

Triplet superconductivity supported by an X_9 high-order Van Hove singularity

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We study a fourfold symmetric dispersion relation of a quantum material, which exhibits a single high-order Van Hove singularity of X_9 type at the Fermi energy. First, we analyze in detail its form, type, and density of states when the energy dispersion is in its canonical form. Subsequently, we study the possibility of a superconducting state when Hubbard repulsive interactions are taken into account. By solving the gap equation, it is shown that triplet state superconductivity with power-law dependence of the critical temperature T_c on the interaction strength can be formed when a single singularity is present in the Brillouin zone. We discuss the effects of fluctuations and provide an upper bound of a possible superconducting critical temperature for the ruthenate $\text{Sr}_3\text{Ru}_2\text{O}_7$, which has been shown to exhibit this type of singularity.

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I. INTRODUCTION

The nontrivial geometry of the electronic band structure plays a crucial role in determining the physical properties of correlated electronic phases. Foundational work by Lifshitz opened the direction of understanding the consequences of the Fermi surface geometry and the associated topological transitions, such as the formation or collapse of pockets and necks [1,2]. These transitions can be induced by varying certain parameters in the system thereby offering greater control over the material properties. Typically, such transitions can be linked to the presence of critical points in the energy dispersion on the Fermi surface.

A critical point $\mathbf{k}_0$ in the energy dispersion $\varepsilon(\mathbf{k})$ is characterized by the vanishing Bloch velocity $\nabla\varepsilon(\mathbf{k}_0) = 0$ and typically corresponds to either a maximum, a minimum, or a saddle point. In two dimensions, the dispersion necessarily features a saddle point, leading to logarithmically divergent density of states (DOS), which defines a conventional Van Hove singularity (VHS) [3]. In this case, the saddle is a non-degenerate point as the determinant of the Hessian is nonzero $\det(\partial_{k_x}\partial_{k_y}\varepsilon(\mathbf{k})) \neq 0$. Therefore, the energy dispersion around this point can be expanded up to the second order, leading to a canonical form $\pm k_x^2 \mp k_y^2$. The conventional Lifshitz transitions and the associated VHS are reported in a plethora of materials including cuprates, Fe-based and ferromagnetic superconductors, heavy fermions, cobaltates, and ruthenates [4–11].

However, if the determinant of the Hessian also vanishes $\det(\partial_{k_x}\partial_{k_y}\varepsilon(\mathbf{k})) = 0$, the quadratic approximation fails, requiring an expansion of the dispersion beyond second order. Consequently, the critical points become degenerate and are referred to as higher-order critical points. As in the conventional case, the Fermi surface becomes singular and the corresponding DOS diverges, exhibiting a higher-order Van Hove singularity (HOVHS). However, HOVHSs are accompanied by a stronger power-law divergence with unique ratios for the prefactors of energy [12–14]. When the Fermi surface hosts HOVHSs, it can have profound consequences as they significantly amplify the electron correlations, potentially leading to interesting emergent phases. Experimentally, HOVHS can be detected as a direct signature in tunneling conductivity measurements, e.g., Refs. [15,16], and the divergent DOS can lead to unusual thermodynamic and transport properties [14].

With advances in experimental methods, HOVHS have attracted considerable attention in recent years [17–21]. Signatures of HOVHS have been detected in several materials. Notable examples include ruthenates, such as Sr_2RuO_4 , where an HOVHS has been observed at the surface [22], and $\text{Sr}_3\text{Ru}_2\text{O}_7$, where an HOVHS (X_9) can be induced by an external magnetic field [12,23]. Other candidate materials include overdoped graphene [24] and kagomé metals [25,26]. In twisted bilayer graphene [27,28], HOVHS are linked with the appearance of almost flat bands around the magic twist angle [15]. They might also play considerable role in the understanding of the phases observed in Bernal bilayer graphene [29,30]. The availability of the materials hosting HOVHSs, along with the ability to detect, characterize, and manipulate these singularities [22,31], makes this a promising research direction in controlling correlated behavior.

Our study focuses on the X_9 singularity, which is directly relevant to the benchmark quantum material $\text{Sr}_3\text{Ru}_2\text{O}_7$ [32–35]. X_9 is a fourth-order saddle of codimension 8, which is the singularity with the lowest codimension that respects fourfold symmetry. It does not fit into the classification

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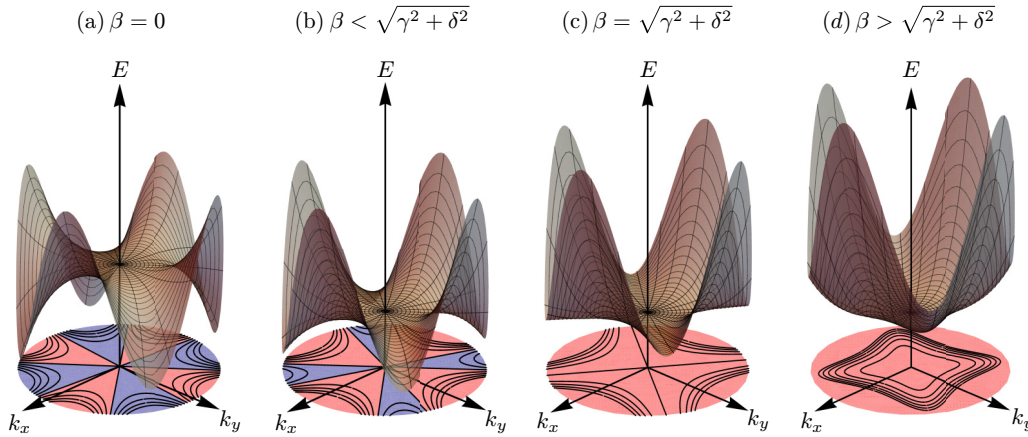


FIG. 1. The general C_4 symmetric X_9 quartic polynomial takes the form $k^4(\beta + \gamma \cos 4\theta + \delta \sin 4\theta)$ in polar coordinates. As $|\beta|$ increases from a value less than $\sqrt{\delta^2 + \gamma^2}$ to greater than it, we go from a saddle to a higher-order maximum/minimum (depending on the sign of β). The transition case is marked by (c) $|\beta| = \sqrt{\delta^2 + \gamma^2}$, wherein the dispersion tangentially intersects the k plane along a pair of straight lines through the origin, but does not change sign. Particle and hole sectors of the k plane are colored blue and red, respectively, and become equal in size in the particle-hole symmetric case, that is, when $\beta = 0$.

scheme of the seventeen based on Thom's theorem of the Catastrophe theory [12], it is unimodal as we will explain below.

We first present the X_9 singularity and analyze its range of influence in real systems. Then, we investigate the problem of the superconducting pairing in the presence of a single X_9 singularity at the Fermi surface in a system with nominally weak repulsive interactions, where we show that triplet superconductivity is possible and derive an expression for the critical temperature. We discuss the effects of fluctuations and the condition to stabilize a triplet superconducting state. We then provide an estimate of T_c for the case of $\text{Sr}_3\text{Ru}_2\text{O}_7$. The remainder of the paper is organized as follows: In Sec. II, we analyze the singularity in a general framework, followed by a detailed study of superconducting pairing in Sec. III, and we conclude in Sec. IV.

II. ANALYSIS OF THE SINGULARITY

A. Form and type of the singularity

We assume a fourfold rotational symmetric dispersion expanded to quartic order that need not have $k_x \leftrightarrow k_y$ reflection symmetry. This takes the form

$$E(\mathbf{k}) = ak^2 + b(k_x^4 + k_y^4) + c(k_x^2 k_y^2) + d(k_x^3 k_y - k_x k_y^3). \quad (1)$$

Let us initially set the quadratic part to zero, i.e., $a = 0$. It is easy to see that the d term breaks the $k_x \leftrightarrow k_y$ reflection symmetry while it preserves C_4 rotation taking the form $(k_x, k_y) \rightarrow (k_y, -k_x)$. Equivalently, the quartic part of the dispersion takes in polar coordinates the form $k^4(\beta + \gamma \cos 4\theta + \delta \sin 4\theta)$ with

$$\beta = \frac{6b + c}{8}, \quad (2a)$$

$$\gamma = \frac{2b - c}{8}, \quad (2b)$$

$$\delta = \frac{d}{4}. \quad (2c)$$

Regarding the conditions for the singularity to be a higher-order saddle, we expect $E(\mathbf{k}) = 0$ to correspond to a set of curves through the origin (rather than being a set that contains just the origin). The curves precisely contain the points where the dispersion changes sign in one direction. Since $k^4 \geq 0$, we need to examine the equation $\beta + \gamma \cos 4\theta + \delta \sin 4\theta = 0$. For a saddle, δ and γ cannot both be simultaneously zero since we will have a trivial maximum or minimum depending on the sign of β . With this constraint, the conditions can be rewritten as

$$\beta + \sqrt{\gamma^2 + \delta^2} \cos(4\theta - \phi_0) = 0, \quad (3)$$

where $\phi_0 \in [-\pi, \pi)$ is the angle for which $\cos \phi_0 = \gamma / \sqrt{\gamma^2 + \delta^2}$ and $\sin \phi_0 = \delta / \sqrt{\gamma^2 + \delta^2}$. There is a unique value of ϕ_0 in $[-\pi, \pi)$ and other values are of the form $2m\pi + \phi_0$ for integer m . Therefore,

$$\phi_0 = \begin{cases} 2 \tan^{-1} \left(\frac{-\gamma + \sqrt{\delta^2 + \gamma^2}}{\delta} \right), & \delta \neq 0, \\ -\pi, & \delta = 0 \text{ and } \gamma < 0, \\ 0, & \delta = 0 \text{ and } \gamma > 0. \end{cases} \quad (4)$$

Equation (3) has solution only when $|\beta| \leq \sqrt{\gamma^2 + \delta^2}$, which is also the condition for the dispersion to become a saddle (when the inequality is strict). The solutions of Eq. (3) for unique values of θ modulo 2π read

$$\theta_{n,\pm} = \frac{n\pi}{2} \pm \frac{1}{4} \cos^{-1} \left(\frac{-\beta}{\sqrt{\gamma^2 + \delta^2}} \right) + \frac{\phi_0}{4}, \quad (5)$$

where $n \in \{0, 1, 2, 3\}$. These solutions for θ represent four straight lines through the origin (each $\theta_{n,\pm}$ for $n \leq 1$ and its $n + 2$ partner given by $\theta_{n+2,\pm} = \theta_{n,\pm} + \pi$, respectively, constitute the two opposite directions of a straight line emanating from the origin).

There is a saddle when $|\beta| < \sqrt{\gamma^2 + \delta^2}$, a higher-order maximum/minimum that tangentially intersects the $k_x k_y$ plane along two straight lines through the origin when $|\beta| =$

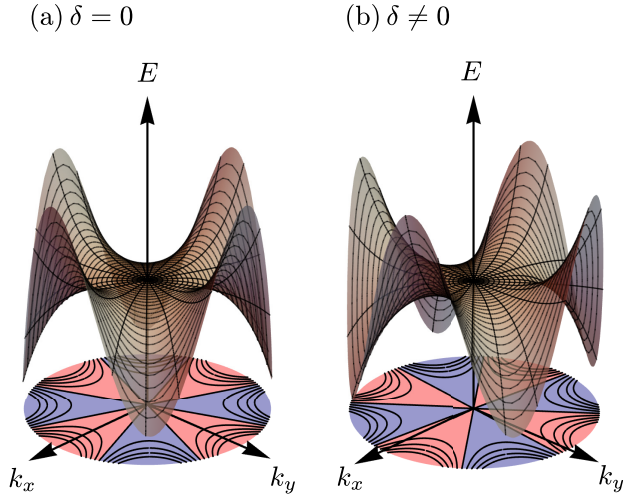


FIG. 2. The δ term in $k^4(\beta + \gamma \cos 4\theta + \delta \sin 4\theta)$ breaks the $k_x \leftrightarrow k_y$ reflection symmetry. This causes a rotation of the saddle and its contours with respect to the k_x and k_y axes when $\delta \neq 0$.

$\sqrt{\gamma^2 + \delta^2}$ and a higher-order maximum/minimum when $|\beta| > \sqrt{\gamma^2 + \delta^2}$. The three cases are illustrated in Fig. 1.

When the $k_x \leftrightarrow k_y$ symmetry is broken, the constant energy contours of the saddle need not be oriented symmetrically with respect to the k_x and k_y axes. In fact, they will be rotated by an angle with respect to the coordinate axes due to the ϕ_0 correction appearing in the polar form of the dispersion [that is, in $\cos(4\theta - \phi_0)$]. We begin by assuming that $|\beta| < \sqrt{\gamma^2 + \delta^2}$, which guarantees a saddle. Let us first consider the case when $\delta = 0$, which ensures that the terms breaking $k_x \leftrightarrow k_y$ reflection are killed off, and $\phi_0 = 0$ or π . We can easily check that the zero-energy contours are straight lines that are rotated by $\pm\theta$ about the k_x and k_y axes, where $\theta = \cos^{-1}(-\beta/\gamma)$. In this situation, the $k_x \leftrightarrow k_y$ reflection is respected. When $\delta \neq 0$, we still have something analogous, with the difference that the entire system is rotated by an angle of $\phi_0/4$. This breaks the $k_x \leftrightarrow k_y$ symmetry, as illustrated in Fig. 2.

B. Determinacy and codimension of the X_9 singularity

First, for clarity, we define the two concepts. Determinacy is the largest power of the lowest-order Taylor expansion that describes the singularity. Codimension is the number of missing directions in polynomial space under any coordinate transformation. For the canonical form of the X_9 singularity, which is $\pm(k_x^4 + c k_x^2 k_y^2 + k_y^4)$, the condition $|c| \neq 2$ ensures a finite determinacy equal to 4 and a finite codimension equal to 8 [36]. The finite determinacy is needed to ensure that we can truncate the Taylor expansion to a finite degree (that is, at least as large as the determinacy). This allows us to then extract the DOS from the low energy dispersion [12]. To analyze the general, C_4 symmetric form that we have been working with [defined in Eq. (1) with a set to zero], we begin with the polar form $k^4(\beta + \sqrt{\gamma^2 + \delta^2} \cos(4\theta - \phi_0))$. For $\beta = 0$, by a simple rotation of the coordinate system by $\phi_0/4$, we can put the dispersion in the form $\sqrt{\gamma^2 + \delta^2}(k_x^4 - 6k_x^2 k_y^2 + k_y^4)$. This

is a particle-hole symmetric saddle with finite determinacy and codimension.

To investigate the case $\beta \neq 0$, we first rotate the (k_x, k_y) coordinate system with the transformation $\theta \rightarrow \theta + (\phi_0 + \Theta(-\beta)\pi)/4$, where $\Theta(x)$ is the Heaviside step function that is zero for negative x and unity for positive x . The dispersion then reads

$$\begin{aligned} E(\mathbf{k}) &= \text{sgn}(\beta) k^4 (|\beta| + \sqrt{\gamma^2 + \delta^2} \cos 4\theta) \\ &= \text{sgn}(\beta) [(|\beta| + \sqrt{\gamma^2 + \delta^2})(k_x^4 + k_y^4) \\ &\quad + (2|\beta| - 6\sqrt{\gamma^2 + \delta^2}) k_x^2 k_y^2]. \end{aligned} \quad (6)$$

By rescaling k by $(|\beta| + \sqrt{\gamma^2 + \delta^2})^{1/4}$, the dispersion in the canonical form becomes

$$E(\mathbf{k}) = \text{sgn}(\beta) \left[k_x^4 + k_y^4 + \left(\frac{2|\beta| - 6\sqrt{\gamma^2 + \delta^2}}{|\beta| + \sqrt{\gamma^2 + \delta^2}} \right) k_x^2 k_y^2 \right]. \quad (7)$$

Thus, the condition for finite determinacy and codimension is

$$\frac{|2|\beta| - 6\sqrt{\gamma^2 + \delta^2}|}{|\beta| + \sqrt{\gamma^2 + \delta^2}} \neq 2. \quad (8)$$

This automatically rules out the previously discussed case of $|\beta| = \sqrt{\gamma^2 + \delta^2}$, which separates the maxima/minima like dispersions from saddlelike ones. For a saddle with $|\beta| < \sqrt{\gamma^2 + \delta^2}$, it is easy to see that the above equation is always satisfied and this guarantees finite determinacy and codimension. Likewise, for $|\beta| > \sqrt{\gamma^2 + \delta^2}$, we will either get the trivial relation $\sqrt{\gamma^2 + \delta^2} = 0$ or the contradiction $|\beta| = \sqrt{\gamma^2 + \delta^2}$, when we set the left-hand side to equal to 2. The former corresponds to the condition $\delta = \gamma = 0$. Therefore, the prohibited cases are $|\beta| = \sqrt{\gamma^2 + \delta^2}$ and $\gamma = \delta = 0$. For all other values of β , γ , and δ , we have a finitely determinate singularity with a finite codimension that is either a higher-order maximum/minimum or a higher-order saddle, depending on the magnitude of β in comparison to $\sqrt{\delta^2 + \gamma^2}$.

It is important to note that a nonzero β breaks particle-hole symmetry as can be seen in Fig. 1.

C. Density of states in the pristine X_9 saddle

We start by analyzing the pristine, C_4 -symmetric X_9 saddle. Note that the analysis of the maximum/minimum scenario is not necessary since it is easy to argue for a power-law DOS using scaling arguments and there is no scope for a universal ratio of prefactors since the DOS is identically zero above/below the maximum/minimum energy. Since the DOS integrals are not affected by rotating the coordinate system, we assume that we have already rotated the (k_x, k_y) coordinate system with the transformation $\theta \rightarrow \theta + (\phi_0 + \Theta(-\beta)\pi)/4$ as before, to obtain the following dispersion relation:

$$\begin{aligned} E(\mathbf{k}) &= \text{sgn}(\beta) k^4 (|\beta| + \sqrt{\gamma^2 + \delta^2} \cos(4\theta)) \\ &= \text{sgn}(\beta) k^4 \sqrt{\gamma^2 + \delta^2} (\cos \eta + \cos(4\theta)), \end{aligned} \quad (9)$$

where $\eta \in (0, \pi/2]$ is a parameter that characterizes the particle-hole asymmetry. When $\eta = \pi/2$, the particle and hole sectors of the $\mathbf{k}$ plane are symmetric, while in the $\eta \rightarrow 0$ limit,

one of the sectors constricts to a set of lines rather than a two-dimensional region. To avoid cumbersome notation, let us assume $\text{sgn}(\beta) = 1$. When it is -1 , we just have to swap the

analysis of the DOS for positive and negative energies worked out below. Up to some constant prefactors, the DOS is given by

$$\nu(\epsilon) = \frac{1}{(2\pi)^2} \int_0^{2\pi} d\theta \int_0^\Lambda dk k \delta(\epsilon - k^4 \sqrt{\gamma^2 + \delta^2} (\cos \eta + \cos(4\theta))). \quad (10)$$

We evaluate $\nu(\epsilon)$ for both $\epsilon > 0$ and $\epsilon < 0$ cases. Assuming that the cutoff Λ is large enough, we set it to infinity without affecting the nature of the infrared divergence. After simplifying the integral, the expression for $\nu(\epsilon)$ in the $\epsilon > 0$ case reads

$$\begin{aligned} \nu(\epsilon) &= \frac{1}{(2\pi)^2} \int_0^{2\pi} d\theta \int_0^\Lambda dk k \frac{\delta(k - (\frac{\epsilon}{\sqrt{\gamma^2 + \delta^2} (\cos \eta + \cos(4\theta))})^{1/4})}{4k^3 \sqrt{\gamma^2 + \delta^2} (\cos \eta + \cos(4\theta))} \Theta(\cos \eta + \cos(4\theta)) \\ &= \frac{|\epsilon|^{-1/2}}{16\pi^2 (\gamma^2 + \delta^2)^{1/4}} \int_0^{2\pi} d\theta \frac{\Theta(\cos \eta + \cos(4\theta))}{\sqrt{\cos \eta + \cos(4\theta)}}, \quad \text{for } \epsilon > 0. \end{aligned} \quad (11)$$

The details of the calculation are in Appendix A. If we proceed similarly for $\epsilon < 0$, then the ratio of the prefactors of $|\epsilon|^{-1/2}$ can be expressed in a closed form in terms of the generalized hypergeometric function ${}_pF_q$ and the elliptic integral of the first kind K :

$$\frac{D_+(\eta)}{D_-(\eta)} = \frac{\sqrt{2\pi} \eta^{-1/2} (4\eta^2 {}_3F_2(\frac{1}{2}, \frac{1}{2}, 1; \frac{3}{4}, \frac{5}{4}; \eta^2) - {}_3F_2(\frac{3}{4}, 1, \frac{5}{4}; \frac{3}{2}, \frac{3}{2}; \frac{1}{\eta^2}) + 2\eta^{\frac{3}{2}} \frac{1}{\sqrt{1+\eta}} K(\frac{2}{1+\eta}))}{\Gamma(\frac{1}{4})^2 \eta {}_2F_1(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}; \eta^2) + 8\Gamma(\frac{3}{4})^2 ((1-\eta^2) {}_2F_1(\frac{3}{4}, \frac{3}{4}, -\frac{1}{2}; \eta^2) + (3\eta^2 - 1) {}_2F_1(\frac{3}{4}, \frac{3}{4}, \frac{1}{2}; \eta^2))}. \quad (12)$$

The limiting value of the ratio is 1 as $\eta \rightarrow \pi/2$ and ∞ as $\eta \rightarrow 0$, which is the expected behavior since $\eta = \pi/2$ gives the particle-hole symmetric saddle and $\eta \rightarrow 0$ gives a higher-order maximum that tangentially intersects the k plane (as discussed earlier), which corresponds to a vanishing D_- . The variation of the ratio with η is shown in Fig. 3.

We also find that by adding a small quadratic term, thus being off the critical tuning, we are still able to recover power-law scaling of the DOS in a suitable energy range. This analysis is in Appendix B.

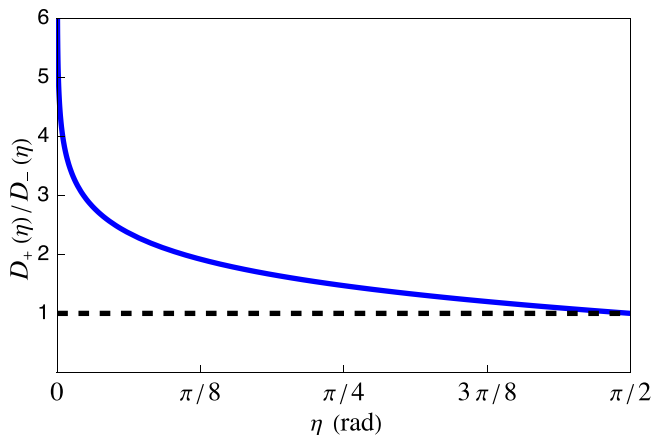


FIG. 3. The ratio of prefactors in the power-law DOS of an X_9 saddle depends on the ratio $|\beta|/\sqrt{\delta^2 + \gamma^2}$, which we define to be $\cos \eta$. When $\eta \rightarrow 0$, we expect the particle sector to shrink to zero size, which should give a diverging ratio of prefactors. By contrast, when $\eta \rightarrow \pi/2$, we get a particle-hole symmetric saddle that has a unit ratio of prefactors.

III. SUPERCONDUCTIVITY

We now study the superconducting pairing instability at a single X_9 Van Hove singularity at the Fermi surface. We consider a Hubbard model of interacting electrons with a nominally weak repulsive interaction U ,

$$\mathcal{H} = \sum_{k\alpha} \varepsilon_k c_{k\alpha}^\dagger c_{k\alpha} + \frac{U}{N} \sum_{kk'q} c_{k+q,\uparrow}^\dagger c_{k'-q,\downarrow}^\dagger c_{k',\downarrow} c_{k,\uparrow}, \quad (13)$$

where ε_k is the energy dispersion around the X_9 point and $c_{k,\alpha}^\dagger$ is a fermion creation operator for momentum k and spin α . Further, it is assumed that the singularity is at the Fermi surface and all variables are measured from the singular point. We choose to work with the fully symmetric form of the dispersion and use the plane polar coordinates $\varepsilon(k) \propto k^4 \cos(4\theta)$ throughout our analysis.

As our system involves a bare repulsive electron-electron interaction, we study the possibility of pairing driven by the Kohn-Luttinger (KL) type mechanism [37–42]. In the KL mechanism, screening of the bare repulsion U can lead to the emergence of effective attractive channels. This process is enhanced by divergent DOS. We adopt the method developed in Ref. [39] where the authors went beyond the one-loop renormalization group (RG) and study the possibility of a superconducting state by solving self-consistently the gap equation. The reason for the need to go beyond one-loop is that due to the power-law divergence of the DOS, the corrections to RG are of the same order as the leading terms. It is worth mentioning that a Stoner analysis would suggest a ferromagnetic (Stoner) instability always, but the question is to explore possible solution of a triplet superconductivity. As a result, solving directly the gap equation and determining T_c , is a necessary step forward. This is a mean-field treatment that provides a mean-field T_c and this is an important point for the discussion about fluctuations later. A general recipe though

for the analysis of potential pairing as mediated by repulsive interactions is to consider the irreducible pairing vertex Γ . This irreducible pairing vertex is the antisymmetrized interaction with zero total incoming and outgoing momenta, dressed by the renormalizations, which in second order in U is given by

$$\Gamma_{\alpha\beta\gamma\nu}(\mathbf{k}; \mathbf{p}) = U(\delta_{\alpha\gamma}\delta_{\beta\nu} - \delta_{\alpha\nu}\delta_{\beta\gamma}) + U^2\Pi_{\text{ph}}(\mathbf{k} + \mathbf{p})\delta_{\alpha\gamma}\delta_{\beta\nu} - U^2\Pi_{\text{ph}}(\mathbf{k} - \mathbf{p})\delta_{\alpha\nu}\delta_{\beta\gamma}, \quad (14)$$

where $\Pi_{\text{ph}}(\mathbf{k})$ is the static particle-hole polarization bubble.

Beyond the random phase approximation technique, the most relevant contributions are crossed diagrams (diagrams with interactions that cross each other), these are represented by insertions of particle-particle bubbles into the particle-hole channel. As previous work [39] shows, the restriction to second order in U of the vertex Γ although seems questionable, is justified because the typical momenta $\mathbf{k}$ and $\mathbf{p}$, which are responsible for pairing, are comparable to the cutoff Λ . For such momenta, the effect of the crossed diagrams on the vertex, leading to its suppression, is small and therefore negligible. In addition, we consider that we work far from another (Stoner) instability. To analyze the pairing problem, we employ the linearized gap equation with the vertex Γ replacing the bare interaction U . Hence, the gap equation is given by

$$\Delta_{\alpha\beta}(\mathbf{k}) = - \int d^2p \frac{\tanh\left(\frac{\epsilon_p}{2(1+Z_p)T}\right)}{2\epsilon_p(1+Z_p)} \Gamma_{\alpha\gamma\beta\nu}(\mathbf{k}; \mathbf{p}) \Delta_{\gamma\nu}(\mathbf{p}), \quad (15)$$

where Z_p is the inverse quasiparticle weight that allows us to consider the self-energy corrections consistently through the relation as $\Sigma_p = i\omega Z_p$.

The gap equation can be further decomposed into spin-singlet and spin-triplet channels,

$$\Delta_{\alpha\beta}(\mathbf{k}) = \Delta_s(\mathbf{k})i\sigma^y_{\alpha\beta} + \Delta(\mathbf{k}) \cdot (i\sigma^y\vec{\sigma})_{\alpha\beta}. \quad (16)$$

It is obvious from Eq. (14) that the interaction is repulsive in the singlet channel and hence the associated pairing instability can be ruled out. Therefore, we search for a potential solution to the gap equation in the triplet channel. The equation in the triplet channel becomes

$$\Delta_i(\mathbf{k}) = -U^2 \int d^2p \frac{\tanh\left(\frac{\epsilon_p}{2(1+Z_p)T}\right)}{2\epsilon_p(1+Z_p)} \times [\Pi_{\text{ph}}(\mathbf{k} + \mathbf{p}) - \Pi_{\text{ph}}(\mathbf{k} - \mathbf{p})] \Delta_i(\mathbf{p}), \quad (17)$$

where the spin indices can be dropped in this case. The aim is to investigate whether the gap equation can lead to a nontrivial solution with a finite critical temperature T_c , which is the central result of this section.

Next, we compute the two key ingredients for this calculation, which are the static particle-hole bubble $\Pi_{\text{ph}}(\mathbf{q}, 0)$ and the inverse quasiparticle weight Z_p at the X_9 point.

A. Static and dynamic particle-hole bubble

The particle-hole bubble at finite momentum and frequency transfer that is needed for the self-energy calculation

reads

$$\Pi_{\text{ph}}(\mathbf{q}, \omega_m) = \frac{1}{4\pi^2} \int d^2k \frac{f(\epsilon_{k+q}) - f(\epsilon_k)}{i\omega_m + \epsilon_{k+q} - \epsilon_k}, \quad (18)$$

where f is the Fermi function. In the limit of zero temperature and no frequency transfer, Eq. (18) reduces to

$$\begin{aligned} \Pi_{\text{ph}}(\mathbf{q}, 0) &= \frac{1}{4\pi^2} \int d^2k \delta(\epsilon(\mathbf{k})) \\ &= \frac{1}{4\pi^2} \int_q^\Lambda dk k \int_0^{2\pi} d\theta \delta(k^4 \cos(4\theta)), \end{aligned} \quad (19)$$

where the cutoff q captures the small- and finite- q behavior of the bubble and Λ is the upper cutoff for the problem. Using the standard properties of the delta function, the bubble is reexpressed as

$$\begin{aligned} \Pi_{\text{ph}}(\mathbf{q}, 0) &= \frac{1}{4\pi^2} \int_q^\Lambda dk \frac{k}{|k^4|} \int_0^{2\pi} d\theta \sum_{n=0}^7 \frac{\delta(\theta - \theta_n)}{|-4 \sin(4\theta_n)|}, \\ \text{s.t. } \epsilon(k, \theta_n) &= 0, \end{aligned} \quad (20)$$

where n runs from 0 to 7 and θ_n corresponds to zero dispersion lines in the patch. The final expression for the bubble becomes

$$\Pi_{\text{ph}}(\mathbf{q}, \omega_m = 0) = \frac{1}{4\pi^2} \left(\frac{1}{q^2} - \frac{1}{\Lambda^2} \right). \quad (21)$$

As a caveat, we note that the polarization bubble is computed without accounting for the angular dependence. However, this approximation correctly captures the leading power of q , and including the complete angular dependence would not alter the result of our analysis in the following.

To make further progress with the dynamical polarization bubble $\Pi_{\text{ph}}(\mathbf{q}, \omega_m)$, it is enough to calculate it (and subsequently the self-energy) to the leading, linear order in ω as justified due to the weak interaction U . Therefore, Eq. (18) can be reexpressed as

$$\Pi_{\text{ph}}(\mathbf{q}, \omega_m) = \frac{1}{4\pi^2} \int d^2k \frac{\delta(\epsilon_k)}{\left(1 + \frac{i\omega_m}{\epsilon_{k+q} - \epsilon_k}\right)}, \quad (22)$$

and perturbatively expand in small parameter $\omega_m/(\epsilon_{k+q} - \epsilon_k)$. To linear order in ω_m , the expression reads

$$\begin{aligned} \Pi_{\text{ph}}(\mathbf{q}, \omega_m) &\approx \frac{1}{4\pi^2} \int d^2k \delta(\epsilon_k) \left(1 - \frac{i\omega_m}{\epsilon_{k+q} - \epsilon_k} \right) \\ &= \Pi_{\text{ph}}(\mathbf{q}, 0) + \Pi_{\text{ph}}^D(\mathbf{q}, \omega_m), \end{aligned} \quad (23)$$

where $\Pi_{\text{ph}}(\mathbf{q}, 0)$ is the static bubble given by Eq. (21) and

$$\Pi_{\text{ph}}^D(\mathbf{q}, \omega_m) \equiv \frac{-i\omega_m}{4\pi^2} \int d^2k \frac{\delta(\epsilon_k)}{\epsilon_{k+q} - \epsilon_k}. \quad (24)$$

The details of the calculations are presented in Appendix C. Here, we present the final result that is needed:

$$\Pi_{\text{ph}}(\mathbf{q}, \omega_m) = \frac{1}{4\pi^2 q^2} \left(1 + \frac{i\omega_m}{32q^4} \sum_{n=0}^7 (-1)^n \cot(\theta_n - \alpha) \right). \quad (25)$$

B. Self-energy

We are now in a position to calculate the self-energy:

$$\Sigma(\omega) \sim U^2 \int d^2q \int d\Omega \Pi(\mathbf{q}, \Omega) G(\mathbf{k} + \mathbf{q}, \Omega + \omega). \quad (26)$$

The integration variable Ω runs over the entire frequency domain, but a fixed momentum cutoff $q_{\min}$ is imposed, determined by the external frequency.

As we are primarily interested in the self-energy dependence on momentum scales rather than its detailed structure, we assume that the interacting momentum q is above a threshold $q_{\min}$ that allows us to treat the particle-hole bubble as static [39]. This minimum momentum $q_{\min}$ can be estimated by the inequality

$$|\omega| < \frac{24q^4}{\left| \sum_{n=0}^7 (-1)^n \cot(\theta_n - \alpha) \right|}. \quad (27)$$

We note that $\cot(\theta_n - \alpha) \neq 0$ in its region of validity. However, there is a competing constraint [Eq. (C19)] that needs to be satisfied simultaneously. For completeness, we rewrite the conditions to be satisfied:

$$0 < |\omega| < 4\Lambda^3 q |\sin(\theta_n - \alpha)|, \quad (28a)$$

$$0 < |\omega| < \frac{24q^4}{\left| \sum_{n=0}^7 (-1)^n \cot(\theta_n - \alpha) \right|}. \quad (28b)$$

The minimum momentum $q_{\min}$ is obtained when we are closer to one of θ_n , where $\sin(x) \sim x$ and $\cot(x) \sim 1/x$. Therefore, to satisfy both constraints simultaneously, the lower limit

can be set as

$$q_{\min} \sim |\omega|^{1/4}. \quad (29)$$

By fixing the momentum scales, $q > q_{\min} \sim |\omega|^{1/4}$, the dynamic part of the bubble becomes negligible and $\Pi_{\text{ph}}(q, \Omega) \approx \Pi_{\text{ph}}(q, 0)$ holds. Therefore, the self-energy reads

$$\Sigma(\omega) = \frac{U^2}{4\pi^2} \int dq \frac{q}{q^2} \int d\alpha \int \frac{d\Omega}{(i\Omega + i\omega - \varepsilon_{\mathbf{k}+\mathbf{q}})}. \quad (30)$$

The details of the calculation are in Appendix D. The final result is

$$\Sigma(\omega) = i|\omega| \left(\frac{2U^2}{q^4} \ln \left(\frac{\Lambda}{q} \right) \right). \quad (31)$$

The quasiparticle weight now reads

$$Z_q = g \left(\frac{\Lambda}{q} \right)^4 \ln \left(\frac{\Lambda}{q} \right), \quad (32)$$

where $g \sim U^2 \Lambda^{-4} / A^2$ is the dimensionless coupling constant and A is related to the dispersion $\varepsilon_k = Ak^4 \cos(4\theta)$. The constant g depends on the momentum upper cutoff explicitly. The value of A is taken to be 1 without loss of generality throughout the calculations.

C. Gap equation

Now, the central question of this work regarding the possibility of a superconducting state with a finite critical temperature can be analyzed. The gap equation, including the self-energy, is written as

$$\Delta(\mathbf{k}) = \frac{U^2}{A^2} \int_0^\Lambda dp p \int_0^{2\pi} d\theta_p \frac{\tanh \left(A \frac{p^4 \cos(4\theta_p)}{2T_c} \right)}{2p^4 \cos(4\theta_p) \left[1 + g \left(\frac{\Lambda}{p} \right)^4 \ln \left(\frac{\Lambda}{p} \right) \right]} \Delta(\mathbf{p}) \times \frac{4kp \cos(\theta_p - \theta_k)}{k^4 + p^4 - 2k^2 p^2 \cos(2\theta_p - 2\theta_k)}, \quad (33)$$

where prefactor A is explicitly retained to keep track of the dimensions.

In principle, the gap can be a combination of all lattice harmonics that are consistent with the odd parity of the gap equation $\Delta(k, \theta_k + \pi) = -\Delta(k, \theta_k)$. However, it is not possible to analyze every channel analytically as the channels become linearly dependent. To gain insight, we analyze the gap equation in two limiting regimes $k \ll p_0$ and $k \gg p_0$, where $p_0 \sim (2T_c/A)^{1/4}$ is a natural momentum scale in the problem due to the presence of the tanh function. In the limit $k \ll p_0$, we expand the integrand of Eq. (33) in k/p , which is a small parameter. We retain only the term $\frac{1}{p^4}$, as it is the leading contribution that can give rise to an attractive interaction. In this regime, the gap is of the form $\Delta(\mathbf{k}) \sim k \cos \theta_k$ or $\Delta(\mathbf{k}) \sim k \sin \theta_k$. In the other limit $k \gg p_0$, we get a decaying form $\Delta(\mathbf{k}) \sim \frac{1}{k^3} \cos \theta_k$ or $\Delta(\mathbf{k}) \sim \frac{1}{k^3} \sin \theta_k$, suggesting a nonmonotonic momentum dependence.

In the limit $k \ll p_0$, the solution $\Delta(\mathbf{k}) \sim k \cos \theta_k$ or $\Delta(\mathbf{k}) \sim k \sin \theta_k$ belongs to the two-dimensional E representation of C_{4v} group, which is the lattice version of p -wave symmetry. In principle, the gap can be a complex combination of both elements of $E(k_x, k_y)$, resulting in chiral p -wave order, which preserves C_4 -rotational symmetry. Alternatively,

the system can choose one of the directions, spontaneously breaking C_4 symmetry to C_2 , leading to nematic order.

For cases discussed above, the equation for T_c reads

$$1 = \frac{2U^2}{A^2} \int_0^\Lambda dp p \int_0^{2\pi} d\theta_p \frac{\tanh \left(A \frac{p^4 \cos(4\theta_p)}{2T_c} \right)}{p^5 \cos(4\theta_p) \left[1 + g \left(\frac{\Lambda}{p} \right)^4 \ln \left(\frac{\Lambda}{p} \right) \right]} \times \cos^2(\theta_p). \quad (34)$$

To extract the dependence of T_c on the coupling constant g or U , we rescale the momentum variables as $x = p \left(\frac{\Lambda}{2T_c} \right)^{1/4}$, which renders the integral dimensionless, and sets the upper cutoff to $\tilde{\Lambda} = \Lambda \left(\frac{\Lambda}{2T_c} \right)^{1/4}$. Thus, the equation for T_c becomes

$$T_c = \frac{U^2}{A} \int_0^{\tilde{\Lambda}} dx \int_0^{2\pi} d\theta_p \frac{\tanh(x^4 \cos(4\theta_p))}{\cos(4\theta_p)} \times \frac{\cos^2(\theta_p)}{x^5 + \frac{U^2}{AT_c} x \ln \left(\frac{\tilde{\Lambda}}{x} \right)}. \quad (35)$$

Before proceeding to the numerical solution, some comments are in order. Regarding the convergence of the integral, in the IR region $\tanh(x^4) \approx x^4$ and the integrand roughly behaves

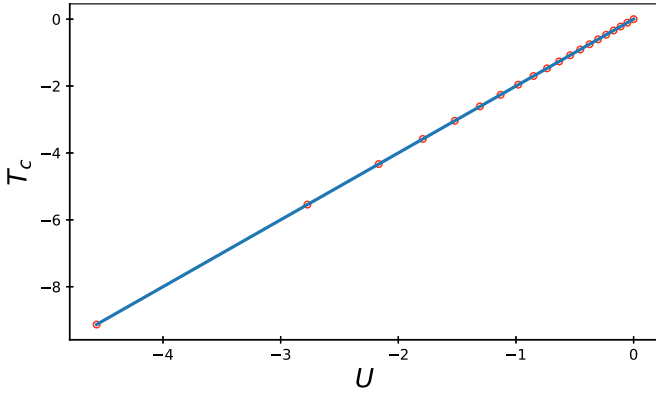


FIG. 4. Log-log plot of normalized T_c vs normalized U . The slope of the linear fit is 2, indicating quadratic dependence of T_c on U . U ranges from 10^{-4} to 10^{-2} eV with A fixed at 0.1 eV.

as $x^3 / \ln(\frac{\tilde{\Lambda}}{x})$, which tends to zero in the limit $x \rightarrow 0$. In the UV region, $\tanh(\Lambda^4) \rightarrow 1$ and the integrand behaves as $1/\Lambda^5$, which is negligible. Additionally, the integrand changes sharply near $\theta_n = \frac{(2n+1)\pi}{8}$. To make the numerical integration accurate, we partition the domain into eight equal parts, evaluate the contribution in each domain (from θ_n to θ_{n+1}) and then add all the contributions. This means that Eq. (35) can be written as

$$T_c = \frac{U^2}{A} \sum_{n=0}^7 \int_0^{\tilde{\Lambda}} dx \int_{\theta_n}^{\theta_{n+1}} d\theta_p \frac{\tanh(x^4 \cos(4\theta_p))}{\cos(4\theta_p)} \times \frac{\cos^2(\theta_p)}{x^5 + \frac{U^2}{AT_c} x \ln\left(\frac{\tilde{\Lambda}}{x}\right)}. \quad (36)$$

We expect T_c to scale as

$$T_c \sim b U^\beta, \quad (37)$$

where b is a constant with the appropriate dimensions. This form of T_c is justified because when the interaction is switched off adiabatically, T_c must decrease and vanish in the limit $U \rightarrow 0$.

Equation (36) provides a self-consistent relation for T_c . This equation can be solved numerically using an iterative procedure for representative values of U (0.1–10 meV) and A (100 meV). The resulting dependence of T_c on U is found to be to a good extent quadratic, as illustrated in Fig. 4. This is because the self-energy correction contributes only over a very small range, whereas the dominant contribution to the integral is independent of the self-energy. Consequently, the U^2 dependence of T_c is preserved. However, the self-energy naturally regularizes the integral.

D. Role of fluctuations

Regarding the role of fluctuations in connection to low dimensionality, it is clear that in purely two dimensions we expect a Berezinskii-Kosterlitz-Thouless (BKT) transition due to proliferation of vortices with algebraic order and finite superfluid stiffness. As discussed in Ref. [43], the BKT transition temperature is then very close to the transition temperature we get from the mean-field solutions of the gap equations. The reason is that the stiffness of the BKT

transition is much smaller than the low temperature stiffness of a weakly coupled superconductor so that the vortex proliferation starts only very near to the mean-field T_c . The crucial point though is that this BKT physics holds for a triplet superconductor only in the presence of spin-orbit interaction; otherwise, fluctuations destroy the mean-field solution and the BKT transition.

IV. DISCUSSION

In this work, we have analyzed in detail the X_9 singularity of a two-dimensional quantum material, which is the singularity of lowest codimension that is allowed by $\pi/2$ rotation. In the catastrophe's theory classification scheme, this singularity has corank 2, codimension 8, determinacy 4, and winding number 8. Symmetry analysis suggests that the X_9 singularity can be realized in the band structure of a variety of lattices at high-symmetry points. For example, at the Γ and M points of square lattices (symmetry groups C_4 and D_4). There is strong evidence that it governs the physics of the correlated material $\text{Sr}_3\text{Ru}_2\text{O}_7$ as thermodynamic measurements suggest. Therefore, it presents a prime example of an HOVHS to be investigated.

We first showed that the power-law scaling of the DOS can be detected in the vicinity of the critical point even if there is a small quadratic term $\propto k^2$ in the dispersion relation. This is also consistent with the study of HOVHS in the presence of disorder [44]. We then took into account a Hubbard repulsive interaction in the weak-coupling regime, to study possible superconducting states formed in the presence of a single singularity at the Fermi surface. The central conclusion is that a triplet superconducting state can be formed, with a critical temperature that scales as a power law of the interaction strength $T_c \propto U^2$ in the weak-coupling limit. Note that this exponent agrees with the exponent found in the numerical calculations of Ref. [39] for a different dispersion relation that provides though the same exponent, $-1/2$, in the divergence of the DOS. This means that the effect of the self-energy corrections is similar in both cases. It is worth emphasizing that the pairing problem has been analyzed for a single singularity. If more singularities are present within the Brillouin zone, there is always the possibility of the development of density waves [18,45].

Regarding $\text{Sr}_3\text{Ru}_2\text{O}_7$, where superconductivity has not been observed yet, first we note that the tuning to the X_9 singularity was achieved through an applied magnetic field as the tuning parameter [23,32–35]. This point, in addition to the fact that there is an appreciable spin-orbit interaction in the material, makes possible the development of superconductivity and reduces the effect of the fluctuations. As a result, only a triplet superconductivity can be a solution, as we have shown in the present work.

By assuming a value of $U \approx 1.5$ meV and $A \approx 0.1$ eV, consistent with the previous studies on $\text{Sr}_3\text{Ru}_2\text{O}_7$ [23,46,47], we obtain a value of $T_c \approx 40$ mK. In the expansion of the energy dispersion relation of the material in the vicinity of the point in the Brillouin zone where the X_9 is [23], there is an additional small quadratic term, which makes the dispersion deviate from the particle-hole symmetry and the canonical form of the X_9 singularity. As a result, due to the fact that the estimate of the T_c is made at the particle-hole symmetric point

as well as the experimental conditions we have described, we can only set an upper bound on the superconducting T_c through the mean-field treatment. It is evident that a possible superconducting state is detectable only at ultralow temperatures. In the case of $\text{Sr}_3\text{Ru}_2\text{O}_7$, there is also the possibility of nesting within the Brillouin zone. Experimental results have shown that at higher temperatures the formation of a spin-density wave with a characteristic incommensurate wave vector [23,48], which connects different parts of the Fermi surface. Therefore, the estimate of the upper bound of T_c is consistent in this case.

As mentioned above, the calculation refers to the situation where a single Van Hove point is present. More broadly, the work serves as an example for other materials that can possess an HOVHS, and especially the fourfold symmetric X_9 , in their band structure.

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DATA AVAILABILITY

The code that was used in this article is available upon reasonable request.

APPENDIX A: DETAILS FOR THE CALCULATION OF DOS FOR THE PRISTINE CASE

Starting from Eq. (12) for $\epsilon > 0$, the integral can be represented in terms of elliptic integrals if the limits of integration are restricted carefully using the step function. In an analogous way, the integral for $\epsilon < 0$ also reads

$$v(\epsilon) = \frac{|\epsilon|^{-1/2}}{16\pi^2(\gamma^2 + \delta^2)^{1/4}} \int_0^{2\pi} d\theta \frac{\Theta(-\cos\eta - \cos(4\theta))}{\sqrt{-\cos\eta - \cos(4\theta)}}. \quad (\text{A1})$$

$$v(\epsilon) = \frac{1}{(2\pi)^2} \int_0^{2\pi} d\theta \int_0^{\Lambda} dk k \delta(\epsilon - \alpha k^2 - \text{sgn}(\beta) k^4 \sqrt{\gamma^2 + \delta^2} (\cos\eta + \cos(4\theta))). \quad (\text{B1})$$

We have to analyze only two distinct cases with $\text{sgn}(\alpha)\text{sgn}(\beta) = 1$ and $\text{sgn}(\alpha)\text{sgn}(\beta) = -1$. Let us assume $\text{sgn}(\beta) = 1$, so that we have to worry only about $\text{sgn}(\alpha)$ and $\text{sgn}(\epsilon)$. We can simplify the integral somewhat by the transformation $t = k^2$. Also notice the $\pi/2$ periodicity in θ of the integrand, which allows us to restrict the limits of the θ integral to $(0, \pi/2)$, picking up an overall factor of 4. This then allows us to use the substitution $v = 4\theta$, yielding a simplified version

$$v(\epsilon) = \frac{1}{2(2\pi)^2} \int_0^{2\pi} dv \int_0^{\Lambda^2} dt \delta(\epsilon - \alpha t - t^2 \sqrt{\gamma^2 + \delta^2} (\cos\eta + \cos v)). \quad (\text{B2})$$

Let us define the following dimensionless ratio:

$$r = \left(\frac{E_s}{|\epsilon|} \right)^{1/2} = \left(\frac{\alpha^2}{|\epsilon| \sqrt{\gamma^2 + \delta^2}} \right)^{1/2}. \quad (\text{B3})$$

The integrands in both cases are periodic with a period equal to $\pi/2$. Solving the equation $\cos\eta + \cos(4\theta) = 0$, we obtain

$$\theta = \frac{(2m+1)\pi}{4} \pm \frac{\eta}{4}, \quad m \in \mathbb{Z}, \quad (\text{A2})$$

with the only distinct solutions for θ relevant being $\pi/4 \pm \eta/4$ as the others lie outside the integration range of $[0, \pi/2]$ since $\eta \in (0, \pi/2]$ leading to

$$0 < \frac{\pi}{4} - \frac{\eta}{4} < \frac{\pi}{4} + \frac{\eta}{4} < \frac{\pi}{2}. \quad (\text{A3})$$

Thus, to compute the effect of the step functions in the integrals, we just have to examine the sign of $\cos\eta + \cos 4\theta$ in each of the regions defined by the inequality above. For this, we just need to look at the sign of the derivative of the expression, given by $-4 \sin 4\theta$. The derivative is negative at $\theta = \pi/4 - \eta/4$ and positive at $\theta = \pi/4 + \eta/4$. Therefore, we have

$$\begin{aligned} \text{sgn}(\cos\eta + \cos 4\theta) \\ = \begin{cases} 1, & \theta \in [0, \pi/4 - \eta/4) \cup (\pi/4 + \eta/4, \pi/2], \\ -1, & \theta \in (\pi/4 - \eta/4, \pi/4 + \eta/4). \end{cases} \end{aligned} \quad (\text{A4})$$

Finally, we can evaluate the integrals with the restricted limits in terms of elliptic integrals F and K to get the ratio of prefactors [Eq. (14)].

APPENDIX B: DOS WITH THE QUADRATIC TERM

We will now include a “small” quadratic term and investigate its effect on the DOS. We will have to find some scales in the problem to quantify this smallness. A natural and obvious energy scale in the problem is $E_s := \alpha^2 / \sqrt{\gamma^2 + \delta^2} = 8a^2 / \sqrt{(2b-c)^2 + 4d^2}$, which in essence compares the quadratic and quartic coefficients. We will now show that for $\epsilon \ll E_s$ the logarithmic divergence dominates, while for $\epsilon \gg E_s$, the power-law regime with $-1/2$ exponent and η -dependent ratio of prefactors dominates:

Our regime of interest is the “high-energy” limit, that is, when $r \rightarrow 0$ or equivalently $|\epsilon| \gg E_s$. In this scenario, we can rescale the t integral using the coordinate transformation $t = u |\epsilon|^{1/2}/(\gamma^2 + \delta^2)^{1/4}$ to obtain

$$v(\epsilon) = \frac{|\epsilon|^{-1/2}}{8\pi^2(\gamma^2 + \delta^2)^{1/4}} \int_0^{2\pi} dv \int_0^R du \delta(\text{sgn}(\epsilon) - r \text{sgn}(\alpha) u - u^2(\cos \eta + \cos v)), \quad (\text{B4})$$

where we have defined another dimensionless ratio appearing in the upper limit of the u integral

$$R = \frac{\Lambda^2(\gamma^2 + \delta^2)^{1/4}}{|\epsilon|^{1/2}} = \left(\frac{\sqrt{\gamma^2 + \delta^2} \Lambda^4}{|\epsilon|} \right)^{1/2}. \quad (\text{B5})$$

The condition that R is much larger than 1 would be tantamount to saying that the energy of interest, ϵ is well within the range where the quartic term adequately describes the dispersion. This is so because the cutoff Λ is chosen to restrict the DOS integration domain in momentum space to the region where the quartic term is important. Therefore, $\sqrt{\gamma^2 + \delta^2} \Lambda^4$ precisely provides the energy equivalent of the momentum cutoff. Finally, when $E_s \ll \epsilon \ll \sqrt{\gamma^2 + \delta^2} \Lambda^4$, we can ignore the small terms and obtain a first approximation

$$v(\epsilon) \approx \frac{|\epsilon|^{-1/2}}{8\pi^2(\gamma^2 + \delta^2)^{1/4}} \int_0^{2\pi} dv \int_0^\infty du \delta(\text{sgn}(\epsilon) - u^2(\cos \eta + \cos v)). \quad (\text{B6})$$

This has the form already treated in Appendix A when we dealt with the DOS for the pure saddle. As stated in the main text, even if we are off the critical tuning, we will still be able to recover power-law scaling of the DOS in a suitable energy range.

APPENDIX C: DETAILS OF THE DYNAMIC PARTICLE-HOLE BUBBLE CALCULATION

In order to obtain a form for self-energy, we need to know the dynamical polarization bubble $\Pi_{\text{ph}}(\mathbf{q}, \omega_m)$. We restrict our calculations in obtaining the bubble and self-energy to linear order in ω , which is justified due to the weak interaction U . Therefore, the knowledge of the leading behavior of the polarization bubble is enough. The full bubble given in Eq. (18) can be reexpressed as

$$\Pi_{\text{ph}}(\mathbf{q}, \omega_m) = \frac{1}{4\pi^2} \int d^2k \delta(\epsilon_k) \left(\frac{\epsilon_{k+q} - \epsilon_k}{i\omega_m + \epsilon_{k+q} - \epsilon_k} \right). \quad (\text{C1})$$

To obtain the bubble to linear order in frequency, we consider a small but finite frequency, i.e., $|\omega_m| \leq |\epsilon_{k+q} - \epsilon_k| \neq 0$. This approximation allows to make a perturbative expansion in the small parameter $\omega_m/(\epsilon_{k+q} - \epsilon_k)$:

$$\Pi_{\text{ph}}(\mathbf{q}, \omega_m) = \frac{1}{4\pi^2} \int d^2k \frac{\delta(\epsilon_k)}{(1 + \frac{i\omega_m}{\epsilon_{k+q} - \epsilon_k})}. \quad (\text{C2})$$

To linear order in ω_m , the expression reads

$$\Pi_{\text{ph}}(\mathbf{q}, \omega_m) \approx \frac{1}{4\pi^2} \int d^2k \delta(\epsilon_k) \left(1 - \frac{i\omega_m}{\epsilon_{k+q} - \epsilon_k} \right) = \Pi_{\text{ph}}(\mathbf{q}, 0) + \Pi_{\text{ph}}^D(\mathbf{q}, \omega_m), \quad (\text{C3})$$

where $\Pi_{\text{ph}}(\mathbf{q}, 0)$ is the static bubble given by Eq. (21) and

$$\Pi_{\text{ph}}^D(\mathbf{q}, \omega_m) \equiv \frac{-i\omega_m}{4\pi^2} \int d^2k \frac{\delta(\epsilon_k)}{\epsilon_{k+q} - \epsilon_k}. \quad (\text{C4})$$

1. Linear in $\mathbf{q}$

To compute the dynamic bubble, we examine Eq. (C4) and expand the denominator to linear order in $\mathbf{q}$ (since $\mathbf{q}$ is assumed to be a small momentum relative to $\mathbf{k}$). Let θ and α be the angles made by $\mathbf{k}$ and $\mathbf{q}$, respectively. Then, the relative angle between interacting momenta is $\phi = \theta - \alpha$. Resolving $\mathbf{q}$ into components along $\hat{k}$ and $\hat{\theta}$, we obtain the required expression

$$\epsilon_{k+q} - \epsilon_k = 4k^3q(\cos(4\theta)\cos(\theta - \alpha) + \sin(4\theta)\sin(\theta - \alpha)). \quad (\text{C5})$$

Evaluating the dynamic bubble, we find that

$$\Pi_{\text{ph}}^D(\mathbf{q}, \omega_m) = \frac{-i\omega_m}{4\pi^2} \int_q^\Lambda dk k \int_0^{2\pi} d\theta \frac{\delta(k^4 \cos(4\theta))}{4k^3q(\cos(4\theta)\cos(\theta - \alpha) + \sin(4\theta)\sin(\theta - \alpha))}. \quad (\text{C6})$$

Using the properties of delta function,

$$\Pi_{\text{ph}}^D(\mathbf{q}, \omega_m) = \frac{-i\omega_m}{16\pi^2} \int_q^\Lambda dk \frac{k^{-6}}{q} \underbrace{\int_0^{2\pi} d\theta \sum_{n=0}^7 \frac{\delta(\theta - \theta_n)/| -4 \sin(4\theta_n)|}{(\cos(4\theta) \cos(\theta - \alpha) + \sin(4\theta) \sin(\theta - \alpha))}}_I. \quad (\text{C7})$$

Performing the θ -integration, I can be evaluated to

$$I = \frac{1}{4} \sum_{n=0}^7 \frac{(-1)^n}{\sin(\theta_n - \alpha)}, \quad (\text{C8})$$

where $\cos(4\theta_n) = 0$ and $\sin(4\theta_n) = (-1)^n$ are used. Expanding explicitly, it is easy to see that $I = 0$ due to pairwise cancellations. The integral I is divergent for $\alpha = \theta_n$; however, the condition imposed $|\omega_m| < \varepsilon_{k+q} - \varepsilon_k \neq 0$ ensures that $\alpha \neq \theta_n$. Therefore, the value of the integral $I = 0$ for any general α . This calculation implies that $\Pi_{\text{ph}}^D(\mathbf{q}, \omega_m) = 0$ to linear order in ω_m , when the dispersion ε_{k+q} is expanded only to first order in $\mathbf{q}$ and θ_{k+q} . As a further step, we include the second-order terms in the expansion of ε_{k+q} .

2. Quadratic in $\mathbf{q}$

To second order in $\mathbf{q}$, the Taylor expansion of ε_{k+q} about $\mathbf{k}$ expressed in terms of gradient and Hessian is of the form

$$\varepsilon_{k+q} = \varepsilon_k + \mathbf{q} \cdot \nabla \varepsilon_k + \frac{1}{2} \mathbf{q}^T \mathbf{H} \mathbf{q} + \mathcal{O}(q^3). \quad (\text{C9})$$

After a straightforward evaluation,

$$\varepsilon_{k+q} - \varepsilon_k \approx 4k^3 q (\cos(4\theta) \cos(\phi) + \sin(4\theta) \sin(\phi)) + 12k^2 q^2 \sin(4\theta) \sin(\phi) \cos(\phi) + 6k^2 q^2 \cos(4\theta) \cos(2\phi). \quad (\text{C10})$$

The terms involving $\cos(4\theta)$ will vanish upon θ -integration and therefore are omitted from subsequent calculations. Retaining only the nonzero terms, we get

$$\begin{aligned} \Pi_{\text{ph}}^D(\mathbf{q}, \omega_m) &= \frac{-i\omega_m}{4\pi^2} \int_q^\Lambda dk k \int_0^{2\pi} d\theta \frac{\delta(k^4 \cos(4\theta))}{4k^3 q \sin(4\theta) \sin(\theta - \alpha) [1 + 3\frac{q}{k} \cos(\theta - \alpha)]} \\ &= \frac{-i\omega_m}{16\pi^2} \int_q^\Lambda dk \frac{k^{-6}}{q} \underbrace{\int_0^{2\pi} d\theta \sum_{n=0}^7 \frac{\delta(\theta - \theta_n)/| -4 \sin(\theta_n)|}{\sin(4\theta) \sin(\theta - \alpha) [1 + 3\frac{q}{k} \cos(\theta - \alpha)]}}_J. \end{aligned} \quad (\text{C11})$$

After the θ -integration, the result for J is

$$J = \frac{1}{4} \sum_{n=0}^7 \frac{1}{\sin(4\theta_n) \sin(\theta_n - \alpha) [1 + 3\frac{q}{k} \cos(\theta_n - \alpha)]}. \quad (\text{C12})$$

To analyze the nature of J , we can expand the denominator in powers of $\frac{q}{k}$, which is a small parameter. To $\mathcal{O}(q/k)$,

$$J = \frac{1}{4} \sum_{n=0}^7 \left[\frac{(-1)^n}{\sin(\theta_n - \alpha)} - \frac{3q}{k} \frac{(-1)^n \cos(\theta_n - \alpha)}{\sin(\theta_n - \alpha)} \right]. \quad (\text{C13})$$

The first term of Eq. (C13) was shown to vanish previously. Upon further expansion of J , the terms with even powers of $\frac{q}{k}$ go to zero and those of odd powers have the following structure:

$$\left(\frac{q}{k}\right)^{(2n+1)} \cot(\theta_n - \alpha) \cos^{2n}(\theta_n - \alpha). \quad (\text{C14})$$

Around $\alpha \approx \theta_n$, which is the region of our interest, J behaves as $\cot(\theta_n - \alpha)$. Therefore, it is sufficient to retain only the $\mathcal{O}(q)$ to describe J . Substituting J in Eq. (C11), we get

$$\Pi_{\text{ph}}^D(\mathbf{q}, \omega_m) = \frac{i\omega_m}{16\pi^2} \frac{3}{4} \int_q^\Lambda dk k^{-7} \sum_{n=0}^7 (-1)^n \cot(\theta_n - \alpha), \quad (\text{C15})$$

$$\Pi_{\text{ph}}^D(\mathbf{q}, \omega_m) = \frac{i\omega_m}{128\pi^2} \left[\frac{1}{q^6} - \frac{1}{\Lambda^6} \right] \sum_{n=0}^7 (-1)^n \cot(\theta_n - \alpha). \quad (\text{C16})$$

Equation (C16) is valid only in the regions that satisfy $|\omega| < |\varepsilon_{k+q} - \varepsilon_k|$, which has been assumed throughout the derivations. This limits the accessibility of regions just around θ_n . To impose this constraint, we examine the dispersion at θ_n :

$$\varepsilon_{k+q} - \varepsilon_k \approx 4k^3 q \sin(4\theta_n) \sin(\phi) + 16k^2 q^2 \sin(4\theta_n) \sin(\phi) \cos(\phi), \quad (\text{C17})$$

which should satisfy $|\omega| < |\varepsilon_{k+q} - \varepsilon_k|$ or equivalently

$$|\omega| < 4k^3 q \sin(\theta_n - \alpha) \left[1 + 4 \frac{q}{k} \cos(\theta_n - \alpha) \right], \quad (\text{C18})$$

and it is assumed that $q \ll k$, this allows us to estimate the constraint around the UV cutoff $k \sim \Lambda$, implying

$$|\sin(\theta_n - \alpha)| > \left(\frac{|\omega|}{4\Lambda^3 q} \right), \quad (\text{C19})$$

which should be satisfied for all θ_n . This puts a restriction on α in determining how close it can be to θ_n (zero-dispersion lines). In other words, the angle between the interacting momenta must be greater than the cutoff imposed in Eq. (C19) to be consistent with the assumptions made to truncate the expansion of $\Pi_{\text{ph}}(\mathbf{q}, \omega)$ to linear ω .

Ignoring the term Λ^{-6} in comparison with q^{-6} in Eq. (C16), the full particle-hole bubble will be

$$\Pi_{\text{ph}}(\mathbf{q}, \omega_m) = \frac{1}{4\pi^2 q^2} \left(1 + \frac{i\omega}{32q^4} \sum_{n=0}^7 (-1)^n \cot(\theta_n - \alpha) \right). \quad (\text{C20})$$

APPENDIX D: DETAILS OF THE SELF-ENERGY CALCULATION

The integration over Ω can be done using contour techniques, the function has a simple pole at $\Omega = -\omega - i\varepsilon_{k+q}$. So, constructing a contour that comprises the real axis and a semicircular arc C_R in the lower half plane enclosing the pole, we can evaluate the integral as

$$\oint \frac{d\Omega}{(i\Omega + i\omega - \varepsilon_{k+q})} = \int_{-\infty}^{\infty} \frac{d\Omega}{(i\Omega + i\omega - \varepsilon_{k+q})} + \int_{C_R} \frac{d\Omega}{(i\Omega + i\omega - \varepsilon_{k+q})} = 2\pi i \text{Res}(\Omega = -\omega - i\varepsilon_{k+q}). \quad (\text{D1})$$

The integral around C_R approaches π , which is a real constant and will be ignored as we are looking for a ω -dependent self-energy $\Sigma(\omega) = i\omega Z_q$, where Z_q is the inverse quasiparticle weight. As a result, the self-energy is

$$\Sigma(\omega) = \frac{iU^2}{2\pi} \int dq \frac{1}{q} \int d\alpha. \quad (\text{D2})$$

The domain for the α -integration is not continuous; there are forbidden regions imposed by the constraint [Eq. (C19)]. Let $d \equiv \arcsin\left(\frac{|\omega|}{4\Lambda^3 q}\right)$ and the constraint $(|\theta_n - \alpha|) > d$ has to be satisfied simultaneously by all θ_n in $(0, \pi)$. Therefore, the valid region of integration will be

$$D = \bigcup_{n=0}^3 (\theta_n + d, \theta_{n+1} - d), \quad (\text{D3})$$

where θ_n 's are ordered and the region from π to 2π provides the same contribution. Thus, the α -integration can be evaluated to

$$\int d\alpha = 2[\alpha]_D = 2\pi - 16d = 2\pi - 16 \arcsin\left(\frac{|\omega|}{4\Lambda^3 q}\right), \quad (\text{D4})$$

where the factor of 2 is multiplied to account for the region $(\pi, 2\pi)$:

$$\Sigma(\omega) = \frac{2iU^2}{\pi} \int_{q_{\min}}^{\Lambda} dq \frac{1}{q} \left(\pi - 8 \arcsin\left(\frac{|\omega|}{4\Lambda^3 q}\right) \right), \quad (\text{D5})$$

within the range of the integration, $|\omega| \ll 4\Lambda^3 q$, which allows us to approximate the arcsin function to linear order of its argument:

$$\Sigma(\omega) \approx \frac{2iU^2}{\pi} \int_{|\omega|^{\frac{1}{4}}}^{\Lambda} dq \left(\frac{\pi}{q} - \frac{2|\omega|}{\Lambda^3 q^2} \right), \quad (\text{D6})$$

$$\Sigma(\omega) = iU^2 \left(\frac{1}{2} \ln\left(\frac{\Lambda^4}{|\omega|}\right) + \frac{4|\omega|}{\pi \Lambda^3 |\omega|^{\frac{1}{4}}} \right). \quad (\text{D7})$$

This can be rewritten in the form

$$\Sigma(\omega) = i|\omega| \left[\frac{U^2}{2} \frac{1}{|\omega|} \ln \left(\frac{\Lambda^4}{|\omega|} \right) + \frac{4U^2}{\pi \Lambda^3 |\omega|^{\frac{1}{4}}} \right]. \quad (\text{D8})$$

Now, using $q_{\min} \approx |\omega|^{1/4}$ and ignoring the second term, which scales as $1/\Lambda^3$, we obtain Eq. (31).

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