} &= -x_0.\end{aligned}\quad (9)$$

For $\mu = 0$, $\omega_0 = -1$ is determined by Eq. (7). Considering the initial condition on the limit cycle $y = 0$, $x = A_0 + \mu A_1 + \mu^2 A_2 + \dots$, where $A_0, A_1, \dots$ are constants to be determined later, the solution to Eq. (9) can be obtained as

$$x_0 = A_0 \cos \theta, \quad y_0 = A_0 \sin \theta, \quad (10)$$

where A_0 is unknown and can be determined by solving the $O(\mu)$ -order ordinary differential equations. The $O(\mu)$ -order ODE system is given by

$$\begin{aligned}\omega_0 \frac{dx_1}{d\theta} + \omega_1 \frac{dx_0}{d\theta} &= y_1, \\ \omega_0 \frac{dy_1}{d\theta} + \omega_1 \frac{dy_0}{d\theta} &= -x_1 + (1 - x_0^2)y_0.\end{aligned}\quad (11)$$

Solving the ODEs and eliminating secular terms yields $A_0 = 2$ and $\omega_1 = 0$. The solution to Eq. (11) is given by

$$\begin{aligned}x_1 &= A_1 \cos \theta - \frac{3}{4} \sin \theta + \frac{1}{4} \sin 3\theta, \\ y_1 &= A_1 \sin \theta + \frac{3}{4} \cos \theta - \frac{3}{4} \cos 3\theta.\end{aligned}\quad (12)$$

Continuing the process, the values of ω_2, ω_3 and A_1, A_2 can be determined by solving the $O(\mu^2)$ -order and $O(\mu^3)$ -order ordinary differential equations, thereby obtaining the expressions for x_1, y_1 and x_2, y_2 . Finally, we obtain $\omega_2 = \frac{1}{16}$, $\omega_3 = 0$,

$$\begin{aligned}x_1 &= -\frac{3}{4} \sin \theta + \frac{1}{4} \sin 3\theta, \\ y_1 &= \frac{3}{4} \cos \theta - \frac{3}{4} \cos 3\theta,\end{aligned}\quad (13)$$

and

$$\begin{aligned}x_2 &= -\frac{1}{8} \cos \theta + \frac{3}{16} \cos 3\theta - \frac{5}{96} \cos 5\theta, \\ y_2 &= -\frac{1}{4} \sin \theta + \frac{9}{16} \sin 3\theta - \frac{25}{96} \sin 5\theta.\end{aligned}\quad (14)$$

Thus, the expressions for the frequency ω and the limit cycle X_c are given by

$$\begin{aligned}\omega &= -1 + \frac{1}{16} \mu^2 + O(\mu^4), \\ x_c &= 2 \cos \theta - \frac{\mu}{4} (3 \sin \theta - \sin 3\theta) \\ &\quad - \frac{\mu^2}{8} \left(\cos \theta - \frac{3}{2} \cos 3\theta + \frac{5}{12} \cos 5\theta \right) + O(\mu^3), \\ y_c &= 2 \sin \theta + \frac{3}{4} \mu (\cos \theta - \cos 3\theta) \\ &\quad - \frac{\mu^2}{8} \left(2 \sin \theta - \frac{9}{2} \sin 3\theta + \frac{25}{12} \sin 5\theta \right) + O(\mu^3).\end{aligned}\quad (15)$$

Next, the expressions for the frequency ω and the limit cycle X_c are used to solve for the phase sensitivity function $Z(\theta)$. Let

$$\begin{aligned}\mathbf{F} &= \begin{pmatrix} y \\ -x + \mu(1 - x^2)y \end{pmatrix}, \\ \mathbf{J}(\mathbf{x}) &= \begin{pmatrix} 0 & 1 \\ -1 - 2\mu xy & \mu(1 - x^2) \end{pmatrix}.\end{aligned}$$

Similarly, using the Poincaré-Lindstedt method, we set

$$\begin{aligned}Z_x &= Z_{x_0} + \mu Z_{x_1} + \mu^2 Z_{x_2} + \dots, \\ Z_y &= Z_{y_0} + \mu Z_{y_1} + \mu^2 Z_{y_2} + \dots\end{aligned}\quad (16)$$

and substitute them into the adjoint equation and the normalization condition:

$$\begin{aligned}\omega \frac{dZ(\theta)}{d\theta} &= -\mathbf{J}^T(X_c(\theta))Z(\theta), \\ Z(\theta)\mathbf{F}(\theta) &= \omega.\end{aligned}\quad (17)$$

By this we can gain ordinary differential equations at each order of μ . Following the same procedure as for deriving the frequency and limit-cycle expressions, we finally obtain

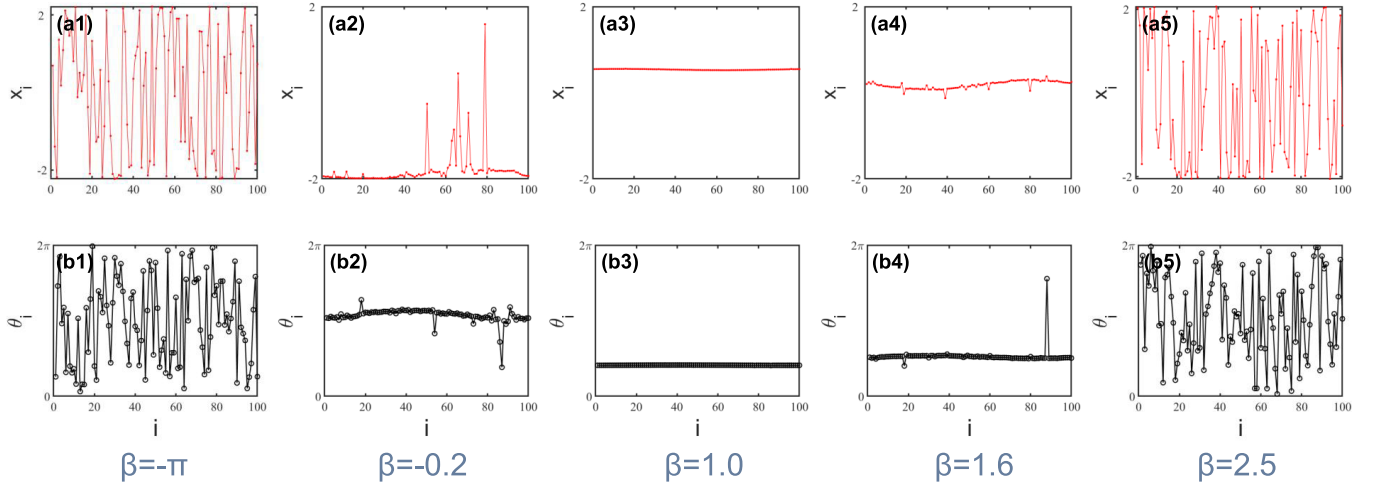


FIG. 8. Snapshots for (ai) original model and (bi) the phase model. Other parameters: $N = 100$, $P = 30$, and $\mu = 0.5$.

the phase sensitivity function as

$$\begin{aligned}
 Z_x(\theta) &= -\frac{1}{2} \sin \theta + \frac{\mu}{16} (15 \cos \theta + 5 \cos 3\theta) \\
 &\quad - \frac{\mu^2}{64} \left(21 \sin \theta + \sin 3\theta - \frac{29}{6} \sin 5\theta \right) + O(\mu^3), \\
 Z_y(\theta) &= \frac{1}{2} \cos \theta - \frac{\mu}{16} (\sin \theta + \sin 3\theta) \\
 &\quad + \frac{\mu^2}{64} \left(3 \cos \theta - 5 \cos 3\theta - \frac{1}{6} \cos 5\theta \right) + O(\mu^3).
 \end{aligned} \tag{18}$$

For the original equation (5), the perturbation is denoted as

$$\sigma p = \frac{\sigma}{2P} \sum_{k=j-P}^{j+P} \begin{pmatrix} (x_k - x_j) \cos \beta \\ (y_k - y_j) \sin \beta \end{pmatrix}^T.$$

Substituting the expressions of the frequency ω , the limit cycle X_c , the phase sensitivity function $Z(\theta)$, and the order parameter equation (6) into the phase reduction equation:

$$\frac{d\theta_j}{dt} = \omega + \sigma_p Z(\theta) p. \tag{19}$$

We perform averaging [50] to remove the fast-oscillating terms and obtain the phase model:

$$\begin{aligned}
 \frac{d\theta_j}{dt} &= -1 + \frac{\mu^2}{16} - \sigma \eta_{1c} + \sigma [\eta_{1s} R_{1,j} \sin(\phi_{1,j} - \theta_j) \\
 &\quad + \eta_{1c} R_{1,j} \cos(\phi_{1,j} - \theta_j) + \eta_{3s} R_{3,j} \sin(\phi_{3,j} - 3\theta_j)],
 \end{aligned} \tag{20}$$

where

$$\begin{aligned}
 \eta_{1c} &= \frac{\mu}{8} (\sin \beta + 9 \cos \beta), \\
 \eta_{1s} &= \frac{1}{2} \left[\left(1 + \frac{\mu^2}{64} \right) \sin \beta + \left(1 - \frac{7\mu^2}{64} \right) \cos \beta \right], \\
 \eta_{3s} &= \frac{\mu^2}{128} (5 \cos \beta - 3 \sin \beta).
 \end{aligned} \tag{21}$$

Numerical simulations of the phase equation (20) yield results that are consistent with those of the original equation (5).

The first row of Fig. 8 illustrates the dynamical transition process of the original system as the rotation parameter β increases at $\sigma = 0.1$, where the system undergoes sequential transitions from desynchronization, isolated state, synchronization, and imperfect synchronization to desynchronization again. The second row of Fig. 8 presents snapshots of the numerical simulation results for the phase model derived from the original system via the phase reduction theory. It can be observed that the transition paths of the phase model are consistent with those of the original system, which further verifies the accuracy of the identified transition routes of the system.

IV. CONCLUSIONS

In summary, this work investigated the dynamical behaviors of a fractional-order coupled Van der Pol oscillator system incorporating a rotation parameter, through which the sign of the coupling strength was governed. The primary focus was placed on the effects of the rotation parameter and the fractional-order derivative on the system's dynamical evolution. This work moved beyond the conventional frameworks of both simple fractional-order models and standard Van der Pol oscillators, revealing a distinctive dynamical mechanism that emerged from the coexistence of rotational coupling modulation and the memory effect inherent in the fractional-order derivative. The key findings were summarized as follows: The rotation parameter distributed the coupling interaction to the x and y variables via trigonometric functions, thereby precisely regulating both the sign and weight of the coupling strength in each direction. Moreover, the memory effect associated with the fractional-order derivative exhibited clear quadrant dependence. As the fractional-order derivative decreased, the onset of amplitude death was delayed when the rotation parameter resided in the second and third quadrants, whereas the transition to an incoherent oscillation death state was accelerated when the rotation parameter lies within the fourth quadrant. Finally, a phase model was derived using phase reduction theory, and numerical simulations based on this reduced model successfully verified the transition routes observed in the original coupled system. It should be noted

that due to the limitations of the relevant theories, we use integer-order approximations instead of fractional-order ones to derive the phase equation, thereby verifying the accuracy of the transition paths. These findings offered insight into the interplay between directional coupling and memory in fractional-order systems, with potential implications for the design of controllable oscillator networks.

ACKNOWLEDGMENTS

The authors would like to thank the anonymous referees for their efforts and valuable comments. This work was supported

by the National Natural Science Foundation of China (Grants No. 12302033 and No. 12272295) and the Fundamental Research Funds for the Central Universities, CHD (China, CN) (Grant No. 300102124206).

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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