

Dual-mode phonon dynamics and lasing in optomechanical cavities

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We use the full nonlinear bifurcation theory as a powerful methodology to thoroughly classify and predict the phonon lasing dynamics in optomechanical cavities. We exemplify its scope in the very relevant and so far vaguely explored dynamics of dual-mode phonon lasing when two independent mechanical modes are directly coupled to one optical field. We uncover a plethora of different lasing regimes in a fixed and realistic cavity geometry, including single- and dual-mode lasing, and an example of chaos. All dynamical regimes are unambiguously associated with bifurcations of different natures, which may exhibit both super- and subcritical natures characterizing the often hysteretic behavior of the systems when they are externally driven by a time-varying laser source. Our results, generalizable to any other optomechanical system, open unprecedented pathways to understand and control the formation and dynamics of optomechanical frequency combs.

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I. INTRODUCTION

Optomechanics studies the interaction between light and mechanical waves in both quantum and classical realms, ranging from individual photons to classical light beams as well as from molecular phonons to macroscopic elastic waves [1–5]. When confined in cavities, light and mechanical waves can interact, affecting the dynamics of each other via the so-called optomechanical (OM) backaction [6]. Under a regime of blue-detuned laser driving, OM backaction yields phonon lasing [6], which in a cascaded effect results in the formation of highly coherent OM frequency combs with spacing equal to the mechanical resonance (typically below 10 GHz, depending on the oscillator size). Such OM frequency combs extend over a bandwidth much narrower than purely optical combs (reaching even more than one octave [7–9]), but the fact that the line spacing is much smaller than in the latter case makes them very appealing for applications such as all-optical frequency conversion [10], sensing [11], or particle levitation [12].

The interaction between optical and mechanical modes is highly nonlinear and complex. However, many phenomena appearing in optomechanics are explained with a linearized model that only involves the interaction between a single mechanical and an optical mode. Effects such as single-mode lasing [13], normal-mode splitting [14], optomechanical

cooling [15–17], or amplification [18] are described analytically using perturbative methods. Beyond this perturbative limit, to unveil the complete nonlinear picture, more intricate techniques such as extensive numerical simulations of the evolution equations are needed, and have led to finding dynamics such as hysteresis [19], and cascaded period-doublings and chaos [20–25]. Yet, many experimental platforms host multiple optical or mechanical modes, giving rise to multimode dynamics beyond the single-mode limit. This multimodal landscape exhibits rich physical effects such as synchronization [26–28], chaos [29,30], and mode competition [31–33]. Such a system has been studied using analytical approximations and methodologically solving the evolution equations. However, these methods cannot fully capture the whole dynamics of the system since highly nonlinear or unstable solutions cannot be found.

In this work, we unveil the dynamics of a multimode OM cavity through rigorous bifurcation analysis and numerical continuation [34]. This methodology yields a precise and comprehensive description of the system's intrinsic nonlinear nature and provides the shape and stability of the resulting OM frequency combs. Commonly used in nonlinear science, these mathematical tools have been used in purely theoretical works in fields such as optical frequency combs and solitons in Kerr resonators [35,36], multidimensional geometries [37–40], vegetation [41,42], and in a wide variety of biological systems [43]. Importantly, this methodology has also proven to be very predictive for experiments involving optical frequency combs [44] or polariton condensates [45]. Even though this methodology has been previously exploited in optomechanics in the single-mode mechanical and optical regimes [46], it remains scant in the context of multimode OM systems, which we study in this work.

In general, the simultaneous parametric amplification of multiple mechanical oscillators coupled to an optical mode

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is often limited by mode competition, resulting in the amplification of a single mode and damping the rest. This effect typically arises unless the mechanical resonators have notably different frequencies [47–49] or the optical driving is modulated [50]. The simultaneous and coherent emission of two phonons with similar frequencies without modulating the optical driving has remained overlooked, as it arises in a regime where the system exhibits strong nonlinear behavior. In our work, we apply bifurcation analysis and numerical continuation to an OM cavity comprising two independent mechanical modes. Remarkably, we identify a configuration where phonon emission occurs at two distinct but similar frequencies.

Our results reveal that dual-mode phonon lasing can emerge in a regime where both mechanical modes oscillate with comparable amplitudes following a period-doubling bifurcation. Notably, we find that the period-doubling bifurcation modifies the frequency comb structure and plays a crucial role in triggering the lasing of the second mechanical mode under specific conditions, unveiling an unexpected pathway to dual-mode phonon lasing. We also identify a distinct dual-mode lasing regime in the limit where the mechanical frequencies differ by more than an order of magnitude. This case has recently been explored numerically and experimentally [49]; however, a complete picture from the bifurcation point of view is still missing. We find that a dual-mode phonon comb arises through a torus bifurcation [51], giving rise to simultaneous lasing at widely separated frequencies. Finally, we also study a route to chaos through cascaded period doubling in this system with two mechanical modes coupled to an optical mode.

II. DUAL-MODE OM EQUATIONS

We consider a general optically driven OM cavity with two movable mirrors, each having its natural mechanical frequency (Ω_i), damping rate (Γ_i), optical losses (κ), and OM coupling rate (g_i) [see Fig. 1(i)]. The normalized equations of motion for the intracavity optical field (a) and the mirror displacements (x_i) are [6]

$$\frac{da}{d\tau} = (i(\Delta + 2x_1 + 2x_2) - 1)a + S, \quad (1a)$$

$$\frac{d^2x_1}{d\tau^2} = -\Omega_1^2x_1 - \Gamma_1\frac{dx_1}{d\tau} + |a|^2, \quad (1b)$$

$$\frac{d^2x_2}{d\tau^2} = -\Omega_2^2x_2 - \Gamma_2\frac{dx_2}{d\tau} + |a|^2, \quad (1c)$$

where Δ corresponds to the frequency detuning between the laser and the closest optical resonance, and S is the input photon flux. All parameters are given in dimensionless form, normalized by κ . The explicit relation between these normalized and physical parameters is provided in Appendix A.

Single-mode phonon lasing occurs when one of the movable mirrors undergoes self-sustained, coherent mechanical oscillations driven by the OM interaction. Such an effect arises at Hopf bifurcations where the stationary states ($\dot{a} = \dot{x}_1 = \dot{x}_2 = 0$) lose stability and give rise to oscillatory behavior (see Appendixes A and B). This transition appears when a

pair of complex-conjugate eigenvalues of the Jacobian matrix crosses the imaginary axis.

Figures 1(a)–1(h) present the first (HB1) and second (HB2) Hopf bifurcations, as well as saddle nodes (SNs), for both high and low mechanical dissipation rates (Γ_i). These bifurcations are mapped across different mechanical frequency configurations, including degenerate cases. Breaking the degeneracy in Γ_i has a negligible effect on the overall structure of the bifurcation diagrams. Notably, increasing mechanical frequencies shifts all bifurcations higher, whereas lower dissipation rates lead to reduced thresholds. Furthermore, SNs—which indicate bistability—occur exclusively in the red-detuned regime ($\Delta < 0$). In contrast, phonon lasing thresholds HB1 and HB2 are lower in the blue-detuned regime ($\Delta > 0$), favoring oscillatory behavior for lower pumps. As seen in Figs. 1(a) and 1(e), by carefully controlling the detuning, we can selectively excite one mechanical mode. This behavior is reminiscent of the mode competition control described in Ref. [33], where the lasing mode is controlled via the pump strength S . In our case, the detuning serves a similar role in selecting the lasing mode. Figure 1(I) shows a bifurcation diagram as a function of Δ for a fixed value $S = 0.7$, with $\Omega_1 = 0.1$ and $\Omega_2 = 0.3$. We identify a pair of HBs. Within the detuning range between these two HBs, the system acts as a phonon laser at Ω_1 . The optical amplitude a and the displacement x_1 oscillate at a frequency near Ω_1 , while x_2 remains stationary. To excite phonon lasing at Ω_2 , we need to increase S . This leads to a new pair of eigenvalues crossing the imaginary axis, resulting in the emergence of another pair of Hopf bifurcations [see Fig. 1(II)]. Increasing S also leads to bistability in the red-detuned region ($\Delta < 0$), as shown in Fig. 1(II). This bistable region appears between two SNs. In addition to these vertical cuts in Fig. 1(c), Figs. 1(III) and 1(IV) display horizontal cuts at fixed detuning values, showing the system's response as a function of the pump strength S , revealing a characteristic S-shaped curve. As a result of this detailed exploration, Fig. 1 reveals the rare coexistence of two Hopf bifurcations associated with qualitatively different dynamical thresholds (internal modes) that emerge due to the multimodal nature of our system and present intertwined paths that are easily traceable with our methodology. This alone is of great importance for cavity design since the loci of the different HBs determine the system's lasing thresholds and features. Note that previous studies typically locate just the one HB threshold [52–54].

III. DUAL-MODE PHONON LASING AND CHAOS

Now, we consider a specific and experimentally accessible blue-detuned case whose normalized and SI parameters are presented in Table I in Appendix C. By performing a parameter sweep of Eqs. (A7a)–(A7c) over S , and obtaining the maximum cavity intensity, $\max(|a|^2)$, we plot Fig. 2(a). The parameters of the system are $\Omega_1 = 2$, $\Omega_2 = 3$, $\Gamma_1 = \Gamma_2 = 10^{-5}$, and $\Delta = 1$. The steady state remains stable until the first HB. From this steady-state solution, two HBs emerge, leading to periodic solution branches P_1 and P_2 . These branches correspond to self-sustained oscillations (phonon lasing) of the first and second mechanical modes, respectively. In some regions, both branches are represented by solid lines,

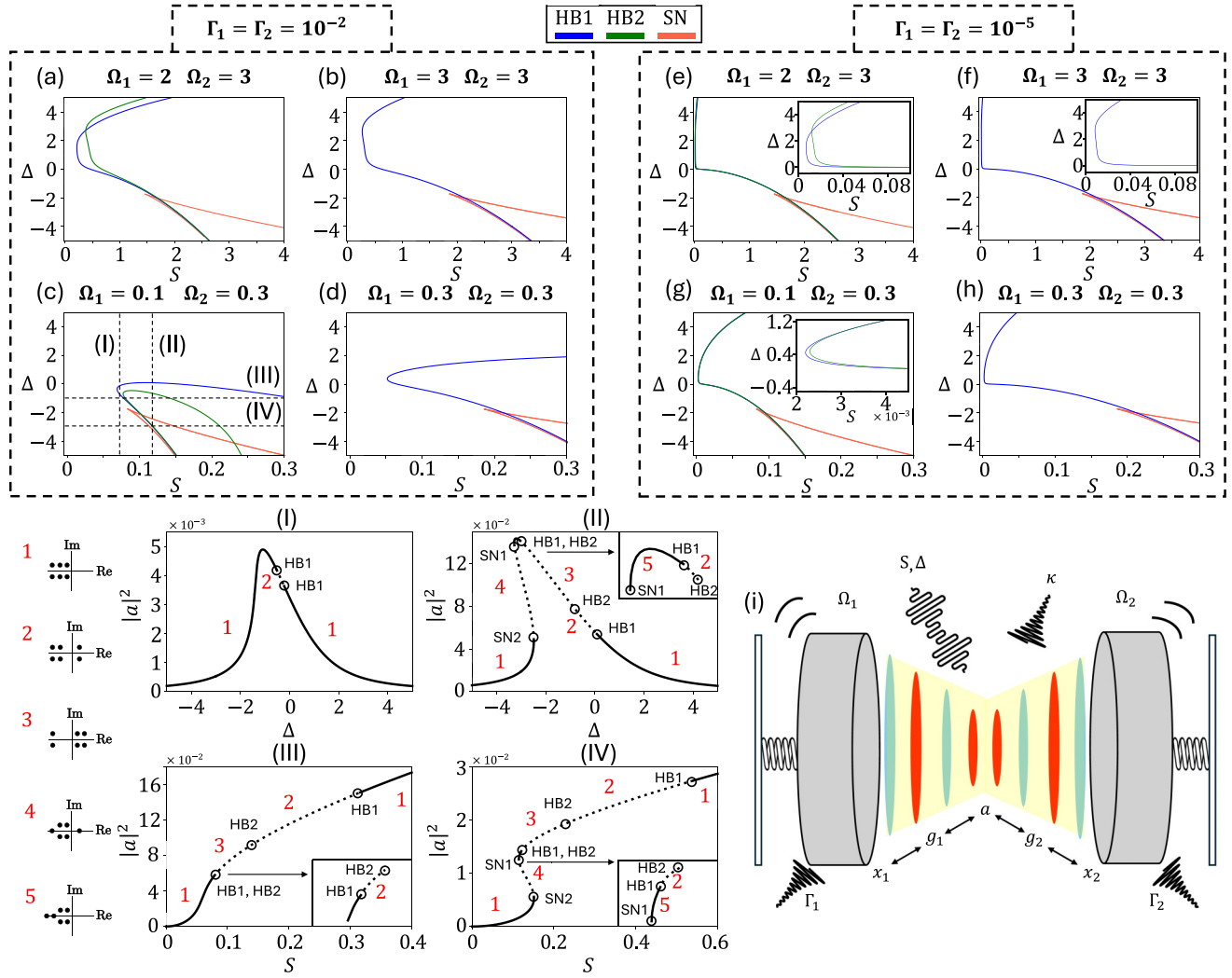


FIG. 1. Panels (a)–(h) show the first (HB1, blue) and second (HB2, green) Hopf bifurcations, along with saddle-node bifurcations (SN, orange), for different mechanical frequency configurations and dissipation rates. The left (right) column corresponds to high (low) mechanical dissipation rates, $\Gamma_1 = \Gamma_2 = 10^{-2}$ ($\Gamma_1 = \Gamma_2 = 10^{-5}$). Panels (I)–(IV) show steady-state solutions of $|a|^2$ as a function of detuning Δ [panels (I) and (II)] and S [panels (III) and (IV)] for selected parameter cuts in panel (c), illustrating bistability and Hopf bifurcations. Red-detuned cases (II) and (IV) exhibit three steady-state branches separated by SN bifurcations, while blue-detuned cases (I) and (III) transition directly to oscillations at lower thresholds. Numbers indicate stability regions that correspond to the eigenvalues: 1 and 5 (stable steady state), 2 and 3 (periodic solutions), and 4 (instability). The diagrams on the bottom left schematically indicate the qualitative location of the eigenvalues in the complex plane for the corresponding regions. Insets in panels (e)–(g) and (II)–(IV) provide a magnified view of the regions where the curves are most densely packed. Panel (i) depicts an OM cavity where a single optical mode is coupled to two mechanical modes (two movable mirrors).

indicating that they are simultaneously stable and thus correspond to a bistable regime. Additionally, a distinct solution, shown in blue, corresponds to a dual-mode lasing branch (P_{DL}), where both mechanical modes are simultaneously in the lasing regime. This blue curve emerges from a period-doubling (PD) bifurcation of the P_1 curve at $S = 0.0314$. The PD bifurcation in P_1 generates new frequency components at half of the mechanical frequency $\Omega_1 = 2$, along with its harmonics, resulting in a frequency comb with integer-spaced lines up to approximately 20. This mechanism effectively drives the second mechanical oscillator at its natural frequency, eventually leading to its lasing. As a result, both mechanical modes enter the lasing regime simultaneously, exhibiting comparable intensities.

Figures 2(b)–2(g) show the time-domain evolution for a fixed pump value $S = 0.035$, illustrating the behavior of the three solution branches identified in Fig. 2(a). Figures 2(b)–2(d) display two periods of the intracavity optical intensity for the P_{DL} , P_1 , and P_2 branches, respectively. Figures 2(e)–2(g) show the oscillation amplitudes of the two mechanical modes, x_1 and x_2 , for each case. In Fig. 2(e), which corresponds to the blue dual-mode lasing branch, both mechanical modes oscillate with comparable amplitudes. In Fig. 2(f), corresponding to P_1 , only the first mechanical mode is in the lasing regime, while the amplitude of the second mode remains negligible. Finally, Fig. 2(g) corresponds to P_2 , where only the second mechanical mode is in the lasing regime while the first mode remains unamplified.

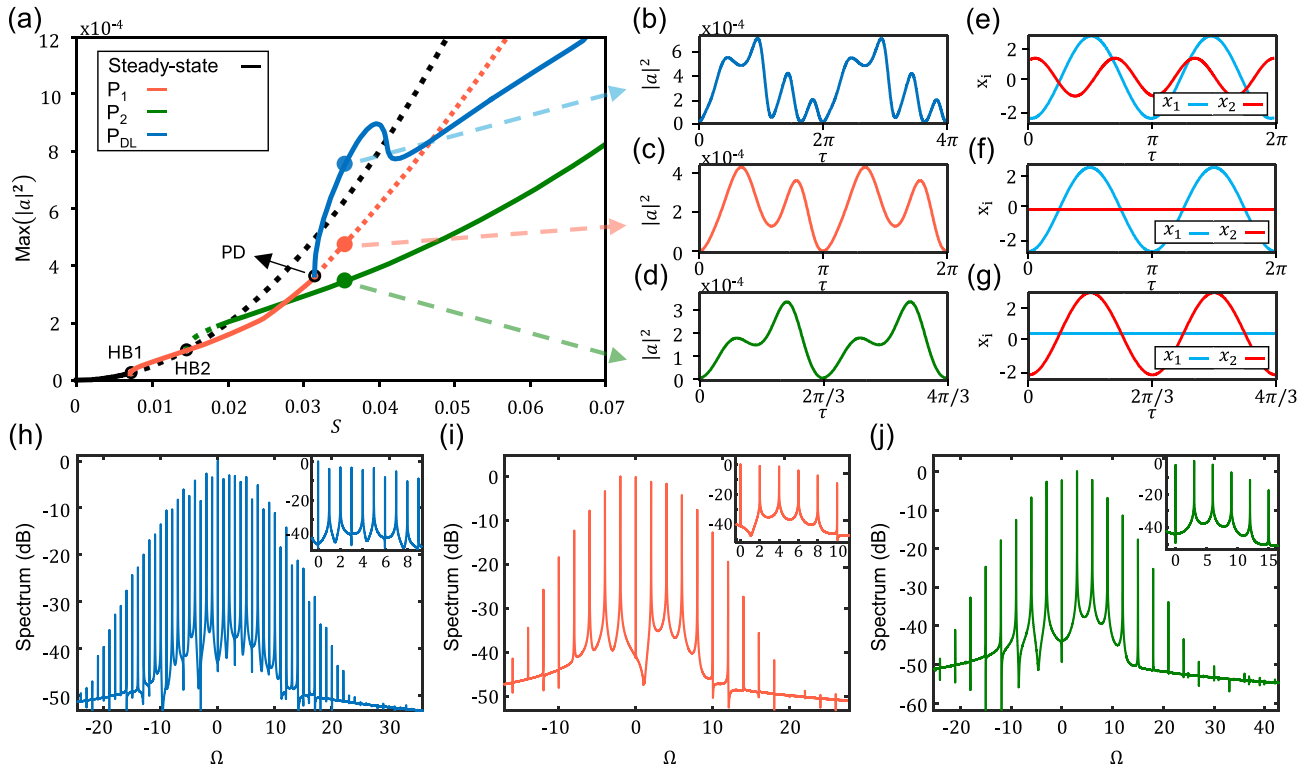


FIG. 2. Periodic solution branches for the values $\Omega_1 = 2$, $\Omega_2 = 3$, $\Gamma_1 = \Gamma_2 = 10^{-5}$, and $\Delta = 1$. (a) Periodic oscillation branches (P_1 and P_2) after two Hopf bifurcations (HB1 and HB2) and dual-mode lasing branch (P_{DL}) after a PD bifurcation. (b)–(d) Optical intensity profiles over two periods for the branches shown in panel (a) at $S = 0.035$. (e)–(g) Corresponding mechanical oscillations for the three branches at $S = 0.035$. Periods in panels (b)–(g) differ as they correspond to distinct solution branches. (h)–(j) Spectra of the optical intensity signals in panels (b)–(d), respectively, in log scale, highlighting the dominant frequency components. The insets show enlarged views to better visualize the spacing between harmonics. These are dual-sideband spectra centered at zero frequency. Dashed lines in panel (a) show unstable solutions.

Figures 2(h)–2(j) present the spectra of the time signals in Figs. 2(b)–2(d), showing the combs. Figures 2(i) and 2(j) correspond to P_1 and P_2 , respectively, and show single comb structures generated by the first and second mechanical modes. In each case, the peaks are centered around the fundamental frequencies Ω_1 or Ω_2 , along with their harmonics. In contrast, Fig. 2(h), corresponding to P_{DL} , reveals multiple peaks associated with both frequencies $\Omega_1 = 2$ and $\Omega_2 = 3$, as well as their harmonics and combinations, such as sum and difference frequencies. This structure forms a dual-frequency comb characteristic of the dual-mode lasing regime. Notably, the solutions in P_1 at $S = 0.035$ lose stability after several periods, as indicated by the dashed line in Fig. 2(a).

Unlike previous studies that stop at Hopf bifurcations [55,56], we reveal in Fig. 2, a dual-mode phonon lasing regime triggered by a period-doubling bifurcation, resulting in a rich dual-frequency comb. Although hybridization [57], mode competition, and anomalous cooling [33] have been reported, phonon lasing in this context remains scarcely explored [49,50]. Our work fills this gap, offering a clear, dynamic roadmap for future experiments. Furthermore, all parameters used are within experimentally feasible ranges.

To delve deeper into the system's dynamics, Fig. 3 presents another experimentally feasible case (see Table II in Appendix C) with parameters set to $\Omega_1 = 0.1$, $\Omega_2 = 1.5$,

$\Gamma_1 = \Gamma_2 = 10^{-3}$, and $\Delta = 2$. In Fig. 3(a), we observe that the first Hopf bifurcation (HB1) is supercritical, while the second (HB2) is subcritical [51,58]. The diagram displays distinct dynamical regimes, labeled 1 to 4. Region 1 corresponds to the steady-state solution. In region 2, the system exhibits a clear single lasing regime, where the second mechanical mode dominates the dynamics. Since this branch emerges from a Hopf bifurcation, it is naturally associated with single-mode lasing. The first mechanical mode remains highly damped and is weakly forced to oscillate at the dominant frequency Ω_2 , but its contribution is negligible. Region 3 is associated with a dual-mode lasing branch that emerges from a torus bifurcation, resulting in a regime where both mechanical modes oscillate at their respective natural frequencies. Finally, region 4 (highlighted in green) corresponds to the enhanced oscillation of the second mechanical mode $\Omega_1 = 0.1$, while the first mode remains strongly damped.

Figures 3(b) and 3(c) show time propagation simulations where the pump parameter S is swept up and down, respectively. In panel (b), as S increases, the system initially resides in the steady state (region 1) until it crosses HB1, entering region 2. Here, the second mechanical mode becomes strongly excited, while the first remains weakly oscillating. Although the first mode initially exhibits two frequency components—its own natural frequency and that of the dominant second mode—it is progressively forced to oscillate

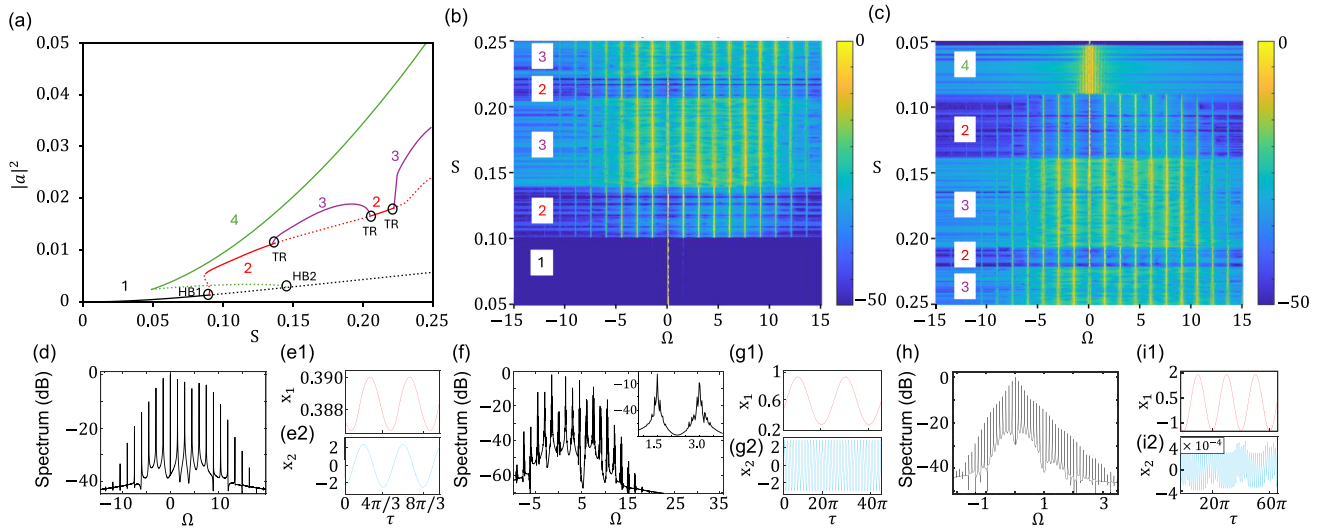


FIG. 3. Bifurcation diagram and dynamical behavior of the system for $\Omega_1 = 0.1$, $\Omega_2 = 1.5$, $\Gamma_1 = \Gamma_2 = 10^{-3}$, and $\Delta = 2$. (a) Bifurcation diagram showing different dynamical regimes labeled 1 to 4. Region 1: steady-state solution. Region 2: Dominant lasing of the first mechanical mode (Ω_1), with the second mode remaining strongly damped and exhibiting a weak, frequency-locked response to the first mode. Region 3: dual-mode lasing originating from a torus (TR) bifurcation. Region 4: enhanced oscillation of the second mechanical mode (Ω_2) while the first mode remains damped. (b) Forward sweep of the pump S showing transitions between regions, including the emergence of a dual-mode frequency comb. (c) Backward sweep showing hysteresis and transition into region 4. (d), (f), (h) Optical spectra for $S = 0.11$, 0.16 , and 0.06 , respectively, showing (d) comb generation from the first mechanical mode, with the second mode highly damped and frequency locked to it; (f) dual-mode comb operation; and (h) single-mode comb generation from the second mechanical mode. (e), (g), (i) Corresponding mechanical oscillations for each case. The inset in panel (f) provides a close-up view that highlights the spacing between harmonics more clearly.

solely at the higher frequency Ω_2 , resulting in a single-mode lasing regime. This behavior stems from the strong dominance of the second mode and the damped nature of the first, which is effectively slaved to the dynamics of the lasing oscillator. Around $S = 0.14$, a TR bifurcation occurs, giving rise to a new branch, where both mechanical modes exhibit enhanced amplitudes at their respective natural frequencies, leading to a dual-mode frequency comb. Interestingly, similar torus and period-doubling bifurcations have been reported in single-mode OM systems [46], but their manifestation in a multimode setup, as shown here, reveals a significantly richer dynamical behavior. This branch eventually rejoins region 2, where only the second mode lases again. When S reaches approximately 0.23, a further transition to region 3 occurs, reinstating the dual-mode lasing regime.

Figure 3(c) reveals the hysteretic nature of the system. When S is decreased from the dual-mode lasing state in region 3, the system retraces its path until $S \approx 0.09$, where it abruptly transitions to region 4. Finally, the system returns to the steady state as the green branch loses stability at $S = 0.05$. Panels (d), (f), and (h) show the spectra for representative points: single-mode comb of the second mechanical mode ($S = 0.11$), dual-mode comb ($S = 0.16$), and single-mode lasing of the first mechanical mode ($S = 0.06$). In each case, the corresponding mechanical oscillations are also plotted, clearly illustrating the distinct dynamical regimes. Beyond the specific parameter configurations discussed here, we conducted a comprehensive analysis to assess the stability and robustness of the dynamical regimes under variations of the system parameters. In particular, we performed extensive parameter sweeps, as presented in Appendix D, where we systematically explored the influence of detuning, mechanical

losses, and pump power. From these simulations, we identified a well-defined range of detuning values where both mechanical modes exhibit comparable oscillation amplitudes, which is essential for achieving the dual-mode lasing regime without requiring external modulation. Furthermore, we observed that reducing the mechanical losses enhances the oscillation amplitudes of the mechanical modes, while higher losses tend to suppress them, especially at lower input power. Notably, we also found that the dual-mode lasing regime persists and remains robust even at elevated pump powers, sustaining coherent oscillations of both mechanical modes.

Additionally, we investigated the role of the PD bifurcation in enabling the transition to dual-mode lasing. Specifically, we varied the mechanical frequencies (not shown). We observed that if, after the PD bifurcation of the lasing mode, the newly generated comb lines coincide with the frequency of the second mechanical mode, this interaction destabilizes the original lasing branch. As a result, a new stable branch emerges from the PD bifurcation, corresponding to the dual-mode lasing regime.

We also performed a dedicated sweep around the torus bifurcation to determine the threshold and characteristics of the dual-mode lasing regime. Altogether, this systematic study provides a parameter map that demonstrates the robustness and versatility of the proposed dynamics.

Finally, chaotic dynamics also emerge in the two-mode optomechanical system under appropriate conditions. When the system is pumped with sufficiently high optical power, it undergoes a sequence of period-doubling bifurcations, eventually leading to chaos.

Figure 4 illustrates this transition through phase portraits (dx_i/dt versus x_i) and frequency spectra. Panels (a1)–(d1)

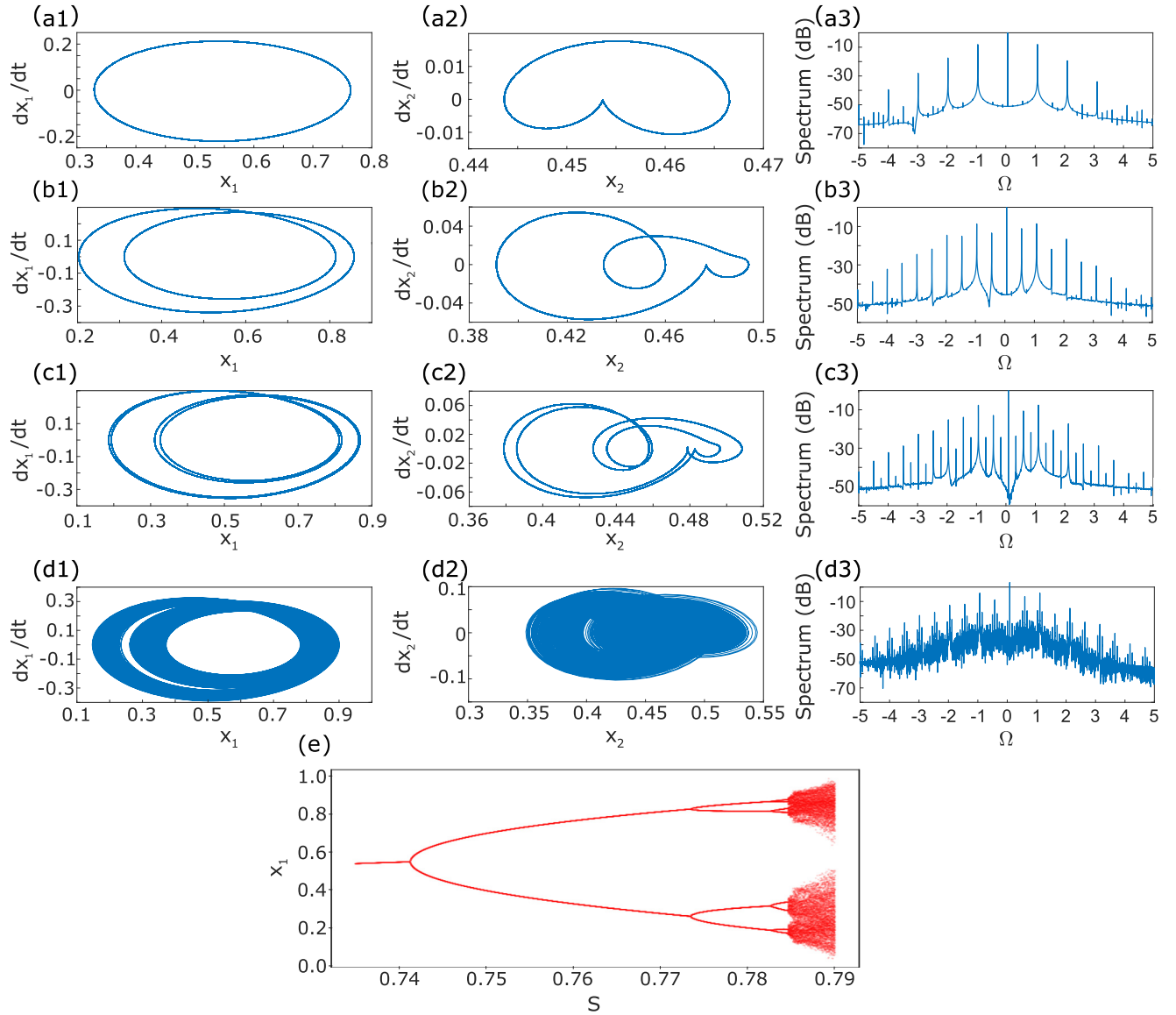


FIG. 4. Chaotic behavior via a cascade of period-doubling bifurcations. Panels (a1)–(d1) and (a2)–(d2) show the phase-space trajectories (dx_i/dt vs x_i) for the first and second mechanical modes, respectively, illustrating the emergence of additional loops with each PD transition. Panels (a3)–(d3) display the corresponding optical spectra, where subharmonics progressively emerge at each bifurcation, signaling the transition to chaos. Panel (e) presents a bifurcation diagram with respect to the driving strength S , showing successive period-doubling bifurcations that correlate with the behavior observed in panels (ai)–(di). Simulation parameters: $\Delta = -2$, $\Omega_1 = 1$, $\Omega_2 = 1.1$, and $\Gamma_1 = \Gamma_2 = 8.2 \times 10^{-3}$. The pump strengths are $S = 0.772$ [panels (a1)–(a3)], $S = 0.78$ [panels (b1)–(b3)], $S = 0.783$ [panels (c1)–(c3)], and $S = 0.785$ [panels (d1)–(d3)].

show the evolution of the first mechanical mode from periodic to chaotic motion. The second mode exhibits similar behavior with a distinct two-lobed structure, as shown in panels (a2)–(d2).

Panels (a3)–(d3) present the optical spectra, where the observed frequency components correspond primarily to harmonics of Ω_1 . This indicates that the optical output is dominated by lasing from the first mechanical mode.

Panel (e) displays a bifurcation diagram as a function of the pump strength S . It reveals a clear sequence of period-doubling bifurcations, with new branches emerging consistently with the transitions shown in panels (ai)–(di).

IV. CONCLUSIONS

To summarize, we used a robust methodology for analyzing the bifurcation structure and dynamical behavior of OM cavities supporting two mechanical modes, targeting the emergence of single- and dual-mode phonon lasing. Our bifurcation analysis, based on steady-state solutions and their stability, reveals a rich sequence of transitions—including Hopf, period-doubling, and torus bifurcations—leading to stable self-sustained oscillations, dual-mode lasing with frequency-comb-like features, and hysteretic behavior (see Figs. 2 and 3). Furthermore, the comprehensive parameter sweep plots shown in Appendix D map the full landscape

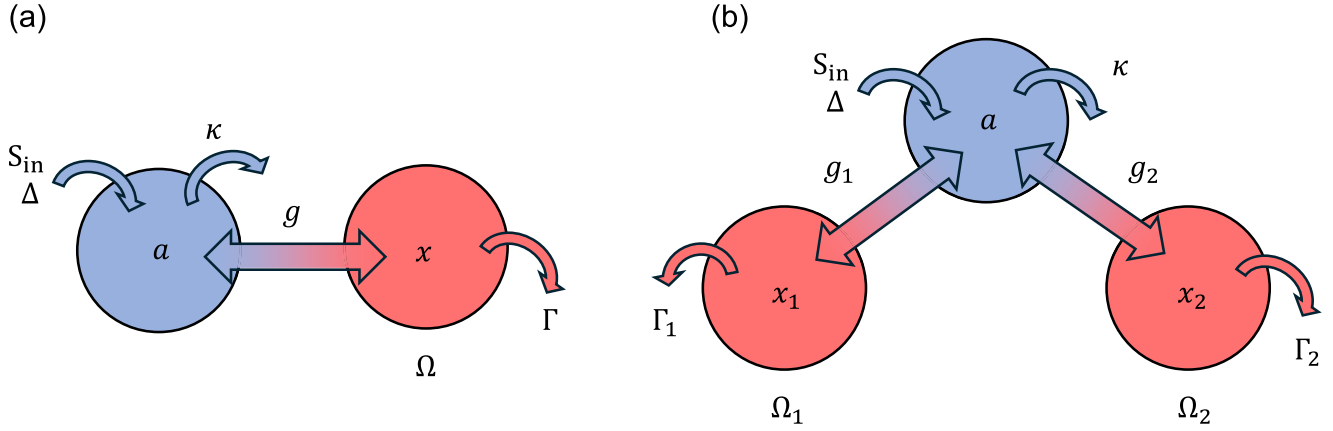


FIG. 5. Schemes of the OM single-mode (a) and two-mode (b) interaction.

of dynamical regimes, providing a powerful tool to identify and control complex behaviors in multimode OM systems. Furthermore, a representative example of chaotic dynamics has also been studied, where the system exhibits a cascade of period-doubling bifurcations leading to chaos—highlighting the potential for rich and intricate nonlinear behavior even in relatively simple multimode configurations, and opening avenues for tailoring nonlinear phonon lasing dynamics and guiding future experimental implementations.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: NORMALIZATION AND STEADY STATE OF THE OM EQUATIONS

1. One optical mode coupled to one mechanical mode

The OM dynamical equations for one optical mode ($\tilde{a}$) coupled to one mechanical mode ($\tilde{x}$) are

$$\frac{d\tilde{a}}{d\tilde{t}} = \left(i(\tilde{\Delta} + \tilde{G}\tilde{x}) - \frac{\tilde{\kappa}}{2} \right) \tilde{a} + \sqrt{\tilde{\kappa}_e} \tilde{S}_{\text{in}}, \quad (\text{A1a})$$

$$\frac{d^2\tilde{x}}{d\tilde{t}^2} = -\tilde{\Omega}^2\tilde{x} - \tilde{\Gamma}\frac{d\tilde{x}}{d\tilde{t}} + \frac{\hbar\tilde{G}}{\tilde{m}}|\tilde{a}|^2. \quad (\text{A1b})$$

Here, $\tilde{\Delta} = \tilde{\omega}_l - \tilde{\omega}_r$ is the detuning of the laser compared to the optical resonance, $\tilde{G}$ is the optical frequency shift per displacement unit, $\tilde{\kappa}$ is the overall optical loss, and $\tilde{\kappa}_e$ represents the input coupling losses. $\tilde{S}_{\text{in}}$ is the input photon flux, $\tilde{m}$

is the mass of the mechanical oscillator, and $\tilde{\Gamma}$ stands for the mechanical losses. Figure 5(a) illustrates the system described by Eqs. (A1a) and (A1b).

To study the stability of these equations, it is numerically convenient to normalize them:

$$\frac{da}{d\tau} = (i(\Delta + 2x) - 1)a + S, \quad (\text{A2a})$$

$$\frac{d^2x}{d\tau^2} = -\Omega^2x - \Gamma\frac{dx}{d\tau} + |a|^2, \quad (\text{A2b})$$

with a and x the optical and mechanical normalized modes. In this normalization we have taken $\tau = \frac{\tilde{\kappa}}{2}\tilde{t}$, $x = \frac{\tilde{G}\tilde{x}}{\tilde{\kappa}}$, and $a = \sqrt{\frac{4\tilde{G}^2\hbar}{\tilde{m}\tilde{\kappa}^3}}\tilde{a}$. On the other hand, $\Delta = \frac{2\tilde{\Delta}}{\tilde{\kappa}}$, $\Omega^2 = \frac{4\tilde{\Omega}^2}{\tilde{\kappa}^2}$, $\Gamma = \frac{2}{\tilde{\kappa}}\tilde{\Gamma}$, and $S = \sqrt{\frac{16\tilde{G}^2\hbar\tilde{\kappa}_e}{\tilde{m}\tilde{\kappa}^5}}\tilde{S}_{\text{in}}$. By using these nondimensional equations, the parameter values become more manageable when considering experimentally feasible ranges.

To solve the steady-state solution, we must take $\frac{da}{d\tau} = \frac{dx}{d\tau} = \frac{d^2x}{d\tau^2} = 0$, so that $a(\tau) = \bar{a}$ and $x(\tau) = \bar{x}$. Now, if we solve Eqs. (A2a) and (A2b), we obtain

$$\bar{x} = \frac{|\bar{a}|^2}{\Omega^2}, \quad (\text{A3a})$$

$$\bar{a} = \frac{-S}{i(\Delta + 2\bar{x}) - 1}. \quad (\text{A3b})$$

Then, if we substitute Eq. (A3a) into Eq. (A3b), we derive the characteristic polynomial:

$$\frac{4}{\Omega^4}|\bar{a}|^6 + \frac{4\Delta}{\Omega^2}|\bar{a}|^4 + (\Delta^2 + 1)|\bar{a}|^2 - |S|^2 = 0. \quad (\text{A4})$$

Equation (A4) contains all the steady-state solution curves as a function of the different OM system parameters. The representation of the steady-state solution is shown for different examples in the main text in Figs. 2(I)–2(IV). Now if we make $\frac{dS}{d|\bar{a}|^2} = 0$, the upper and lower saddle node points for a red-detuned case have the following expressions:

$$|\bar{a}|^2 = \Omega^2 \left(-\frac{1}{3}\Delta \pm \sqrt{\left(\frac{\Delta^2}{36} - \frac{1}{12} \right)} \right), \quad (\text{A5a})$$

$$\Delta = -\frac{4|\bar{a}|^2}{\Omega^2} \pm \sqrt{\frac{4|\bar{a}|^4}{\Omega^4} - 1}. \quad (\text{A5b})$$

From Eqs. (A5a) and (A5b), we obtain the bistability condition $\Delta < -\sqrt{3}$.

2. One optical mode coupled to two mechanical modes

In this case, we add a second mechanical mode ($\tilde{x}_2$) coupled to the same optical mode:

$$\frac{d\tilde{a}}{d\tilde{t}} = \left(i(\tilde{\Delta} + \tilde{G}_1\tilde{x}_1 + \tilde{G}_2\tilde{x}_2) - \frac{\tilde{\kappa}}{2} \right) \tilde{a} + \sqrt{\tilde{\kappa}_e}\tilde{S}_{\text{in}}, \quad (\text{A6a})$$

$$\frac{d^2\tilde{x}_1}{d\tilde{t}^2} = -\tilde{\Omega}_1^2\tilde{x}_1 - \tilde{\Gamma}_1 \frac{d\tilde{x}_1}{d\tilde{t}} + \frac{\hbar\tilde{G}_1|\tilde{a}|^2}{\tilde{m}_1}, \quad (\text{A6b})$$

$$\frac{d^2\tilde{x}_2}{d\tilde{t}^2} = -\tilde{\Omega}_2^2\tilde{x}_2 - \tilde{\Gamma}_2 \frac{d\tilde{x}_2}{d\tilde{t}} + \frac{\hbar\tilde{G}_2|\tilde{a}|^2}{\tilde{m}_2}. \quad (\text{A6c})$$

In principle, we assume that the second mechanical mode is different from the first mechanical mode to generally solve the problem. Figures 1(b) and 1(i) show the scheme of the OM two-mode interaction between one optical and two mechanical modes. Under these circumstances, the equations are normalized as follows:

$$\frac{da}{d\tau} = (i(\Delta + 2x_1 + 2x_2) - 1)a + S, \quad (\text{A7a})$$

$$\frac{d^2x_1}{d\tau^2} = -\Omega_1^2x_1 - \Gamma_1 \frac{dx_1}{d\tau} + |a|^2, \quad (\text{A7b})$$

$$\frac{d^2x_2}{d\tau^2} = -\Omega_2^2x_2 - \Gamma_2 \frac{dx_2}{d\tau} + \frac{\gamma_2}{\gamma_1}|a|^2. \quad (\text{A7c})$$

In this case, the transformations that we used are $\tau = \frac{\tilde{\kappa}}{2}\tilde{t}$, $x_i = \frac{\tilde{G}_i}{\tilde{\kappa}}\tilde{x}_i$, $\gamma_i = \frac{4\hbar\tilde{G}_i^2}{\tilde{m}_i\tilde{\kappa}^3}$, $a = \sqrt{\gamma_1}\tilde{a}$, $\Delta = \frac{2\tilde{\Delta}}{\tilde{\kappa}}$, $\Omega_i^2 = \frac{4\tilde{\Omega}_i^2}{\tilde{\kappa}^2}$, $\Gamma_i = \frac{2}{\tilde{\kappa}}\tilde{\Gamma}_i$, and $S = \frac{2}{\tilde{\kappa}}\sqrt{\gamma_1\tilde{\kappa}_e}\tilde{S}_{\text{in}}$. It should be noted that each mechanical mode is normalized using parameters from both mechanical modes.

Although the normalized equations with two modes can be studied in their most general form, in this work, we focus on the particular case where $\gamma_1 = \gamma_2 \rightarrow \frac{\tilde{G}_1^2}{\tilde{m}_1} = \frac{\tilde{G}_2^2}{\tilde{m}_2} \rightarrow \tilde{\Omega}_1\tilde{g}_{0_1} = \tilde{\Omega}_2\tilde{g}_{0_2}$. This choice simplifies the numerical analysis without loss of generality, as both the methodology employed and the results obtained remain applicable to the general case, preserving the qualitative validity of the conclusions. Hence, the equations are

$$\frac{da}{d\tau} = (i(\Delta + 2x_1 + 2x_2) - 1)a + S, \quad (\text{A8a})$$

$$\frac{d^2x_1}{d\tau^2} = -\Omega_1^2x_1 - \Gamma_1 \frac{dx_1}{d\tau} + |a|^2, \quad (\text{A8b})$$

$$\frac{d^2x_2}{d\tau^2} = -\Omega_2^2x_2 - \Gamma_2 \frac{dx_2}{d\tau} + |a|^2. \quad (\text{A8c})$$

The new normalizations are $x_i = \frac{\tilde{G}_i}{\tilde{\kappa}}\tilde{x}_i$, $\gamma = \frac{4\hbar\tilde{G}_i^2}{\tilde{m}_i\tilde{\kappa}^3}$, $a = \sqrt{\gamma}\tilde{a}$, and $S = \frac{2\sqrt{\gamma\tilde{\kappa}_e}}{\tilde{\kappa}}\tilde{S}_{\text{in}}$.

Conversely, the steady-state solutions are

$$\bar{x}_i = \frac{|\bar{a}|^2}{\Omega_i^2}, \quad (\text{A9a})$$

$$\bar{a} = \frac{-S}{i(\Delta + 2\bar{x}_1 + 2\bar{x}_2) - 1}. \quad (\text{A9b})$$

The expressions in Eqs. (A9a) and (A9b) are closely resembling to the single-mode ones [Eqs. (A3a) and (A3b)], but we have to take into account that the new normalizations of a and x_i are different.

Now, substituting Eq. (A9a) into Eq. (A9b), we derive the characteristic polynomial:

$$4\left(\frac{1}{\Omega_1^4} + \frac{1}{\Omega_2^4} + \frac{2}{\Omega_1^2\Omega_2^2}\right)|\bar{a}|^6 + 4\Delta\left(\frac{1}{\Omega_1^2} + \frac{1}{\Omega_2^2}\right)|\bar{a}|^4 + (\Delta^2 + 1)|\bar{a}|^2 - |S|^2 = 0. \quad (\text{A10a})$$

Following the same steps as in the single mechanical mode model, the saddle nodes have the following expressions:

$$|\bar{a}|^2 = \left(\frac{\Omega_1^2\Omega_2^2}{\Omega_1^2 + \Omega_2^2}\right)\left[-\frac{1}{3}\Delta \pm \sqrt{\left(\frac{\Delta^2}{36} - \frac{1}{12}\right)}\right], \quad (\text{A11a})$$

$$\Delta = -\frac{4|\bar{a}|^2(\Omega_1^2 + \Omega_2^2)}{\Omega_1^2\Omega_2^2} \pm \sqrt{\frac{4|\bar{a}|^4(\Omega_1^2 + \Omega_2^2)^2}{(\Omega_1^2\Omega_2^2)^2} - 1}. \quad (\text{A11b})$$

Equations (A11a) and (A11b) are equivalent to the case of one mechanical mode [Eqs. (A5a) and (A5b)] but taking the transformation $\Omega^2 \rightarrow (\frac{1}{\Omega_1^2} + \frac{1}{\Omega_2^2})^{-1}$. It is also remarkable that we preserve the condition $\Delta < -\sqrt{3}$ for bistability.

3. One mechanical mode limit case

We can simplify from the two-mode to the single-mode case by setting one mechanical frequency, Ω_i , much larger than the other, Ω_j . This becomes apparent in Eqs. (A9a) and (A9b), where for large Ω_i , the contribution from $\tilde{x}_i$ becomes negligible.

Nonetheless, when the two modes are degenerate (i.e., when they have identical mechanical properties), the two-mode equations can be reformulated to resemble the single-mode case.

Let us first consider the normalized two-mode equations, Eqs. (A8a)–(A8c). We define a new variable $X = x_1 + x_2$. Hence, we consolidate the mechanical equations into a single equation:

$$\frac{da}{d\tau} = (i(\Delta + 2X) - 1)a + S, \quad (\text{A12a})$$

$$\frac{d^2X}{d\tau^2} = -\Omega^2X - \Gamma \frac{dX}{d\tau} + 2|a|^2. \quad (\text{A12b})$$

Now, if we rewrite $2|a|^2 = |A|^2$ and $\sqrt{2}S = C$, we finally reach the single-mode case:

$$\frac{dA}{d\tau} = (i(\Delta + 2X) - 1)A + C, \quad (\text{A13a})$$

$$\frac{d^2X}{d\tau^2} = -\Omega^2X - \Gamma \frac{dX}{d\tau} + |A|^2. \quad (\text{A13b})$$

APPENDIX B: STABILITY ANALYSIS

In the previous section, we have obtained the steady-state solutions. Nevertheless, the behavior of the equations is quite more complex. For sufficiently large input powers ($S \gg 1$), the steady-state solutions become unstable, leading to a range of

nonlinear phenomena such as oscillations, period doublings, multistability, and chaos.

The stability analysis of the equations identifies the bifurcations where stability changes occur [51]. This formalism involves considering the steady-state solutions with a time-dependent, sufficiently small perturbation. This approach leads to the calculation of a Jacobian matrix, whose eigenvalues provide information about the stability at each point. The necessary condition for stability is that all eigenvalues' real parts must be negative. If at least one of them becomes positive, the solution is no longer stable and will collapse to a stable attractor [58].

Conversely, if the real part of a pair of eigenvalues becomes positive and one is the complex conjugate of the other, while the real parts of the remaining eigenvalues stay negative, the system will exhibit oscillations. The transition of stability is always marked by a bifurcation; thus, identifying these points is essential for fully characterizing the system's stability.

With this brief introduction, we can show the theoretical treatment of the stability analysis. First, we will derive the single mechanical mode model and then we will develop the model for two mechanical modes.

Let us consider this ansatz: $a(\tau) \approx \bar{a} + \delta a(\tau)$; $x(\tau) \approx \bar{x} + \delta x(\tau)$. We can take the Taylor series expansion of both variables with this general formula:

$$\begin{aligned} \frac{d(\bar{V} + \delta V)}{dt} &= f(\bar{V} + \delta V) \approx f(\bar{V}) + \left(\frac{\partial f(V)}{\partial V_k} \right)_{V=\bar{V}} \delta V \\ &= \left(\frac{\partial f(V)}{\partial V_k} \right)_{V=\bar{V}} \delta V, \end{aligned} \quad (\text{B1a})$$

V being either a or x . Equation (B1a) shows that the time evolution of the optical and mechanical modes is proportional to the Jacobian matrix $\mathcal{J} = \left(\frac{\partial f(V)}{\partial V_k} \right)_{V=\bar{V}}$.

With this principle, we can construct the Jacobian matrix of the OM equations. To linearize the equations, we define the vector $\Psi = (\delta a, \delta a^*, \delta x, \delta P)^T$ with $P = \frac{dx}{dt}$.

Now, introducing the first ansatz into Eqs. (A2a) and (A2b), we can define the following equations:

$$f_1 \equiv \frac{d\delta a}{d\tau} = (i(\Delta + 2(\bar{x} + \delta x) - 1)\delta a + 2i\bar{a}\delta x), \quad (\text{B2a})$$

$$f_2 \equiv \frac{d\delta a^*}{d\tau} = (-i(\Delta + 2(\bar{x} + \delta x) - 1)\delta a^* - 2i\bar{a}^*\delta x), \quad (\text{B2b})$$

$$f_3 \equiv \frac{d\delta x}{d\tau} = \delta P, \quad (\text{B2c})$$

$$f_4 \equiv \frac{d\delta P}{d\tau} = -\Omega^2\delta x - \Gamma\delta P + (\bar{a}\delta a^* + \bar{a}^*\delta a). \quad (\text{B2d})$$

Naturally, the time evolution of the steady-state solutions is null, i.e., $\frac{d\bar{a}}{d\tau} = \frac{d\bar{x}}{d\tau} = 0$, and we have neglected the O^2 term $|\delta a|^2$. The $\mathcal{J}$ matrix is calculated as follows:

$$\mathcal{J} = \begin{pmatrix} \frac{\partial f_1}{\partial \delta a} & \frac{\partial f_1}{\partial \delta a^*} & \frac{\partial f_1}{\partial \delta x} & \frac{\partial f_1}{\partial \delta P} \\ \frac{\partial f_2}{\partial \delta a} & \frac{\partial f_2}{\partial \delta a^*} & \frac{\partial f_2}{\partial \delta x} & \frac{\partial f_2}{\partial \delta P} \\ \frac{\partial f_3}{\partial \delta a} & \frac{\partial f_3}{\partial \delta a^*} & \frac{\partial f_3}{\partial \delta x} & \frac{\partial f_3}{\partial \delta P} \\ \frac{\partial f_4}{\partial \delta a} & \frac{\partial f_4}{\partial \delta a^*} & \frac{\partial f_4}{\partial \delta x} & \frac{\partial f_4}{\partial \delta P} \end{pmatrix}. \quad (\text{B3})$$

Hence, the time evolution of the optical and mechanical modes is $\frac{d}{d\tau}\Psi = \mathcal{J}\Psi$:

$$\frac{d}{d\tau} \begin{pmatrix} \delta a \\ \delta a^* \\ \delta x \\ \delta P \end{pmatrix} = \begin{pmatrix} i(\Delta + 2\bar{x}) - 1 & 0 & 2i\bar{a} & 0 \\ 0 & -i(\Delta + 2\bar{x}) - 1 & -2i\bar{a}^* & 0 \\ 0 & 0 & 0 & 1 \\ \bar{a}^* & \bar{a} & -\Omega^2 & -\Gamma \end{pmatrix} \begin{pmatrix} \delta a \\ \delta a^* \\ \delta x \\ \delta P \end{pmatrix}. \quad (\text{B4})$$

For the case of two mechanical modes, the procedure is similar. We assume the same ansatz for Eqs. (A8a)–(A8c): $a(\tau) \approx \bar{a} + \delta a(\tau)$; $x_1(\tau) \approx \bar{x}_1 + \delta x_1(\tau)$; $x_2(\tau) \approx \bar{x}_2 + \delta x_2(\tau)$. In this case, the equations for the $\mathcal{J}$ matrix are

$$f_1 \equiv \frac{d\delta a}{d\tau} = (i(\Delta + 2(\bar{x}_1 + \delta x_1) + 2(\bar{x}_2 + \delta x_2)) - 1)\delta a + 2i\bar{a}(\delta x_1 + \delta x_2), \quad (\text{B5a})$$

$$f_2 \equiv \frac{d\delta a^*}{d\tau} = (-i(\Delta + 2(\bar{x}_1 + \delta x_1) + 2(\bar{x}_2 + \delta x_2)) - 1)\delta a^* - 2i\bar{a}^*(\delta x_1 + \delta x_2), \quad (\text{B5b})$$

$$f_3 \equiv \frac{d\delta x_1}{d\tau} = \delta P_1, \quad (\text{B5c})$$

$$f_4 \equiv \frac{d\delta P_1}{d\tau} = -\Omega_1^2\delta x_1 - \Gamma_1\delta P_1 + (\bar{a}\delta a^* + \bar{a}^*\delta a), \quad (\text{B5d})$$

$$f_5 \equiv \frac{d\delta x_2}{d\tau} = \delta P_2, \quad (\text{B5e})$$

$$f_6 \equiv \frac{d\delta P_2}{d\tau} = -\Omega_2^2\delta x_2 - \Gamma_2\delta P_2 + (\bar{a}\delta a^* + \bar{a}^*\delta a). \quad (\text{B5f})$$

TABLE I. Feasible normalized and SI OM parameters corresponding to Fig. 2.

Normalized Values	Ω_1 2	Ω_2 3	Γ_1 10^{-5}	Γ_2 10^{-5}	Δ 1	S 0.03	
SI Values	$\tilde{\Omega}_1/2\pi$ (GHz) 2 $\tilde{\kappa}/2\pi$ (GHz) 2	$\tilde{\Omega}_2/2\pi$ (GHz) 3 $\tilde{\kappa}_e/2\pi$ (GHz) 1.5	$\tilde{\Gamma}_1/2\pi$ (kHz) 10 $\tilde{g}_{01}/2\pi$ (MHz) 2.60	$\tilde{\Gamma}_2/2\pi$ (kHz) 10 $\tilde{g}_{02}/2\pi$ (MHz) 2.12	$\tilde{\Delta}/2\pi$ (GHz) 1 $\tilde{m}_1$ (kg) 10^{-16}	$\tilde{S}$ ($s^{-1/2}$) 5.3×10^5 $\tilde{m}_2$ (kg) 10^{-16}	$\tilde{P}$ (nW) 36

Here, the vector of parameters is $\Psi = (\delta a, \delta a^*, \delta x_1, \delta P_1, \delta x_2, \delta P_2)^T$, with $P_1 = \frac{dx_1}{d\tau}$ and $P_2 = \frac{dx_2}{d\tau}$. Finally, the system of time-evolution equations is

$$\frac{d}{d\tau} \begin{pmatrix} \delta a \\ \delta a^* \\ \delta x_1 \\ \delta P_1 \\ \delta x_2 \\ \delta P_2 \end{pmatrix} = \begin{pmatrix} i(\Delta + 2\bar{x}_1 + 2\bar{x}_2) - 1 & 0 & 2i\bar{a} & 0 & 2i\bar{a} & 0 \\ 0 & -i(\Delta + 2\bar{x}_1 + 2\bar{x}_2) - 1 & -2i\bar{a}^* & 0 & -2i\bar{a}^* & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \bar{a}^* & \bar{a} & -\Omega_1^2 & -\Gamma_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \bar{a} & \bar{a} & 0 & 0 & -\Omega_2^2 & -\Gamma_2 \end{pmatrix} \begin{pmatrix} \delta a \\ \delta a^* \\ \delta x_1 \\ \delta P_1 \\ \delta x_2 \\ \delta P_2 \end{pmatrix}. \quad (\text{B6})$$

The stability of the solutions can be investigated by evaluating the eigenvalues of the Jacobian matrices, as we explained at the beginning of this Appendix. Equations (B7a) and (B7b) show the solutions of the eigenvalues ($|\mathcal{J} - \lambda I| = 0$) for one (a) and two (b) mechanical modes. I accounts for the associated 4×4 (single-mode) and 6×6 (two-mode) rank identity matrices.

$$\lambda^4 + \lambda^3(2 + \Gamma) + \lambda^2(1 + 2\Gamma + (2\bar{x} + \Delta)^2 + \Omega^2) + \lambda(\Gamma + \Gamma(2\bar{x} + \Delta)^2 + 2\Omega^2) + \Omega^2 + (2\bar{x} + \Delta)^2\Omega^2 + 4\bar{a}(2\bar{x} + \Delta)\bar{a}^*, \quad (\text{B7a})$$

$$\begin{aligned} &\lambda^6 + \lambda^5(2 + \Gamma_1 + \Gamma_2) + \lambda^4(1 + 2\Gamma_1 + (2 + \Gamma_1)\Gamma_2 + (2(\bar{x}_1 + \bar{x}_2) + \Delta)^2 + \Omega_1^2 + \Omega_2^2) + \lambda^3(\Gamma_1(1 + (2(\bar{x}_1 + \bar{x}_2) + \Delta)^2) \\ &+ 2\Omega_1^2 + \Gamma_2(1 + 2\Gamma_1 + (2(\bar{x}_1 + \bar{x}_2) + \Delta)^2 + \Omega_1^2) + (2 + \Gamma_1)\Omega_2^2) + \lambda^2((1 + (2(\bar{x}_1 + \bar{x}_2) + \Delta)^2)\Omega_1^2 \\ &+ \Gamma_2(\Gamma_1(1 + (2(\bar{x}_1 + \bar{x}_2) + \Delta)^2) + 2\Omega_1^2) + (1 + 2\Gamma_1 + (2(\bar{x}_1 + \bar{x}_2) + \Delta)^2 + \Omega_1^2)\Omega_2^2 + 8\bar{a}(2(\bar{x}_1 + \bar{x}_2) + \Delta)\bar{a}^*) \\ &+ \lambda(\Gamma_2(1 + (2(\bar{x}_1 + \bar{x}_2) + \Delta)^2)\Omega_1^2 + (\Gamma_1(1 + (2(\bar{x}_1 + \bar{x}_2) + \Delta)^2) + 2\Omega_1^2)\Omega_2^2 + 4\bar{a}(\Gamma_1 + \Gamma_2)(2(\bar{x}_1 + \bar{x}_2) + \Delta)\bar{a}^*) \\ &+ (1 + (2(\bar{x}_1 + \bar{x}_2) + \Delta)^2)\Omega_1^2\Omega_2^2 + 4\bar{a}(2(\bar{x}_1 + \bar{x}_2) + \Delta)(\Omega_1^2 + \Omega_2^2)\bar{a}^*. \end{aligned} \quad (\text{B7b})$$

Due to the complexity of both polynomial equations, it is necessary to solve them numerically. For that purpose, we use numerical time-evolution simulations and AUTO [59], a continuation software that efficiently tracks bifurcations and periodic solutions by following steady-state branches as parameters are varied. Unlike direct time-evolution simulations, AUTO allows for the precise identification of bifurcation points and stable periodic solutions without requiring long simulation times, providing a more efficient and robust analysis of the stability of the system.

APPENDIX C: NORMALIZED TO SI PARAMETERS

The experimentally achievable parameters used in Fig. 2 (Fig. 3), along with their normalized values, are shown in Table I (Table II).

The values of $\tilde{\kappa}$, $\tilde{\kappa}_e$, $\tilde{g}_{01}$, $\tilde{g}_{02}$, $\tilde{m}_1$, and $\tilde{m}_2$ have been arbitrarily chosen, as they are not uniquely determined by the selected set of normalized parameters. This selection considers the condition $\tilde{\Omega}_1\tilde{g}_{01}^2 = \tilde{\Omega}_2\tilde{g}_{02}^2$ (see transformations in Sec. I). Nevertheless, these values remain realistic and experimentally feasible.

APPENDIX D: THRESHOLD ANALYSIS FOR DUAL-MODE LASING VIA PARAMETER SWEEPS

In this Appendix, we explore the thresholds for dual-phonon lasing by tracking the continuation of the bifurcations (PD and TR) that lead to this solution from Figs. 2 and 3 across different system parameters. This analysis demonstrates the robustness of the solutions beyond specific parameter values.

TABLE II. Feasible normalized and SI OM parameters corresponding to Fig. 3.

Normalized Values	Ω_1 0.1	Ω_2 1.5	Γ_1 10^{-3}	Γ_2 10^{-3}	Δ 2	S 0.15	
SI Values	$\tilde{\Omega}_1/2\pi$ (GHz) 0.1 $\tilde{\kappa}/2\pi$ (GHz) 2	$\tilde{\Omega}_2/2\pi$ (GHz) 1.5 $\tilde{\kappa}_e/2\pi$ (GHz) 1.5	$\tilde{\Gamma}_1/2\pi$ (MHz) 1 $\tilde{g}_{01}/2\pi$ (MHz) 1.84	$\tilde{\Gamma}_2/2\pi$ (MHz) 1 $\tilde{g}_{02}/2\pi$ (MHz) 0.47	$\tilde{\Delta}/2\pi$ (GHz) 2 $\tilde{m}_1$ (kg) 10^{-16}	$\tilde{S}$ ($s^{-1/2}$) 1.7×10^7 $\tilde{m}_2$ (kg) 10^{-16}	$\tilde{P}$ (μ W) 36

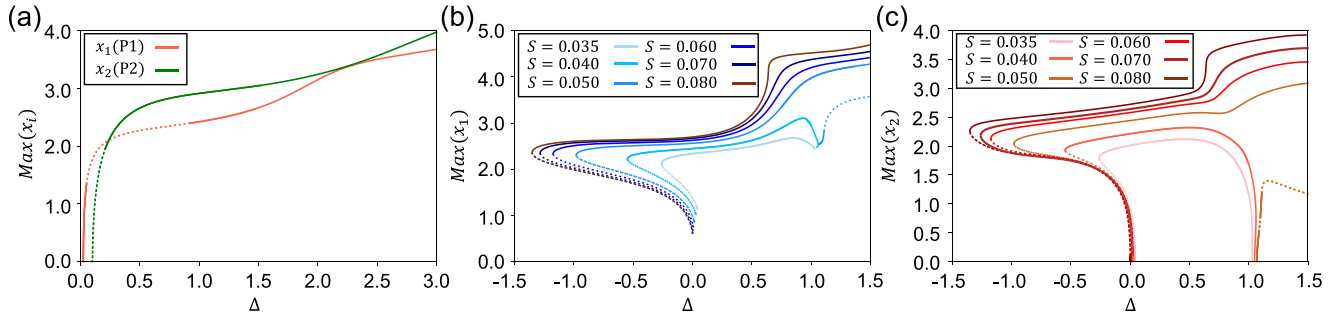


FIG. 6. Amplitude of mechanical oscillations as a function of Δ for the case shown in Fig. 2. (a) Maximum amplitude of the mechanical oscillations for the P_1 and P_2 periodic branches. (b), (c) Corresponding results for the dual-mode lasing branch P_{DL} , using the same study as in panel (a) for x_1 and x_2 , respectively.

The dual-phonon lasing branch remains robust at higher values of S as shown in Fig. 2. However, it is particularly interesting to examine its stability when varying different system parameters. In particular, Fig. 6(a) illustrates the oscillation amplitudes of both mechanical modes as a function of Δ for a fixed value of $S = 0.05$ in P_1 and P_2 (see Fig. 2). As shown, comparable amplitudes for the two mechanical modes are observed only within a limited range of Δ . However, since they belong to the P_1 and P_2 curves, which represent different solution branches, they do not coexist, meaning that only one of the mechanical modes can be enhanced at a given time.

Notably, for this specific value of S , the oscillations become negligible for $\Delta < 0$. It is important to point out that x_1 in P_2 and x_2 in P_1 are not displayed, as their amplitudes remain negligible throughout.

Conversely, Figs. 6(b) and 6(c) show the amplitudes of x_1 and x_2 , respectively, for varying values of S in the dual-phonon lasing branch, P_{DL} . As S increases, the curves broaden, encompassing a wider range of detunings over which both modes oscillate with comparable amplitudes.

A similar analysis for Γ_1 and Γ_2 is presented in Fig. 7. Figures 7(a) and 7(b) illustrate parameter sweeps over Γ_1

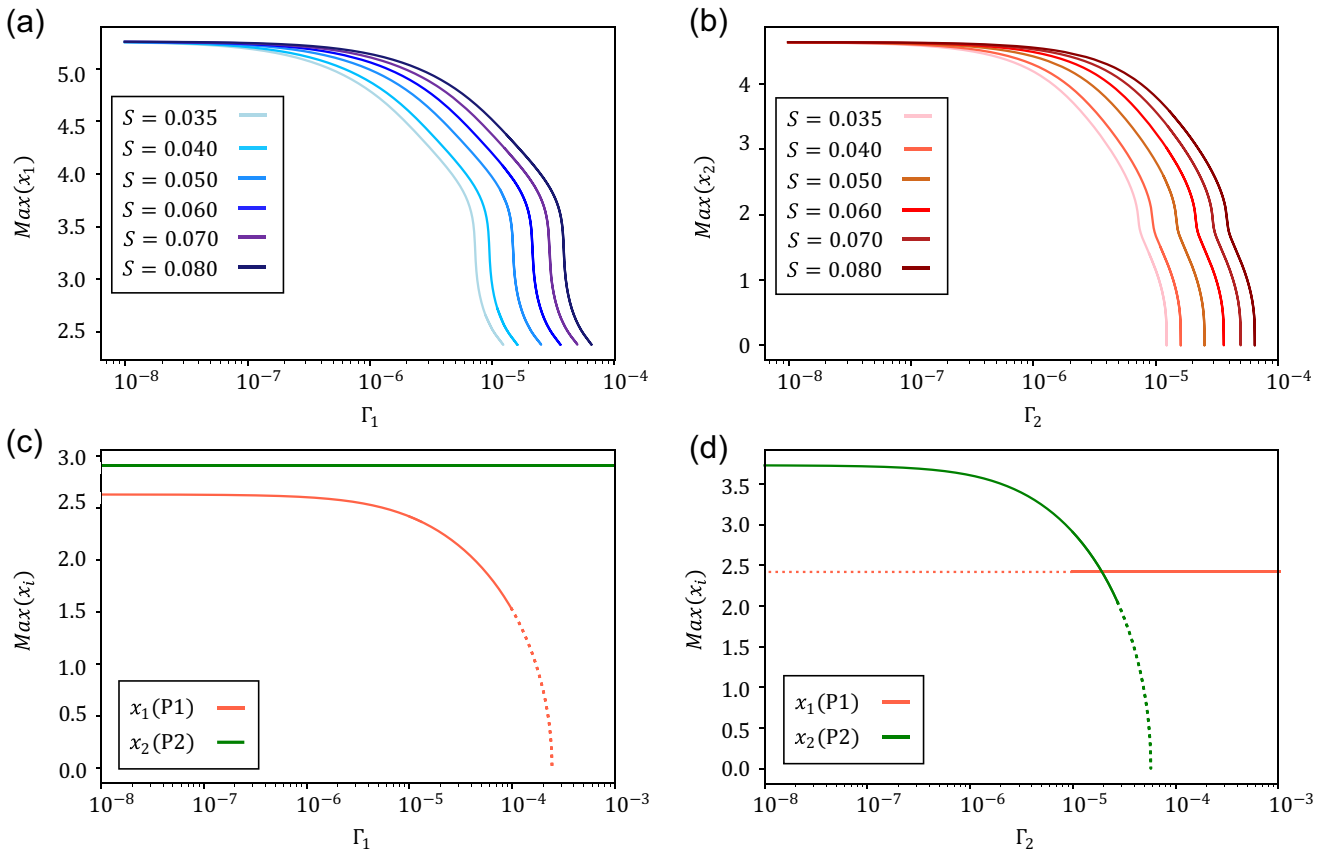


FIG. 7. Maximum amplitude of the mechanical oscillations as a function of Γ_1 and Γ_2 for the case presented in Fig. 2. (a), (b) Sweeps over Γ_1 and Γ_2 , respectively, for different values of S corresponding to the P_{DL} branch solution in Fig. 2(a). (c), (d) Sweeps over Γ_1 and Γ_2 , respectively, for the periodic solution branches P_1 and P_2 , also shown in Fig. 2(a).

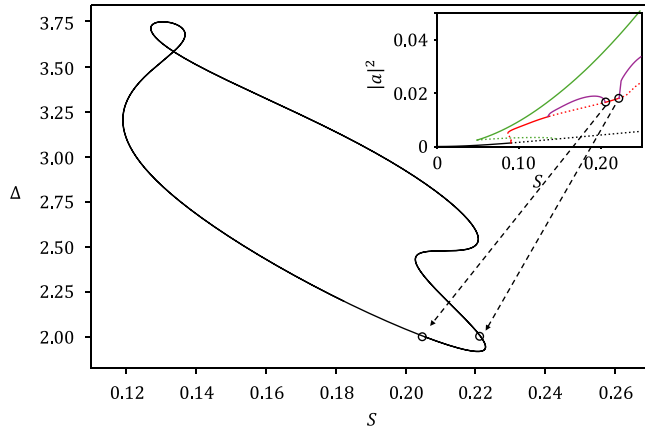


FIG. 8. Sweep of the TR branch over Δ and S , corresponding to Fig. 3. Inset: Fig. 3(a) highlighting the TR bifurcations.

and Γ_2 , respectively, for various values of S . Notably, for sufficiently low mechanical losses, the oscillation amplitudes converge to a specific value regardless of S . In contrast, at higher mechanical losses, the amplitudes decrease, with the rate of decrease depending on the pump strength S . Specifically, higher pump values sustain large oscillation amplitudes over a broader range of losses, while lower pump values result in a steeper decline in oscillation amplitude as losses increase. Figures 7(c) and 7(d) focus on the periodic solutions of P_1 and P_2 at $S = 0.05$. As expected, for P_1 (P_2), only the first mechanical mode x_1 (second mechanical mode x_2) exhibits dominant oscillations. In particular, Fig. 7(c) shows that as Γ_1 increases, the amplitude of x_1 decreases, while x_2 remains unaffected. This behavior is consistent with the dynamics of the P_2 branch, where x_2 predominates, and changes in the losses of x_1 have minimal impact due to the highly attenuated nature of the first mechanical mode in this regime. In Fig. 7(d), we can see the same behavior for x_2 . Notably, in this case, the amplitude of x_1 in P_1 remains unaffected whereas the second mode amplitude gets modified by Γ_2 .

On the other hand, TR bifurcations occur more frequently within the studied parameter range. This bifurcation leads to

dual-phonon lasing in distinct regions, with each mode oscillating near its natural frequency. Figure 8 shows the threshold curve of a TR bifurcation with respect to S and Δ for the case presented in Fig. 3 and depicted in the inset of Fig. 8(c). As shown in the inset, the two consecutive TR bifurcations correspond to the same curve. It is also important to note that this curve does not delimit the dual-phonon lasing from other solutions, as additional bifurcations may appear both inside and outside the delimited region; it simply marks the threshold at which a specific TR bifurcation occurs.

We have also performed calculations by sweeping the mechanical frequencies (Ω_1 and Ω_2). Unlike other parameters such as Δ or Γ_i , where bifurcations shift smoothly with continuous variations, sweeping Ω_1 and Ω_2 does not allow for infinitesimal continuation of bifurcations. Instead, bifurcations like PD and TR still occur for different Ω_i configurations, but their precise locations are not easily predictable, requiring an independent search for each case.

Notably, we have discovered that if the new comb lines generated by the PD bifurcation of the first lasing mode coincide with the natural frequency of the second mechanical mode, this can destabilize the original lasing branch. As a result, the period-doubling bifurcation gives rise to a new stable branch corresponding to dual-mode lasing. This finding reveals a direct mechanism through which the PD bifurcation can mediate energy transfer between mechanical modes, enabling simultaneous lasing under specific conditions.

APPENDIX E: SYMMETRY BREAKING OF γ_2/γ_1

So far, we have studied the limit where both mechanical couplings are equal ($\gamma_2/\gamma_1 = 1$). In physical systems, however, this condition may not be satisfied, leading to a broken symmetry. In Fig. 9, we present the bifurcation diagram as a function of S for a fixed detuning of $\Delta = 1$, for two different values of γ_2/γ_1 . The existence of the period-doubling bifurcation, from which the dual-mode branch arises, depends on γ_2/γ_1 . Note that the asymmetry of the coupling coefficient precludes the existence of dual-mode lasing $\gamma_2/\gamma_1 = 0.5$. Nevertheless, for moderate asymmetry ($\gamma_2/\gamma_1 = 0.9$), it still persists.

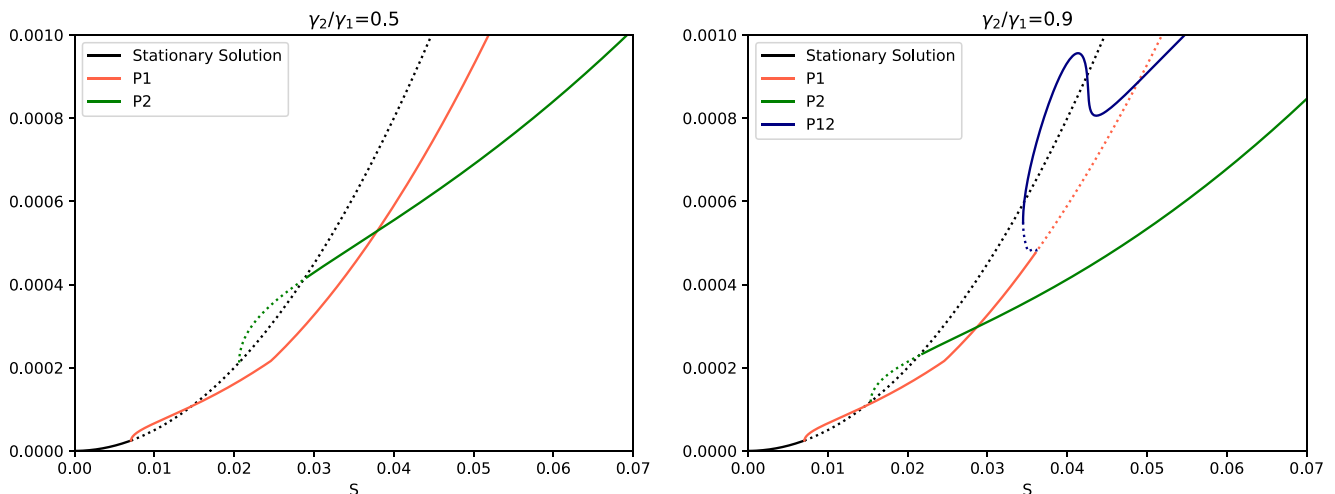


FIG. 9. Bifurcation diagrams for nonsymmetric coupling coefficients γ_2/γ_1 . Parameters: $\Delta = 1$, $\Gamma_m = 10^{-5}$, $\Omega_1 = 3$, and $\Omega_2 = 2$.

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