

Fermi-Bose crossover in superconductors: Disorder in control

Madhuparna Karmakar^{1,*} and Rajesh Narayanan²

¹Department of Physics and Nanotechnology, *SRM Institute of Science and Technology*, Kattankulathur, Chennai 603203, India

²Department of Physics, *Indian Institute of Technology Madras*, Chennai 600036, India



(Received 25 June 2024; revised 6 March 2025; accepted 1 August 2025; published 3 November 2025)

In contrast to ultracold atomic gases, Fermi-Bose crossover in solid-state superconductors is rare and a relatively recent observation. We report the first theoretical proposal of disorder-tuned Fermi-Bose crossover in superconductors with a customized spin-selective disorder potential serving as the control parameter. Our analysis of the relevant spectroscopic signatures unambiguously brings forth the evolution of the Fermi surface topology with disorder, as the system traverses through the Fermi-Bose crossover. We qualitatively analyze our inferences in the light of the recent observations of Fermi-Bose crossover in doped iron chalcogenide superconductors and organic superconductors, as well as propose an experimental setup of two-dimensional electron gas at oxide interfaces where such disorder-tuned Fermi-Bose crossover could be observed.

DOI: [10.1103/hmyb-2b8q](https://doi.org/10.1103/hmyb-2b8q)

I. INTRODUCTION

In two dimensions, deep inside the Bardeen-Cooper-Schrieffer (BCS) regime the Cooper pair formation takes place for an infinitesimal attractive interaction. At the other end of the interaction axis, strong attraction between the fermions gives rise to tightly bound molecules that undergo condensation to the Bose-Einstein condensate (BEC). This continuous tuning of quantum statistics from the weakly coupled BCS to the molecular BEC phase is deeply relevant not only to strongly correlated materials (as it underpins intriguing physics in the form of pseudogap phase, strange metals, and preformed Cooper pairs in the regime of intermediate coupling) [1–4] but also to other diverse areas such as nuclear physics [5] and neutron stars [6].

Initially realized in ultracold atomic gases via Feshbach resonance [7–13], the Fermi-Bose crossover has been recently reported in organic superconductors κ -(BEDT-TTF)₂X [14–18], iron chalcogenides [19–25], magic angle twisted bilayer graphene [26–29], magic angle twisted trilayer graphene [29,30], two-dimensional (2D) gated semiconductors [31–33], magnetoexciton condensates [34–37], and cuprates [38]. The required evolution of the Fermi surface topology across the crossover in these materials is brought about by pressure, chemical doping, strain engineering, ion gating, etc. [14–40].

In this paper, we propose an alternate protocol wherein the Fermi-Bose crossover is achieved via the manipulation of the random disorder potential. More specifically, we tune the fermionic dispersion via the implementation of a

“spin-selective” disorder wherein the different spin species are subjected to dissimilar disorder distributions [41–43]. This asymmetry in disorder distributions witnessed by the two spin species serves as a control parameter to continuously tune through the Fermi-Bose crossover via impurity-induced band structure engineering.

The impact of quenched impurities on the superconducting systems is a subject matter that has been well investigated. However, much of these studies have centered on the impact of conventional spin-independent disorder on the superconducting transition (and on the putative phases). These investigations have unraveled, for instance, the mechanism behind the localization phenomena [44,45] in these low-dimensional systems and also the nature of the disorder-induced superconductor-insulator transition (SIT) [46–53]. Furthermore, these studies have revealed the existence of an anomalous metallic phase in certain low-dimensional disordered superconductors (see, e.g., Ref. [54] for a recent review), and also the observation of exotic disorder-induced Griffiths effects in such systems [55–59]. In contrast, barring a few exceptions [41–43], impact of spin-dependent disorder on the superconducting transition and the putative phases has not been well investigated.

This paper presents a complete elucidation of the rich ground-state phase diagram [see Fig. 1(a)] of such a spin-selectively disordered superconductor highlighting the hitherto mentioned crossover of statistics (from Fermi to Bose) as a function of the asymmetry in the disorder strengths experienced by the two spin species. More specifically, the phase diagram of Fig. 1(a) represented in the interaction (U)-disorder strength (V) plane encodes the existence of (1) a disorder-induced crossover quantified by the scale V_{Bose} that demarcates a fermionic superconductor from its bosonic counterpart, (2) disorder-dependent crossover scale V_g that delineates the gapless and gapped superconducting phases, and (3) a nonmonotonic disorder-controlled transition scale V_{SC} that quantifies the loss of superconducting order and its

*Contact author: madhuparna.k@gmail.com

Published by the American Physical Society under the terms of the *Creative Commons Attribution 4.0 International* license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI.

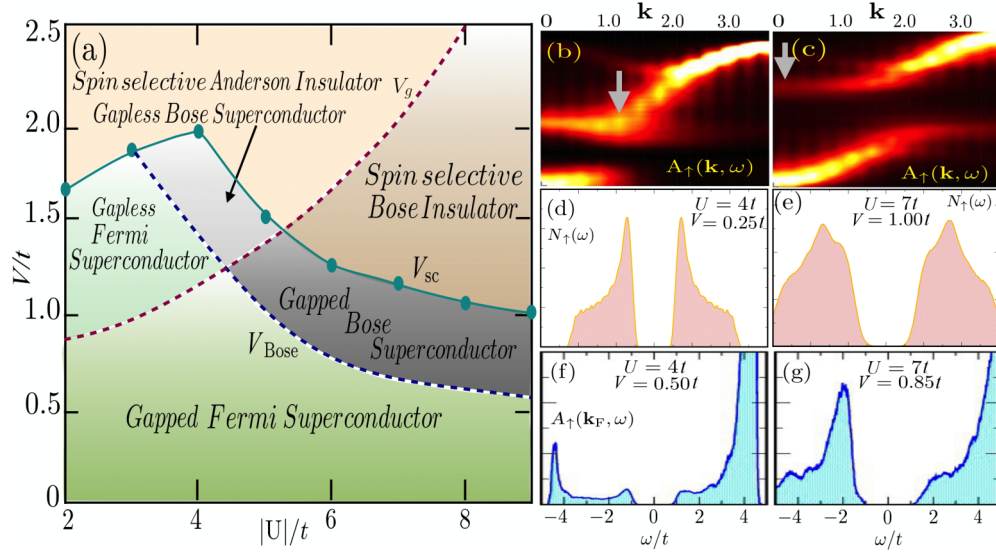


FIG. 1. (a) Ground-state (at $T = 0.01t$) phase diagram in the U - V plane, showing the thermodynamic phases and the relevant crossover and transition scales. (b), (c) Electronic spectral function ($A^{\dagger}(\mathbf{k}, \omega)$) at $U = 4t, V = 0.25t$ and $U = 7t, V = t$, respectively, representative of the fermionic and bosonic superconducting phases. The arrows indicate the gap minima. The corresponding single-particle densities of states (DOS) ($N_{\uparrow}(\omega)$) are shown in panels (d) and (e), respectively. (f), (g) Spectral line shape ($A(\mathbf{k}_F, \omega)$) at the Fermi momentum at $U = 4t, V = 0.50t$ and $U = 7t, V = 0.85t$, respectively, prototypical of Fermi and Bose superconductors.

transition to a correlated Anderson insulator (gapped bosonic insulator) in weak (strong) coupling regime.

Spin-selective disorder-tuned Fermi-Bose crossover should be experimentally observable in ultracold atomic gases hosted on an optical lattice [60–64], using spin-selective speckle disorder [41,42,65]. Among the solid-state systems, such disorder-tuned Fermi-Bose crossover is expected to be realizable in two-dimensional electron gas at the interface of oxide heterostructure, such as the interface between non-magnetic band insulators LaAlO_3 (LAO) and SrTiO_3 (STO) [66–71]: Magnetic mapping via superconducting quantum interference device [72] and magnetic force microscopy [73] have established the inhomogeneous spatial coexistence of ferromagnetism with superconducting correlations. Such inhomogeneous regions of local magnetic polarization are modeled via an inherently random local Zeeman field. The interplay between this random local Zeeman field and the conventional chemical potential disorder that is obtained due to the presence of quenched impurities results in an asymmetric renormalization of the disorder potential seen by the two spin species, thus leading to spin-selective disorder. Such asymmetric renormalization of the disorder potential has also, for instance, been reported in the material $\text{Fe}_{1-y}\text{Co}_y\text{Si}$, wherein the spin splitting due to the exchange interactions leads to the fact that the depth of the potential wells at the Co sites is increased for the majority carriers and decreased for the minority carriers [74]. Another interesting example for a possible realization of the spin selectivity is provided by the family of organic superconductors κ -(ET) $_2\text{X}$, etc. [75], wherein the interplay of chemical potential disorder (via nonstoichiometric doping) and antiferromagnetic fluctuations could lead to the spin-selective disorder.

Our paper is organized along the following lines. In Sec. II, we describe the spin-selectively disordered attractive Hubbard model and the physical indicators that we use to construct the

phase diagram. The phase diagram is elucidated in Sec. III along with a discussion of the various phases and also of the myriad crossover scales in the problem. In Sec. IV, we end the main part of this paper with discussions and conclusions, while supporting results that highlight the spin selectiveness of the Bose-Fermi crossover are relegated to Appendix A. In a similar vein, Appendix B is devoted to the discussion of the spatial maps that ensues as we transition from the gapped to the gapless phase as a function of disorder.

II. MODEL AND INDICATORS

Our starting point is the attractive Hubbard model defined on a square lattice, described by the Hamiltonian

$$H = -t \sum_{\langle ij \rangle, \sigma} (c_{i\sigma}^{\dagger} c_{j\sigma} + c_{j\sigma}^{\dagger} c_{i\sigma}) - |U| \sum_i \hat{n}_{i\uparrow} \hat{n}_{i\downarrow} + \sum_{i\sigma} (V_{i\sigma} - \mu) \hat{n}_{i\sigma}. \quad (1)$$

Here, the hopping amplitude is set at $t = 1$ for the nearest neighbors, the reference energy scale of the system. Spin selectivity is obtained via rendering the random disorder potential $V_{i\sigma}$ to be spin dependent, such that $V_{i\uparrow} = 0$ and $V_{i\downarrow} \neq 0 = V$, selected randomly from a box distribution of $[-V/2, V/2]$. $|U| > 0$ is the attractive on-site interaction giving rise to the s -wave superconducting pairing. We work in the limit of the half filling ($n = 1$) and the global chemical potential μ is adjusted to maintain the same. The degeneracy between the superconducting and charge density wave (CDW) orders at the half filling is lifted in the presence of potential disorder.

The model is made numerically tractable via Hubbard-Stratonovich decomposition of the interaction term and introducing a complex $\Delta_i(\tau) = |\Delta_i(\tau)|e^{i\theta_i(\tau)}$ and a real $\phi_i(\tau)$

scalar (bosonic) auxiliary field at each site, where the former couples to the pairing and the latter to the charge channels [76,77]. The problem is addressed via static path approximated (SPA) Monte Carlo technique wherein the model is envisaged as an effective spin-fermion model with the random, fluctuating, “classical” background of the auxiliary fields coupled to the free fermions [53,78,79]. At the ground state ($T = 0$) where the thermal fluctuations of the bosonic auxiliary fields are irrelevant, SPA reduces to the Hartree-Fock-Bogoliubov de Gennes (HFBdG) approach, which is effectively an inhomogeneous mean field theory (MFT) that accounts for the spatial inhomogeneities in the pairing field amplitude [80,81].

To characterize the ground-state phase diagram seen in Fig. 1(a), we lay recourse to the observables: (1) superconducting pairing field structure factor ($S(\mathbf{q}) = (1/N^2) \sum_{ij} \langle \Delta_i \Delta_j^* \rangle e^{i\mathbf{q} \cdot (\mathbf{r}_i - \mathbf{r}_j)}$); (2) spin-resolved single-particle DOS ($N_\uparrow(\omega) = (1/N) \langle \sum_{i,n} |u_n^i|^2 \delta(\omega - E_n) \rangle$, $N_\downarrow(\omega) = (1/N) \langle \sum_{i,n} |v_n^i|^2 \delta(\omega + E_n) \rangle$) and the associated energy gap E_g^σ ; (3) spin-resolved spectral line shape ($A^\sigma(\mathbf{k}_F, \omega)$) at the Fermi momentum $\mathbf{k} = \mathbf{k}_F$; and (4) spin-resolved spectral function ($A^\sigma(\mathbf{k}, \omega) = (-1/\pi) \text{Im} G^\sigma(\mathbf{k}, \omega)$), where u_n^i and v_n^i are the Bogoliubov eigenvectors with the eigenvalues E_n , σ is the spin label, and $G^\sigma(\mathbf{k}, \omega)$ is the interacting Green’s function, with ω being the real frequency corresponding to the single-particle energy scale $\hbar\omega$ ($\hbar = 1$). The results presented here are for a system size of 24×24 and are robust against finite system size effects. The observables are averaged over ten disorder realizations and are invariant to further increase in the number of disorder realizations.

III. PHASE DIAGRAM AND CROSSOVER SCALES

The phase diagram in Fig. 1(a) hosts a gapped fermionic superconductor along with its gapped bosonic counterpart, both characterized with $S(\mathbf{q}) \neq \mathbf{0}$ and $E_g \neq 0$. They are distinguished with respect to each other via the behavior of the DOS at the band edges: The fermionic superconductor displays the usual inverse square root van Hove singularity [see Fig. 1(d)], whereas the bosonic superconductor displays a jump discontinuity at the gap edges [Fig. 1(e)]. This difference between the fermionic and bosonic statistics can be understood by comparing Figs. 1(b) and 1(c), which tracks the evolution of the Fermi surface topology via the spectral function $A^\uparrow(\mathbf{k}, \omega)$. With increasing disorder strength, the minimum gap shifts from a finite wave vector (or a contour in $\mathbf{k}$ -space) in the fermionic regime (indicating the presence of a Fermi surface) to a situation wherein the minimum is obtained at a single point $\mathbf{k} = \mathbf{0}$, resulting in the DOS and the spectral line shape $A^\sigma(\mathbf{k}_F, \omega)$ [see Figs. 1(f) and 1(g) and Appendix A for more details], developing discontinuities at the gap edges in the bosonic regime as opposed to the inverse square root van Hove singularity in the fermionic description [39].

The phase diagram also hosts (1) a gapless Fermi superconductor ($S(\mathbf{q}) \neq \mathbf{0}$, $E_g = 0$, with inverse square root singularity), (2) a gapless Bose superconductor ($S(\mathbf{q}) \neq \mathbf{0}$, $E_g = 0$, jump discontinuity), (3) spin-selective Anderson insulator ($S(\mathbf{q}) = \mathbf{0}$, $E_g = 0$, featureless spectra), and finally (4) a spin-selective Bose insulator ($S(\mathbf{q}) = \mathbf{0}$, $E_g \neq 0$, with residual

jump discontinuity). The Bose to Fermi crossover is encoded via the disorder scale V_{Bose} , whereas the impurity scale V_g separates the gapped and gapless phases. Finally, the SIT scale mapped out based on $S(\mathbf{q})$ is controlled by V_{SC} . We now set out to elucidate the phase diagram in more detail.

A. Thermodynamic, spectroscopic signatures and gapless superconductivity

Figures 2(a) and 2(b) show the variation of the energy gap $E_g^\uparrow$ ($E_g^\downarrow$ behaves similarly, except for strong disorder) at the Fermi level and the pairing field structure factor ($S(\mathbf{q})$) with disorder at selected interactions. The former is tied to the pairing field amplitude $|\Delta_i|$, leading to the scale $V_g \propto |U|$, while the latter tracks the (quasi-)long-range superconducting phase coherence and gives rise to the nonmonotonic scale $V_{\text{SC}} \propto t^2/|U|$, shown in Fig. 1(a). The behavior of $N_\uparrow(\omega)$ at selected interactions of $U = 4t$ and $U = 7t$ is shown in Figs. 2(c) and 2(d), while the corresponding real-space signatures in terms of the spatial maps of the pairing field amplitude $|\Delta_i|$ and the phase coherence $\cos(\theta_0 - \theta_i)$ (where θ_0 corresponds to the pairing field phase at the reference site and θ_i is the phase at every site other than the reference site) are presented in Figs. 2(e) and 2(f).

At $U = 4t$, the system is a gapped Fermi superconductor upto $V_g \sim 1.10t$ with a robust spectral gap at the Fermi level and inverse square root singularities at the gap edges, typified by a uniform, (quasi-)long-range phase-coherent superconducting state, depicted in Fig. 2(e). The gapless Fermi superconductor in the regime $1.10t \lesssim V < V_{\text{Bose}} \sim 1.25t$ is characterized by spectral weight accumulation at the Fermi level and suppression of the coherence peaks at the gap edges. The corresponding superconducting state is spatially inhomogeneous, as shown in Fig. 2(e), at $U = 4t$ and $V = 1.15t$.

Gapless superconductivity is well established in the context of magnetic impurities [82–86], magnetic superconductors [87–92], Fulde-Ferrell-Larkin-Ovchinnikov superconductors [78,93–98], helical superconductors, etc. [99–106], wherein magnetic correlations serve as pair breaking perturbations in s -wave superconductors, breaking the time-reversal degeneracy of the pairing electrons and leading to suppression of the superconducting correlations via creation of “in-gap” states. The key to understanding the gapless superconductor is the realization that spin-selective randomness can be recast into a local random chemical potential that is (anti)correlated with a Zeeman field counterpart [43]. This random Zeeman field couples to the itinerant fermions as $\sim h(r)\sigma_i$, with $h(r)$ being the spatially inhomogeneous Zeeman field and σ_i the spin of the itinerant fermions. This is a form akin to the scenario wherein magnetic impurities are embedded randomly in a host superconductor: The time-reversal symmetry-breaking disorder gives rise to the in-gap states and the eventual closure of the hard superconducting gap, i.e., a gapless superconducting phase [104,106].

At $V_{\text{Bose}} \sim 1.3t$, discontinuity appears at the gap edges of the single-particle spectra, indicating bosonic correlations. The finite spectral weight at the Fermi level indicates a gapless Bose superconductor. The corresponding spatial signatures at $U = 4t$, $V = 1.65t$ include an inhomogeneous, strongly suppressed $|\Delta_i|$ and loss of (quasi-)long-range phase coherence.

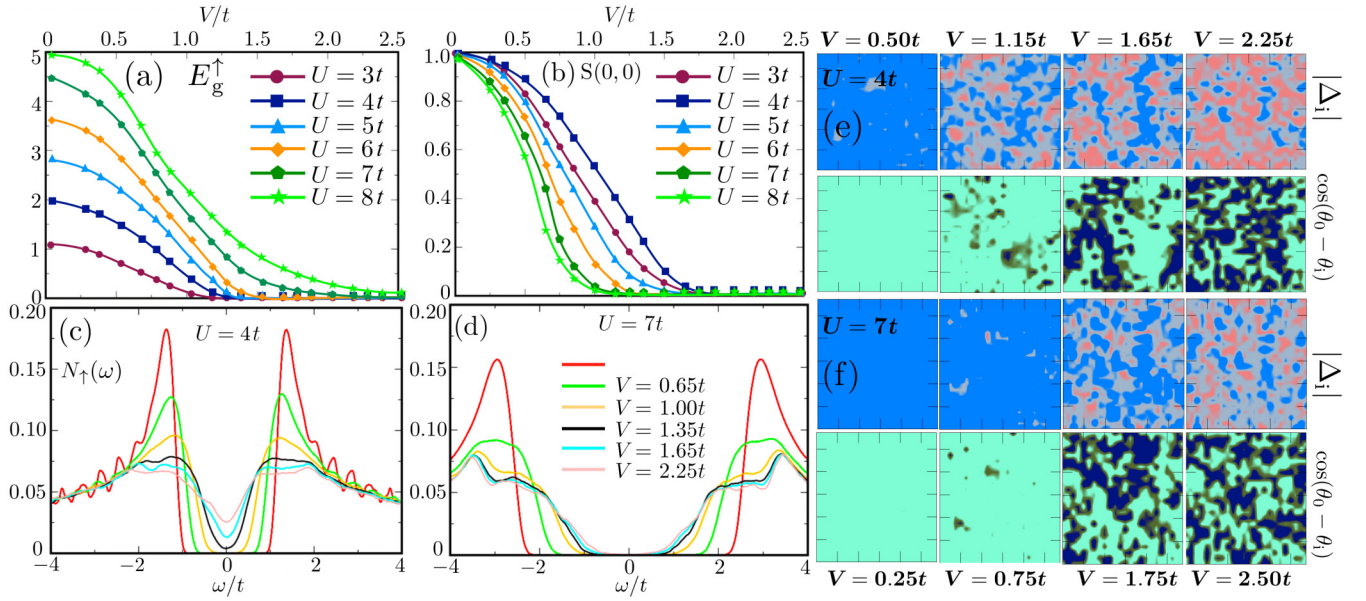


FIG. 2. Thermodynamic and spectroscopic indicators at $T = 0.01t$. (a) Disorder dependence of $E_g^\dagger$, as determined from $N_\dagger(\omega)$ at different interactions. (b) Disorder-induced suppression of $S(0,0)$ corresponding to the loss of (quasi-)long-range phase coherence. (c), (d) Disorder dependence of $N_\dagger(\omega)$ at the representative interactions of $U = 4t$ and $U = 7t$. (e), (f) The corresponding real-space maps for the superconducting pairing field amplitude ($|\Delta_i|$) (blue: high amplitude; red: low amplitude) and the superconducting phase coherence $\cos(\theta_0 - \theta_i)$ (green: high magnitude; black: low magnitude), at selected disorders.

The SIT to the gapless correlated Anderson insulating phase takes place at $V \gtrsim V_{SC}$, characterized by the spectral weight accumulation at the Fermi level, absence of coherence peaks, large transfer of spectral weight away from the Fermi level, and absence of superconducting correlations in the pairing field amplitude and phase, as shown at $U = 4t$ and $V = 2.25t$. The real-space signature of the correlated Anderson insulator, typified by the case at $U = 4t$ and $V = 2.25t$ in Fig. 2(e), shows the lack of superconducting correlations in terms of both the pairing field amplitude and phase, as observed via the corresponding spatial maps.

In the strong coupling regime, the system is a gapped Fermi superconductor upto $V_{Bose} \lesssim 0.65t$ with homogeneous pairing field amplitude and (quasi-)long-range phase coherence, shown at $U = 7t$ and $V = 0.25t$, in Fig. 2(f). Similar real-space signatures are observed at $U = 7t$ and $V = 0.75t$ representative of a gapped Bose superconductor, realized in the regime $V_{Bose} \leq V < V_{SC} \sim 1.10t$ and typified by a spectral gap with gap edge discontinuities in $N_\dagger(\omega)$. The regime $V_{SC} \lesssim V < V_g \sim 2t$ is demarcated as a spin-selective Bose insulator with a robust spectral gap and absence of global superconducting phase coherence. The corresponding real-space signatures, at $U = 7t$ and $V = 1.75t$, include phase uncorrelated puddles of superconducting pairing field with large but randomly distributed $|\Delta_i|$. $V > V_g$, as evidenced by $U = 7t$ and $V = 2.50t$, is a gapless correlated Anderson insulator.

B. Disorder-tuned Fermi-Bose crossover

Figure 3 shows the spin-resolved $A^\dagger(\mathbf{k}, \omega)$ exhibiting disorder-tuned renormalization of the Fermi surface across the Fermi-Bose crossover. With increasing disorder, the electronic dispersion band undergoes renormalization from downward

convex to being independent of $\mathbf{k}$. The topology evolves as the Fermi surface becomes increasingly notional and eventually vanishes in the Bose regime. At the intermediate disorder, a gapless superconducting state is realized, with the corresponding dispersion spectra being transient between a fermionic and a bosonic description. The disorder-tuned evolution of the Fermi surface topology is in agreement with the recent angle resolved photo emission spectroscopy (ARPES) measurements on the hole-doping-tuned Fermi-Bose crossover in $\text{FeSe}_{1-x}\text{S}_x$ [24,25].

C. Spin selectivity

At a representative disorder of $V = 1.35t$, we show the spin-resolved single-particle DOS in Figs. 4(a) and 4(b). At this choice of disorder, the system is a gapless Fermi superconductor at $U = 3t$, which crosses over to a gapless Bose superconductor at $U = 4t$ and $5t$, quantified by the broad coherence peaks with weak gap edge discontinuities. The

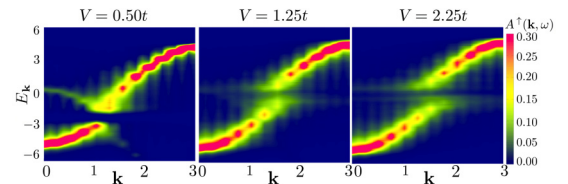


FIG. 3. Disorder-tuned Fermi-Bose crossover as demonstrated via the spectral function ($A^\dagger(\mathbf{k}, \omega)$) at $T = 0.01t$, for $U = 4t$. The weak ($V = 0.25t$) and strong ($V = 2.25t$) disorder limits are typical of the fermionic and bosonic phases, while the intermediate disorder ($V = 1.25t$) regime hosts a (gapless) mixed phase with fermionic and bosonic characteristics.

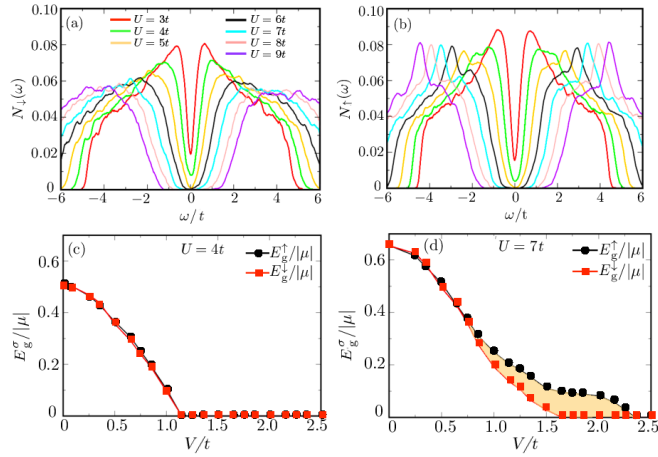


FIG. 4. Interaction dependence of the single-particle DOS at $V = 1.35t$, corresponding to (a) $N_{\uparrow}(\omega)$ and (b) $N_{\downarrow}(\omega)$, respectively. (c), (d) Disorder dependence of the ratio between the spin-selective spectral gap at the Fermi level (E_g^{σ}) and the global chemical potential (μ) adjusted to the Fermi energy, at $U = 4t$ and $U = 7t$, respectively (see text). The shaded region in panel (d) corresponds to the regime where only one of the spin species is rendered gapless by disorder.

spectral gap opens up at $U \geq 6t$ with spin-selective gap edge discontinuities. While for the $\uparrow$ -spin species the discontinuities at the gap edges get prominent with interaction, the $\downarrow$ -spin species exhibit large transfer of spectral weight away from the Fermi level with the corresponding gap edge discontinuities becoming more rounded, owing to the direct impact of the disorder potential.

Figures 4(c) and 4(d) depict the disorder dependence of the ratio of E_g^{σ} to the global chemical potential μ (set to the Fermi energy) at $U = 4t$ and $U = 7t$. In the weak coupling regime [Fig. 4(c)], the ratio $E_g^{\sigma}/|\mu|$ is spin independent. However, in the strong coupling regime ($U \gg V$), $E_g^{\sigma}/|\mu|$ shows asymmetric renormalization as a function of the disorder strength [Fig. 4(d)]: The superconducting gap completely collapses for the $\downarrow$ -spin fermions as a function of V/t , while for the $\uparrow$ -spin species it is still robust. Evidently, the superconducting correlations are suppressed much faster in the $\downarrow$ -spin species owing to the direct influence of the disorder potential, as compared to the $\uparrow$ -spin species that sees the renormalized disorder via U .

IV. DISCUSSION AND CONCLUSIONS

The results elucidated in this paper hinge on the spin selectiveness of the disorder potential, which is obtained, for instance, from an (anti)correlation between a local random chemical potential and a Zeeman field [43]. Such type of effective spin-dependent disorder has been reported in the case of Co-doped FeSi [74], wherein the interplay of spin polarization and disorder conspires such that the two spin species making up the majority and minority bands see different disorder profiles.

The signatures of the disorder-tuned crossover can be gleaned from the spectral function (Fig. 3) and the spectral line shape (see Appendix A) via ARPES [24,25]. Also, the

spin-resolved DOS [Figs. 2(c) and 2(d)] can be extracted from the corresponding tunneling conductance [74].

The disorder-induced renormalization of the Fermi surface that we observe in the spectral function (Fig. 3) including the characteristic backbending feature is reminiscent of the case of $\text{FeSe}_{1-x}\text{S}_x$ [24,107,108], wherein the substitution of Se by isovalent S leads to an analogous evolution of the Bogoliubov quasiparticle band dispersion. In fact, apart from FeSe, other iron-based superconductors such as $\text{Ba}_{1-x}\text{K}_x\text{Fe}_2\text{As}_2$ [107], $\text{LiFe}_{1-x}\text{Co}_x\text{As}$ [108], etc. are also potential candidates wherein the Fermi-Bose crossover can be observed as a function of chemical doping. Our results open up the intriguing possibility of quenched randomness acting as the operative mechanism behind the Fermi-Bose crossover seen in some of the quantum materials discussed above.

While it is possible that the interplay of disorder (induced by doping) and competing phases (for instance, the spin density wave) in the $\text{FeSe}_{1-x}\text{S}_x$ system might lead to a realization of a spin-dependent disorder, one should be cautious in applying our results directly to the Fe-based superconductors. This stems from the fact that the Fe-based systems typically are understood via a multiorbital setup, the physics of which is missing in our simple single-band Hubbard model. However, it is by now well known that certain phenomenon in these systems like the pseudogap phase (at finite temperatures) is orbital selective and gives rise to rich Fermi-Bose crossover physics [109].

Another interesting potential candidate for an experimental realization of the spin-selectively disordered single-band two-dimensional system is provided by the family of organic superconductors $\kappa\text{-(ET)}_2\text{X}$ [75], wherein the interplay of chemical potential disorder (via nonstoichiometric doping) and antiferromagnetic fluctuations could lead to the spin-selective disorder, providing an alternate route to the BEC-BCS (pressure-tuned) crossover realized in these materials.

Finally, we turn our attention to a very important issue, namely, the interplay of charge and superconducting fluctuations in our system. At half filling for a translationally invariant system, the CDW and superconducting order parameters are degenerate. This degeneracy is broken in the disordered system wherein the CDW melts via an Imry-Ma-type effect [110,111]. However, it could be argued that large charge fluctuations could severely depress the superconducting order parameter that could impact the nature of the phase diagram elucidated in Fig. 1(a). To obviate this criticism, we stress that our results are obtained from an SPA-based Monte Carlo analysis [53,78–81], which at the lowest temperatures (at $T = 0.01t$) becomes equivalent to the inhomogeneous HFBdG-type MFT. Thus, the methodology that we have adopted to arrive at the MFT actually takes into account the charge fluctuations as well as the superconducting fluctuations. This fact is well represented in Appendix B where we present the real-space maps of the pairing field, the phase coherence function, and the charge density [earmarked by the variable $\phi(r)$] as a function of disorder (for a fixed interaction strength $U = 4t$), as we anneal in temperature from $T = 0.30t$ to $T = 0.01t$ (note we have only displayed data from $T = 0.20t$ to $T = 0.01t$ in Fig. 6). At the lowest temperatures, with increasing disorder we see that the su-

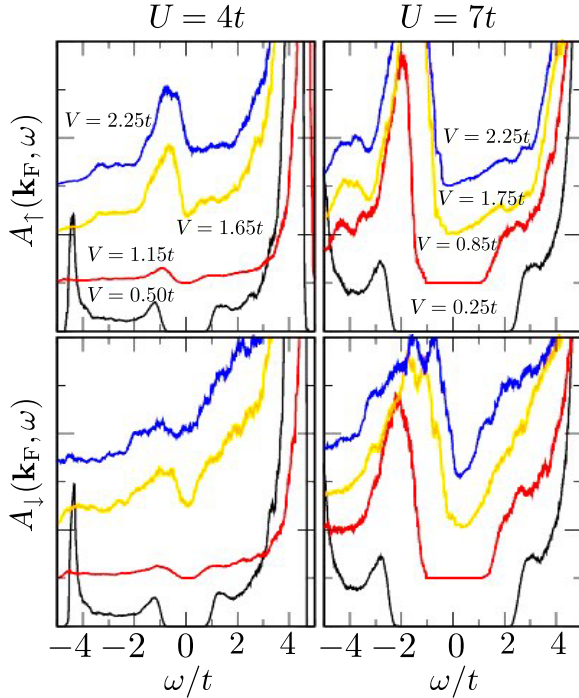


FIG. 5. Spin-resolved spectral line shape at $U = 4t$ for $V = 0.50t, 1.15t, 1.65t$, and $2.25t$ and at $U = 7t$ for $V = 0.25t, 0.85t, 1.75t$, and $2.25t$, at $T = 0.01t$. Note the asymmetric jump discontinuity at the gap edges salient to the bosonic correlations. The curves are uniformly shifted along the y axis for clarity.

perconducting order survives even in the presence of strong charge fluctuations.

In conclusion, we have numerically demonstrated for the first time that customized disorder potential can be used as a tuning knob to realize Fermi-Bose crossover in solid-state superconductors. Based on our analysis of the spectroscopic and thermodynamic signatures of the system, we have shown that engineered chemical potential disorder can give rise to the required renormalization of the Fermi surface architecture of a superconductor so as to catalyze a Fermi-Bose crossover physics. We expect our results to initiate important experiments, particularly in iron chalcogenide and 2D organic superconductors for the solid-state realization of the disorder-tuned orbital-selective and band-selective Fermi-Bose crossover.

ACKNOWLEDGMENTS

The authors acknowledge the use of high-performance computing cluster Aqua, at IIT Madras, India. R.N. acknowledges funding from the Center for Quantum Information Theory in Matter and Space-time, IIT Madras, and from the Department of Science and Technology, Government of India, under Grant No. DST/ICPS/QuST/Theme-3/2019/Q69, as well as support from the Mphasis F1 Foundation via the Centre for Quantum Information, Communication, and Computing (CQuICC). We thank Professor G. Baskaran (IIT Madras) for valuable discussions. We also thank Dr. Prabudha Chakraborty (ISI Bangalore) for a collaboration in the early part of this project.

APPENDIX A: SPIN-SELECTIVE SPECTRAL LINE SHAPE

We show the prominent spin selectivity in terms of the spin-resolved spectral line shape at the selected interactions $U = 4t$ and $7t$, corresponding to the weak and the strong coupling regimes, respectively, in Fig. 5. At $U = 4t$ and weak disorder ($V = 0.5t$), the system is a gapped Fermi superconductor independent of the choice of the spin species. The disorder cross section $V = 1.15t$ is the boundary between the gapped and the gapless Fermi superconducting states at $U = 4t$, and the corresponding spectral line shape exhibits a suppressed gap at the Fermi level. The pronounced asymmetry at the gap edges for the $\uparrow$ -spin species is due to the short-range bosonic correlations, with the system being proximate to the Bose superconducting state. For the $\downarrow$ -spin species, disorder quickly suppresses the bosonic correlations leading to largely symmetric line shape. The gapless Bose superconductor at $V = 1.65t$ is ascertained by the pronounced asymmetry in the spectral line shape for both the spin species along with finite spectral weight at the Fermi level, while the strong disorder regime at $V = 2.25t$ is spin selective with the $\uparrow$ -spin species containing signatures of bosonic correlations, whereas the $\downarrow$ -spin species is relatively featureless.

In the strong coupling ($U = 7t$) and weak disorder ($V = 0.25t$) regime, the spectral line shape pertaining to both the spin species corresponds to the gapped Fermi superconductor. The Bose superconducting phase at the intermediate disorder ($V = 0.85t$) is established by the asymmetric gap edge discontinuities in the spectral line shape. $V = 1.75t$ is a (spin-selective) Bose insulator phase characterized by the signatures of bosonic correlations in the $\uparrow$ -spin species line shape, while the $\downarrow$ -spin spectra are featureless. The loss of bosonic correlations at stronger disorder is via a spin-dependent scale, as observed at the representative disorder of $V = 2.25t$. The spectral line shape along with exhibiting spin selectivity provides clear distinction between the fermionic and bosonic topologies of the Fermi surface in terms of pronounced asymmetry in the gap edge features.

APPENDIX B: SPATIAL MAPS

Figure 6 shows the spatial maps corresponding to the pairing field amplitude, phase coherence, and the auxiliary field $\phi(r)$ coupled to the charge density distribution at weak ($V = 0.25t$), intermediate ($V = 0.75t$), and strong ($V = 1.75t$) disorder regimes for $|U| = 4t$ that is obtained from a full SPA-based Monte Carlo analysis [53,78–81]. In the weak disorder regime, the system is a homogeneous superconductor with long-range phase coherence at the lowest temperature. Now, as the disorder strength increases into the intermediate disorder regime, there is a fragmentation effect at play as the system breaks into large local islands, which are phase locked across the entire sample, as evidenced by the behavior of the coherence function $\cos(\theta_0 - \theta_r)$. As the disorder strength increases further, as exemplified by the case $V = 1.75t$, the fragmentation effect is accentuated and the system breaks into domains where the pairing field amplitude as well as the energy gap is suppressed. Further, the system has lost its long-range phase coherence in this regime as shown by the $\cos(\theta_0 - \theta_r)$ map at $T = 0.01t$, thus entering into the

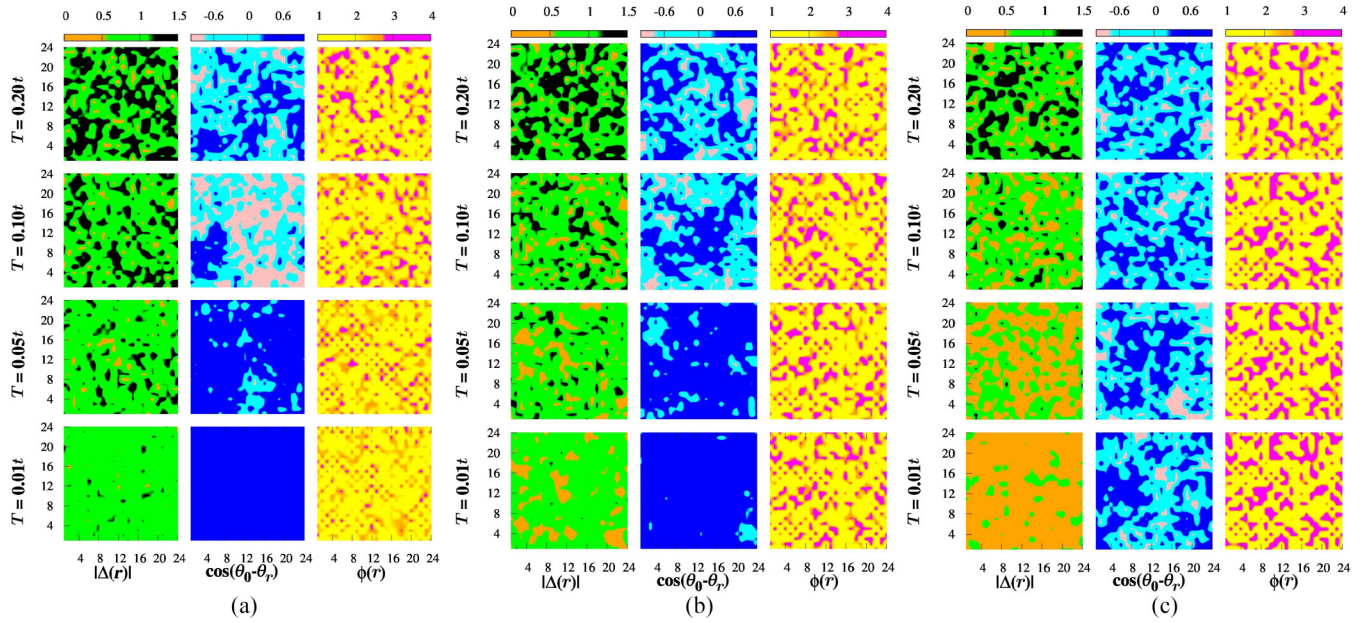


FIG. 6. Thermal evolution of the real-space indicators at three different cross sections corresponding to (a) weak ($V = 0.25t$), (b) intermediate ($V = 0.75t$), and (c) strong ($V = 1.75t$) disorder regimes, at $U = 4t$. In each case, the leftmost and the middle column correspond to the distribution of the superconducting pairing field amplitude ($|\Delta(r)|$) and phase coherence ($\cos(\theta_0 - \theta_r)$), while the rightmost column corresponds to the distribution of the auxiliary field $\phi(r)$ coupled to the charge density. At weak and intermediate disorder regimes, phase coherence survives up to temperatures higher than the one corresponding to the temperature at which pairing field amplitude undergoes fragmentation. At strong disorder, however, the pairing field amplitude is strongly suppressed even at the lowest temperature, while there are strong fluctuations in phase. Note that while in the regimes of weak and intermediate disorder the transition is driven by amplitude fluctuations, the fluctuations in phase drive the transition in the regime of strong disorder.

disordered phase. As evidenced in these spatial maps via the behavior of the quantity $\phi(r)$, the loss of phase coherence is a combined effect of the phase and charge fluctuations.

Now, for the sake of completeness, we briefly comment on the impact of thermal fluctuations on this spin-selectively disordered attractive Hubbard model: Temperature fluctuations also act as a disordering mechanism once again leading to the fragmentation effect that we saw above. As seen in Fig. 6, the maps corresponding to $V = 0.25t$ at $T = 0.10t$ show signatures of gapless regime. Though the superconducting state has undergone weak fragmentation, the regions of local order in both the pairing field amplitude and phase are still weakly correlated. Further increase in temperature leads these fragments to disconnect themselves and loss of global phase coherence and thus superconducting order.

The behavior at the intermediate disorder ($V = 0.75t$) is qualitatively similar to that observed in the weak disorder regime. There are, however, certain quantitative differences that are observable even at low temperatures. The lowest temperature state as we discussed above is now fragmented, with the size of the islands being larger than those observed at $V = 0.25t$. Thermal evolution gives rise to similar behavior as discussed above. However, as evidenced from Fig. 6, the thermal transition into the disordered or nonsuperconducting state is obtained at a lower temperature.

At the larger disorder strength of $V = 1.75t$, even at the lowest temperature the fragmentation of the system is complete (as elucidated in the beginning of this section). However, with increasing temperature the system phase segregates into islands of local order that are uncorrelated with respect to each other.

- [1] V. J. Emery and S. A. Kivelson, Importance of phase fluctuations in superconductors with small superfluid density, *Nature (London)* **374**, 434 (1995).
- [2] J. Corson, R. Mallozzi, J. Orenstein, J. N. Eckstein, and I. Bozovic, Vanishing of phase coherence in underdoped $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+\delta}$, *Nature (London)* **398**, 221 (1999).
- [3] L. Li, Y. Wang, S. Komiyama, S. Ono, Y. Ando, G. D. Gu, and N. P. Ong, Diamagnetism and Cooper pairing above T_c in cuprates, *Phys. Rev. B* **81**, 054510 (2010).
- [4] B. Keimer, S. A. Kivelson, M. R. Norman, S. Uchida, and J. Zaanen, From quantum matter to high-temperature superconductivity in copper oxides, *Nature (London)* **518**, 179 (2015).
- [5] G. C. Strinati, P. Pieri, G. Röpke, P. Schuck, and M. Urban, The BCS–BEC crossover: From ultra-cold Fermi gases to nuclear systems, *Phys. Rep.* **738**, 1 (2018).
- [6] S. Heckel, P. P. Schneider, and A. Sedrakian, Light nuclei in supernova envelopes: A quasiparticle gas model, *Phys. Rev. C* **80**, 015805 (2009).
- [7] M. Randeria and E. Taylor, Crossover from Bardeen-Cooper-Schrieffer to Bose-Einstein condensation and the unitary Fermi gas, *Annu. Rev. Condens. Matter Phys.* **5**, 209 (2014).

- [8] W. Ketterle and M. W. Zwierlein, Making, probing and understanding ultracold Fermi gases, *La Rivista del Nuovo Cimento* **31**, 247 (2008).
- [9] J. T. Stewart, J. P. Gaebler, and D. S. Jin, Using photoemission spectroscopy to probe a strongly interacting Fermi gas, *Nature (London)* **454**, 744 (2008).
- [10] J. P. Gaebler, J. T. Stewart, T. E. Drake, D. S. Jin, A. Perali, P. Pieri, and G. C. Strinati, Observation of pseudogap behaviour in a strongly interacting Fermi gas, *Nat. Phys.* **6**, 569 (2010).
- [11] C. Chin, M. Bartenstein, A. Altmeyer, S. Riedl, S. Jochim, J. H. Denschlag, and R. Grimm, Observation of the pairing gap in a strongly interacting Fermi gas, *Science* **305**, 1128 (2004).
- [12] C. Chin, R. Grimm, P. Julienne, and E. Tiesinga, Feshbach resonances in ultracold gases, *Rev. Mod. Phys.* **82**, 1225 (2010).
- [13] S. Giorgini, L. P. Pitaevskii, and S. Stringari, Theory of ultracold atomic Fermi gases, *Rev. Mod. Phys.* **80**, 1215 (2008).
- [14] S. Imajo, T. Kobayashi, Y. Matsumura, T. Maeda, Y. Nakazawa, H. Taniguchi, and K. Kindo, Pseudogap formation in organic superconductors, *Phys. Rev. Mater.* **7**, 124803 (2023).
- [15] Y. Suzuki, K. Wakamatsu, J. Ibuka, H. Oike, T. Fujii, K. Miyagawa, H. Taniguchi, and K. Kanoda, Mott-driven BEC-BCS crossover in a doped spin liquid candidate κ -(BEDT-TTF)₄Hg_{2.89}Br₈, *Phys. Rev. X* **12**, 011016 (2022).
- [16] H. Oike, Y. Suzuki, H. Taniguchi, Y. Seki, K. Miyagawa, and K. Kanoda, Anomalous metallic behaviour in the doped spin liquid candidate κ -(ET)₄Hg_{2.89}Br₈, *Nat. Commun.* **8**, 756 (2017).
- [17] S. Imajo, S. Sugiura, H. Akutsu, Y. Kohama, T. Isono, T. Terashima, K. Kindo, S. Uji, and Y. Nakazawa, Extraordinary π -electron superconductivity emerging from a quantum spin liquid, *Phys. Rev. Res.* **3**, 033026 (2021).
- [18] R. Boyack, Z. Wang, Q. Chen, and K. Levin, Unified approach to electrical and thermal transport in high- T_c superconductors, *Phys. Rev. B* **104**, 064508 (2021).
- [19] S. Kasahara, T. Watashige, T. Hanaguri, Y. Kohsaka, T. Yamashita, Y. Shimoyama, Y. Mizukami, R. Endo, H. Ikeda, K. Aoyama, T. Terashima, S. Uji, T. Wolf, H. von Löhneysen, T. Shibauchi, and Y. Matsuda, Field-induced superconducting phase of FeSe in the BCS-BEC cross-over, *Proc. Natl. Acad. Sci. USA* **111**, 16309 (2014).
- [20] S. Kasahara, T. Yamashita, A. Shi, R. Kobayashi, Y. Shimoyama, T. Watashige, K. Ishida, T. Terashima, T. Wolf, F. Hardy, C. Meingast, H. v. Lohneysen, A. Levchenko, T. Shibauchi, and Y. Matsuda, Giant superconducting fluctuations in the compensated semimetal FeSe at the BCS-BEC crossover, *Nat. Commun.* **7**, 12843 (2016).
- [21] T. Hanaguri, S. Kasahara, J. Böker, I. Eremin, T. Shibauchi, and Y. Matsuda, Quantum vortex core and missing pseudogap in the multiband BCS-BEC crossover superconductor FeSe, *Phys. Rev. Lett.* **122**, 077001 (2019).
- [22] B. L. Kang, M. Z. Shi, S. J. Li, H. H. Wang, Q. Zhang, D. Zhao, J. Li, D. W. Song, L. X. Zheng, L. P. Nie, T. Wu, and X. H. Chen, Preformed Cooper pairs in layered FeSe-based superconductors, *Phys. Rev. Lett.* **125**, 097003 (2020).
- [23] Y. Mizukami, M. Haze, O. Tanaka, K. Matsuura, D. Sano, J. Böker, I. Eremin, S. Kasahara, Y. Matsuda, and T. Shibauchi, Unusual crossover from Bardeen-Cooper-Schrieffer to Bose-Einstein-condensate superconductivity in iron chalcogenides, *Commun. Phys.* **6**, 183 (2023).
- [24] T. Hashimoto, Y. Ota, A. Tsuzuki, T. Nagashima, A. Fukushima, S. Kasahara, Y. Matsuda, K. Matsuura, Y. Mizukami, T. Shibauchi, S. Shin, and K. Okazaki, Bose-Einstein condensation superconductivity induced by disappearance of the nematic state, *Sci. Adv.* **6**, eabb9052 (2020).
- [25] S. Rinott, K. B. Chashka, A. Ribak, E. D. L. Rienks, A. Taleb-Ibrahimi, P. L. Fevre, F. Bertran, M. Randeria, and A. Kanigel, Tuning across the BCS-BEC crossover in the multiband superconductor Fe_{1+y}Se_xTe_{1-x}: An angle-resolved photoemission study, *Sci. Adv.* **3**, e1602372 (2017).
- [26] Y. Cao, V. Fatemi, S. Fang, K. Watanabe, T. Taniguchi, E. Kaxiras, and P. Jarillo-Herrero, Unconventional superconductivity in magic-angle graphene superlattices, *Nature (London)* **556**, 43 (2018).
- [27] X. Lu, P. Stepanov, W. Yang, M. Xie, M. A. Aamir, I. Das, C. Urgell, K. Watanabe, T. Taniguchi, G. Zhang, A. Bachtold, A. H. MacDonald, and D. K. Efetov, Superconductors, orbital magnets and correlated states in magic-angle bilayer graphene, *Nature (London)* **574**, 653 (2019).
- [28] M. Oh, K. P. Nuckolls, D. Wong, R. L. Lee, X. Liu, K. Watanabe, T. Taniguchi, and A. Yazdani, Evidence for unconventional superconductivity in twisted bilayer graphene, *Nature (London)* **600**, 240 (2021).
- [29] H. Kim, Y. Choi, C. Lewandowski, A. Thomson, Y. Zhang, R. Polski, K. Watanabe, T. Taniguchi, J. Alicea, and S. Nadj-Perge, Evidence for unconventional superconductivity in twisted trilayer graphene, *Nature (London)* **606**, 494 (2022).
- [30] J. M. Park, Y. Cao, K. Watanabe, T. Taniguchi, and P. Jarillo-Herrero, Tunable strongly coupled superconductivity in magic-angle twisted trilayer graphene, *Nature (London)* **590**, 249 (2021).
- [31] Y. Saito, T. Nojima, and Y. Iwasa, Highly crystalline 2D superconductors, *Nat. Rev. Mater.* **2**, 16094 (2016).
- [32] Y. Nakagawa, Y. Saito, T. Nojima, K. Inumaru, S. Yamanaka, Y. Kasahara, and Y. Iwasa, Gate-controlled low carrier density superconductors: Toward the two-dimensional BCS-BEC crossover, *Phys. Rev. B* **98**, 064512 (2018).
- [33] Y. Nakagawa, Y. Kasahara, T. Nomoto, R. Arita, T. Nojima, and Y. Iwasa, Gate-controlled BCS-BEC crossover in a two-dimensional superconductor, *Science* **372**, 190 (2021).
- [34] J. P. Eisenstein, L. N. Pfeiffer, and K. W. West, Precursors to exciton condensation in quantum Hall bilayers, *Phys. Rev. Lett.* **123**, 066802 (2019).
- [35] I. Sodemann, I. Kimchi, C. Wang, and T. Senthil, Composite fermion duality for half-filled multicomponent Landau levels, *Phys. Rev. B* **95**, 085135 (2017).
- [36] G. Wagner, D. X. Nguyen, S. H. Simon, and B. I. Halperin, s -wave paired electron and hole composite fermion trial state for quantum Hall bilayers with $\nu = 1$, *Phys. Rev. Lett.* **127**, 246803 (2021).
- [37] X. Liu, J. I. A. Li, K. Watanabe, T. Taniguchi, J. Hone, B. I. Halperin, P. Kim, and C. R. Dean, Crossover between strongly coupled and weakly coupled exciton superfluids, *Science* **375**, 205 (2022).
- [38] K. M. Bastiaans, D. Chatzopoulos, J.-F. Ge, D. Cho, W. O. Tromp, J. M. van Ruitenbeek, M. H. Fischer, P. J. de Visser, D. J. Thoen, E. F. C. Driessen, T. M. Klapwijk, and M. P. Allan, Direct evidence for Cooper pairing without a spectral gap in a disordered superconductor above T_c , *Science* **374**, 608 (2021).

- [39] Y. L. Loh, M. Randeria, N. Trivedi, C.-C. Chang, and R. Scalettar, Superconductor-insulator transition and Fermi-Bose crossovers, *Phys. Rev. X* **6**, 021029 (2016).
- [40] T. Hazra, N. Trivedi, and M. Randeria, Spectral functions across an insulator to superconductor transition, [arXiv:2011.06598](https://arxiv.org/abs/2011.06598).
- [41] M. Jiang, R. Nangneri, N. Trivedi, G. G. Batrouni, and R. T. Scalettar, Gapless inhomogeneous superfluid phase with spin-dependent disorder, *New J. Phys.* **15**, 023023 (2013).
- [42] R. Nangneri, M. Jiang, T. Cary, G. G. Batrouni, and R. T. Scalettar, Interplay of superconductivity and spin-dependent disorder, *Phys. Rev. B* **85**, 134506 (2012).
- [43] M. Karmakar, S. Roy, S. Mukherjee, and R. Narayanan, Disorder stabilized breached-pair phase in an *s*-wave superconductor, *Phys. Rev. Res.* **4**, 043159 (2022).
- [44] B. Sacépé, T. Dubouchet, C. Chapelier, M. Sanquer, M. Ovdia, D. Shahar, M. Feigel'man, and L. Ioffe, Localization of preformed Cooper pairs in disordered superconductors, *Nat. Phys.* **7**, 239 (2011).
- [45] B. Sacépé, C. Chapelier, T. I. Baturina, V. M. Vinokur, M. R. Baklanov, and M. Sanquer, Pseudogap in a thin film of a conventional superconductor, *Nat. Commun.* **1**, 140 (2010).
- [46] M. P. Fisher, Quantum phase transitions in disordered two-dimensional superconductors, *Phys. Rev. Lett.* **65**, 923 (1990).
- [47] T. Vojta, Rare region effects at classical, quantum, and non-equilibrium phase transitions, *J. Phys. A: Math. Gen.* **39**, R143 (2006).
- [48] T. Vojta, Phases and phase transitions in disordered quantum systems, *AIP Conf. Proc.* **1550**, 188 (2013).
- [49] T. Vojta, Disorder in quantum many-body systems, *Annu. Rev. Condens. Matter Phys.* **10**, 233 (2019).
- [50] J. A. Hoyos, C. Kotabage, and T. Vojta, Effects of dissipation on a quantum critical point with disorder, *Phys. Rev. Lett.* **99**, 230601 (2007).
- [51] T. Vojta, C. Kotabage, and J. A. Hoyos, Infinite-randomness quantum critical points induced by dissipation, *Phys. Rev. B* **79**, 024401 (2009).
- [52] A. Ghosal, M. Randeria, and N. Trivedi, Inhomogeneous pairing in highly disordered *s*-wave superconductors, *Phys. Rev. B* **65**, 014501 (2001).
- [53] Y. Dubi, Y. Meir, and Y. Avishai, Nature of the superconductor-insulator transition in disordered superconductors, *Nature (London)* **449**, 876 (2007).
- [54] A. Kapitulnik, S. A. Kivelson, and B. Spivak, *Colloquium: Anomalous metals: Failed superconductors*, *Rev. Mod. Phys.* **91**, 011002 (2019).
- [55] S. Kaur, H. K. Kundu, S. Kumar, A. Dogra, R. Narayanan, T. Vojta, and A. Bid, Novel emergent phases in a two-dimensional superconductor, *New J. Phys.* **26**, 083001 (2024).
- [56] Y. Xing, H.-M. Zhang, H.-L. Fu, H. Liu, Y. Sun, J.-P. Peng, F. Wang, X. Lin, X.-C. Ma, Q.-K. Xue *et al.*, Quantum Griffiths singularity of superconductor-metal transition in Ga thin films, *Science* **350**, 542 (2015).
- [57] S. Shen, Y. Xing, P. Wang, H. Liu, H.-L. Fu, Y. Zhang, L. He, X. Xie, X. C. and Lin, J. Nie, and J. Wang, Observation of quantum Griffiths singularity and ferromagnetism at the superconducting LaAlO₃/SrTiO₃ interface, *Phys. Rev. B* **94**, 144517 (2016).
- [58] N. A. Lewellyn, I. M. Percher, J. Nelson, J. Garcia-Barriocanal, I. Volotsenko, A. Frydman, T. Vojta, and A. M. Goldman, Infinite-randomness fixed point of the quantum superconductor-metal transitions in amorphous thin films, *Phys. Rev. B* **99**, 054515 (2019).
- [59] P. Reiss, D. Graf, A. A. Haghighirad, T. Vojta, and A. I. Coldea, Signatures of a quantum Griffiths phase close to an electronic nematic quantum phase transition, *Phys. Rev. Lett.* **127**, 246402 (2021).
- [60] D. McKay and B. DeMarco, Thermometry with spin-dependent lattices, *New J. Phys.* **12**, 055013 (2010).
- [61] O. Mandel, M. Greiner, A. Widera, T. Rom, T. W. Hansch, and I. Bloch, Controlled collisions for multi-particle entanglement of optically trapped atoms, *Nature (London)* **425**, 937 (2003).
- [62] A. E. Feiguin and M. P. A. Fisher, Exotic paired states with anisotropic spin-dependent Fermi surfaces, *Phys. Rev. Lett.* **103**, 025303 (2009).
- [63] W. V. Liu, F. Wilczek, and P. Zoller, Spin-dependent Hubbard model and a quantum phase transition in cold atoms, *Phys. Rev. A* **70**, 033603 (2004).
- [64] O. Mandel, M. Greiner, A. Widera, T. Rom, T. W. Hänsch, and I. Bloch, Coherent transport of neutral atoms in spin-dependent optical lattice potentials, *Phys. Rev. Lett.* **91**, 010407 (2003).
- [65] J. Skolimowski, D. Vollhardt, and K. Byczuk, Spin-selective localization of correlated lattice fermions, *Phys. Rev. B* **92**, 094202 (2015).
- [66] A. Ohtomo and H. Y. Hwang, A high-mobility electron gas at the LaAlO₃/SrTiO₃ heterointerface, *Nature (London)* **427**, 423 (2004).
- [67] M. Zhang, K. Du, T. Ren, H. Tian, Z. Zhang, H. Y. Hwang, and Y. Xie, A termination-insensitive and robust electron gas at the heterointerface of two complex oxides, *Nat. Commun.* **10**, 4026 (2019).
- [68] K. Han, K. Hu, X. Li, K. Huang, Z. Huang, S. Zeng, D. Qi, C. Ye, J. Yang, H. Xu, A. Ariando, J. Yi, W. Lae, S. Yan, and X. R. Wang, Erasable and recreatable two-dimensional electron gas at the heterointerface of SrTiO₃ and a water-dissolvable overlayer, *Sci. Adv.* **5**, eaaw7286 (2019).
- [69] P. Xu, W. Han, P. M. Rice, J. Jeong, M. G. Samant, K. Mohseni, H. L. Meyerheim, S. Ostanin, I. V. Maznichenko, I. Mertig, E. K. U. Gross, A. Ernst, and S. S. P. Parkin, Reversible formation of 2D electron gas at the LaFeO₃/SrTiO₃ interface via control of oxygen vacancies, *Adv. Mater.* **29**, 1604447 (2017).
- [70] W. Niu, K. Du, S. Wang, M. Zhang, M. Gao, Y. Chen, H. Liu, W. Zhou, F. Song, P. Wang, Y. Xu, X. Wang, J. Shen, and R. Zhang, Intrinsic ferromagnetism and quantum transport transition in individual Fe-doped Bi₂Se₃ topological insulator nanowires, *Nanoscale* **9**, 12372 (2017).
- [71] Y. Chen, N. Pryds, J. E. Kleibecker, G. Koster, J. Sun, E. Stamate, B. Shen, G. Rijnders, and S. Linderöth, Metallic and insulating interfaces of amorphous SrTiO₃-based oxide heterostructures, *Nano Lett.* **11**, 3774 (2011).
- [72] J. A. Bert, B. Kalisky, C. Bell, M. Kim, Y. Hikita, H. Y. Hwang, and K. A. Moler, Direct imaging of the coexistence of ferromagnetism and superconductivity at the LaAlO₃/SrTiO₃ interface, *Nat. Phys.* **7**, 767 (2011).

- [73] F. Bi, M. Huang, S. Ryu, H. Lee, C.-W. Bark, C.-B. Eom, P. Irvin, and J. Levy, Room-temperature electronically-controlled ferromagnetism at the $\text{LaAlO}_3/\text{SrTiO}_3$ interface, *Nat. Commun.* **5**, 5019 (2014).
- [74] F. P. Mena, J. F. DiTusa, D. van der Marel, G. Aeppli, D. P. Young, A. Damascelli, and J. A. Mydosh, Suppressed reflectivity due to spin-controlled localization in a magnetic semiconductor, *Phys. Rev. B* **73**, 085205 (2006).
- [75] T. Furukawa, K. Miyagawa, M. Matsumoto, T. Sasaki, and K. Kanoda, Microscopic evidence for preformed Cooper pairs in pressure-tuned organic superconductors near the Mott transition, *Phys. Rev. Res.* **5**, 023165 (2023).
- [76] J. Hubbard, Calculation of partition functions, *Phys. Rev. Lett.* **3**, 77 (1959).
- [77] H. J. Schulz, Effective action for strongly correlated fermions from functional integrals, *Phys. Rev. Lett.* **65**, 2462 (1990).
- [78] M. Karmakar and P. Majumdar, Population-imbalanced lattice fermions near the BCS-BEC crossover: Thermal physics of the breached pair and Fulde-Ferrell-Larkin-Ovchinnikov phases, *Phys. Rev. A* **93**, 053609 (2016).
- [79] W. E. Evenson, J. R. Schrieffer, and S. Q. Wang, New approach to the theory of itinerant electron ferromagnets with local-moment characteristics, *J. Appl. Phys.* **41**, 1199 (1970).
- [80] S. Tarat and P. Majumdar, A real space auxiliary field approach to the BCS-BEC crossover, *Eur. Phys. J. B* **88**, 68 (2015).
- [81] S. Tarat and P. Majumdar, Charge dynamics across the disorder-driven superconductor-insulator transition, *Europhys. Lett.* **105**, 67002 (2014).
- [82] A. A. Abrikosov and L. P. Gor'kov, Contribution to the theory of superconducting alloys with paramagnetic impurities, *JETP* **12**, 1243 (1961).
- [83] K. Maki, On persistent currents in a superconducting alloy. II, *Prog. Theor. Phys.* **29**, 333 (1963).
- [84] K. Maki, The behavior of superconducting thin films in the presence of magnetic fields and currents, *Prog. Theor. Phys.* **31**, 731 (1964).
- [85] K. Maki and P. Fulde, Equivalence of different pair-breaking mechanisms in superconductors, *Phys. Rev.* **140**, A1586 (1965).
- [86] P. G. De Gennes and G. Sarma, Some relations between superconducting and magnetic properties, *J. Appl. Phys.* **34**, 1380 (1963).
- [87] M. Karmakar and P. Majumdar, Noncollinear order and gapless superconductivity in s -wave magnetic superconductors, *Phys. Rev. B* **93**, 195147 (2016).
- [88] S. Rex, I. V. Gornyi, and A. D. Mirlin, Majorana bound states in magnetic skyrmions imposed onto a superconductor, *Phys. Rev. B* **100**, 064504 (2019).
- [89] M. Garnier, A. Mesaros, and P. Simon, Topological superconductivity with deformable magnetic skyrmions, *Commun. Phys.* **2**, 126 (2019).
- [90] J. Spethmann, S. Meyer, K. von Bergmann, R. Wiesendanger, S. Heinze, and A. Kubetzka, Discovery of magnetic single- and triple- $\mathbf{q}$ states in $\text{Mn}/\text{Re}(0001)$, *Phys. Rev. Lett.* **124**, 227203 (2020).
- [91] A. I. Buzdin, Proximity effects in superconductor-ferromagnet heterostructures, *Rev. Mod. Phys.* **77**, 935 (2005).
- [92] B. Lechtenberg, R. Peters, and N. Kawakami, Interplay between charge, magnetic, and superconducting order in a Kondo lattice with attractive Hubbard interaction, *Phys. Rev. B* **98**, 195111 (2018).
- [93] P. Fulde and R. A. Ferrell, Superconductivity in a strong spin-exchange field, *Phys. Rev.* **135**, A550 (1964).
- [94] A. I. Larkin and Yu N. Ovchinnikov, Nonuniform state of superconductors, *Zh. Eksp. Teor. Fiz.* **47** (1964).
- [95] S. Gerber, M. Bartkowiak, J. L. Gavilano, E. Ressouche, N. Egetenmeyer, C. Niedermayer, A. D. Bianchi, R. Movshovich, E. D. Bauer, J. D. Thompson, and M. Kenzelmann, Switching of magnetic domains reveals spatially inhomogeneous superconductivity, *Nat. Phys.* **10**, 126 (2014).
- [96] D. Y. Kim, S.-Z. Lin, F. Weickert, M. Kenzelmann, E. D. Bauer, F. Ronning, J. D. Thompson, and R. Movshovich, Intertwined orders in heavy-fermion superconductor CeCoIn_5 , *Phys. Rev. X* **6**, 041059 (2016).
- [97] M. O. J. Heikkinen, D.-H. Kim, M. Troyer, and P. Törmä, Nonlocal quantum fluctuations and fermionic superfluidity in the imbalanced attractive Hubbard model, *Phys. Rev. Lett.* **113**, 185301 (2014).
- [98] J. Gukelberger, S. Lienert, E. Kozik, L. Pollet, and M. Troyer, Fulde-Ferrell-Larkin-Ovchinnikov pairing as leading instability on the square lattice, *Phys. Rev. B* **94**, 075157 (2016).
- [99] N. F. Q. Yuan and L. Fu, Supercurrent diode effect and finite-momentum superconductors, *Proc. Natl. Acad. Sci. USA* **119**, e2119548119 (2022).
- [100] A. Akbari and P. Thalmeier, Fermi surface segmentation in the helical state of a Rashba superconductor, *Phys. Rev. Res.* **4**, 023096 (2022).
- [101] A. Daido, Y. Ikeda, and Y. Yanase, Intrinsic superconducting diode effect, *Phys. Rev. Lett.* **128**, 037001 (2022).
- [102] R. Wakatsuki, Y. Saito, S. Hoshino, Y. M. Itahashi, T. Ideue, M. Ezawa, Y. Iwasa, and N. Nagaosa, Nonreciprocal charge transport in noncentrosymmetric superconductors, *Sci. Adv.* **3**, e1602390 (2017).
- [103] E. Strambini, M. Spies, N. Ligato, S. Ilić, M. Rouco, C. González-Orellana, M. Ilyn, C. Rogero, F. S. Bergeret, J. S. Moodera, P. Virtanen, T. T. Heikkilä, and F. Giazotto, Superconducting spintronic tunnel diode, *Nat. Commun.* **13**, 2431 (2022).
- [104] Z. Zhu, M. Papaj, X.-A. Nie, H.-K. Xu, Y.-S. Gu, X. Yang, D. Guan, S. Wang, Y. Li, C. Liu, J. Luo, Z.-A. Xu, H. Zheng, L. Fu, and J.-F. Jia, Discovery of segmented Fermi surface induced by Cooper pair momentum, *Science* **374**, 1381 (2021).
- [105] M. Smidman, M. B. Salamon, H. Q. Yuan, and D. F. Agterberg, Superconductivity and spin-orbit coupling in noncentrosymmetric materials: A review, *Rep. Prog. Phys.* **80**, 036501 (2017).
- [106] M. Karmakar, Temperature-tuned Fermi-surface topology and segmentation in noncentrosymmetric superconductors, *Phys. Rev. B* **107**, 064503 (2023).
- [107] T. Shimojima, W. Malaeb, A. Nakamura, T. Kondo, K. Kihou, C.-H. Lee, A. Iyo, H. Eisaki, S. Ishida, M. Nakajima, S. ichi Uchida, K. Ohgushi, K. Ishizaka, and S. Shin, Antiferroic electronic structure in the nonmagnetic superconducting state of the iron-based superconductors, *Sci. Adv.* **3**, e1700466 (2017).
- [108] H. Miao, T. Qian, X. Shi, P. Richard, T. K. Kim, M. Hoesch, L. Y. Xing, X. C. Wang, C. Q. Jin, J. P. Hu, and H. Ding,

- Observation of strong electron pairing on bands without Fermi surfaces in $\text{LiF}_{1-x}\text{Co}_x\text{As}$, [Nat. Commun. **6**, 6056 \(2015\)](#).
- [109] H. Tajima, Y. Yerin, A. Perali, and P. Pieri, Enhanced critical temperature, pairing fluctuation effects, and BCS-BEC crossover in a two-band Fermi gas, [Phys. Rev. B **99**, 180503 \(2019\)](#).
- [110] C. Huscroft and R. T. Scalettar, Effect of disorder on charge-density wave and superconducting order in the half-filled attractive Hubbard model, [Phys. Rev. B **55**, 1185 \(1997\)](#).
- [111] Y. Imry and S.-K. Ma, Random-field instability of the ordered state of continuous symmetry, [Phys. Rev. Lett. **35**, 1399 \(1975\)](#).