

Asymptotically exact solution of the non-Hermitian disordered interacting Hatano-Nelson chain

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We present an asymptotically exact solution of a paradigmatic non-Hermitian model: the disordered interacting fermionic Hatano-Nelson model, or equivalently, the non-Hermitian spin-1/2 XXZ model. We use a renormalization group method suited for disordered systems and show that non-Hermitian couplings are relevant perturbations to the Hermitian model, which ultimately leads to a quantum-to-classical crossover. The ground state of the model consists of a collection of strongly coupled pairs of spins of arbitrary size at random positions that, unlike the Hermitian case, do not form singlets, but a mixture of the singlet and the $M = 0$ triplet state. As a result, the magnetic susceptibility in the x - y directions becomes negative and diverges at a finite small temperature. Additionally, in sharp contrast to the $\ln(L)$ increase observed in disordered Hermitian chains, the entanglement entropy of a partition of size L saturates for large L , as the strongly coupled pairs become classical and stop contributing at large length scales.

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Introduction. The study of open quantum systems originated in the pursuit of a description of the effects of various external perturbations, especially those coming from external baths (see, e.g., Ref. [1]). It has found applications in acoustics [2], optics [3], photonics [4], and other areas. More recently, it has seen a resurgence in situations where the quantum system is submitted to intermittent measurements [5]. In either case, the system is often described by a Lindblad equation [6,7], part of which relies on a non-Hermitian (nH) Hamiltonian dynamics (for a review, see Ref. [8]).

Although most nH studies have focused on clean systems, some attention has been given to the effects of disorder [9–22]. Moreover, the effects of interparticle interactions have mostly been looked at in the absence of disorder [23–34]. Finally, when both disorder and interactions are present, there has been a lot of numerical work [35–38], with little analytical insight (for a notable exception, see Ref. [39]). In this work, we intend to fill this gap with an asymptotically exact low-energy solution of the disordered, nH antiferromagnetic spin-1/2 XXZ chain, or equivalently, the disordered interacting nH fermionic Hatano-Nelson model [28,33,40–44]. We will show that (i) it is governed by an infinite-disorder low-energy fixed point [45], (ii) the fixed-point distributions can be obtained analytically and demonstrate that disorder in the nH couplings is relevant in the renormalization group (RG) sense, leading to a quantum-to-classical crossover, (iii) the

entanglement entropy (EE) of a partition of size L acts as a witness for said crossover, and finally, (iv) the spin susceptibility (of the spin model) becomes highly anisotropic at low temperatures, in sharp contrast to the Hermitian case, where all directions show the same power-law behavior. Other known properties of the noninteracting case can also be easily inferred from the solution.

The model. We focus on the disordered nH spin-1/2 XXZ chain

$$H = \sum_i J_i \left[\frac{e^{\gamma_i}}{2} S_i^+ S_{i+1}^- + \frac{e^{-\gamma_i}}{2} S_i^- S_{i+1}^+ + \Delta_i S_i^z S_{i+1}^z \right]. \quad (1)$$

The couplings J_i , Δ_i , and γ_i are spatially uncorrelated and chosen from three independent probability distributions with the restrictions $J_i > 0$ and $\Delta_i > -1/2$ to avoid ferromagnetic phases. Within these assumptions, this represents the most general parametrization of the three independent terms that leads to a real spectrum (see the Supplemental Material [46]), which allows us to work only in the unbroken PT -symmetric phase [47]. Equation (1) reduces to the XXZ model in the Hermitian limit $\gamma_i = 0$. It can also be mapped onto fermions through a Jordan-Wigner transformation,

$$H = \sum_i J_i \left[\frac{e^{\gamma_i}}{2} c_i^\dagger c_{i+1} + \frac{e^{-\gamma_i}}{2} c_{i+1}^\dagger c_i + \Delta_i \delta \hat{n}_i \delta \hat{n}_{i+1} \right], \quad (2)$$

where $\delta \hat{n}_i = \hat{n}_i - 1/2$. This is the disordered interacting fermionic Hatano-Nelson model [40–42]. Note that the ground state of this model corresponds to the half-filled case with $\langle n_i \rangle = 1/2$. We analyze the model with the help of the strong disorder renormalization group (SDRG) [48,49]. The SDRG, since its conception, has provided deep insights into many disordered systems (for a review, see Ref. [50]), including the Hermitian version of Eq. (2) [51]. It has been

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recently used in an nH setting [39]. The procedure amounts to rescaling states and physical quantities to lower energies and longer wavelengths. In the Hermitian XXZ chain (and, as we will show, also in the nH one), this leads asymptotically to infinitely large effective disorder and, thus, to exact statements about the behavior of the system. In this paper, we employ it to solve the model above.

Method and procedure. The SDRG procedure works by considering the hierarchy of local excitation gaps obtained from solving all the problems of two adjacent spins. We focus on the most strongly coupled pair (SCP), the one with the largest gap (which we call Ω) between the ground and first excited state, keep only the ground state, and discard the excited states. We will refer to these pairs of spins as SCPs. This approach is justifiable as long as the energy or temperature scale T of interest and the neighboring couplings are significantly smaller than Ω . It can be shown (see the Supplemental Material [46]) that the local two-spin gap is $J_i(1 + \Delta_i)/2$, if $\Delta_i \in [-1/2, 1]$, and J_i , if $\Delta_i > 1$. Because of non-Hermiticity, right and left eigenstates (i.e., eigenstates of H and $H^\dagger$) are different. Irrespective of the couplings, the local right ground state is always nondegenerate and written as $|\text{GS}^{(R)}\rangle = \cosh \frac{\gamma}{2} |0, 0\rangle + \sinh \frac{\gamma}{2} |1, 0\rangle$, where $|J, M\rangle$ denote the standard total angular momentum states of the pair. The left ground state $|\text{GS}^{(L)}\rangle$ is obtained by changing $\gamma \rightarrow -\gamma$. Note that they are a mixture of the singlet and the $M = 0$ triplet states, but reduce to pure singlets only in the Hermitian case ($\gamma_i = 0$).

After freezing the SCP in its ground state, we treat its coupling to the adjacent spins perturbatively. Importantly, perturbation theory has to be implemented using the biorthogonal basis [52] so that the right and left eigenstates are orthonormalized as $\langle \psi_i^L | \psi_j^R \rangle = \delta_{i,j}$ (see the Supplemental Material [46]). Assuming that the largest local gap Ω is between sites i and $i + 1$, new renormalized couplings appear between spins $i - 1$ and $i + 2$ given by [see the Supplemental Material [46] and Fig. 1(a)]

$$\tilde{J}_{i-1} = \frac{J_{i-1}J_{i+1}}{\Omega} \frac{1}{1 + \Delta_i}, \quad \tilde{\Delta}_{i-1} = \frac{\Delta_{i-1}\Delta_{i+1}(1 + \Delta_i)}{2}, \quad (3)$$

$$\tilde{\gamma}_{i-1} = \gamma_{i-1} + \gamma_i + \gamma_{i+1}, \quad \tilde{\ell}_{i-1} = \ell_{i-1} + \ell_i + \ell_{i+1}. \quad (4)$$

We have added, for later convenience, the decimation rule of the bond size ℓ_i , which is geometrically obvious and identical to that for γ_i . Note the decoupling of the decimation rule of γ_i from those of J_i and Δ_i . This facilitates greatly the analysis of the SDRG flow and justifies the parametrization chosen in Eq. (1). The procedure is iterated until we reach either the ground state or the target temperature $T = \Omega \ll \Omega_0$, where Ω_0 is the largest local gap of the bare distributions.

From the structure of the SDRG iterations, we can already conclude what the structure of the ground state is. It will consist of nested SCPs of all sizes formed from spins at random positions. We refer to this phase as a random strongly coupled pair phase (RSCPP). This nomenclature sets it apart from the random singlet phase of the Hermitian model, the XXZ chain [51].

SDRG flows. We now analyze the SDRG flow and construct the phase diagram. Due to the decoupling of the γ_i variables, the flow of the J_i and Δ_i distributions is *identical* to that of the

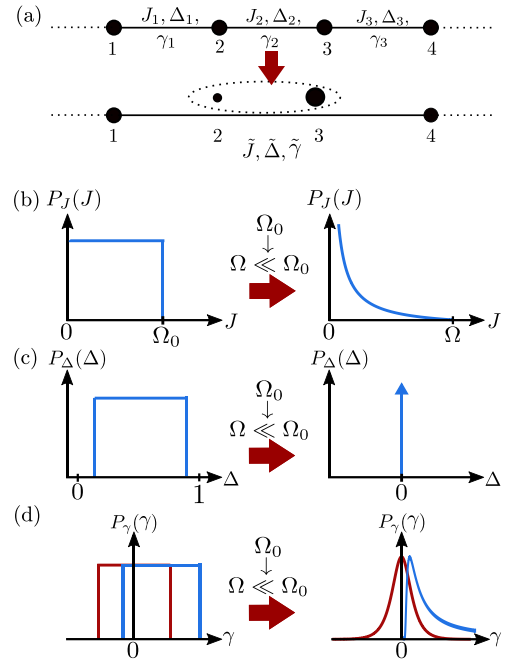


FIG. 1. (a) The SDRG decimation step. The strongest-coupled pair is assumed to be formed by spins 2 and 3. This pair is decimated, and quantum fluctuations lead to effective couplings between spins 1 and 4, initially not directly coupled. The flows of the distributions of the couplings (b) J , (c) Δ , and (d) γ , at two different stages of the RG flow. The fixed point of the J distribution is universal and tends toward infinite width. If the anisotropy distribution, $P_\Delta(\Delta)$, has support between zero and one, it flows to a delta-function fixed-point distribution at the origin. The distribution of γ is shown for both symmetric (red) and asymmetric (blue) initial distributions. Under the flow, both distribution widths increase without limit, but the asymmetric one has support only for $\gamma > 0$.

Hermitian XXZ model, which we now summarize [51]. The relative width of the J distribution increases without limit as the cutoff Ω is lowered, flowing toward a universal infinite-randomness fixed point of the form

$$P_J(J) \sim \frac{1}{J^{1-1/\Gamma}}, \quad \Gamma = \ln \left(\frac{\Omega_0}{\Omega} \right) \rightarrow \infty. \quad (5)$$

The increasing width of the distribution guarantees that the perturbation theory of Eq. (3) is increasingly more accurate and asymptotically exact at low energies. The flow of the J distribution is schematically shown in Fig. 1(b).

The infrared stable fixed points of the Δ distribution are $\Delta_i = 0$, corresponding to the XX model, and $\Delta_i \rightarrow \infty$, corresponding to the Ising fixed point. Separating them is the unstable fixed point of the Heisenberg model $\Delta_i = 1$ [51]. For concreteness, we show schematically the first kind of flow in Fig. 1(c). From now on, we will focus only on flows to the XX and Heisenberg fixed points.

The effects of non-Hermiticity are contained in the flow of γ_i . The structure of Eq. (4) might suggest that γ_i would follow a standard random walk. This is not the case, however, because the renormalized γ_i become strongly correlated with J_i , as the latter primarily define the local gaps Ω_i and thus, which bond will be decimated next. Therefore, as very

weak bonds are generated after many RG steps [see Eq. (5)], their corresponding γ_i will be accordingly very large. The asymptotic form of the joint distribution of J_i and bond lengths ℓ_i was obtained analytically for the Hermitian XXZ chain [51]. That result can be used to immediately determine the joint distribution of J_i and γ_i , due to their identical decimation rules. For later calculations, however, we will also need the joint distributions of J_i , γ_i , and ℓ_i , which is given in the Supplemental Material [46].

Let us describe briefly the qualitative flow of γ_i . From Eq. (4), one can easily see that the Hermitian XXZ chain with $\gamma_i = 0$ is a fixed point. Also from the additive form of Eq. (4), it is simple to see that this fixed point is unstable: For any small initial randomness in γ_i , the distribution will broaden. Some features of this initial distribution crucially influence the flow. If the initial distribution is an even function of γ , it remains so under the SDRG, but its width increases without limit. More precisely, the fixed-point distribution is a universal even function of the scaled variable $y = C_\gamma \gamma / \Gamma$, where $\Gamma \rightarrow \infty$ [see Eq. (5)] and C_γ is a nonuniversal constant. If the initial distribution is not an even function of γ , this initial asymmetry grows with the flow. The asymptotic distribution also becomes increasingly broader, but its support is confined to *either positive or negative values of γ* . More precisely, if the first moment of the distribution is positive (negative), the fixed-point distribution is a universal function of the scaled variable $y = C_\gamma \gamma / \Gamma^2$ with strict support for positive (negative) values of y . This asymmetric case applies to the flow of bond lengths ℓ_i , which remain strictly positive [51]. The schematic evolution of the γ distribution for both cases is illustrated in Fig. 1(d). This extremely asymmetric form of the γ distribution at low energies translates into a dominant one-sidedness of the hopping amplitudes in Eq. (2). This provides a natural explanation for the nH skin effect of the model [11–13,15], *even in the interacting case*, and its complete absence in the symmetric case [20].

The ground state of the RSCPP is formed out of localized SCPs with arbitrarily long sizes, a complete parallel to the Hermitian case [51]. This is a property of the flow and is not spoiled by non-Hermiticity. The localization of the SCPs has a subexponential character, however, as observed numerically [20]. This can be seen from the *typical* spin-spin correlation function $\langle S_i^z S_j^z \rangle_{\text{typ}} \sim \exp -\sqrt{|i-j|/\xi}$ [51,53]. This was also known for the noninteracting Hermitian fermionic model [54].

Thermodynamic properties. We focus first on the magnetic susceptibility of the spin model along the x , y , and z directions. There are in general two contributions to the magnetic susceptibility χ at temperature T ,

$$\chi^\alpha(T) = \chi_{\text{free}}^\alpha(T) + \chi_{\text{SCP}}^\alpha(T), \quad \alpha = x, y, z, \quad (6)$$

the first term coming from spins not yet decimated and the second one from the already decimated SCPs at scale $\Omega = T$. Because the J distribution is so broad at low T , nondecimated spins are only weakly coupled to their active neighbors and can be accurately treated as free spins. Their contribution is Curie-like, $\sim 1/T$, times the density of active spins at T , $n_{\text{free}}(T) \sim 1/[\ln(1/T)]^2$, so that [51]

$$[\chi_{\text{free}}^\alpha(T)]^{-1} \sim T[\ln(1/T)]^2. \quad (7)$$

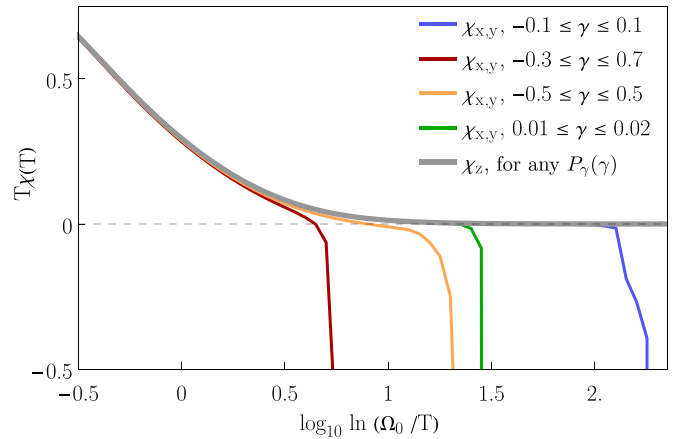


FIG. 2. Magnetic susceptibility multiplied by the temperature for different initial uniform distributions of γ , as a function of $\ln(\Omega_0/T)$, in log scale. The z component, $T\chi_z(T)$, shown in gray, is independent of $P_\gamma(\gamma)$ and the same as for the Hermitian chain, approaching zero as $T \rightarrow 0$. In red, yellow, green, and blue are $T\chi_{x,y}(T)$, which depend strongly on the distribution of γ . For small T , it becomes negative and diverges at a small but finite T , a consequence of a negative contribution from each SCP.

This contribution is the same for any component of the susceptibility, even if the SU(2) symmetry is broken, $\Delta \neq 1$. This is a consequence of an emergent symmetry of the model [51,55]. The contribution of a *single* SCP, $\chi_{\text{SCP}}^\alpha(T)$, is given by (see the Supplemental Material [46])

$$T\chi_{\text{SCP}}^\alpha(T) = \begin{cases} -\sinh^2\left(\frac{\gamma}{2}\right), & \text{if } \alpha = x, y, \\ 0, & \text{if } \alpha = z. \end{cases} \quad (8)$$

Note how this contribution only exists in the x - y directions and in the nH case. This is because the ground state of an SCP is a superposition of the two-spin singlet and $M = 0$ triplet states. Thus, $\chi_{\text{SCP}}(T)$ involves summing over all the decimated SCPs, weighted by the γ distribution at the scale at which they were decimated. A key feature is that $\chi_{\text{SCP}}^{x,y}(T)$ *diverges at a finite T* . For the symmetric case, for instance (see the Supplemental Material [46]),

$$\chi_{\text{SCP}}^{x,y}(T) \sim -\frac{1}{4T} \ln \left[\frac{2C_\gamma}{\Gamma_T} \tan \left(\frac{\Gamma_T}{2C_\gamma} \right) \right], \quad (9)$$

where $\Gamma_T = \ln(\Omega_0/T)$, which diverges at $T = \Omega_0 \exp(-\pi C_\gamma)$. There is also a divergence in the asymmetric case (see the Supplemental Material [46]). This behavior can be traced back to the solution of the nH two-site problem [see Eq. (8)] and is not a consequence of any collective phase transition. The susceptibility in the z direction is the same in both Hermitian and nH cases and remains positive and nondivergent, since no SCP contributes. Figure 2 shows the total susceptibility for a numerical implementation of the SDRG procedure in chains with periodic boundary conditions, in several different cases. It is worth mentioning that the susceptibility is *not* a self-averaging quantity, showing extremely large sample-to-sample fluctuations, which is discussed in more detail in the Supplemental Material [46].

Finally, at a low temperature T , the specific entropy receives no contributions from the frozen SCPs, and only a

density $n_{\text{free}}(T)$ of nondecimated spins contributes $k_B \ln 2$. From this, one obtains the low-temperature specific heat as $c_V(T) \sim k_B / [\ln(1/T)]^3$, which is the same as in the Hermitian case [51].

Entanglement entropy. Another remarkable effect of non-Hermiticity is captured by the ground-state entanglement entropy of a partition of size L [56,57]. It is defined by

$$S(L) = -\text{Tr}[\hat{\rho}(L) \log_2 \hat{\rho}(L)], \quad (10)$$

where $\hat{\rho}(L)$ is the reduced density matrix of the partition, fixing the full system in its ground state. Within the SDRG, only spins inside the partition that form SCPs with spins outside of it contribute to the EE. We thus need to add the contributions from those SCPs that cross the partition boundaries [58–60].

There are two possible ways of defining the density matrix of an nH system: $\hat{\rho}^{\text{RL}} = |\psi^{\text{R}}\rangle\langle\psi^{\text{L}}|$ [26,61–64] or $\hat{\rho}^{\text{RR}} = |\psi^{\text{R}}\rangle\langle\psi^{\text{R}}|$ [56,65,66] (the case of LL, with left eigenvectors, is identical to RR). Interestingly, these two schemes lead to remarkably different results for the EE of a single SCP, S_{pair} . For the RL case, there is always maximal entanglement in an SCP, $S_{\text{pair}} = 1$, and the EE does not depend on the nH coupling γ (see the Supplemental Material [46]). In the RR case, in contrast, the EE depends on γ (see the Supplemental Material [46]),

$$S_{\text{pair}} = |\gamma|(1 - \tanh |\gamma|) \log_2 e + \log_2(1 + e^{-2|\gamma|}). \quad (11)$$

It is equal to 1 only when $\gamma = 0$, recovering the behavior of singlets, but decreases to zero as $|\gamma| \rightarrow \infty$, as the SCP becomes separable. As the distribution width of γ grows without limit, long, low-energy SCPs stop contributing to the EE, which saturates for large enough L .

The EE of a size- L partition, $S(L)$, is shown in Fig. 3 for different distributions of γ , obtained by numerical simulations of chains of size $N = 220$, using periodic boundary conditions (circles). Note that $S(L)$ is symmetric with respect to the chain half size $N/2$. Nevertheless, it is clear, especially for broader initial γ distributions, that the EE saturates before L reaches $N/2$. This is confirmed by analytical calculations, see the lines in Fig. 3. The calculation requires the use of the fixed-point joint distribution of couplings, bond lengths, and γ , which we have explicitly obtained (see the Supplemental Material [46]).

Conclusions. While a lot of recent work on disordered nH systems has been performed numerically, we have obtained an asymptotically exact analytical description of a disordered nH spin chain, equivalent to the fermionic interacting Hatano-Nelson model. We have mapped out its phase diagram and described some of its physical properties. The main result is that the non-Hermiticity is *relevant* in the RG sense; i.e., it grows in both strength and distribution width at low energies.

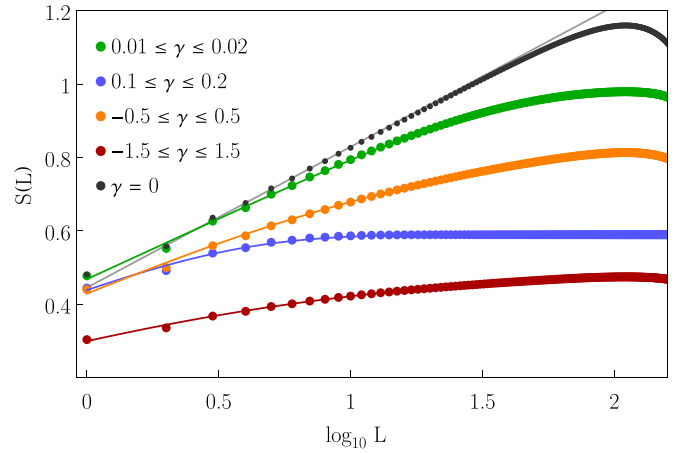


FIG. 3. Entanglement entropy as a function of the partition size L . The points were obtained by numerically decimating chains of length $N = 220$ and averaging over 2000 disorder realizations. The lines correspond to fits to the analytical expressions derived in the asymptotic large- L regime. The RSCPP shows a saturation at large L , a consequence of the SCPs formed by spins far apart not contributing as they become classical. In black is the entanglement entropy of the Hermitian model, with the gray line showing the analytical result. In this case, each pair of spins is maximally entangled, which leads to a logarithmic increase of $S(L)$. In all cases, there is a downturn as $L \rightarrow N/2$, because $S(L)$ is symmetric with respect to the chain half size.

In our case, this is manifested explicitly as a divergence in the magnetic susceptibility at a finite small temperature (in contrast to a quasi-Curie-like behavior in the Hermitian case) and a saturation of the entanglement entropy as a function of partition size (in contrast to a log increase in the Hermitian case). Our work opens up the possibility of exploring several other cases, such as nH versions of random transverse-field Ising model.

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Data availability. The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

- [1] I. Rotter and J. P. Bird, A review of progress in the physics of open quantum systems: Theory and experiment, *Rep. Prog. Phys.* **78**, 114001 (2015).
- [2] Z. Gu, H. Gao, P.-C. Cao, T. Liu, X.-F. Zhu, and J. Zhu, Controlling sound in non-Hermitian acoustic systems, *Phys. Rev. Appl.* **16**, 057001 (2021).

- [3] Z. Zhang, D. Ma, J. Sheng, Y. Zhang, Y. Zhang, and M. Xiao, Non-Hermitian optics in atomic systems, *J. Phys. B: At. Mol. Opt. Phys.* **51**, 072001 (2018).
- [4] L. Feng, R. El-Ganainy, and L. Ge, Non-Hermitian photonics based on parity-time symmetry, *Nat. Photon.* **11**, 752 (2017).

- [5] H. M. Wiseman and G. J. Milburn, *Quantum Measurement and Control* (Cambridge University Press, Cambridge, UK, 2009).
- [6] A. J. Daley, Quantum trajectories and open many-body quantum systems, *Adv. Phys.* **63**, 77 (2014).
- [7] F. Minganti, A. Miranowicz, R. W. Chhajlany, and F. Nori, Quantum exceptional points of non-Hermitian Hamiltonians and Liouvillians: The effects of quantum jumps, *Phys. Rev. A* **100**, 062131 (2019).
- [8] Y. Ashida, Z. Gong, and M. Ueda, Non-Hermitian physics, *Adv. Phys.* **69**, 249 (2020).
- [9] F. Hébert, M. Schram, R. Scalettar, W. Chen, and Z. Bai, Hatano-Nelson model with a periodic potential, *Eur. Phys. J. B* **79**, 465 (2011).
- [10] R. Hamazaki, K. Kawabata, N. Kura, and M. Ueda, Universality classes of non-Hermitian random matrices, *Phys. Rev. Res.* **2**, 023286 (2020).
- [11] S. Longhi, Stochastic non-Hermitian skin effect, *Opt. Lett.* **45**, 5250 (2020).
- [12] J. Claes and T. L. Hughes, Skin effect and winding number in disordered non-Hermitian systems, *Phys. Rev. B* **103**, L140201 (2021).
- [13] Z.-Q. Zhang, H. Liu, H. Liu, H. Jiang, and X. Xie, Bulk-boundary correspondence in disordered non-Hermitian systems, *Sci. Bull.* **68**, 157 (2023).
- [14] E. T. Kokkinakis, K. G. Makris, and E. N. Economou, Anderson localization versus hopping asymmetry in a disordered lattice, *Phys. Rev. A* **110**, 053517 (2024).
- [15] B. Midya, Topological phase transition in fluctuating imaginary gauge fields, *Phys. Rev. A* **109**, L061502 (2024).
- [16] W. Wang, X. Wang, and G. Ma, Anderson transition at complex energies in one-dimensional parity-time-symmetric disordered systems, *Phys. Rev. Lett.* **134**, 066301 (2025).
- [17] C. Wang, W. He, X. R. Wang, and H. Ren, Unified one-parameter scaling function for Anderson localization transitions in nonreciprocal non-Hermitian systems, *Phys. Rev. Lett.* **134**, 176301 (2025).
- [18] J. Shang and H. Hu, Spreading dynamics in the Hatano-Nelson model with disorder, *Phys. Rev. B* **112**, 014205 (2025).
- [19] K. Sun and H. Hu, Lyapunov formulation of band theory for disordered non-Hermitian systems, *arXiv:2507.09447*.
- [20] S. Longhi, Erratic non-Hermitian skin localization, *Phys. Rev. Lett.* **134**, 196302 (2025).
- [21] B. Li, C. Chen, and Z. Wang, Universal non-Hermitian transport in disordered systems, *Phys. Rev. Lett.* **135**, 033802 (2025).
- [22] T. Yoshida, S.-B. Zhang, T. Neupert, and N. Kawakami, Non-Hermitian Mott skin effect, *Phys. Rev. Lett.* **133**, 076502 (2024).
- [23] G. Albertini, S. R. Dahmen, and B. Wehefritz, Phase diagram of the non-Hermitian asymmetric XXZ spin chain, *J. Phys. A: Math. Gen.* **29**, L369 (1996).
- [24] U. Bilstein and B. Wehefritz, Spectra of non-Hermitian quantum spin chains describing boundary induced phase transitions, *J. Phys. A: Math. Gen.* **30**, 4925 (1997).
- [25] T. Fukui and N. Kawakami, Breakdown of the Mott insulator: Exact solution of an asymmetric Hubbard model, *Phys. Rev. B* **58**, 16051 (1998).
- [26] R. Couvreur, J. L. Jacobsen, and H. Saleur, Entanglement in nonunitary quantum critical spin chains, *Phys. Rev. Lett.* **119**, 040601 (2017).
- [27] X. Z. Zhang and Z. Song, η -pairing ground states in the non-Hermitian Hubbard model, *Phys. Rev. B* **103**, 235153 (2021).
- [28] G. Chen, F. Song, and J. L. Lado, Topological spin excitations in non-Hermitian spin chains with a generalized kernel polynomial algorithm, *Phys. Rev. Lett.* **130**, 100401 (2023).
- [29] S. Sayyad and J. L. Lado, Topological phase diagrams of exactly solvable non-Hermitian interacting Kitaev chains, *Phys. Rev. Res.* **5**, L022046 (2023).
- [30] H. Li, X.-H. Yu, M. Nakagawa, and M. Ueda, Yang-Lee zeros, semicircle theorem, and nonunitary criticality in Bardeen-Cooper-Schrieffer superconductivity, *Phys. Rev. Lett.* **131**, 216001 (2023).
- [31] P.-Y. Yang and Y.-C. Tzeng, Entanglement Hamiltonian and effective temperature of non-Hermitian quantum spin ladders, *SciPost Phys. Core* **7**, 074 (2024).
- [32] F. C. Alcaraz and L. M. Ramos, Conformally invariant free-parafermionic quantum chains with multispin interactions, *Phys. Rev. E* **109**, 044138 (2024).
- [33] L. Mao, X. Yang, M.-J. Tao, H. Hu, and L. Pan, Liouvillian skin effect in a one-dimensional open many-body quantum system with generalized boundary conditions, *Phys. Rev. B* **110**, 045440 (2024).
- [34] G. Akemann, F. Balducci, A. Chenu, P. Pöbller, F. Roccati, and R. Shir, Two transitions in complex eigenvalue statistics: Hermiticity and integrability breaking, *Phys. Rev. Res.* **7**, 013098 (2025).
- [35] J. Liu and Z. Xu, From ergodicity to many-body localization in a one-dimensional interacting non-Hermitian Stark system, *Phys. Rev. B* **108**, 184205 (2023).
- [36] C.-Z. Lu, X. Deng, S.-P. Kou, and G. Sun, Many-body phase transitions in a non-Hermitian Ising chain, *Phys. Rev. B* **110**, 014441 (2024).
- [37] K. Suthar, Boundary-driven many-body phase transitions in a non-Hermitian disordered fermionic chain, *Phys. Rev. B* **111**, 064202 (2025).
- [38] P. Brighi, M. Ljubotina, F. Roccati, and F. Balducci, Finite steady-state current defies non-Hermitian many-body localization, *Phys. Rev. Res.* **7**, L042014 (2025).
- [39] S. Tiwary and J. E. Moore, Measurement-induced phase transition in a disordered XX spin chain: A real-space renormalization group study, *Phys. Rev. B* **113**, 104203 (2026).
- [40] N. Hatano and D. R. Nelson, Localization transitions in non-Hermitian quantum mechanics, *Phys. Rev. Lett.* **77**, 570 (1996).
- [41] N. Hatano and D. R. Nelson, Vortex pinning and non-Hermitian quantum mechanics, *Phys. Rev. B* **56**, 8651 (1997).
- [42] N. Hatano and D. R. Nelson, Non-Hermitian delocalization and eigenfunctions, *Phys. Rev. B* **58**, 8384 (1998).
- [43] B. Dóra and C. P. Moca, Full counting statistics in the many-body Hatano-Nelson model, *Phys. Rev. B* **106**, 235125 (2022).
- [44] T. Orito and K.-I. Imura, Entanglement dynamics in the many-body Hatano-Nelson model, *Phys. Rev. B* **108**, 214308 (2023).
- [45] D. S. Fisher, Critical behavior of random transverse-field Ising spin chains, *Phys. Rev. B* **51**, 6411 (1995).
- [46] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/jbb1-gcbh>, which includes details about other parametrizations of the Hamiltonian, the derivation of the

- SDRG decimation rules, a discussion of the joint distributions of J , γ , and ℓ , the details of the calculation of the magnetic susceptibility and the entanglement entropy, which also includes Refs. [67,68].
- [47] C. M. Bender and S. Boettcher, Real spectra in non-Hermitian Hamiltonians having $\mathcal{PT}$ symmetry, *Phys. Rev. Lett.* **80**, 5243 (1998).
 - [48] C. Dasgupta and S. K. Ma, Low-temperature properties of the random Heisenberg antiferromagnetic chain, *Phys. Rev. B* **22**, 1305 (1980).
 - [49] S. K. Ma, C. Dasgupta, and C. K. Hu, Random Antiferromagnetic Chain, *Phys. Rev. Lett.* **43**, 1434 (1979).
 - [50] F. Iglói and C. Monthus, Strong disorder RG approach—A short review of recent developments, *Eur. Phys. J. B* **91**, 290 (2018).
 - [51] D. S. Fisher, Random antiferromagnetic quantum spin chains, *Phys. Rev. B* **50**, 3799 (1994).
 - [52] D. C. Brody, Biorthogonal quantum mechanics, *J. Phys. A: Math. Theor.* **47**, 035305 (2014).
 - [53] J. C. Getelina and J. A. Hoyos, The correlation functions of certain random antiferromagnetic spin-1/2 critical chains, *Eur. Phys. J. B* **93**, 2 (2020).
 - [54] M. Inui, S. A. Trugman, and E. Abrahams, Unusual properties of midband states in systems with off-diagonal disorder, *Phys. Rev. B* **49**, 3190 (1994).
 - [55] V. L. Quito, J. A. Hoyos, and E. Miranda, Emergent SU(3) symmetry in random spin-1 chains, *Phys. Rev. Lett.* **115**, 167201 (2015).
 - [56] L. Herviou, N. Regnault, and J. H. Bardarson, Entanglement spectrum and symmetries in non-Hermitian fermionic non-interacting models, *SciPost Phys.* **7**, 069 (2019).
 - [57] L.-M. Chen, Y. Zhou, S. A. Chen, and P. Ye, Quantum entanglement and non-Hermiticity in free-fermion systems, *Chin. Phys. Lett.* **41**, 127302 (2024).
 - [58] G. Refael and J. E. Moore, Entanglement entropy of random quantum critical points in one dimension, *Phys. Rev. Lett.* **93**, 260602 (2004).
 - [59] J. A. Hoyos, A. P. Vieira, N. Laflorencie, and E. Miranda, Correlation amplitude and entanglement entropy in random spin chains, *Phys. Rev. B* **76**, 174425 (2007).
 - [60] G. Refael and J. E. Moore, Criticality and entanglement in random quantum systems, *J. Phys. A: Math. Theor.* **42**, 504010 (2009).
 - [61] M. Fossati, F. Ares, and P. Calabrese, Symmetry-resolved entanglement in critical non-Hermitian systems, *Phys. Rev. B* **107**, 205153 (2023).
 - [62] G. Cipolloni and J. Kudler-Flam, Entanglement entropy of non-Hermitian eigenstates and the Ginibre ensemble, *Phys. Rev. Lett.* **130**, 010401 (2023).
 - [63] Y.-B. Guo, Y.-C. Yu, R.-Z. Huang, L.-P. Yang, R.-Z. Chi, H.-J. Liao, and T. Xiang, Entanglement entropy of non-Hermitian free fermions, *J. Phys.: Condens. Matter* **33**, 475502 (2021).
 - [64] C.-T. Hsieh and P.-Y. Chang, Relating non-Hermitian and Hermitian quantum systems at criticality, *SciPost Phys. Core* **6**, 062 (2023).
 - [65] R. Modak and B. P. Mandal, Eigenstate entanglement entropy in a $\mathcal{PT}$ -invariant non-Hermitian system, *Phys. Rev. A* **103**, 062416 (2021).
 - [66] K. Kawabata, T. Numasawa, and S. Ryu, Entanglement phase transition induced by the non-Hermitian skin effect, *Phys. Rev. X* **13**, 021007 (2023).
 - [67] M. Abramowitz and I. A. Stegun, eds., *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, 9th ed. (Dover, New York, 1972).
 - [68] M. M. Sternheim and J. F. Walker, Non-Hermitian Hamiltonians, decaying states, and perturbation theory, *Phys. Rev. C* **6**, 114 (1972).