

Strong correlations in the dynamical evolution of lowest Landau level bosons

Y. Yang^{✉*} and N. R. Cooper[✉]

T.C.M. Group, Cavendish Laboratory, University of Cambridge, J.J. Thomson Avenue, Cambridge CB3 0US, United Kingdom



(Received 18 February 2026; accepted 14 July 2026; published 18 August 2026)

Recent experiments with rotating Bose gases have demonstrated the interaction-driven hydrodynamic instability of an initial extended striplike state in the lowest Landau level. We investigate this phenomenon in the low-density limit, where the mean-field Gross-Pitaevskii theory becomes inadequate, using exact diagonalization studies and analytic arguments. We show that the behavior can be understood in terms of weakly interacting repulsively bound few-body clusters. Signatures of cluster behavior are observed in the expectation values of observables that oscillate at frequencies characterized by the energies of few-body bound states. Using a semiclassical theory for interacting clusters, we predict the long-time growth of the cloud width to be a power law in the logarithm of time. This slow thermalization of bound clusters represents a form of quantum many-body scars.

DOI: [10.1103/j1zq-ztnv](https://doi.org/10.1103/j1zq-ztnv)

I. INTRODUCTION

The behavior of Bose-Einstein condensates (BECs) under rotation is drastically different from that of a normal fluid [1–13]. Rotating BECs host vortices carrying quantized vorticity, and at moderate rotation rates the ground state is a vortex lattice similar to those in type-II superconductors. At very high rotation rates, theoretical studies predict that the vortex lattice is destabilized by quantum fluctuations, and the rotating Bose gas enters a strongly correlated regime [14,15]. This regime is the bosonic analog of the fractional quantum Hall effect (FQHE), featuring many phases and states familiar from the FQHE of electrons, including Laughlin states, composite fermions, and non-Abelian phases. Furthermore, recent experimental advances demonstrated the tantalizing possibility of realizing such states with ultracold atoms [16], paving the way for the quantum simulation of the FQHE.

Turning to the dynamical aspects of rotating BECs, a separate very interesting series of experiments was recently carried out [17–19]. In these experiments, bosons were prepared in a rotating condensate of a striplike shape. This state was shown to be dynamically unstable and to spontaneously form a density modulation. This behavior was explained in both the lowest Landau level (LLL) regime and the hydrodynamic regime as a Kelvin-Helmholtz instability [19]. On the other hand, given that these results were developed with either the Bogoliubov approximation or the mean-field Gross-Pitaevskii (GP) theory [20], one naturally wonders how the

dynamics changes as the density is lowered, in the same way that the vortex lattice state transitions into the bosonic FQHE.

In this work, we investigate theoretically the dynamics of a striplike condensate in this low-density regime where GP theory becomes inadequate. A schematic of the setup is shown in Fig. 1, depicting the creation of pairs of particles with opposite momenta from the condensate due to the interactions between the bosons. We use both exact diagonalization (ED) methods and analytical approaches to obtain a consistent understanding of the key features. We focus on observables that are most easily accessible in experiments: the particle density, the width of the cloud strip, and the density-density correlation function. Interestingly, we find that these observables oscillate at frequencies that are inconsistent with the mean-field treatment. Rather, as we explain, they can be understood in terms of a picture where the dynamics is dominated by few-body clusters: each cluster being a composite object of a few bosons with zero internal angular momentum. We also show that the full excitation spectrum of contact-interacting bosons in the LLL can be cast in a similar light, showing peaks in the densities of states at energies corresponding to noninteracting clusters. Furthermore, we propose a semiclassical theory of cluster interactions, which explains further detailed structures of the density of states. The semiclassical theory also predicts the width of the striplike cloud to grow as a power law in the logarithm of time. This differs from diffusive or subdiffusive behaviors sometimes seen in the literature that are power laws in time itself [21]. We also show how the logarithmic timescales also apply to the thermalization of the cloud.

II. SYSTEM AND OBSERVABLES

We start with the second quantized Hamiltonian for dilute Bose gases, with field operators $\hat{\psi}^{(\dagger)}(\mathbf{r})$. We consider particles of mass M in two dimensions and impose an effective magnetic field via minimal substitution $\mathbf{p} \rightarrow \mathbf{p} + e\mathbf{A}$ [1].

*Contact author: yy490@cam.ac.uk

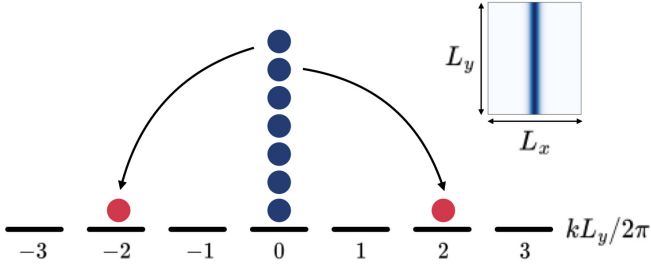


FIG. 1. A schematic of the system in question. The initial state is the condensate (dark blue) in the zero-momentum state in the Landau gauge in the LLL. Interactions cause the condensate to deplete, producing pairs of atoms in opposite momentum states (red). These give rise to the dynamic instabilities. The inset shows the initial state in real space on a torus of size $L_x \times L_y$.

We choose the Landau gauge where $\mathbf{A} = (0, Bx)$ so that the Hamiltonian becomes

$$\hat{H} = \int d^2r \left\{ \hat{\psi}^\dagger(\mathbf{r}) \left[\frac{p_x^2}{2M} + \frac{1}{2} M \omega_c^2 (x + k_y \ell_B^2)^2 \right] \hat{\psi}(\mathbf{r}) + \frac{g}{2} \hat{\psi}^\dagger(\mathbf{r}) \hat{\psi}^\dagger(\mathbf{r}) \hat{\psi}(\mathbf{r}) \hat{\psi}(\mathbf{r}) \right\}, \quad (1)$$

where we define the cyclotron frequency $\omega_c \equiv eB/M$ and the magnetic length $\ell_B \equiv \sqrt{\hbar/eB}$. (This describes harmonically trapped gases rotating at the trap frequency Ω when $\omega_c = 2\Omega$ [1].) g is the strength of contact interactions between the bosons. For the rest of this paper, we work in units $\hbar = 1$.

The single-particle part of the Hamiltonian is that of a harmonic oscillator, giving quantized Landau levels $E_n = \omega_c(n + 1/2)$ and eigenstates labeled by momentum k along the y direction $\phi_{nk}(\mathbf{r}) \propto e^{-(x/\ell_B + k\ell_B)^2/2} H_n(x/\ell_B + k\ell_B) e^{iky}$, where H_n are Hermite polynomials. Imposing periodic boundary conditions (PBCs) along the y direction gives quantized values of $k = 2\pi m/L_y$ for integer m . PBCs along the x direction lead to a finite number of states $N_\phi = L_x L_y / 2\pi \ell_B^2$ per Landau level, which is also the number of magnetic flux quanta threaded through the system [22].

Assuming that the cyclotron energy is larger than the interaction energy (i.e., very fast rotation), the gas will reside fully in the LLL ($n = 0$) [1,14,23]. Projecting the field operators onto the LLL and accounting for the periodicity, i.e. $\hat{\psi}(\mathbf{r}) \rightarrow \sum_k \hat{a}_k [\sum_m \phi_{0,k+mN_\phi}(\mathbf{r})]$ [22], we get a Hamiltonian

$$\hat{H} = \frac{\ell_B}{L_y} \sum_{k_1 k_2 k_3 k_4} V_{k_1 k_2 k_3 k_4} \hat{a}_{k_1}^\dagger \hat{a}_{k_2}^\dagger \hat{a}_{k_3} \hat{a}_{k_4}, \quad (2)$$

where the potential energy is given by

$$V_{k_1 k_2 k_3 k_4} = \frac{g}{2\sqrt{2\pi} \ell_B^2} \sum_{m_1 m_2 m_3 m_4} \delta_{q_1 + q_2 - q_3 - q_4} \times \exp \left[-\frac{\ell_B^2}{2} \sum_{i=1}^4 q_i^2 + \frac{\ell_B^2}{8} \left(\sum_{i=1}^4 q_i \right)^2 \right]. \quad (3)$$

δ is the Kronecker delta and q_i is a shorthand for $q_i \equiv k_i + m_i N_\phi$. The kinetic energy in the LLL is a constant and has been removed from the Hamiltonian. These matrix

elements provide the scattering on the torus arising from the contact interactions (as seen in Fig. 1), the strength of which is conveniently represented by the Haldane pseudopotential for two particles with zero relative angular momentum, $V_0 = g/4\pi \ell_B^2$. The delta function conserves the total momentum modulo the additions of integer multiples of $2\pi N_\phi/L_y = L_x/\ell_B^2$ [24].

Our main numerical approach is the exact diagonalization of this Hamiltonian for N particles and N_ϕ flux quanta. We also study the time evolution of the striplike condensate $|\Psi(t=0)\rangle \propto (\hat{a}_0^\dagger)^N |0\rangle$ by integrating Schrödinger's equation $i\partial_t |\Psi(t)\rangle = \hat{H} |\Psi(t)\rangle$.

To analyze the numerical results, we choose the occupation numbers for each momentum state $\hat{\rho}_k \equiv \hat{a}_k^\dagger \hat{a}_k$ and their correlations as our observables. Additionally, we study the squared cloud width $\hat{x}^2/\ell_B^2 \equiv (\frac{1}{2} + \frac{1}{N} \sum_k (k\ell_B)^2 \hat{\rho}_k)$, which helps to clarify the long-term behavior of the cloud strip.

III. OVERVIEW OF NUMERICAL RESULTS

We first discuss the energy scales and densities determining the validity of mean-field descriptions: One expects GP theory to break down when the density of the condensate is sufficiently low and quantum effects dominate.

In equilibrium systems, the two-dimensional (2D) density n_{2D} and equivalently the filling fraction $\nu = 2\pi n_{2D} \ell_B^2$ are used as control parameters of the system. For instance, in studies of the ground state, it is known that the uncertainty in the position of a vortex is $\propto 1/\sqrt{n_{2D}}$. In a low-density regime where these quantum fluctuations become comparable to the vortex spacing, or $\nu \sim n_{2D} \ell_B^2 \lesssim \mathcal{O}(1)$, the vortices can no longer be treated as classical pointlike objects as in GP theory [1], instead transitioning into a strongly correlated regime [15].

In out-of-equilibrium systems, the density and the filling fraction are neither homogeneous nor constant, so they cannot be the sole control parameters of the system. Instead, we must also consider timescales or energy scales. In the mean-field GP theory, the parameter used is gn_{2D} , which is the typical energy per particle when the density is n_{2D} . However, when the gas becomes so dilute that each particle interacts only with a few others [again $n_{2D} \ell_B^2 \lesssim \mathcal{O}(1)$], the granularity becomes important. Indeed, the zeroth Haldane pseudopotential V_0 is the fundamental energy scale of two-body interactions. Thus, in the low-density regime where the mean-field energy gn_{2D} becomes comparable to V_0 , or equivalently when $n_{2D} \ell_B^2 \lesssim \mathcal{O}(1)$, GP results can no longer be expected to hold. Instead, the exact dynamics is governed by both V_0 and the density n_{2D} separately. Note that, when discussing the evolution of the striplike condensate, the density is more conveniently expressed in terms of a one-dimensional (1D) filling fraction $\nu_{1D} \equiv N \ell_B / L_y$ along the strip, which is also the initial density of the cloud.

Interestingly, these two considerations lead to essentially the same parameter: GP will not hold either in or out of equilibrium when the density is so dilute that the typical particle spacing locally cannot be considered to be much less than the magnetic length.

We present the simulated squared cloud width in Fig. 2 for two systems, which are representative of the low-density

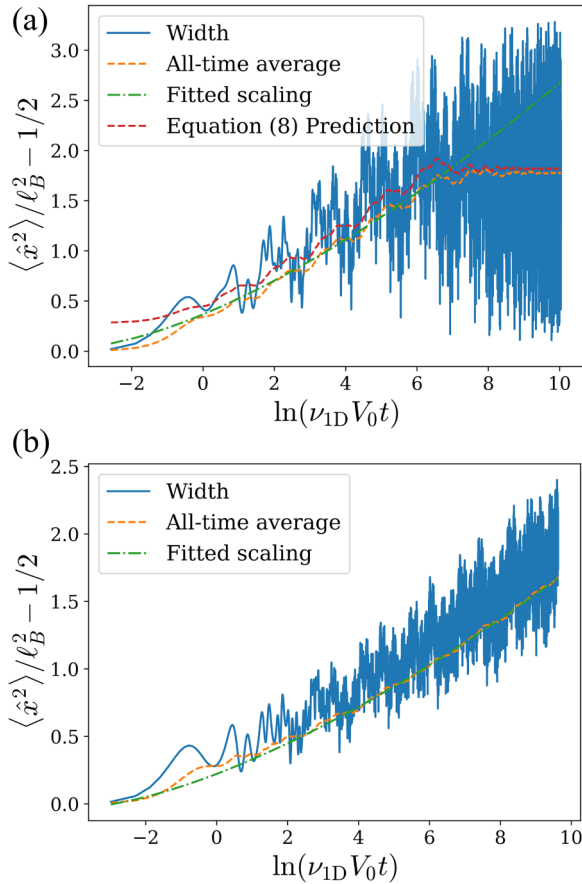


FIG. 2. The squared cloud width in (a) $N = 3$, $v_{\text{ID}} = 0.24$ and (b) $N = 4$, $v_{\text{ID}} = 0.16$ systems. The blue is the raw squared width, showing fast oscillations. The dashed orange curve is the averaged width for times from 0 to t , showing a long-term growth that is consistent with our $(\ln t)^{1.5}$ prediction (dash-dotted green curve) described in Sec. V C. The scaling is obtained from numerical fits. In $N = 3$, we have direct calculations based on Eq. (8) yielding the dashed red curves.

(strongly correlated) regime for which GP theory is inadequate. For the system to qualify as low density, the energy per particle $g(N-1)/2\sqrt{2\pi}L_y\ell_B$ must be comparable to V_0 , so $v_{\text{ID}} \lesssim 1/\sqrt{2\pi} \approx 0.40$. In the two examples, $v_{\text{ID}} = 0.24$ and 0.16 , respectively. Note that a logarithmic scale is used for time, while the width is displayed on a linear scale.

There are several interesting features to analyze and understand.

(1) The cloud width ($\langle \Psi(t) | \hat{x}^2 | \Psi(t) \rangle$, blue curve) shows rapid oscillations in time.

(2) These rapid oscillations can be removed with time averaging, which we do by defining an “all-time average” $\overline{\langle \hat{x}^2 \rangle}(T) \equiv \frac{1}{T} \int_0^T \langle \Psi(t) | \hat{x}^2 | \Psi(t) \rangle dt$. As opposed to a seemingly more natural time-averaging procedure in which one averages over windows in time, this quantity is used because our system contains exponentially different timescales (Sec. V C), and fixed window sizes cannot fully capture the evolution in all timescales. The all-time-averaged width shows periodic undulations in the logarithm of time while also showing an overall growth. This is particularly clear for the $N = 3$

system (with a period of ≈ 1.4) [Fig. 2(a)] but also present in all cases.

(3) There is an eventual saturation of the cloud width due to finite-size effects in the system. This is evident for $N = 3$, but not for $N = 4$ for the timescale shown in Fig. 2(b).

Here, we propose a theory of repulsively bound few-boson clusters that explains all of these features.

IV. NONINTERACTING CLUSTERS

A set of c contact-interacting bosons in the LLL always has an exact eigenstate with zero angular momentum (all c particles overlap completely) and energy $V_0 c(c-1)/2$. This is what we refer to as a cluster of size c : an object composed of a few particles and zero internal angular momentum.

Exact diagonalization studies have also shown that excited states can be viewed as being composed of separate clusters of this type. An example is the $N = 3$ three-body system, in which exact eigenstates feature a pair (a cluster of size $c = 2$) and a singlet (a cluster of size $c = 1$) and their energies approach V_0 as their angular momentum increases [25].

In the following sections, we will begin with a model of the N -particle systems as a set of noninteracting clusters. Concretely, a state with a definite cluster composition—a cluster state—satisfies

$$N = \sum_c c n_c, \quad E = \sum_c n_c \frac{c(c-1)}{2} V_0, \quad (4)$$

where E is the total energy and n_c is the number of c clusters. Each set of cluster numbers $\{n_c\}$ —with $c = 1, 2, 3, \dots$ representing what we will refer to as “singlets,” “doublets” (also “pairs”), “triplets,” etc.—describes a distinct decomposition, and we denote the cluster state by $|\{n_c\}\rangle$.

A. Fast oscillations

Within the cluster picture, the fast oscillations of the cloud width arise from the fact that the initial state is a superposition of cluster states of different $\{n_c\}$, and therefore of different energies [Eq. (4)].

The power spectra of the central density $\langle \hat{\rho}_0 \rangle$ and the density squared $\langle \hat{\rho}_0 \hat{\rho}_0 \rangle$ are shown in Fig. 3, showing sharp peaks in frequency space. The prominent frequency peak corresponds to the energy of the repulsively bound two-particle cluster, which is given by the Haldane pseudopotential $2\pi f = V_0$. Since this characterizes the energy of a completely overlapping pair of particles, it suggests that the relevant state has one pair of bosons and one isolated singlet, i.e., $|\{n_1 = 1, n_2 = 1\}\rangle$. Some other peaks can be explained as differences between cluster energies. For instance, the peak at $2V_0$ is obtained as the difference between the aforementioned $\{n_1 = 1, n_2 = 1\}$ state and the $\{n_3 = 1\}$ state, whose energy is $c(c-1)V_0/2 = 3V_0$ for $c = 3$. These observations give strong evidence that the dynamics of the cloud strip should be described in terms of weakly interacting few-body clusters. Clusters account for the main peaks at integer multiples of V_0 , while others must be explained with cluster-cluster interactions that we explore in Sec. V.

Curiously, despite the above interpretation of the $2V_0$ peak implying the existence of the $\{n_3 = 1\}$ state, the $3V_0$

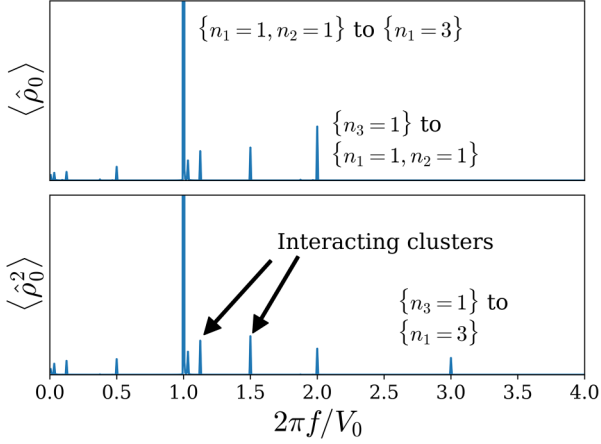


FIG. 3. The power spectra of $\langle \hat{\rho}_0 \rangle$ and $\langle \hat{\rho}_0^2 \rangle$ in the $N = 3$, $\nu_{1D} = 0.12$ system. Peaks corresponding to transitions between different cluster states can be seen, including the $3V_0 \rightarrow V_0$ transition. The $3V_0 \rightarrow 0$ transition is missing in $\langle \hat{\rho}_0 \rangle$ but is present in the $\langle \hat{\rho}_0^2 \rangle$ spectrum.

peak itself is not observed. This is because of “selection rules” that eliminate certain frequencies from the spectra. Consider the V_0 peak that comes from the matrix element $\langle \{n_1 = 3\} | \hat{\rho}_k | \{n_1 = 1, n_2 = 1\} \rangle$. The density operator has one creation and one annihilation operator each, so it can “break apart” the pair. The matrix element is thus nonzero, giving the transition frequency of V_0 . In contrast, the matrix element governing the $3V_0$ transition $\langle \{n_1 = 3\} | \hat{\rho}_k | \{n_3 = 1\} \rangle = 0$ because two particles must be moved to break up the triplet, but only one annihilation operator is available. Hence, we do not see a frequency peak at $3V_0$. On the other hand, when we examine the square of the densities, the $3V_0$ peak is present, because $\hat{\rho}_k^2$ can move two particles. In other words, the matrix element $\langle \{n_1 = 3\} | \hat{\rho}_k^2 | \{n_3 = 1\} \rangle \neq 0$.

The $N = 4$ spectra tell a similar story, with a $3V_0$ peak present in the $\langle \hat{\rho}_0 \rangle$ spectrum due to the $\{n_4 = 1\}$ to $\{n_1 = 1, n_3 = 1\}$ transition. The $4V_0$ and $5V_0$ peaks are present in the $\langle \hat{\rho}_0^2 \rangle$ spectrum due to the $\{n_4 = 1\}$ to $\{n_1 = 2, n_2 = 1\}$ and $\{n_2 = 2\}$ transitions. The $6V_0$ peak is absent in both spectra.

These selection rules further support the cluster picture, as shown in the lower panel of Fig. 3.

B. Energy and energy density correlator spectra

Given that the dynamical features of the cloud’s time evolution are explained from a cluster perspective, it is natural to ask whether all eigenstates of the Hamiltonian are cluster states.

In the density of states obtained via ED, the major peaks can be identified with the decompositions in Eq. (4), e.g., in Fig. 4(a). There are peaks at energies $0, V_0, 3V_0, 6V_0$, corresponding to four singlets ($\{n_1 = 4\}$, labeled A), pairs ($\{n_1 = 2, n_2 = 1\}$, labeled B), triplets ($\{n_1 = 1, n_3 = 1\}$, labeled D), and quadruplets ($\{n_4 = 1\}$, labeled E) of particles. The $2V_0$ peaks can be interpreted as two pairs ($\{n_2 = 2\}$, labeled C), a possibility in an $N = 4$ system.

Furthermore, given Eq. (4), the expectation value of local observable $\hat{O}$ in a state should be $\sum_c n_c \langle c | \hat{O} | c \rangle$, where

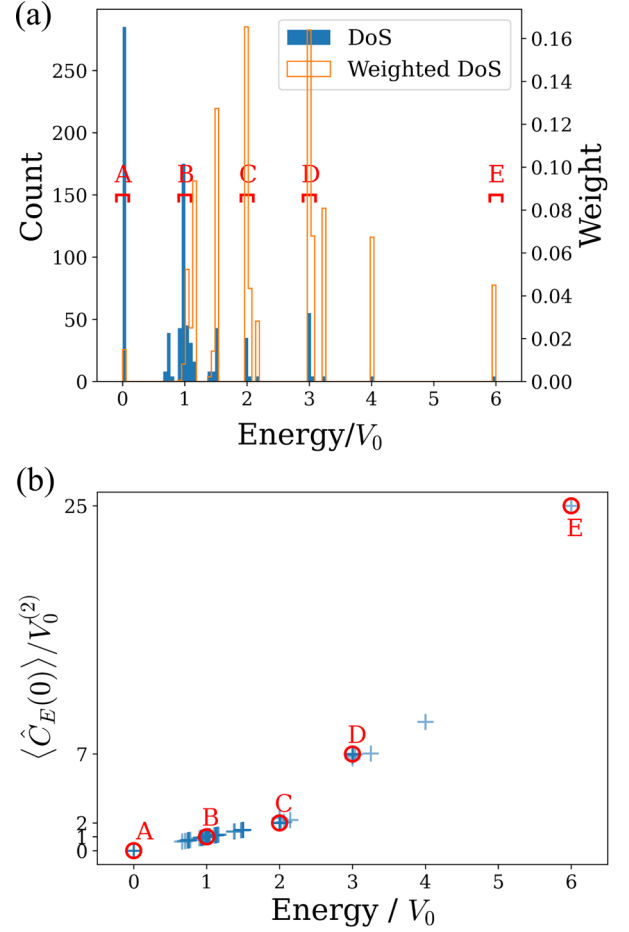


FIG. 4. The density of states for $N = 4$, $\nu_{1D} = 0.32$. In panel (a), the solid blue shows the density of states, while the hollow orange shows the density of states weighted by the their overlaps with the initial state $|\langle n | \Psi(t=0) \rangle|^2$. In panel (b), we plot expectation values of $\hat{C}_E(0)$ in the energy eigenstates. Each point is plotted with low opacity, so thick color indicates the presence of many points in close proximity. The circled states (labeled A–E) in panel (b) correspond to the energy brackets shown in red in panel (a) and cover the majority of the eigenstates. These states conform well to the decomposition into independent clusters as described by Eq. (4) and satisfy the prediction $\langle \hat{C}_E(0) \rangle = \sum_c n_c \langle c | \hat{C}_E(0) | c \rangle$. In particular, the states labeled C have energy $2V_0$ and $\langle \hat{C}_E(0) \rangle = 2V_0^{(2)}$, so can be interpreted as $n_2 = 2$ states. Note that, in this system, $\langle \hat{C}_E(0) \rangle$ depends linearly on energy even in the interacting cluster states (such as those at $1.5V_0$). This observation is explained in Appendix C.

$|c\rangle \propto (\hat{\psi}^\dagger)^c |0\rangle$. As a demonstration, we calculate the values of the (zero separation) energy density correlations $\hat{C}_E(0) \equiv \int d^2r \langle \frac{g}{2} \hat{\psi}^\dagger \hat{\psi}^\dagger \hat{\psi} \hat{\psi} \rangle^2$, where the field operators are projected onto the LLL. Denoting the expectation value in a pair state as $V_0^{(2)} \equiv V_0^2 / \pi \ell_B^2$, we find by explicit computation the expectation values in triplets to be $7V_0^{(2)}$ and those in quadruplets to be $25V_0^{(2)}$. Importantly, in a state of two pairs, the expectation value $\langle \hat{C}_E(0) \rangle$ should be $2V_0^{(2)}$, since we have $n_2 = 2$.

These predictions are confirmed numerically. In Fig. 4(b), the states labeled C, which we identify to be $\{n_2 = 2\}$ states, have $\langle \hat{C}_E(0) \rangle = 2V_0^{(2)}$. Similarly, the states labeled D, which

we identify to be $\{n_1 = 1, n_3 = 1\}$, have $\langle \hat{C}_E(0) \rangle = 7V_0^{(2)}$, as expected for the triplet.

This shows that many energy eigenstates can indeed be considered as combinations of independent clusters.

V. INTERACTING CLUSTERS

In the preceding sections, we showed how the energy spectra of a rotating Bose gas in the LLL with contact interactions can be deciphered with a picture of few-body clusters. This explains the fast oscillations in Fig. 2. Nonetheless, there are eigenstates whose energies cannot be explained in terms of energy differences of the noninteracting clusters [Eq. (4)]. To account for these, we must now include interactions between clusters. In fact, noninteracting clusters have no dynamics beyond the rapid oscillatory behavior described previously. In particular, there could be no gradual expansion of the cloud: If all energies are multiples of V_0 , the system must be periodic in time with a period $2\pi/V_0$. In this section, we construct a simple theory for the interactions between the clusters and use it to predict the long-term dynamics of the cloud.

A. Classical and semiclassical cluster interactions

We start by constructing a classical Hamiltonian from the interaction energy between clusters that captures their dynamics at low densities. The interaction energy between two singlets positioned at $\mathbf{r}$ and $\mathbf{r}'$ can be shown to be $V_0 \text{sech}[(\mathbf{r} - \mathbf{r}')^2/4\ell_B^2]$, while that between two larger clusters can be approximated by multiplying this formula by the sizes of the clusters. Importantly, the interaction energy is Gaussian: $\sim V_0 e^{-r^2/4\ell_B^2}$ as $|\mathbf{r} - \mathbf{r}'| \rightarrow \infty$.

We focus on interactions between pairs of clusters in subsequent analyses, since they are dominant in the low-density limit we are considering and are sufficient to explain the numerical results presented in Fig. 2. While interactions between three or more clusters are in principle present, they merely produce higher-order corrections and add fine structure to pairwise interactions.

From the interaction energy, we see classically that the dynamics depends on the logarithm of time. Consider two clusters separated by a distance r . The force between them is $F = -\nabla E \sim V_0 r e^{-r^2/4\ell_B^2}/\ell_B^2$ for large r , and the velocity due to the force is $F/eB \sim V_0 r e^{-r^2/4\ell_B^2}/\hbar$. For the motion to be appreciable on the scale of r , a time of $V_0 t/\hbar \sim e^{r^2/4\ell_B^2}$ is required, meaning that $r^2/\ell_B^2 \sim \ln(V_0 t/\hbar)$. Thus, a dependence on the logarithm of time arises naturally in the classical analysis. We shall see later how this emerges in the quantum picture too.

This classical interaction energy also manifests itself in the exact quantum spectrum. ED reveals that, for each independent cluster energy E in Eq. (4), the peak in the density of states (DoS) in fact comprises a series of states with energies approaching E exponentially. As an analytic example, in an infinite boundaryless $N = 3$ system, the nonzero energies appearing at angular momentum l are given by $[1 + 2(-1/2)^l]V_0$ for $l \neq 1$. The energy difference to V_0 halves for each subsequent state [25]. For larger numbers of particles, the ratios between subsequent energy differences are no longer exactly the same but still stay approximately constant (see

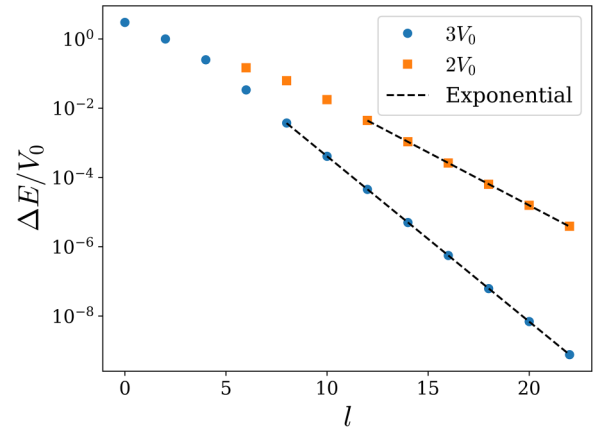


FIG. 5. The series of states as functions of CoM angular momenta l converging to $3V_0$ and $2V_0$ with $N = 4$ in an infinite system. Their energies approach the asymptotic values exponentially (linearly on a log plot). The black lines are straight lines to guide the eye.

Fig. 5). In finite systems, there is necessarily a cutoff set by the system size, since clusters cannot be further apart than the system size itself.

To explain this exponential trend, we invoke the semiclassical approach of Bohr-Sommerfeld quantization. In the LLL, this means that the area of closed orbits $r^2/\ell_B^2 \propto n$, with $n \in \mathbb{N}$. Therefore, the quantized interaction energies are $\sim \Delta_0 e^{-\alpha n}$ for some constant α and Δ_0 , correctly predicting the exponential series of energies.

The quantization of the orbit area can also be related to the quantization of angular momentum l by noting that in boundaryless systems

$$\langle r^2/\ell_B^2 \rangle = 2 + \frac{2l}{N}. \quad (5)$$

For our purposes, l should be interpreted as the angular momentum in the center-of-mass (CoM) frame.

In summary, interactions produce corrections to Eq. (4). The eigenstates form series around $\{n_c\}$ and can be labeled with their CoM angular momentum l as $\{|n_c, l\rangle$. Their energies are given by $E_{\{n_c, l\}} - E_{\{n_c\}} \sim \Delta_{0, \{n_c\}} e^{-\alpha l}$.

For later use, we compute the matrix elements $\langle \{n_c\}, l | \hat{x}^2 | \{n_c\}, l \rangle$ and $\langle \{n_c\}, l | \hat{x}^2 | \{n_c\}, l + 2 \rangle$. The former is analytically given by Eq. (5) and $\langle x^2 \rangle = \langle r^2 \rangle/2$ (with a constant offset on a torus). The latter too was numerically found to be very closely linear in l . The dependence is linear because the matrix element should be proportional to the areas of the clusters, which is linear in l [Eq. (5)].

B. The overlap of the initial state with cluster states

The semiclassical theory set out above thus accounts for the states at nonintegral multiples of V_0 in energy and predicts their properties, such as their sizes. Returning now to the dynamical problem, in order to analytically predict the dynamics of the cloud strip, we also need to know the weights of different cluster states in the initial state.

The total weight in the cluster series described with $\{n_c\}$ can be computed by taking the overlap between the initial

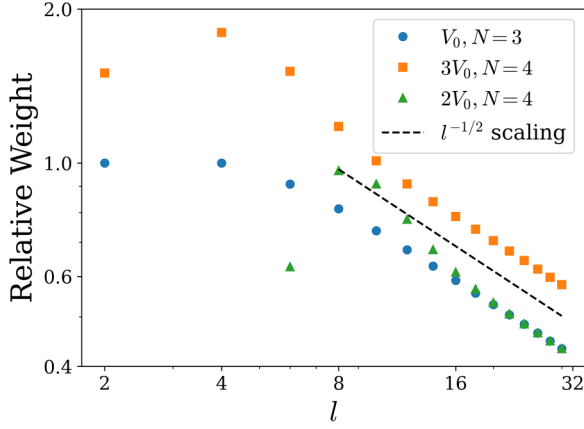


FIG. 6. The relative weights of cluster states at angular momenta l obtained numerically. Log-log scales are used. We see that the V_0 series in $N = 3$ and the $3V_0$ and $2V_0$ series in $N = 4$ all eventually approach a $l^{-0.5}$ scaling. The black line is to guide the eye.

state and $|\{n_c\}, \{\mathbf{r}_{c,i}\}\rangle \propto \prod_c \prod_{i=1}^{n_c} \hat{\psi}^\dagger(\mathbf{r}_{c,i})^c |0\rangle$, where $\mathbf{r}_{c,i}$ are the positions of the clusters. Also denote with $n = \sum_c n_c$ the total number of clusters. In particular, the predicted weight of the N -body peak in an N -particle system can be shown to be exactly $(2\sqrt{\pi}/L_y)^{N-1}\sqrt{N}$ by taking $n_N = 1$. For $n > 1$, the total weight in each cluster series is predicted to be

$$|w_{\{n_c\}}|^2 = \frac{N!}{\prod_c n_c! (c!)^{n_c}} \left(\frac{2\sqrt{\pi}\ell_B}{L_y} \right)^{N-n} \sqrt{\prod_c c^{n_c}}, \quad (6)$$

which is accurate in the low-density limit to leading order in $1/L_y$. This weight must be slightly modified in the thermodynamic limit where v_{1D} is held constant and $L_y \rightarrow \infty$, as discussed in Appendix A along with its derivation.

Equation (6) is also intuitive: The $(1/L_y)^{N-n}$ factor accounts for the fact that while n bosons can be placed arbitrarily, each additional boson after that has only a probability $\propto 1/L_y$ to be placed at one of those positions to form clusters. This formula predicts the formation of $\sqrt{2\pi}Nv_{1D}$ pairs in the initial state in the thermodynamic limit (correctly conserving energy), which is another intuitive result since the probability of two particles overlapping at random is $\propto v_{1D}$. These results apply to leading order in v_{1D} .

Within each cluster series, the weights as a function of angular momentum can be obtained numerically. Doing so produces the l dependence of relative weights in the V_0 series in $N = 3$ and $2V_0, 3V_0$ series in $N = 4$ shown in Fig. 6. The relative weight decreases approximately as $1/\sqrt{l}$ for sufficiently large l . This can be understood as follows: The initial state produces a uniform distribution of distances r between clusters, but $l \sim r^2$, so the distribution of l is $\sim 1/\sqrt{l}$.

When the cluster sizes become comparable to either L_x/N or L_y/N , these predictions break down.

C. Spreading dynamics

We can now explain the actual spreading dynamics of the cloud. The initial state is given by an expansion of the

form

$$|\Psi(t)\rangle = \sum_{\{n_c\}} \sum_l w_{\{n_c\},l} e^{-iE_{\{n_c\},l}t} |\{n_c\}, l\rangle. \quad (7)$$

Calculating the time-averaged squared width then yields

$$\begin{aligned} & \frac{1}{T} \int_0^T \langle \Psi(t) | \hat{x}^2 | \Psi(t) \rangle dt \\ &= \sum_{\{n_c\}} \left\{ \sum_{l=0}^L \langle l | \hat{x}^2 | l \rangle |w_l|^2 + 2 \sum_{l=0}^{L-2} \langle l | \hat{x}^2 | l+2 \rangle w_l w_{l+2} \right. \\ & \quad \times \left. \text{sinc} \left(\frac{1 - e^{-2\alpha}}{e^{\alpha l}} \Delta_0 T \right) \right\}, \end{aligned} \quad (8)$$

where the $\{n_c\}$ dependence inside the curly braces is suppressed. Note that we have only retained terms that are diagonal in $\{n_c\}$. Off-diagonal terms that involve matrix elements between $\{n_c\}$'s with different energies lead to fast oscillations and quickly integrate to zero. Off-diagonal terms that involve matrix elements between $\{n_c\}$'s with the same energy are forbidden by the selection rules, as $\hat{x}^2$ is a one-body operator. Extra terms involving these matrix elements will appear when the long-term behavior of two-body operators is considered.

This sum of sinc functions with exponentially different frequencies produces the periodic structure seen in Fig. 2: The main features of these functions are equally spaced in logarithmic time.

The presence of a maximum L in this formula is a finite-size effect; L depends on the size of the initial cloud as $L \sim (L_y/\ell_B)^2$. Taken at face value, this gives the saturation in cloud width in Fig. 2. One may object that angular momentum is not conserved on a torus, so the cloud should keep spreading. However, the typical size of the cloud after time evolution is $\sim \sqrt{L_y \ell_B} \ll L_y$, so boundary effects are exponentially suppressed and hardly felt.

The semiclassical theory thus predicts that the cloud strip grows by gradually losing coherences between different interacting states, and it saturates at a width given by the incoherent sum over the different angular momenta. For $N = 3$, plotting the prediction with Eq. (8) using calculated weights and matrix elements in the $\{n_1 = 1, n_2 = 1\}$ series gives the red curve in Fig. 2, showing excellent agreement with numerical data.

The scaling of the growth rate is given by the following analysis. The matrix elements were numerically found to scale as $\langle l | \hat{x}^2 | l+2 \rangle \sim l$, and we argued that the weights follow $|w_l w_{l+2}| \sim 1/\sqrt{l}$. We can think of the sinc functions as 1 for $\Delta_0 T \ll e^{\alpha l}$ and 0 for $\Delta_0 T \gg e^{\alpha l}$, with the transition happening when $l \sim \ln T$. Performing the sum up to $\ln T$ gives $\langle \hat{x}^2 \rangle(T) \sim (\ln T)^{1.5}$. The green curves in Fig. 2 show this scaling, agreeing very well with the simulation data.

VI. LONG-TIME DYNAMICS AND THERMALIZATION

When considering the thermalization of the gas cloud, one expects the ultimate long-time state to be a maximal entropy state for large numbers of particles. This is because the classical dynamics of guiding center drift is chaotic for $N \geq 4$ particles [26], so one expects the long time dynamics to be

ergodic. When the volume accessible to the gas is large, the maximal entropy state at fixed energy density is a state with one “megacluster” with all of the energy and many singlets freely distributed in space. To carry all the initial energy $E \sim \nu_{1D} N V_0$, this megacluster must have a size $N_f \sim \sqrt{\nu_{1D} N}$. The remaining $N - N_f$ particles freely explore the phase space and provide the entropy. This is analogous to the distribution of Onsager vortices, which shows clustering with diffuse tails [27]. Numerically, this was confirmed by noting that $\langle \hat{C}_E(0) \rangle$ at energies with multiple cluster decompositions (such as $3V_0$ in $N = 6$ can be made of $n_2 = 3$ or $n_1 = 3, n_3 = 1$) are much more concentrated around the megacluster values (such as $7V_0^{(2)}$ in this example). See Appendix B for more details. GP simulations also qualitatively bear this out by showing the coalescence of cloud fragments.

During thermalization, one may think of smaller clusters gradually merging into larger ones, eventually resulting in the megacluster with all the energy of the system. At each step, due to the large number of particles that need to be rearranged, the time required for the merger of c clusters on a length scale of R can be estimated to be $\sim e^{cR^2}$, with some numerical factors in the exponent. This can be understood as a Fermi golden rule type result from the matrix element between different cluster decompositions. The rate-limiting step in the entire thermalization process is the last step: the formation of an $N_f \sim \sqrt{N\nu_{1D}}$ cluster on a scale of $R \sim L_y$, taking $e^{\sqrt{N\nu_{1D}}L_y^2}$ time.

The logarithmic growth of observables and rates of thermalization presented in our study is very different from the power-law dependence predicted in Ref. [21] based on a model of fracton hydrodynamics. The reason is that the geometries are very different: Our system involves a strip condensate expanding into the vacuum, whereas systems of fixed density were studied in Ref. [21]. To recover our results, one should allow for a density-dependent diffusion coefficient in the hydrodynamic description, and, in particular, one that decays exponentially as density lowers (which can be developed along the same lines as our classical cluster interaction picture). This would then predict a “hydrodynamics” with our logarithmic timescales.

VII. COMPARISON WITH GROSS-PITAEVSKII THEORY

It is instructive to compare the ED results with the time-dependent GPE. It is known that the GP theory is exact when the filling fraction tends to infinity while the interaction energy remains finite [28]. For the striplike condensate, the limits are $\nu_{1D} \rightarrow \infty$ and $gN/(L_y \ell_B) = g\nu_{1D}/\ell_B^2 \rightarrow \text{constant}$.

One can immediately note that, since $g/\ell_B^2 \sim 1/\nu_{1D} \rightarrow 0$ in these limits and $V_0 \propto g/\ell_B^2$ sets the energy scale of the dynamics of the clusters, in the GP regime, there must be no signature of the microscopic clusters of the form we found in exact diagonalization. This is unsurprising because the initial particle density $N/(L_y \ell_B)$ merely determines the timescale of the GP dynamics. This was confirmed numerically. Consider the evolution from low density (where the cluster picture applies) to the regime of high density (where GP theory is accurate). One expects this to be a gradual crossover as behaviors such as the fragmentation of the condensate persist into the GP regime. We may attempt to estimate the densities

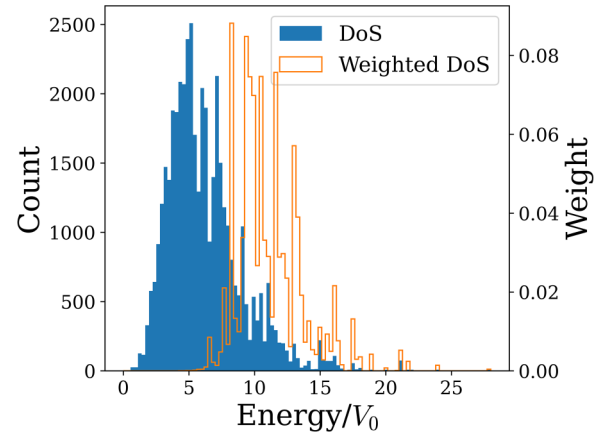


FIG. 7. The ED DoS in $N = 8, N_\phi = 17, L_y = 4\pi \ell_B$ ($\nu_{1D} \approx 0.64$), similar to Fig. 4. Peaks are all smeared out due to interactions. The DoS increases with energy below $\approx 5V_0$ because the number of possible positional arrangements is limited on the small torus for states with a large number of clusters like $\{n_1 = 8\}$.

at which the crossover happens. Numerically, with an $N = 8$ system, we obtained a characteristic GP frequency of around $\approx 0.063 \times gN/(L_y \ell_B)$. Since the GP frequency scales as ν_{1D} , they become comparable to the lowest cluster frequencies of order V_0 when the density becomes $\nu_{1D} \approx 1$.

One sees a hint of how the cluster picture breaks down as the density increases in Fig. 7, which shows the density of states for a system with $N = 8, \nu_{1D} \approx 0.6$. The peaks at lower energies $E \leq 5V_0$ become completely washed out, and the identifiable peaks at $6V_0$ and $7V_0$ are also broadened. This can be qualitatively understood as the peaks being broadened by interactions and eventually overlapping.

VIII. EXPERIMENTAL SIGNATURES

We are now in a position to discuss the prospects of experimentally observing the beyond-mean-field signatures of the clusters. The most natural experimental probe employed in current experiments is the quantum gas microscope, with which one can take snapshots of the positions of all atoms with single-atom resolution. This enables, for instance, direct measurements of the time evolution of the squared width $\langle x^2 \rangle$ from a series of destructive measurements, all with the same initial condition.

A quantum gas microscope also straightforwardly yields the correlation functions $g_2, g_3, \dots$ through the distribution of the atoms. Here, we study the correlation function

$$g_2(\Delta \mathbf{r}) \equiv \int \langle \hat{\psi}^\dagger(\mathbf{r}) \hat{\psi}^\dagger(\mathbf{r} + \Delta \mathbf{r}) \hat{\psi}(\mathbf{r} + \Delta \mathbf{r}) \hat{\psi}(\mathbf{r}) \rangle d^2 r. \quad (9)$$

Note that while $g_2(0)$ is usually a helpful diagnostic tool for the state of the system (e.g., condensed versus noncondensed states), in our system it is a constant given by the system's energy. We therefore have to look at the position dependence of $g_2(\Delta \mathbf{r})$ to get more information.

Some cross sections of g_2 as functions of time are shown in Fig. 8. Several qualitative comments can be made.

(1) The gradual widening of $g_2(\Delta x, 0)$ corroborates the general growth of the cloud width.

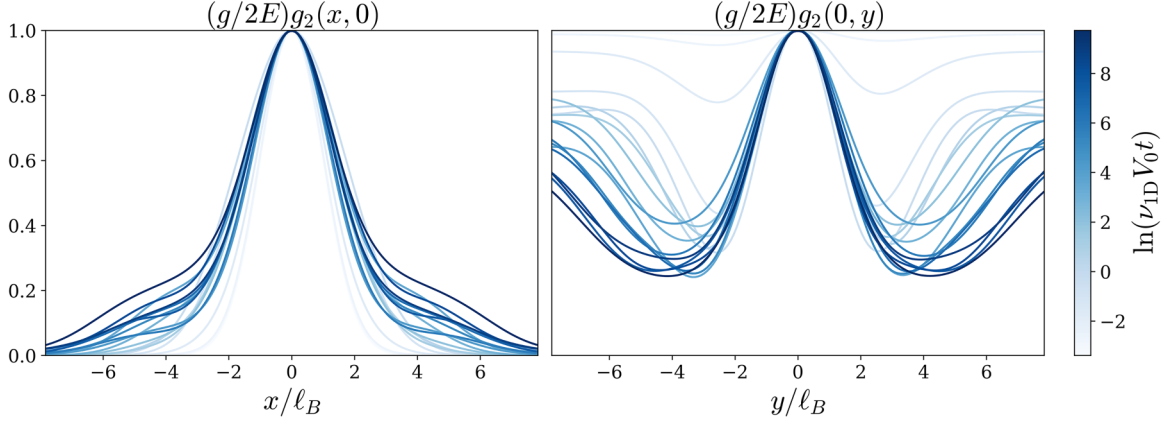


FIG. 8. Sections of $g_2(\Delta \mathbf{r})$ for $N = 4$, $N_\phi = 47$, $L_y = 6\pi \ell_B$ along the x and y directions as functions of time. The correlations are normalized by the total energy. Darker colors indicate later times on a logarithmic scale, as shown in the color bar. The x cross section starts as a Gaussian and expands outward in time. The y cross section starts as a constant, develops minima as predicted by Bogoliubov theory, and plateaus at large Δy . The minima also expands outward with time.

(2) The initial development of minima in $g_2(0, \Delta y)$ agrees with existing experimental findings and the Bogoliubov analysis [19]. The plateaus at large Δy can be cast in the light of independent clusters—The clusters do not care about their mutual separations once they form.

(3) The outward expansion of the minima in $g_2(0, \Delta y)$ at later times is a sign of both the growth of the cluster and their separations. Indeed, one expects that at late times the clusters become well separated in space and are easily identifiable in quantum gas microscopy.

Additionally, while not obvious from the cross sections in logarithmic time, the peak around $\Delta \mathbf{r} = 0$ has deformations (originating from the initial anisotropy) that rotate, producing the same frequencies as described in Sec. IV A. Hence, $g_2(\Delta \mathbf{r})$ can also reveal information about the clusters.

Nevertheless, a key difficulty in the experimental detection of the beyond-mean-field signals is that V_0 is small in current systems. The original experiment [19] used a scattering length of $a_s = 3.3$ nm and $M = 3.82 \times 10^{-26}$ kg for sodium atoms, giving an interaction strength of $g = \frac{4\pi\hbar^2 a_s}{M} \approx 10^{-50}$ J m³. The trap had an axial frequency $\omega_z = 2.8\omega$, $\omega = 2\pi \times 88.6$ Hz, so the effective 2D interaction strength is $g_{2D} = 3.6 \times 10^{-45}$ J m². The experiment also had a magnetic length of $\ell_B = 1.6$ μ m. Hence, we have $V_0 = \hbar \times 1.1$ Hz. This means that the period of oscillations associated with the cluster energy V_0 is $T = 2\pi\hbar/V_0 = 5.9$ s, which could be difficult to achieve.

Doubling both the interaction strength and the rotation speed would reduce it by a factor of 4. More interestingly, using the theory of clusters, we can consider higher frequency peaks, say $10V_0$ (e.g., between a quintuplet and five singlets). According to the selection rules, this could be observed with the correlation function g_5 . While a full measurement of g_5 would be very laborious, one can extract only relevant information by measuring the fifth moment of observables, such as $\langle (\hat{a}_k^\dagger \hat{a}_k)^5 \rangle$. Combining these measures gives a period of ≈ 150 ms, which is more accessible in experiments.

Another difficulty is that a low density $\nu_{1D} \lesssim 1$ is required for cluster behavior to become apparent. However, since the

system is far from being homogeneous, it will be possible to observe clusters by restricting one's attention to the edges of the cloud where the density is lower. There, signatures such as cluster frequencies at $V_0/\hbar$ should be present. Note that, because the energy density is proportional to the particle density squared, in the low-density region where the energy is primarily held in pairs, the density of pairs becomes the square of the density of particles. However if one has the imaging resolution to detect the pairs in the low-density regions, these would behave similarly to the ones explored in this work.

Knowing the states are clusters, one could also envisage applications such as spectroscopically addressing precisely n -body clusters. For instance, one potential way to address the pairs with rf pulses is to use scattering lengths' dependence on the hyperfine states of the atom. Since $V_0 \propto a_s$, the cluster energy of two atoms in hyperfine state F is different from two atoms in state F' by up to $\sim 0.6V_0$ [29,30]. This means that, when using an rf pulse to drive the $F \rightarrow F'$ transition, the atoms in pair states could be off-resonance and stay in the F state. By choosing F' to be an untrapped state, it is possible to create a state where only pairs remain.

IX. DISCUSSIONS AND CONCLUSIONS

In conclusion, we have shown that the dynamical evolution of interacting bosons in the lowest Landau level at low filling fractions can be understood in terms of largely noninteracting clusters and that observables show oscillations at a frequency V_0 characteristic of two-body contact interactions. Furthermore, long-term dynamical features can be understood with semiclassical interactions between pairs of clusters.

From our considerations of the striplike condensate, three different processes and timescales emerge. First, when the filling fraction $\nu_{1D} \gtrsim 1$, the mean-field GP timescale gn_{2D} is the fastest and can be thought of as the internal timescale of large clusters with typical particle number $\nu_{1D}\ell_B$. However, when $\nu_{1D}\ell_B \lesssim 1$, mean-field theory breaks down and this timescale becomes irrelevant. Instead, there is the fast timescale τ given by $V_0\tau \sim 1$ corresponding to the internal motion of (small)

clusters. Second, there is the timescale of interactions between pairs of clusters, which produces their motions. Motions on the length scale R have timescales $\ln V_0 \tau \sim R^2$ up to some constant power, giving us the spreading dynamics we explored. Finally, there is the timescale of the merger of clusters and of thermalization. Clusters of size c separated by a length scale R merge on the timescale $\ln V_0 \tau \sim cR^2$ (subject to energy conservation), so clusters merge on increasingly large scales as time goes on. By examining the number and mean separation of c clusters with quantum gas microscopy, this aspect of the thermalization of clusters could be probed. Full thermalization takes a time of $\ln V_0 \tau \sim \sqrt{N\nu_{\text{ID}}} L_y^2$. This, of course, corresponds to an exceedingly long time, so in realistic scenarios, the clusters may not merge fully.

Interesting connections can also be made with quantum many-body scars. The clusters can be viewed as classical bound states that are stable over time $T \gg \hbar/V_0$. These are thus reminiscent of the “unstable classical periodic orbits” central to quantum scarring [31–33]. Indeed, the behavior we saw in logarithmic time speaks to the clusters’ robustness. The almost classical motion of the clusters also bears resemblance to the “quantum trails” in phase space [34].

Our work also invites further questions, such as how to address and manipulate the clusters in the LLL and how to understand exchange effects between the clusters. It is also interesting to study the behavior of the bosons on a lattice, similar to the Harper-Hofstadter model, since the required densities and interaction strengths could be easier to achieve in optical lattices [16].

ACKNOWLEDGMENTS

We thank Zoran Hadzibabic, Claudio Castelnovo, and Andrea Pizzi for helpful discussions. This work was supported by the EPSRC (Grants No. EP/V062654/1 and No. EP/Y01510X/1) and a Simons Investigator Award (Grant No. 511029). Y.Y. acknowledges support from the Cambridge International Scholarship by the Cambridge Trust.

DATA AVAILABILITY

The code used to generate and analyze the data is available [35].

APPENDIX A: WEIGHTS IN CLUSTER SERIES

Here, we provide a derivation of Eq. (6).

As described in the main text, the total weight in a given cluster configuration is given by

$$w = \frac{\langle \Psi_0 | \{n_c\}, \{\mathbf{r}\} \rangle \langle \{n_c\}, \{\mathbf{r}\} | \Psi_0 \rangle}{\langle \Psi_0 | \Psi_0 \rangle \langle \{n_c\}, \{\mathbf{r}\} | \{n_c\}, \{\mathbf{r}\} \rangle} \mathcal{D}\mathbf{r}, \quad (\text{A1})$$

where $\Psi_0 \propto (a_0^\dagger)^N |0\rangle$ is the initial state, $|\{n_c\}, \{\mathbf{r}\}\rangle \propto \prod_c \prod_{i=1}^{n_c} \hat{\psi}^\dagger(\mathbf{r}_{c,i})^c |0\rangle$, and $\mathcal{D}\mathbf{r}$ indicates an integration over all positional configurations.

We see that in the dilute limit, $\langle \{n_c\}, \{\mathbf{r}\} | \{n_c\}, \{\mathbf{r}\} \rangle \sim \frac{1}{(2\pi)^N} \prod_c (c!)^{n_c}$ because the leading contribution comes from configurations where $\mathbf{r}_{c,i}$ are all distinct. The factors of 2π arise because the field operators projected onto the LLL are not normalized. Similarly, $\langle \Psi_0 | \Psi_0 \rangle = N!$. The numerator is

easily evaluated using Wick’s theorem, yielding

$$w = \frac{(2\pi)^N N!}{\prod_c (c!)^{n_c}} \int d^2 r_1 d^2 r_2 \dots |\phi_0(\mathbf{r}_1)|^{2c_1} |\phi_0(\mathbf{r}_2)|^{2c_2} \dots \quad (\text{A2})$$

The Gaussian integrals can be performed, resulting in $\frac{1}{(\sqrt{\pi} L_y)^N} \prod_c (L_y \sqrt{\pi/c})^{n_c}$. Now, the states created by $\hat{\psi}^\dagger$ are overcomplete, so to compensate for that, one requires that a c cluster must have a weight of 1 to be in a c cluster. This gives a correction of $c/2\pi$ per cluster, leading to

$$w = \frac{N!}{\prod_c (c!)^{n_c}} \left(\frac{2\sqrt{\pi} \ell_B}{L_y} \right)^{N-n} \sqrt{\prod_c c^{n_c}}. \quad (\text{A3})$$

Finally, when integrating over cluster configurations, we must note that different clusters (of the same size) are indistinguishable, giving the $1/\prod_c n_c!$ factor, proving Eq. (6).

From this equation, we see that in the limit where $L_y \rightarrow \infty$, the $n_1 = N$ and the $n_1 = N - 2, n_2 = 1$ terms alone reproduce the total energy of the system, while other terms are negligible.

In the thermodynamic limit, one needs to account for the fact that different clusters cannot overlap, or they would count as one cluster. This produces a correction factor to the volume integrals when integrating over $\mathcal{D}\mathbf{r}$. Alternatively, the normalization factor $\langle \{n_c\}, \{\mathbf{r}\} | \{n_c\}, \{\mathbf{r}\} \rangle$ is larger whenever two clusters overlap, reducing the overall overlap. Taking the former approach and stipulating that clusters must not overlap, the numerator is modified. Instead of n independent integrations along y producing L_y^n , a back-of-the-envelope calculation produces a factor of the following type:

$$\begin{aligned} & L_y (L_y - \ell_B) \dots [L_y - (n-1)\ell_B] \\ &= L_y^n \exp \left[\sum_{k=0}^{n-1} \ln \left(1 - \frac{k\ell_B}{L_y} \right) \right] \approx L_y^n \exp \left(-\frac{n^2 \ell_B}{2L_y} \right). \end{aligned} \quad (\text{A4})$$

Here, the “excluded volume” of each cluster is assumed to be ℓ_B . There could be numerical factors within the exponent. One then sees that this suppresses the divergent “probabilities” in the limit the number of clusters $n \rightarrow \infty$ while n/L_y remains constant. Such a term does not change the predicted number of pairs in the low ν limit.

APPENDIX B: MERGER OF CLUSTERS

One way to study the merger of clusters is through the operator $\hat{C}_E(0)$ as we did in Fig. 4. An example for $N = 6$ is shown in Fig. 9.

This $N = 6$ system differs from the $N = 4$ system previously considered in that there are multiple possible cluster decompositions $\{n_c\}$ at the same energy. Consequently, instead of the one-to-one correspondence between the energy and $\langle C_E(0) \rangle$, we see a mixture between the different possibilities. This is an indication that, subject to energy conservation, clusters will merge and change their identities eventually.

The higher energy states shown in the figure consist of clusters that are closer together, and as described in the main

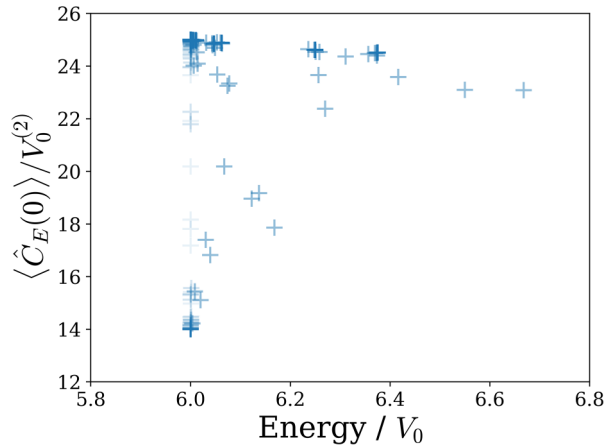


FIG. 9. Eigenstates in a six-particle system around $E = 6V_0$. Plotted quantities are the same as in Fig. 4(b). At this energy, the state could be comprised of two triplets ($n_3 = 2$) or one quadruplet and two singlets ($n_2 = 1, n_4 = 1$). The former leads to $\langle \hat{C}_E(0) \rangle = 14V_0^{(2)}$; the latter leads to $\langle \hat{C}_E(0) \rangle = 25V_0^{(2)}$. Indeed, results for $\langle \hat{C}_E(0) \rangle$ do appear to be grouped around these values. The presence of states with $\langle \hat{C}_E(0) \rangle$ in between these two values indicate mixing of the cluster decompositions in the energy eigenstates. At higher energies, the clusters are closer together, and tunneling between the different cluster configurations is easier. Consequently, the mixing of cluster configurations is thorough, giving at each energy one single $\langle \hat{C}_E(0) \rangle$ value between the two noninteracting values.

text, the matrix elements between different decompositions are greater. The mixing between decompositions appears quite

thorough for the high-energy states, as there is a one-to-one correspondence between energy and $\langle \hat{C}_E(0) \rangle$ at higher energies. As the clusters become increasingly separated (higher CoM angular momentum), the matrix element falls off as a power of a Gaussian. This means that the numerical algorithms cannot be trusted to produce the true eigenstates, and numerically we obtain states at the unmixed $\langle \hat{C}_E(0) \rangle$ values. We still expect the clusters to merge even for these eigenstates despite the smallness of the matrix elements, and the eigenstates will obey the eigenstate thermalization hypothesis based on the very general arguments laid out in the main text.

APPENDIX C: ZERO-SEPARATION ENERGY DENSITY CORRELATIONS IN THREE-BODY SYSTEMS

Here, we explain why $\langle \hat{C}_E(0) \rangle \propto E$ for the states between 0 and $2V_0$ in Fig. 4(b). In the language of Sec. V, those states are essentially $|n_1 = 1, n_2 = 1, l\rangle$ with one more irrelevant singlet.

Consider the operator $\hat{C}_E(0) \equiv \int d^2r (\frac{g}{2} \hat{\psi}^\dagger \hat{\psi}^\dagger \hat{\psi} \hat{\psi})^2$. Rearranging the field operators, we find that

$$\frac{\hat{C}_E(0)}{4\pi^3 \ell_B^6 V_0^{(2)}} = \hat{P}_4 + \frac{4}{2\pi \ell_B^2} \hat{P}_3 + \frac{2}{(2\pi \ell_B^2)^2} \hat{P}_2, \quad (C1)$$

where we defined $\hat{P}_n \equiv \int d^2r (\hat{\psi}^\dagger)^n \hat{\psi}^n$. In a three-particle state, $\langle \hat{P}_4 \rangle = 0$, while $\langle \hat{P}_3 \rangle \neq 0$ if and only if the internal angular momentum $l = 0$. We also have $\langle \hat{P}_2 \rangle = E/2\pi \ell_B^2 V_0$, so for the $l > 0$ states, $\langle \hat{C}_E(0) \rangle / V_0^{(2)} = E/V_0$. In the $l = 0$ state (the $3V_0$ state), the $\langle \hat{P}_3 \rangle$ term also contributes, giving the result $\langle \hat{C}_E(0) \rangle = 7V_0^{(2)}$ quoted in the main text.

-
- [1] N. R. Cooper, Rapidly rotating atomic gases, *Adv. Phys.* **57**, 539 (2008).
 - [2] A. L. Fetter, Rotating trapped Bose-Einstein condensates, *Rev. Mod. Phys.* **81**, 647 (2009).
 - [3] S. Viefers, Quantum Hall physics in rotating Bose-Einstein condensates, *J. Phys.: Condens. Matter* **20**, 123202 (2008).
 - [4] I. Bloch, J. Dalibard, and W. Zwerger, Many-body physics with ultracold gases, *Rev. Mod. Phys.* **80**, 885 (2008).
 - [5] M. R. Matthews, B. P. Anderson, P. C. Haljan, D. S. Hall, C. E. Wieman, and E. A. Cornell, Vortices in a Bose-Einstein condensate, *Phys. Rev. Lett.* **83**, 2498 (1999).
 - [6] K. W. Madison, F. Chevy, W. Wohlleben, and J. Dalibard, Vortex formation in a stirred Bose-Einstein condensate, *Phys. Rev. Lett.* **84**, 806 (2000).
 - [7] P. C. Haljan, I. Coddington, P. Engels, and E. A. Cornell, Driving Bose-Einstein-condensate vorticity with a rotating normal cloud, *Phys. Rev. Lett.* **87**, 210403 (2001).
 - [8] A. S. Bradley, C. W. Gardiner, and M. J. Davis, Bose-Einstein condensation from a rotating thermal cloud: Vortex nucleation and lattice formation, *Phys. Rev. A* **77**, 033616 (2008).
 - [9] C. Pethick, *Bose-Einstein Condensation in Dilute Gases*, 2nd ed. (Cambridge University Press, Cambridge, 2008).
 - [10] V. Bretin, S. Stock, Y. Seurin, and J. Dalibard, Fast rotation of a Bose-Einstein condensate, *Phys. Rev. Lett.* **92**, 050403 (2004).
 - [11] V. Schweikhard, I. Coddington, P. Engels, V. P. Mogendorff, and E. A. Cornell, Rapidly rotating Bose-Einstein condensates in and near the lowest Landau level, *Phys. Rev. Lett.* **92**, 040404 (2004).
 - [12] N. R. Cooper, E. H. Rezayi, and S. H. Simon, Vortex lattices in rotating atomic Bose gases with dipolar interactions, *Phys. Rev. Lett.* **95**, 200402 (2005).
 - [13] X.-C. Yao, H.-Z. Chen, Y.-P. Wu, X.-P. Liu, X.-Q. Wang, X. Jiang, Y. Deng, Y.-A. Chen, and J.-W. Pan, Observation of coupled vortex lattices in a mass-imbalance Bose and Fermi superfluid mixture, *Phys. Rev. Lett.* **117**, 145301 (2016).
 - [14] N. K. Wilkin, J. M. F. Gunn, and R. A. Smith, Do attractive bosons condense? *Phys. Rev. Lett.* **80**, 2265 (1998).
 - [15] N. R. Cooper, N. K. Wilkin, and J. M. F. Gunn, Quantum phases of vortices in rotating Bose-Einstein condensates, *Phys. Rev. Lett.* **87**, 120405 (2001).
 - [16] J. Léonard, S. Kim, J. Kwan, P. Segura, F. Grusdt, C. Repellin, N. Goldman, and M. Greiner, Realization of a fractional quantum Hall state with ultracold atoms, *Nature (London)* **619**, 495 (2023).
 - [17] R. Yao, S. Chi, B. Mukherjee, A. Shaffer, M. Zwierlein, and R. J. Fletcher, Observation of chiral edge transport in a rapidly rotating quantum gas, *Nat. Phys.* **20**, 1726 (2024).
 - [18] R. J. Fletcher, A. Shaffer, C. C. Wilson, P. B. Patel, Z. Yan, V. Crépel, B. Mukherjee, and M. W. Zwierlein, Geometric

- squeezing into the lowest Landau level, *Science* **372**, 1318 (2021).
- [19] B. Mukherjee, A. Shaffer, P. B. Patel, Z. Yan, C. C. Wilson, V. Crépel, R. J. Fletcher, and M. Zwierlein, Crystallization of bosonic quantum Hall states in a rotating quantum gas, *Nature (London)* **601**, 58 (2022).
- [20] S. Sinha and G. V. Shlyapnikov, Two-dimensional Bose-Einstein condensate under extreme rotation, *Phys. Rev. Lett.* **94**, 150401 (2005).
- [21] C. Zerba, A. Seidel, F. Pollmann, and M. Knap, Emergent fracton hydrodynamics of ultracold atoms in partially filled Landau levels, *PRX Quantum* **6**, 020321 (2025).
- [22] D. Yoshioka, B. I. Halperin, and P. A. Lee, Ground state of two-dimensional electrons in strong magnetic fields and 1/3 quantized Hall effect, *Phys. Rev. Lett.* **50**, 1219 (1983).
- [23] M. Lewin and R. Seiringer, Strongly correlated phases in rapidly rotating Bose gases, *J. Stat. Phys.* **137**, 1040 (2009).
- [24] F. D. M. Haldane, Many-particle translational symmetries of two-dimensional electrons at rational Landau-level filling, *Phys. Rev. Lett.* **55**, 2095 (1985).
- [25] S. Mashkevich, S. Matveenko, and S. Ouvry, Exact results for the spectra of bosons and fermions with contact interaction, *Nucl. Phys. B* **763**, 431 (2007).
- [26] A. Botero and F. Leyvraz, The two-dimensional three-body problem in the large magnetic field limit is integrable, *Phys. Lett. A* **380**, 2211 (2016).
- [27] M. T. Reeves, K. Goddard-Lee, G. Gauthier, O. R. Stockdale, H. Salman, T. Edmonds, X. Yu, A. S. Bradley, M. Baker, H. Rubinsztein-Dunlop, M. J. Davis, and T. W. Neely, Turbulent relaxation to equilibrium in a two-dimensional quantum vortex gas, *Phys. Rev. X* **12**, 011031 (2022).
- [28] E. H. Lieb and R. Seiringer, Derivation of the Gross-Pitaevskii equation for rotating Bose gases, *Commun. Math. Phys.* **264**, 505 (2006).
- [29] E. Tiesinga, C. Williams, P. Julienne, K. Jones, P. Lett, and W. Phillips, A spectroscopic determination of scattering lengths for sodium atom collisions, *J. Res. Natl. Inst. Stand. Technol.* **101**, 505 (1996).
- [30] M. Egorov, B. Opanchuk, P. Drummond, B. V. Hall, P. Hannaford, and A. I. Sidorov, Measurement of *s*-wave scattering lengths in a two-component Bose-Einstein condensate, *Phys. Rev. A* **87**, 053614 (2013).
- [31] A. Pizzi, L.-H. Kwan, B. Evrard, C. B. Dag, and J. Knolle, Genuine quantum scars in many-body spin systems, *Nat. Commun.* **16**, 6722 (2025).
- [32] C. J. Turner, Weak ergodicity breaking from quantum many-body scars, *Nat. Phys.* **14**, 745 (2018).
- [33] M. Serbyn, D. A. Abanin, and Z. Papić, Quantum many-body scars and weak breaking of ergodicity, *Nat. Phys.* **17**, 675 (2021).
- [34] A. Pizzi, Quantum trails and memory effects in the phase space of chaotic quantum systems, *Phys. Rev. Lett.* **134**, 140402 (2025).
- [35] Y. Yang, LLLBosonDynamics, Apollo—University of Cambridge Repository, (2026) <https://doi.org/10.17863/CAM.127485>.