

Stable spin squeezing of nitrogen-vacancy centers via Floquet-driven magnons

Anying Feng, Ziyu Zhou, Jun Xu^{✉,*} and Xiangming Hu^{✉,†}*College of Physical Science and Technology, Central China Normal University, Wuhan 430079, People's Republic of China*

(Received 5 August 2025; accepted 26 January 2026; published 13 February 2026)

We propose an efficient scheme for generating a stable spin-squeezed state in a hybrid magnon-nitrogen-vacancy (NV) centers system via magnon-Floquet driving. The system consists of a yttrium iron garnet (YIG) film directly coupled to NV centers that are equidistant from the film surface, and at the same time the YIG film is driven by a Floquet field. The magnon dissipation is exploited to distribute the NV centers into the dark state, manifesting as the stable spin-squeezed state. We show that the system exhibits the low- or high-energy dark state, which is determined by whether the lowering or the raising spin operator plays the dominant role in evolution. Furthermore, we also show that the spin squeezing is robust against thermal effects and decoherence, and it is possible to detect the change in N at the single-spin level, since it exhibits a distinct odd-even effect. This scheme provides a feasible way for experimental implementation and thus has potential applications in quantum information processing, quantum simulation, and precision metrology.

DOI: [10.1103/hkc6-jxrs](https://doi.org/10.1103/hkc6-jxrs)

I. INTRODUCTION

In recent years, spin squeezing [1–3] has attracted widespread attention in theoretical and experimental studies. In fundamental quantum physics and quantum information processing [4–6], spin squeezing has been utilized to probe quantum entanglement [7–9], enhance the efficiency of quantum computation and quantum communication [10–12], and improve the stability of quantum storage and quantum gate operations [13]. In quantum simulation, spin squeezing has been widely used to study quantum phase transitions [14], quantum chaos [15–18], and complex many-body interactions [5,19–27], owing to its relative ease of generation and experimental measurement. In precision metrology, spin squeezing can also enhance measurement precision, such as in Ramsey spectroscopy [1,3,28–35], as well as in the development of more precise atomic clocks [3,24,36–42] and gravitational wave interferometers [43–45]. It is worth noting that the definition of spin squeezing is not unique. The most extensively studied squeezing parameters were originally proposed by Kitagawa and Ueda [2], and by Wineland *et al.* [1,3]. The latter, introduced naturally in standard Ramsey spectroscopy experiments, is defined as the ratio of the phase resolution when using a correlated state to that when using a coherent spin state [46].

There are many approaches for generating spin squeezing. One approach is based on the nonlinear atom-atom interac-

tion [5,22,47–52]. For example, Estève *et al.* experimentally demonstrated the generation of spin-squeezed, entangled states in a Bose-Einstein condensate by splitting it in an optical lattice, achieving a 3.8 dB precision gain beyond the standard quantum limit [22]. Liu *et al.* proposed a simplified theoretical scheme to generate spin-squeezed states via one- and two-axis twisting using a single off-resonance stimulated Raman scattering process in an optical cavity [49]. Another approach is to transfer the squeezing of photons to the atomic ensembles [19,53–56]. For instance, Hald *et al.* demonstrated the transfer of quantum entanglement from light to atoms by mapping polarization-squeezed light onto a collective atomic spin, resulting in spin squeezing and noise reduction below the standard quantum limit [56]. The last approach is to perform the quantum nondemolition (QND) measurement of the output field by utilizing the dispersive interaction between the light field and atoms [24,25,39,41,57–69]. In particular, Shah *et al.* demonstrated that the bandwidth of the high-sensitivity atomic magnetometer had increased by four times via continuous QND measurements [69]. However, the spin squeezing produced by these approaches is instantaneous, meaning that this squeezing exists only within a specific time window of evolution and then disappears rapidly due to decoherence effects. Therefore, achieving stable and highly robust spin squeezing is the main challenge that experimentalists are facing.

Yttrium iron garnet (YIG), as a ferromagnetic material, has a high spin density and low magnetic loss, which can effectively prevent energy loss caused by Joule heating [70]. Additionally, the magnon features a wide frequency range and strong coupling strength. This enables it to couple coherently various quantum systems, including microwaves [71–76], optical photons [77,78], phonons [79–81], solid spin defects [52,82,83], and superconducting qubits [84–86]. Furthermore, it supports efficient quantum control at room temperature, significantly superior to other quantum systems [70]. On the

*Contact author: junxu@ccnu.edu.cn†Contact author: xmhu@ccnu.edu.cn

other hand, the nitrogen-vacancy (NV) centers in diamond have a relatively long coherence time [87,88] and extremely high controllability [89], which enables them to serve as quantum bits that can maintain quantum state coherence for more than several seconds [90]. In the hybrid magnon-NV centers quantum systems, two different physical platforms are coupled together to overcome the limit of individual systems and combine their advantages [70,82,91]. Recently, Fukami *et al.* theoretically showed that YIG magnons can mediate strong long-range entanglement between NV centers at low temperatures via virtual-magnon-exchange protocols [82]. Jiang *et al.* experimentally achieved high-fidelity room-temperature quantum logic operations using a hybrid magnon-NV center system, enabled by coherent magnon control with significantly reduced energy consumption and thermal noise [70].

In this paper, we propose a method using magnon-Floquet driving to achieve a stable spin-squeezed state in a hybrid magnon-NV centers quantum system. The hybrid system is composed of a YIG film directly coupled to the spin of NV centers. All the NV centers are positioned at the same distance from the film surface to ensure that the coupling strength between them is identical. At the same time, the YIG film is driven through a Floquet-driven field, as in Ref. [74]. By leveraging the periodic temporal modulation from a Floquet drive, the composite interaction of magnon and NV centers leads to the existence of a collective spin dark state, which manifests as the stable spin-squeezed state. By applying the Bogoliubov transformation to the spin operators and the mean-field approximation in the large- N limit, we obtain an analytical solution of the spin squeezing parameter at steady state, which is verified to be consistent with the dark-state results. In our scheme, the magnon dissipation is exploited to produce and stabilize the spin-squeezed state. When the lowering spin operator plays the dominant role in evolution, the system relaxes to the low-energy dark state. Conversely, if the raising spin operator dominates, the system evolves toward the high-energy dark state. It should be noted that the existence of the high-energy dark state originates from the finite dimensions of the spin system's Hilbert space. In addition, the system exhibits a distinct odd-even effect, which makes it possible to detect the change in N at the single-spin level. Our scheme has potential applications in quantum information processing, quantum simulation, and precision metrology.

In comparison with previous studies [92–98], our scheme exhibits the following significant characteristics: (1) The present scheme is an all-solid-state platform, featuring superior stability, high integration, and strong resistance to environmental temperature variations and magnetic noise interference. This significantly enhances system robustness and experimental feasibility. (2) Unlike the Raman scheme, which requires atoms with a specific four-level configuration [92,94,96], and the dissipative scheme by interacting with a squeezed reservoir [95,97,98], our scheme only utilizes magnon-Floquet engineering, eliminating the need for multilevel structures and an initial squeezed reservoir. (3) More importantly, we find that stable spin squeezing can be generated in both low- and high-energy dark state subspace. This stems from the finite-dimensional Hilbert space of the spin system, which inherently restricts phase-space divergence.

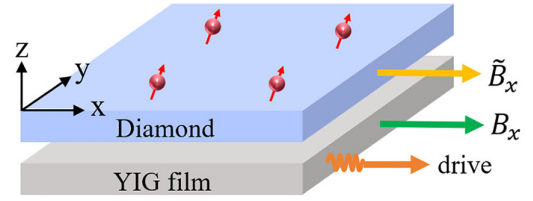


FIG. 1. Schematic representation of the system structure. The hybrid system consists of a YIG film directly coupled to NV center spins. All NV center spins are positioned at the same distance from the film surface, and the magnons are driven by a Floquet field with frequency Ω and normalized amplitude η . A bias magnetic field B_x and a Floquet driving field are applied along the x direction on the YIG film, while a magnetic field $\tilde{B}_x$ is applied along the x direction on the NV centers.

The remaining part of this paper is organized as follows. In Sec. II, we describe the system model, the derivation of the effective Hamiltonian, and the master equation. In Sec. III, the dark state is presented first, and then the spin squeezing is discussed via analytical solutions, numerical results, and thermal effects and decoherence. In Sec. IV, we discuss the experimental feasibility of this scheme. Finally, a conclusion is given in Sec. V.

II. MODEL AND EQUATIONS

As depicted in Fig. 1, we investigate a hybrid system where a YIG film is magnetically coupled to an ensemble of N NV centers [70]. An external magnetic field $\tilde{B}_x$ is applied along the axis of symmetry of the NV centers, inducing a Zeeman effect. The bias magnetic field B_x and the Floquet driving field are applied along the x direction on the YIG film. Then, the excited magnon propagates along the surface and couples to the spin modes of the NV centers. All the NV centers are positioned at the same distance from the surface of the film to ensure that the coupling strength between them is identical. We assume that the system can be described by the collective spin operators $\hat{J}_{x,y,z} = \frac{1}{2} \sum_j \hat{\sigma}_{x,y,z}^j$, where $\hat{\sigma}_{x,y,z}^j$ are the Pauli matrices for the j th NV center. The magnon-NV centers coupling system is governed by the time-dependent Hamiltonian ($\hbar = 1$):

$$\hat{H}(t) = \hat{H}_0 + \hat{H}_1 + \hat{H}_d(t), \quad (1)$$

where the first term

$$\hat{H}_0 = \omega_m \hat{m}^\dagger \hat{m} + \omega_{NV} \hat{J}_z \quad (2)$$

represents the free energy of the magnon and NV centers. Here, the magnon angular frequency is given by $\omega_m = |\gamma_e| B_x$ with $\gamma_e/2\pi = 28 \text{ GHz/T}$, which can be tuned by the external magnetic field B_x . The operator $\hat{m}$ ($\hat{m}^\dagger$) represents the annihilation (creation) operator of the magnon. For the NV centers, the spin-triplet ground states $\{|0\rangle, |\pm 1\rangle\}$ are the eigenstates of the spin operator $\hat{J}_z$, where the degeneracy between the $|\pm 1\rangle$ states is split by an external magnetic field $\tilde{B}_x$ via the Zeeman effect, resulting in transition frequencies $\omega_\pm = D_0 \pm |\gamma_e| \tilde{B}_x$, with $|0\rangle$ taken as the energy reference. Here, $D_0 = 2\pi \times 2.87 \text{ GHz}$ denotes the zero-field splitting. Under an appropriate magnetic field $\tilde{B}_x$, the states $|0\rangle$ and $|-1\rangle$

form a spin qubit and we therefore set $\omega_{\text{NV}} = \omega_-$ and define $\hat{\sigma}_z = |-1\rangle\langle -1| - |0\rangle\langle 0|$. The second term

$$\hat{H}_I = g(\hat{m} + \hat{m}^\dagger)\hat{J}_x \quad (3)$$

describes the interaction between the magnon and NV centers, indicating the exchange of energy and quantum states. g denotes the magnon-NV centers coupling strength. The raising and lowering operators for the collective spin of the NV centers are defined as $\hat{J}_\pm = \sum_j \hat{\sigma}_\pm^j$. According to Ref. [82], at a 5 nm distance between the YIG film and the NV centers, $g/2\pi$ is roughly 0.5 MHz. With a sinusoidal drive, the last term reads

$$\hat{H}_d(t) = \Omega\eta \cos(\Omega t)\hat{m}^\dagger\hat{m}, \quad (4)$$

where Ω and η are the frequency and normalized amplitude of the Floquet driving field, respectively. In our system, the Floquet driving is achieved by modulating the frequency of the magnon mode [74].

In order to focus on resonant interactions, we follow the standard technique [99] and transform to the interaction picture. The transform is performed by using the unitary transformation: $\hat{H}'_I = \hat{U}^\dagger(t)\hat{H}_I\hat{U}(t)$, where $\hat{U}(t) = e^{-i\hat{H}_0 t - i\int_0^t \hat{H}_d(\tau)d\tau}$. The Hamiltonian in the interaction picture can be written as

$$\hat{H}'_I = \frac{g}{2}\hat{m}^\dagger[\hat{J}_+e^{i\Delta_1(t)} + \hat{J}_-e^{i\Delta_2(t)}] + \text{H.c.}, \quad (5)$$

where $\Delta_1(t) = (\omega_m + \omega_{\text{NV}})t + \eta \sin(\Omega t)$ and $\Delta_2(t) = (\omega_m - \omega_{\text{NV}})t + \eta \sin(\Omega t)$. Next, we use the Bessel functions to expand the complex exponential terms, and then obtain the Hamiltonian expressed with an infinite sum

$$\begin{aligned} \hat{H}'_I = \frac{g}{2}\hat{m}^\dagger \left[\sum_{p=-\infty}^{+\infty} J_p(\eta)e^{i(\omega_m + \omega_{\text{NV}} + p\Omega)t} \hat{J}_+ \right. \\ \left. + \sum_{q=-\infty}^{+\infty} J_q(\eta)e^{i(\omega_m - \omega_{\text{NV}} + q\Omega)t} \hat{J}_- \right] + \text{H.c.} \end{aligned} \quad (6)$$

Here, $J_p(\eta)$ and $J_q(\eta)$ are the Bessel function coefficients for the p th and q th order expansions, respectively. The Hamiltonian contains $\hat{m}^\dagger\hat{J}_+$ and $\hat{m}^\dagger\hat{J}_-$, both oscillating with complex exponential. To simplify the analysis, we focus on the slow oscillation terms while neglecting the fast oscillation terms. Specifically, the terms that meet condition $|\omega_m + \omega_{\text{NV}} + p\Omega| \ll g$ and $|\omega_m - \omega_{\text{NV}} + q\Omega| \ll g$ can be retained. By assuming $\omega_m = -\omega_{\text{NV}}$ and $\Omega = \omega_{\text{NV}} - \omega_m$, only the terms with $p = 0$ and $q = 1$ make a significant contribution to the system dynamics (see Appendix A). Thus, in this case, the effective interaction Hamiltonian can be rewritten as

$$\hat{H}_{\text{eff}} \approx \frac{g}{2}\hat{m}^\dagger[J_0(\eta)\hat{J}_+ + J_1(\eta)\hat{J}_-] + \text{H.c.} \quad (7)$$

The effective Hamiltonian describes the transitions between the ground and excited states, accompanied by the creation or annihilation of the magnon. Assuming that the characteristic timescale of the system Hamiltonian is significantly longer than the reservoir correlation time, the environment can be treated as a memoryless reservoir. Consequently, the Markovian master equation is given by

$$\frac{d}{dt}\hat{\rho} = -i[\hat{H}_{\text{eff}}, \hat{\rho}] + \kappa\mathcal{L}[\hat{m}]\hat{\rho}, \quad (8)$$

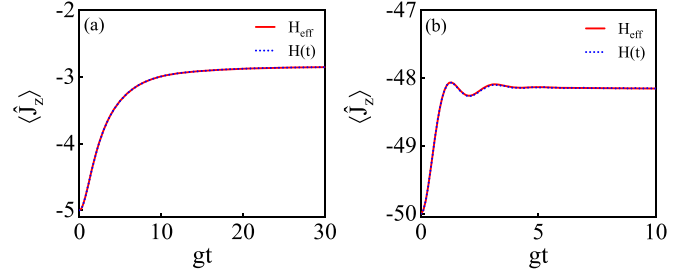


FIG. 2. Time evolution of $\langle \hat{J}_z^2 \rangle$ with $\hat{H}(t)$ in Eq. (1) and $\hat{H}_{\text{eff}}$ in Eq. (7) for (a) $N = 10$ and (b) $N = 100$. The initial state is $|\psi(0)\rangle = |J = \frac{N}{2}, m_J = -\frac{N}{2}\rangle_{\text{spin}}|0\rangle_m$, and the other parameters are $\Omega = 300g$, $\kappa = 2g$, and $\eta = 1.6$.

where $\mathcal{L}[\hat{o}]\hat{\rho} = \hat{o}\hat{\rho}\hat{o}^\dagger - \frac{1}{2}(\hat{o}^\dagger\hat{o}\hat{\rho} + \hat{\rho}\hat{o}^\dagger\hat{o})$ is the Lindblad operator and κ is the magnon dissipation rate. This master equation describes the time evolution of the system's density matrix $\hat{\rho}$, including both unitary evolution and dissipative dynamics. It is noted that the Lindblad term $\kappa\mathcal{L}[\hat{m}]\hat{\rho}$ remains unchanged after performing the same unitary transformation.

To verify the validity of our approximations, we perform numerical simulations of the system dynamics by using the full Hamiltonian in Eq. (1) and the effective Hamiltonian in Eq. (7), and then compare the results of the two approaches. Figure 2 shows the time evolution of $\langle \hat{J}_z^2 \rangle$ for different numbers N of NV centers. The initial state is set to $|\psi(0)\rangle = |J = \frac{N}{2}, m_J = -\frac{N}{2}\rangle_{\text{spin}}|0\rangle_m$ and the other parameters are taken as $\Omega = 300g$, $\kappa = 2g$, and $\eta = 1.6$. As shown in Fig. 2(a), when $N = 10$, $\langle \hat{J}_z^2 \rangle$ increases rapidly over time and stabilizes after a certain period of time. In Fig. 2(b), when N is increased to 100, the time evolution of $\langle \hat{J}_z^2 \rangle$ initially exhibits weak oscillations, but eventually stabilizes as well. As can be seen from Fig. 2, the evolution of $\langle \hat{J}_z^2 \rangle$ governed by the effective Hamiltonian in Eq. (7) is in excellent agreement with the evolution obtained directly from the original full Hamiltonian in Eq. (1), confirming that our approximations are valid.

III. DARK STATE AND SPIN SQUEEZING

We are now investigating the preparation of dark states and their corresponding quantum dynamical behaviors. References [92–98] indicate that for even N , the combined effects of the unitary and dissipative dynamics drive the system to the dark state, defined as $|D\rangle = |\psi\rangle_{\text{spin}}|0\rangle_m$, where $|\psi\rangle_{\text{spin}}$ is the steady state of the NV centers and $|0\rangle_m$ is the vacuum state of the magnon. Since we work in the interaction picture, the eigenstate $|D\rangle$ of the effective Hamiltonian $\hat{H}_{\text{eff}}$ is defined in the interaction picture. When the system reaches a steady state, the steady state of the NV centers should satisfy the equation

$$(J_0(\eta)\hat{J}_+ + J_1(\eta)\hat{J}_-)|\psi\rangle_{\text{spin}} = 0. \quad (9)$$

Since total angular momentum J is conserved in Eq. (8), a dark state exists for each permissible value of J . Each $|\psi\rangle_{\text{spin}}$ has an average spin polarization along the z direction and exhibits squeezing (antisqueezing) in the $\hat{J}_x$ ($\hat{J}_y$) direction. To quantify this, we calculate the ratio of spin fluctuation in the x and y directions, given by $\langle \hat{J}_x^2 \rangle / \langle \hat{J}_y^2 \rangle = (1 - \varepsilon)^2 / (1 + \varepsilon)^2$. Here, $\varepsilon = J_0(\eta)/J_1(\eta)$ is defined as the ratio of the

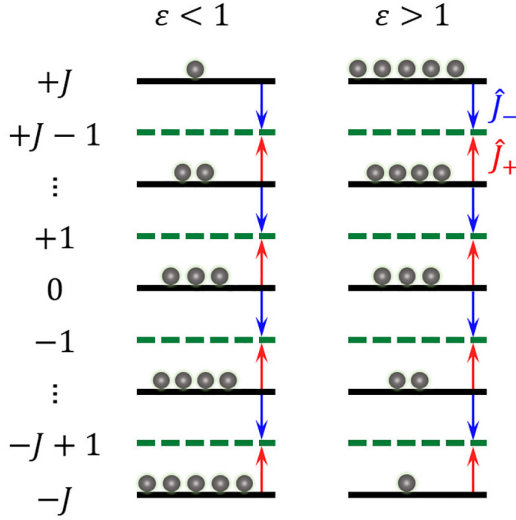


FIG. 3. Sketch of the steady state for $\varepsilon < 1$ and $\varepsilon > 1$. The number of gray spheres represents the population of a level $|J, m_J\rangle$. When $\varepsilon < 1$, the system is confined to the low-energy dark state subspace. The raising and lowering operators (red and blue arrows) induce destructive interference between adjacent energy levels such that every intermediate energy level (dashed green lines) is unoccupied. When $\varepsilon > 1$, the system enters the high-energy dark state subspace, where destructive interference between adjacent energy levels still occurs. While the population distribution is completely symmetrical in the case of $\varepsilon < 1$.

zeroth-order and first-order Bessel functions. Clearly, for all cases where $\varepsilon > 0$, the ratio of spin fluctuation is always less than 1, indicating that the dark state is a spin-squeezed state. We focus on symmetric states with total spin $J = \frac{N}{2}$ and expand $|\psi\rangle_{\text{spin}}$ in terms of the eigenstates of $\hat{J}_z$,

$$|\psi\rangle_{\text{spin}} = \sum_m c_m \left| J = \frac{N}{2}, m_J = -\frac{N}{2} + m \right\rangle. \quad (10)$$

Substituting the expanded steady state into Eq. (9) and using a recursive method, we obtain a relationship between c_m and c_{m+2} . Assuming $c_1 = 0$, only the even terms are left

$$c_{m=2n} = (-1)^n \varepsilon^n \binom{\frac{N}{2}}{n} \binom{N}{2n}^{-\frac{1}{2}} c_0, \quad (11)$$

where $(:)$ denotes the binomial coefficient and c_0 is determined by the normalization condition $\sum_m |c_m|^2 = 1$.

It is well known that the characteristic of a pure bosonic squeezed state is its pairing structure, being a superposition of states with even photon numbers [100]. An analogous paired structure holds in our spin system. The fully polarized state $|J, m_J = -J\rangle$ can be regarded as the “vacuum,” while the state $|J, m_J = -J + m\rangle$ corresponds to having m excitations (i.e., m flipped spins). Thus, it can be seen from Eq. (11) that the spin dark state $|\psi\rangle_{\text{spin}}$ also involves only even numbers of excitations m . Formally speaking, this paired structure will induce destructive interference in both bosons and spin systems, thereby generating a dark state.

Figure 3 shows the population distribution of the dark state for $\varepsilon < 1$ and $\varepsilon > 1$. The number of gray spheres represents the population of a level $|J, m_J\rangle$. According to Eq. (9), when

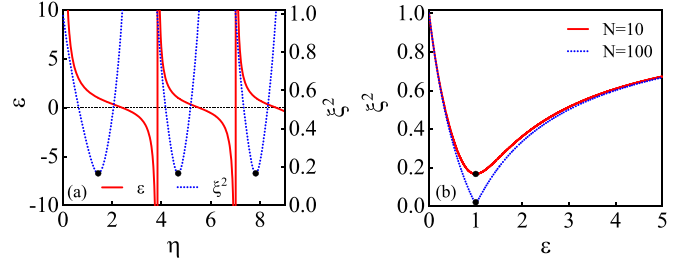


FIG. 4. (a) The ratio ε and the spin squeezing parameter ξ^2 vs the normalized amplitude η for $N = 10$. (b) The spin squeezing parameter ξ^2 vs the ratio ε for $N = 10, 100$. When $\varepsilon = 1$, $J_0(\eta) = J_1(\eta)$ and $\langle \hat{J}_z \rangle = 0$. These singular points are represented by the black dots.

$\varepsilon < 1$, the lowering operator $\hat{J}_-$ dominates and the ground state $|J, m_J = -J\rangle$ has a greater number of occupancies. From the ground state $|J, m_J = -J\rangle$ to the excited state $|J, m_J = J\rangle$, the occupancies decrease successively. However, when $\varepsilon > 1$, the situation is completely the opposite. The combined effect of the raising and lowering operators causes destructive interference between adjacent energy levels, resulting in each intermediate energy level remaining unoccupied. Specifically, when $(J_0(\eta)\hat{J}_+ + J_1(\eta)\hat{J}_-)$ acts on a paired state, it generates a state containing only an odd number of excitations. To achieve a dark state for a given odd excitation number m_{odd} , destructive interference must occur between two pathways leading to m_{odd} : $\hat{J}_-$ acts on a state with $(m_{\text{odd}} + 1)$ excitations, and $\hat{J}_+$ acts on a state with $(m_{\text{odd}} - 1)$ excitations. This is shown schematically in Fig. 3.

To quantify the degree of spin squeezing, we introduce the spin squeezing parameter [1,3]:

$$\xi^2 = \frac{N \min(\delta \hat{J}_{\vec{n}_\perp})^2}{|\langle \vec{J} \rangle|^2} = \frac{N \langle \hat{J}_x^2 \rangle}{|\langle \hat{J}_z \rangle|^2}, \quad (12)$$

where $\min(\delta \hat{J}_{\vec{n}_\perp})^2$ is the minimum variance perpendicular to the direction of the mean total spin $\vec{n}_0 = \vec{J}/|\langle \vec{J} \rangle|$. If $\xi^2 < 1$, it means that the quantum state is squeezed. The smaller ξ^2 indicates the stronger spin squeezing.

Figure 4(a) shows the variation of the ratio ε and the spin squeezing parameter ξ^2 concerning the normalized amplitude η . It can be seen that as η increases, both ε and ξ^2 exhibit periodic variations, due to the periodic modulation from a Floquet driving. The two curves reveal that the spin squeezing occurs only when $\varepsilon > 0$, which is consistent with the previous analysis. However, when $\varepsilon = 1$, $J_0(\eta) = J_1(\eta)$ and $\langle \hat{J}_z \rangle = 0$. This leads to the existence of singular points in the spin squeezing, which are represented by the black dots. Figure 4(b) shows the spin squeezing parameter ξ^2 versus the ratio ε for different N . As ε increases, the degree of spin squeezing first increases and then decreases. Specifically, as $\varepsilon \rightarrow 1$, the spin squeezing approaches the Heisenberg limit, i.e., $\xi^2 \rightarrow 2/(N + 2)$, indicating that the dark state is close to the optimal spin-squeezed state. Similarly, if $\varepsilon = 1$, the spin squeezing has singular points, which are marked by the black dots. When N increases from 10 to 100, the trend of the spin squeezing remains unchanged, but the spin squeezing becomes stronger.

A. Analytical solutions

In order to deeply understand the interaction mechanisms within the system, we introduce the Bogoliubov mode $\hat{\beta} = \sinh r \hat{J}_+ + \cosh r \hat{J}_-$. Then, the effective Hamiltonian can be rewritten as

$$\hat{H}_{\text{eff}} = \chi \hat{m}^\dagger \hat{\beta} + \text{H.c.}, \quad \text{for } \varepsilon < 1, \quad (13)$$

$$\hat{H}_{\text{eff}} = \chi \hat{m}^\dagger \hat{\beta}^\dagger + \text{H.c.}, \quad \text{for } \varepsilon > 1, \quad (14)$$

where the effective coupling constant is given as $\chi = \frac{\varepsilon}{2} \sqrt{|J_1^2(\eta) - J_0^2(\eta)|}$. From the above equations, it is evident that when $\varepsilon < 1$, the interaction between the magnon mode and the Bogoliubov mode takes the form of a beam-splitter type coupling, whereas when $\varepsilon > 1$, it becomes a parametric-amplifier type interaction that induces correlated excitations of magnon and spins. The squeezing parameter r in the two cases is defined through the hyperbolic tangent function as

$$\tanh r = \varepsilon, \quad \text{for } \varepsilon < 1, \quad (15)$$

$$\tanh r = \frac{1}{\varepsilon}, \quad \text{for } \varepsilon > 1. \quad (16)$$

Substituting Eqs. (13) and (14) into Eq. (8), the Heisenberg-Langevin equation for the magnon mode can be derived as

$$\dot{\hat{m}} = -\frac{\kappa}{2} \hat{m} - i\chi \hat{\beta}, \quad \text{for } \varepsilon < 1, \quad (17)$$

$$\dot{\hat{m}} = -\frac{\kappa}{2} \hat{m} - i\chi \hat{\beta}^\dagger, \quad \text{for } \varepsilon > 1. \quad (18)$$

Setting $\dot{\hat{m}} = 0$, the steady state of the magnon mode can be solved. If the magnon dissipation rate is sufficiently large, the magnon mode can be eliminated adiabatically. Then, the reduced master equation for the NV centers is derived as

$$\frac{d}{dt} \hat{\rho} = \Gamma \mathcal{L}[\hat{\beta}] \hat{\rho}, \quad \text{for } \varepsilon < 1, \quad (19)$$

$$\frac{d}{dt} \hat{\rho} = \Gamma \mathcal{L}[\hat{\beta}^\dagger] \hat{\rho}, \quad \text{for } \varepsilon > 1, \quad (20)$$

where $\Gamma = 4\chi^2/\kappa$ is the collective decay rate. For the regime of $\varepsilon < 1$, the spin energy is dissipated through the Lindblad channel $\mathcal{L}[\hat{\beta}] \hat{\rho}$, and the system is cooled into the low-energy dark state subspace defined by $\hat{\beta}|\psi\rangle_{\text{spin}} = 0$. In contrast, when $\varepsilon > 1$, the master equation involves the dissipator $\mathcal{L}[\hat{\beta}^\dagger] \hat{\rho}$, which represents the spin excitation process that drives the system into the high-energy dark state subspace defined by $\hat{\beta}^\dagger|\psi\rangle_{\text{spin}} = 0$. Figure 3 also provides a good illustration of these two physical processes.

It is noteworthy that the dark state defined by Eq. (9) is the unique steady-state solution of the Lindblad master equations (19) and (20). Under strong magnon dissipation, the magnon mode is adiabatically eliminated, leaving the NV spins dynamic governed by the effective dissipators $\mathcal{L}[\hat{\beta}] \hat{\rho}$ or $\mathcal{L}[\hat{\beta}^\dagger] \hat{\rho}$. Only the state that satisfies $\hat{\beta}|\psi\rangle_{\text{spin}} = 0$ or $\hat{\beta}^\dagger|\psi\rangle_{\text{spin}} = 0$ is a steady state of the master equation, indicating that the Hamiltonian dark state is equivalent to the unique steady state of the master equation, thereby ensuring its robustness against dissipation.

To evaluate the degree of squeezing for a large number of NV centers, we derive approximate analytical expressions for the spin squeezing parameters based on the reduced master

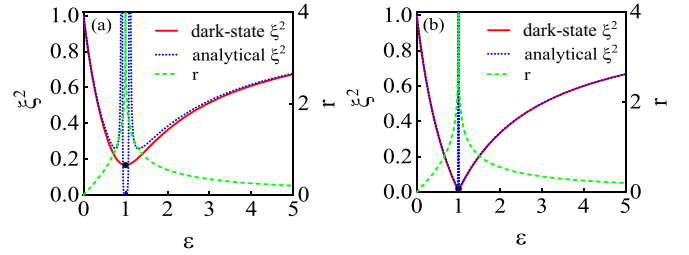


FIG. 5. The spin squeezing parameter ξ^2 and the squeezing parameter r vs ε for (a) $N = 10$ and (b) $N = 100$. The dark-state results (red solid lines) and analytical solutions (blue dotted lines) are according to Eqs. (10) and (25), respectively.

equation. By neglecting the spin decoherence effects, we solve for the Heisenberg-Langevin equations for $\hat{J}_z$ and $\hat{J}_x^2$ from Eq. (20):

$$\frac{d}{dt} \langle \hat{J}_z \rangle = -\Gamma (\sinh^2 r \langle \hat{J}_+ \hat{J}_- \rangle - \cosh^2 r \langle \hat{J}_- \hat{J}_+ \rangle), \quad (21)$$

$$\frac{d}{dt} \langle \hat{J}_x^2 \rangle = -\frac{\Gamma}{2} (\cosh r - \sinh r) \times (\langle \hat{J}_x \hat{J}_z + \hat{J}_z \hat{J}_x \rangle \times (\sinh r \hat{J}_- + \cosh r \hat{J}_+) + \text{H.c.}). \quad (22)$$

Here, we take the high-energy dark state subspace for $\varepsilon > 1$ as an example. Equations (21) and (22) involve higher-order moments of the spins, whose time evolution should be calculated independently. In the large- N limit, the spin ensemble remains close to its maximal polarization along the z direction, as shown in Fig. 2(b), the mean spin component satisfies $\langle \hat{J}_z \rangle \approx N/2$, and the quantum fluctuations around this mean value are small. In this case, we can linearize the equations of motion around $\langle \hat{J}_z \rangle \approx N/2$ by introducing small fluctuations $\delta \hat{J}_z = \hat{J}_z - N/2$, and neglecting higher-order terms of $\langle \delta \hat{J}_z \rangle$. Further, using the approximation $\langle \hat{J}_x^2 + \hat{J}_y^2 \rangle = \langle \hat{J}^2 \rangle - \langle \hat{J}_z^2 \rangle \approx -N \langle \delta \hat{J}_z \rangle + \frac{N}{2}$, the simplified equations can be obtained as

$$\frac{d}{dt} \langle \delta \hat{J}_z \rangle \approx -N\Gamma (\langle \delta \hat{J}_z \rangle + \sinh^2 r), \quad (23)$$

$$\frac{d}{dt} \langle \hat{J}_x^2 \rangle \approx -N\Gamma \left(\langle \hat{J}_x^2 \rangle - \frac{N}{4} e^{-2r} \right). \quad (24)$$

It can be seen that both quantities decay exponentially in time, and the effective decay rate is $N\Gamma$. By substituting the steady-state solutions of $\langle \delta \hat{J}_z \rangle$ and $\langle \hat{J}_x^2 \rangle$ into Eq. (12), we get the spin squeezing parameter

$$\xi^2 \approx \frac{N^2 e^{-2r}}{(N - 2 \sinh^2 r)^2}. \quad (25)$$

For the low-energy dark state subspace of $\varepsilon < 1$, we linearize the equation of motion around $\langle \hat{J}_z \rangle \approx -\frac{N}{2}$ and use a similar method to derive the spin squeezing parameter. We find that the expression of the spin squeezing parameter is the same as Eq. (25).

In Fig. 5, we plot the spin squeezing parameter ξ^2 and the squeezing parameter r versus ε for (a) $N = 10$ and (b) $N = 100$. Furthermore, we also compare the spin squeezing based on the dark state by Eq. (10) to the analytical solutions by Eq. (25). The former is precise, while the latter is approximate. As shown in Fig. 5(a), when $N = 10$, the analytical results are only valid when ε deviates significantly from

$\varepsilon = 1$. This is because the mean-field approximation is valid only when r is relatively small. If ε is gradually increased from 0 to 1, $|\langle \delta \hat{J}_z \rangle| = \sinh^2 r$ will rapidly increase to infinity, while $|\langle \hat{J}_z \rangle|$ will first decrease to 0 and then rapidly increase to infinity. This results in the spin squeezing parameter first decreasing, then increasing, and finally dropping rapidly to 0, as shown by the blue dotted line in Fig. 5(a). However, when we increase N to 100, the mean-field approximation is valid over a wide range and the analytical results are very in agreement with the dark-state results except for $\varepsilon = 1$, as shown in Fig. 5(b).

It is worth noting that the stable spin squeezing in our proposed scheme is established through the magnon dissipation, which induces the spin relaxation and excitation process. Generally speaking, the excitation process in traditional bosonic systems usually leads to instability of the system, making it difficult to generate a squeezed state. Therefore, in order to achieve squeezing in a bosonic system, it is often limited by the bosonic relaxation process. However, for the spin system, such a restriction is not necessary. The reason is that the conventional bosonic system has an infinite-dimensional Hilbert space, while the spin system has a finite-dimensional Hilbert space. Even during the excitation process, the spin system can maintain its stability, thereby achieving the steady-state spin squeezing. Our scheme also reveals that the stable state can be generated in not only low-energy dark state but also high-energy dark state, which has not been mentioned in previous studies [92–98].

B. Numerical results

The previous subsection mainly introduced steady-state spin squeezing. In this subsection, we will discuss the dynamic behavior of spin squeezing. Figure 6 shows the time evolution of the spin squeezing parameter ξ^2 from the full master equation (8), along with the Husimi-Q function for the spin distribution on the Bloch sphere at selected time points. The initial state is $|\psi(0)\rangle = |J = \frac{N}{2}, m_J = \frac{N}{2}\rangle_{\text{spin}}|0\rangle_m$ and the other parameters are $N = 40$, $\kappa = 2g$, and $\eta = 1.3$. As seen from Fig. 6(a), the spin distribution on the Bloch sphere evolves, exhibiting squeezing along the x axis and antisqueezing along the y axis. The spin squeezing parameter ξ^2 first drops rapidly over time and then gradually reaches a stable value, as shown in Fig. 6(b).

Figure 7(a) presents the time evolution of the spin squeezing parameter ξ^2 obtained from Eq. (8) for different initial states. The other parameters are taken as $N = 10$, $\kappa = 2g$, and $\eta = 1.3$. The system almost simultaneously evolves into the unique steady spin-squeezed state, independent of the initial excitation number of the magnon. Therefore, when the magnon dissipation is sufficiently large, the adiabatic elimination of the magnon mode is a good approximation. Figure 7(b) shows the time evolution of the spin squeezing parameter ξ^2 for different numbers of NV centers. All the curves exhibited oscillations before stabilizing. Different N lead to different steady spin-squeezed states. The larger the N is, the stronger the steady-state spin squeezing will be, and the shorter the time required to reach the steady state will be. Figure 7(c) gives the time evolution of the spin squeezing parameter ξ^2 for different total spins. Although the system was initially in an

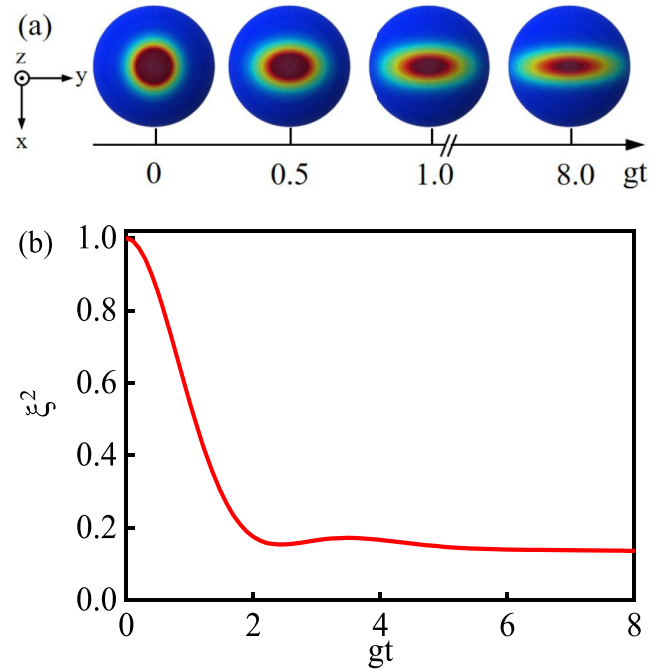


FIG. 6. (a) The Husimi-Q function for the spin distribution on the Bloch sphere at different times. (b) Time evolution of the spin squeezing parameter ξ^2 . The initial state is chosen as $|\psi(0)\rangle = |J = \frac{N}{2}, m_J = \frac{N}{2}\rangle_{\text{spin}}|0\rangle_m$ and the other parameters are $N = 40$, $\kappa = 2g$, and $\eta = 1.3$.

unsqueezed state, it eventually reached a stable squeezed state over time. Different initial total spins will result in different stable spin-squeezed states. Figure 7(d) plots the time evolution of the spin squeezing parameter ξ^2 for different initial spin projections. The system evolves into the unique steady spin-squeezed state over time, independent of the initial spin projection.

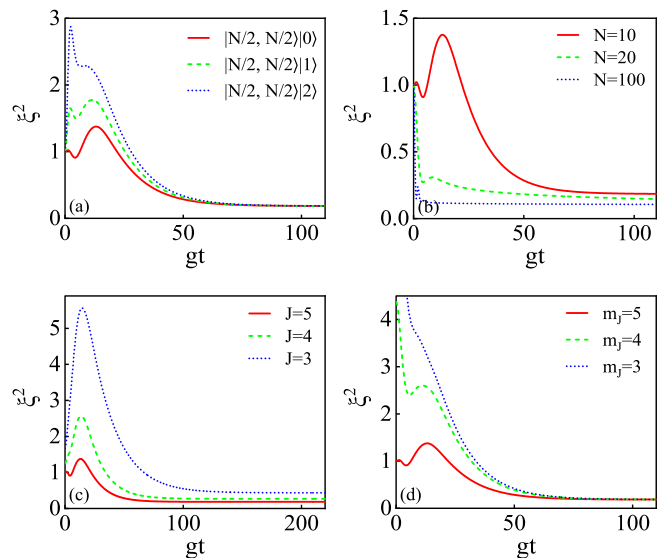


FIG. 7. Time evolution of the spin squeezing parameter ξ^2 obtained from Eq. (8) for (a) different initial states, (b) different numbers of NV centers, (c) different total spins, and (d) different initial spin projections. The other parameters are the same as in Fig. 6 except for $N = 10$.

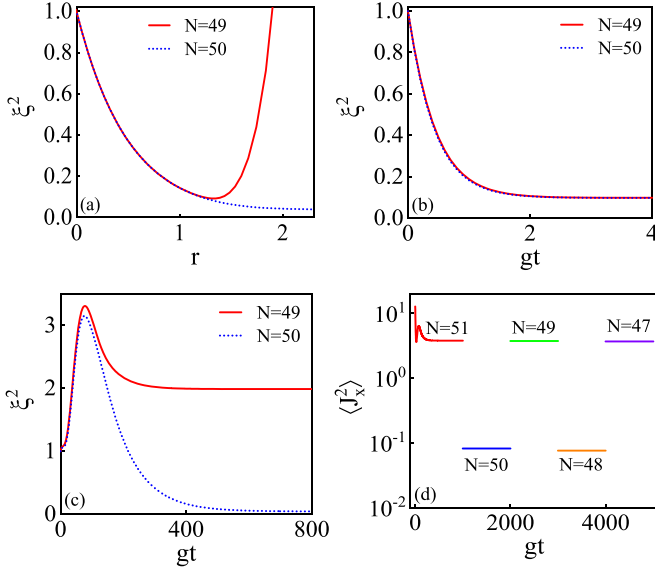


FIG. 8. (a) The spin squeezing parameter ξ^2 vs the squeezing parameter r for odd N (red solid line) and even N (blue dotted line). Time evolution of ξ^2 for odd and even N when (b) $\eta = 1.3$ ($r \approx 1.2$) and (c) $\eta = 1.41$ ($r \approx 2.1$). (d) Illustration of how the odd-even sensitivity of the steady state can be used for sensing. The initial state is chosen as $|\psi(0)\rangle = |J = \frac{N}{2}, m_J = \frac{N}{2}\rangle_{\text{spin}}|0\rangle_m$ and the other parameters are taken as $\kappa = 2g$ and $\eta = 1.41$.

In addition, we find that the spin squeezing exhibits remarkable sensitivity to the parity of the total spin number N [95]. Under certain conditions, the steady states of systems with N spins and those with $N + 1$ spins exhibit significant differences. When the $\hat{\beta}^\dagger$ mode acts on a paired state, it generates states with odd excitation numbers m_{odd} . In order to achieve the dark state, these odd excitation states must cancel each other through destructive interference. For the spin system, the maximum excitation number is the total particle number, i.e., $m_{\text{max}} = N$. When N is even, m_{max} is also even. All possible odd excitation numbers can form destructive interference through the actions of $\hat{J}_\pm$. When N is odd, m_{max} is also odd. In this case, the needed destructive interference is impossible to obtain. Starting with a fully paired state, a state $|J, -J + m_{\text{max}}\rangle$ can be created by acting with $\hat{J}_+$ on $|J, -J + (m_{\text{max}} - 1)\rangle$. However, there is no complementary $\hat{J}_-$ process, since there is no state with $m_{\text{max}} + 1$ excitations. Therefore, the interference condition cannot be satisfied for all states, and the steady state is mixed.

Figure 8(a) plots the spin squeezing parameter ξ^2 versus the squeezing parameter r for $N = 49$ (red solid line) and $N = 50$ (blue dotted line). It can be seen that when r is small, there is no appreciable even-odd effect. However, for large r and odd N , the steady state is impure and exhibits no spin squeezing. Specifically, for odd N , the steady-state squeezing diverges in the large- r limit. Figure 8(b) gives the time evolution of the spin squeezing parameter ξ^2 for odd and even N when $r \approx 1.2$. The two curves show almost the same evolution, meaning that the odd- N steady state has the same squeezing as the even- N case. This is because when r is sufficiently small, the probability of high excitation numbers is strongly suppressed, and hence the frustration caused by

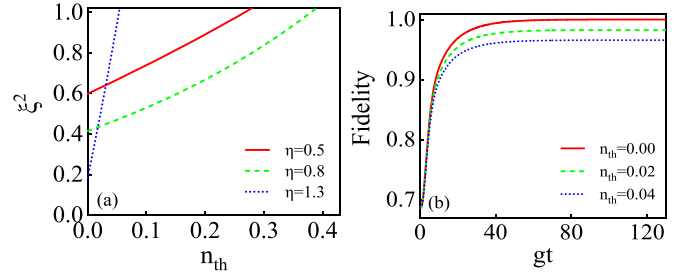


FIG. 9. (a) The spin squeezing parameter ξ^2 vs the magnon thermal occupation number n_{th} for different normalized amplitudes η . (b) Time evolution of the fidelity for different magnon thermal occupation numbers n_{th} at $\eta = 1.3$. The initial state is chosen as $|\psi(0)\rangle = |J = \frac{N}{2}, m_J = \frac{N}{2}\rangle_{\text{spin}}|0\rangle_m$, and the other parameters are taken as $N = 10$ and $\kappa = 2g$.

truncating the maximum excitation number has almost no effect on the system. However, when r is large, the situation will be completely different and the system exhibits a distinct odd-even effect, as shown in Fig. 8(c). In Fig. 8(d), we plot the time evolution of the variance $\langle J_x^2 \rangle$. The evolution is interrupted at randomly selected times, where a single randomly chosen spin is removed from the system. When a spin is removed ($N \rightarrow N - 1$), the parity of the system changes, leading to a corresponding change in the dark state subspace. This change is nonperturbative, as it modifies the interference conditions among collective spin excitations and thereby affects the squeezing properties. As shown in Fig. 8(d), each spin-loss event is accompanied by a sudden jump in the value of $\langle J_x^2 \rangle$, which originates from a real physical transition between two distinct steady states. The significant odd-even sensitivity in the steady state offers the possibility of detecting variations in N at the single-spin level.

C. Thermal effects and decoherence

We now consider the effects of thermal excitations and single-spin dephasing on our spin-squeezing protocol. To this end, we modify the master equation (8) by including the magnon relaxation terms with thermal occupation number n_{th} and the spin dephasing term with a rate γ_ϕ , as follows:

$$\begin{aligned} \frac{d}{dt}\hat{\rho}' = & -i[\hat{H}_{\text{eff}}, \hat{\rho}'] + \frac{\gamma_\phi}{2} \sum_j \mathcal{L}[\hat{\sigma}_z^j]\hat{\rho}' \\ & + \kappa(n_{\text{th}} + 1)\mathcal{L}[\hat{m}]\hat{\rho}' + \kappa n_{\text{th}}\mathcal{L}[\hat{m}^\dagger]\hat{\rho}', \end{aligned} \quad (26)$$

where single-particle processes may break the conservation of total spin.

Figure 9(a) demonstrates that although the magnon thermal effect has a destructive effect on the spin squeezing, the spin squeezing is still possible under room temperature conditions. For instance, when $\eta = 0.8$, the spin squeezing parameter ξ^2 increases monotonically with n_{th} until it disappears at $n_{\text{th}} \approx 0.4$. The spin squeezing becomes less pronounced as thermal noise grows, since the larger n_{th} corresponds to the stronger decoherence of the collective spin mode. In particular, for $\eta = 1.3$, the spin squeezing is significant only in the low-temperature regime, showing that the preservation of quantum correlations requires a nearly thermal-vacuum environment.

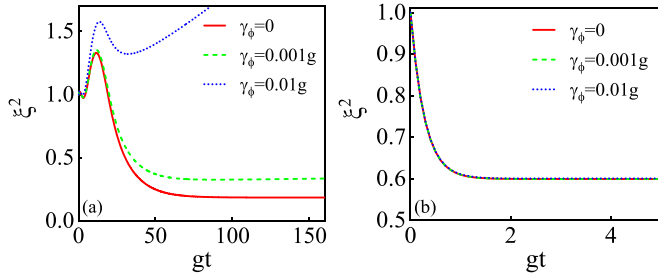


FIG. 10. Time evolution of the spin squeezing parameter ξ^2 for different dephasing rates γ_ϕ . Taking the parameter (a) $\eta = 1.3$ and (b) $\eta = 0.5$. The other parameters are the same as in Fig. 9.

In Fig. 9(b), we give the time evolution of the fidelity for different magnon thermal occupation numbers n_{th} at $\eta = 1.3$. The fidelity rapidly approaches unity in the ideal case ($n_{th} = 0$), showing that the system can evolve into the desired target state with high accuracy. However, as n_{th} increases, the steady-state fidelity gradually decreases, reflecting the degradation of quantum coherence caused by magnon thermal effects. These results confirm that suppressing thermal excitations is crucial for maintaining high-fidelity quantum states and robust spin squeezing in hybrid magnonic systems.

Figure 10 shows the time evolution of the spin squeezing parameter ξ^2 for different dephasing rates γ_ϕ . We choose different normalized amplitudes: (a) $\eta = 1.3$ and (b) $\eta = 0.5$. When $\eta = 1.3$, even extremely weak dephasing effects can significantly weaken the steady-state spin squeezing. In contrast, for $\eta = 0.5$, since the squeezing degree is inherently not very high, it has strong robustness against the dephasing effect. This means that an excessively high spin-squeezing strength will significantly increase the sensitivity of the spin squeezing to the dephasing effect. In our scheme, the dephasing effect can be suppressed by tuning the amplitude of the Floquet driving field.

The magnon-mediated approach presented in this work offers distinct advantages over conventional cavity-mediated spin-squeezing schemes. First, magnon modes in ferromagnetic systems like YIG can be readily integrated on-chip and coupled to solid-state spins via near-field interactions, offering superior scalability and integrability [84,86,101]. Conversely, cavity-mediated schemes typically employ bulky three-dimensional superconducting microwave or optical cavities, which also require precise resonance matching with the spins [24]. Second, magnon systems enable strong and tunable interactions with large spin ensembles through their collective coupling strength [102,103]. In contrast, the coupling in cavity systems is typically constrained by the mode volume and density. Third, magnon systems are more robust to temperature fluctuations and can function at or near room temperature, whereas cavity-mediated schemes typically rely on cryogenic environments to suppress thermal noise [86].

IV. EXPERIMENTAL FEASIBILITY

Experimentally, our model can be realized by using a YIG film and bulk diamond containing an array of NV centers, as illustrated in Fig. 1. According to Ref. [82], when the distance between the YIG film and the NV center is

5 nm, the coupling strength $g/2\pi$ is approximately 0.5 MHz. The single-crystal (100) diamond samples used in Ref. [70] are laser-cut and polished into a chip with dimensions of $2\text{ mm} \times 2\text{ mm} \times 0.15\text{ mm}$. An ensemble of the NV centers is generated through 15-keV ^{14}N ion implantation at a fluence of 1×10^{13} ions/cm². The low-energy manifold of the NV center consists of $S = 1$ spin triplet states, where the ground state corresponds to $m_S = 0$ and the excited states correspond to $m_S = \pm 1$. The optical excitation and state readout of the NV centers can be achieved by using a 532-nm laser.

The permanent magnet provides a bias magnetic field of strength B_x , which is along the x axis and perpendicular to the y - z plane, maximizing the interaction between the magnon and the spin. A small coil is tightly wound around the YIG film in the y - z plane to provide the Floquet driving, thereby modulating the bias magnetic field and consequently tuning the frequency of the magnon mode [74]. When the excitation frequency of the magnon mode matches the electron spin resonance frequency of the NV centers, the nearby magnon drives the transitions of the NV centers. This can be detected by the intensity of the electron spin-related photoluminescence of the NV centers [70]. In our proposed scheme, the effective Hamiltonian in Eq. (7) is obtained under the conditions $\omega_m = -\omega_{NV}$ and $\Omega = \omega_{NV} - \omega_m$. These conditions can be implemented by tuning the magnetic field B_x . Using the relations $\omega_m = |\gamma_e|B_x$ and $\omega_{NV} = D_0 - |\gamma_e|\tilde{B}_x$, the magnetic field applied to the NV centers can be determined as $\tilde{B}_x = B_x + D_0/|\gamma_e|$. The frequency and amplitude of the Floquet drive are given by $\Omega = -2|\gamma_e|B_x$ and $\eta\Omega$, respectively. To ensure that the magnon couples exclusively to the $|-\rangle$ spin state of the NV centers, we choose $0 < B_x \leq 32\text{ mT}$ (see Appendix B).

At a room temperature of $T = 300\text{ K}$, the coherence time of a single NV center is $T_2 \sim 2\text{ ms}$ [104,105], corresponding to the spin-dephasing rate of $\gamma_\phi/2\pi \sim 500\text{ Hz}$, i.e., $\gamma_\phi/2\pi \sim 0.001g/2\pi$. The magnon decay rate is set as $\kappa/2\pi \sim g/\pi \sim 1\text{ MHz}$, giving a magnon lifetime of approximately $1\text{ }\mu\text{s}$. As shown in Fig. 6(b), the system reaches a steady spin-squeezed state within about $10\text{ }\mu\text{s}$, which is much shorter than the spin coherence time, ensuring the feasibility of dissipative squeezing generation under realistic experimental conditions.

V. CONCLUSION

In conclusion, we have utilized the magnon-Floquet-driven method to generate the steady-state spin squeezing in the hybrid magnon-NV centers quantum system. By using Floquet-driven periodic time modulation, an effective coupling between magnons and NV centers can be established, thereby revealing the existence of the collective dark state. This dark state plays a crucial role in generating the stable spin-squeezed state. We have analyzed the physical mechanism under the mean-field approximation and presented the dynamical evolution by numerical calculations. Furthermore, we find that the magnon dissipation can induce the spin relaxation or excitation process, thereby causing the spin system to be in either a low-energy dark state or a high-energy dark state. In addition, the system exhibits a distinct odd-even effect, which makes it possible to detect the change in N at the single-spin level. The all-solid-state approach can enhance the stability of the system and does not require the initial

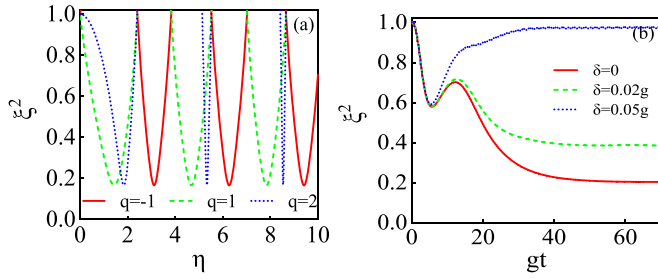


FIG. 11. (a) The spin squeezing parameter ξ^2 vs the normalized amplitude η for $p = 0$ and different values of q . (b) Time evolution of the spin squeezing parameter ξ^2 for different detuning δ . The other parameters are the same as in Fig. 2(a).

preparation of nonclassical states, providing a promising direction for effective spin control and quantum state modulation.

ACKNOWLEDGMENTS

This work was supported by the National Natural Science Foundation of China (Grant No. 12274164) and Fundamental Research Funds for the Central Universities (Grants No. CCNU25JCPT030 and No. CCNU24JC017).

DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: FREQUENCY SELECTION

In the main text, to simplify the analysis, we focus on the slow oscillation terms while neglecting the fast oscillation terms. Specifically, the terms that meet condition $|\omega_m + \omega_{NV} + p\Omega| \ll g$ and $|\omega_m - \omega_{NV} + q\Omega| \ll g$ can be retained. Since $|\omega_m, \omega_{NV}, \Omega| \gg g$, the following two cases need to be further discussed.

First of all, we consider the case of exact resonance, i.e., $|\omega_m + \omega_{NV} + p\Omega| = 0$ and $|\omega_m - \omega_{NV} + q\Omega| = 0$. If we assume $\omega_m = \omega_{NV}$, then $q = 0$ and p is any nonzero integer. If assuming $\omega_m = -\omega_{NV}$, then $p = 0$ and q is any nonzero integer. Figure 11(a) plots the spin squeezing parameter ξ^2 versus the normalized amplitude η for $p = 0$ and different values of q . For different values of q , the spin squeezing parameter ξ^2 exhibits periodic oscillations, which are mainly due to the modulation of the coupling strength by Bessel functions $J_q(\eta)$. Only the position of the optimal squeezing has shifted, while the degree of optimal squeezing has not

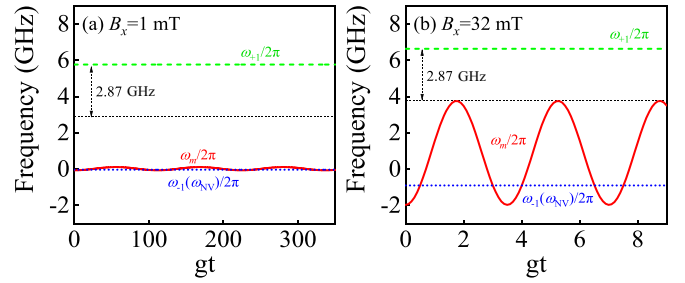


FIG. 12. Time evolution of the magnon and NV center frequencies for different magnetic field: (a) $B_x = 1$ mT and (b) $B_x = 32$ mT. The normalized amplitude is set to $\eta = 1.6$.

been affected. Without loss of generality, we choose $p = 0$ and $q = 1$ in the main text.

Second, we consider the case of nonexact resonance. Choosing the parameters $p = 0$, $q = 1$, and $\omega_m = -\omega_{NV}$, then $|\omega_m + \omega_{NV} + p\Omega| = 0$ and defining a small detuning $\delta = \omega_m - \omega_{NV} + \Omega$. Figure 11(b) shows the time evolution of ξ^2 for different detuning δ . When $\delta = 0$, the spin ensemble rapidly reaches a steady squeezed state with minimal ξ^2 . As the detuning increases, the spin squeezing progressively diminishes, indicating that the detuning weakens the effective spin-magnon coupling, thereby degrading the spin squeezing. Consequently, even a small detuning can significantly suppress the spin squeezing, underscoring that the optimal squeezing is achieved under exact resonant conditions.

APPENDIX B: THE RANGE OF THE MAGNETIC FIELD B_x

In our proposed scheme, the effective Hamiltonian in Eq. (7) is achieved by assuming the conditions $\omega_m = -\omega_{NV}$ and $\Omega = \omega_{NV} - \omega_m$. Using the relations $\omega_m = |\gamma_e|B_x$ and $\omega_{NV} = D_0 - |\gamma_e|\tilde{B}_x$, the magnetic field applied to the NV centers can be determined as $\tilde{B}_x = B_x + D_0/|\gamma_e|$. The frequency and amplitude of the Floquet drive are given by $\Omega = -2|\gamma_e|B_x$ and $\eta\Omega$, respectively. Figure 12 shows the time evolution of the magnon and NV center frequencies for different magnetic field B_x at $\eta = 1.6$. Here, we set the zero-field splitting $D_0 = 2\pi \times 2.87$ GHz to be the lower boundary of the detuning between the NV center $|+1\rangle$ state and the magnon frequency, ensuring a sufficiently large off-resonance condition. When the magnetic field is applied in the range of $0 < B_x \leq 32$ mT, the $|+1\rangle$ state can be safely excluded due to its off-resonance with the magnon, and in this case, the $|0\rangle$ and $|-1\rangle$ states of the NV center can together encode a two-level spin qubit.

[1] D. J. Wineland, J. J. Bollinger, W. M. Itano, F. L. Moore, and D. J. Heinzen, Spin squeezing and reduced quantum noise in spectroscopy, *Phys. Rev. A* **46**, R6797(R) (1992).
[2] M. Kitagawa and M. Ueda, Squeezed spin states, *Phys. Rev. A* **47**, 5138 (1993).
[3] D. J. Wineland, J. J. Bollinger, W. M. Itano, and D. J. Heinzen, Squeezed atomic states and projection noise in spectroscopy, *Phys. Rev. A* **50**, 67 (1994).
[4] K. Hammerer, A. S. Sørensen, and E. S. Polzik, Quantum interface between light and atomic ensembles, *Rev. Mod. Phys.* **82**, 1041 (2010).

[5] M. F. Riedel, P. Böhi, Y. Li, T. W. Hänsch, A. Sinatra, and P. Treutlein, Atom-chip-based generation of entanglement for quantum metrology, *Nature (London)* **464**, 1170 (2010).
[6] T. E. Lee, F. Reiter, and N. Moiseyev, Entanglement and spin squeezing in non-Hermitian phase transitions, *Phys. Rev. Lett.* **113**, 250401 (2014).
[7] L. Amico, R. Fazio, A. Osterloh, and V. Vedral, Entanglement in many-body systems, *Rev. Mod. Phys.* **80**, 517 (2008).
[8] O. Guehne and G. Tóth, Entanglement detection, *Phys. Rep.* **474**, 1 (2009).

- [9] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Quantum entanglement, *Rev. Mod. Phys.* **81**, 865 (2009).
- [10] J. Q. You and F. Nori, Superconducting circuits and quantum information, *Phys. Today* **58**(11), 42 (2005).
- [11] J. Q. You and F. Nori, Atomic physics and quantum optics using superconducting circuits, *Nature (London)* **474**, 589 (2011).
- [12] S. N. Shevchenko, S. Ashhab, and F. Nori, Landau-Zener-Stückelberg interferometry, *Phys. Rep.* **492**, 1 (2010).
- [13] Y.-H. Kang and Y. Xia, Entanglement creations and quantum gate implementations of spin qubits with Lyapunov control, *IEEE J. Sel. Top. Quantum Electron.* **26**, 1 (2019).
- [14] S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, UK, 1999).
- [15] E. J. Heller, Bound-state eigenfunctions of classically chaotic Hamiltonian systems: Scars of periodic orbits, *Phys. Rev. Lett.* **53**, 1515 (1984).
- [16] R. Schack, G. M. D'Ariano, and C. M. Caves, Hypersensitivity to perturbation in the quantum kicked top, *Phys. Rev. E* **50**, 972 (1994).
- [17] F. Haake, *Quantum Signature of Chaos* (Springer, Berlin, 1999).
- [18] S. Chaudhury, A. Smith, B. E. Anderson, S. Ghose, and P. S. Jessen, Quantum signatures of chaos in a kicked top, *Nature (London)* **461**, 768 (2009).
- [19] J. Hald, J. Sørensen, C. Schori, and E. Polzik, Spin squeezed atoms: A macroscopic entangled ensemble created by light, *Phys. Rev. Lett.* **83**, 1319 (1999).
- [20] C. Orzel, A. Tuchman, M. Fenselau, M. Yasuda, and M. Kasevich, Squeezed states in a Bose-Einstein condensate, *Science* **291**, 2386 (2001).
- [21] B. Julsgaard, A. Kozhekin, and E. Polzik, Experimental long-lived entanglement of two macroscopic objects, *Nature (London)* **413**, 400 (2001).
- [22] J. Estève, C. Gross, A. Weller, S. Giovanazzi, and M. K. Oberthaler, Squeezing and entanglement in a Bose-Einstein condensate, *Nature (London)* **455**, 1216 (2008).
- [23] T. Fernholz, H. Krauter, K. Jensen, J. F. Sherson, A. S. Sørensen, and E. S. Polzik, Spin squeezing of atomic ensembles via nuclear-electronic spin entanglement, *Phys. Rev. Lett.* **101**, 073601 (2008).
- [24] J. Appel, P. J. Windpassinger, D. Oblak, U. B. Hoff, N. Kjaergaard, and E. S. Polzik, Mesoscopic atomic entanglement for precision measurements beyond the standard quantum limit, *Proc. Natl. Acad. Sci. USA* **106**, 10960 (2009).
- [25] T. Takano, M. Fuyama, R. Namiki, and Y. Takahashi, Spin squeezing of a cold atomic ensemble with the nuclear spin of one-half, *Phys. Rev. Lett.* **102**, 033601 (2009).
- [26] C. Gross, T. Zibold, E. Nicklas, J. Estève, and M. K. Oberthaler, Nonlinear atom interferometer surpasses classical precision limit, *Nature (London)* **464**, 1165 (2010).
- [27] A. Louchet-Chauvet, J. Appel, J. J. Renema, D. Oblak, N. Kjaergaard, and E. S. Polzik, Entanglement-assisted atomic clock beyond the projection noise limit, *New J. Phys.* **12**, 065032 (2010).
- [28] J. Bollinger, W. Itano, D. Wineland, and D. Heinzen, Optimal frequency measurements with maximally correlated states, *Phys. Rev. A* **54**, R4649 (1996).
- [29] G. Agarwal and M. Scully, Ramsey spectroscopy with non-classical light sources, *Phys. Rev. A* **53**, 467 (1996).
- [30] P. R. Berman, *Atom Interferometry* (Academic Press, New York, 1997).
- [31] G. Xu and D. Heinzen, State-selective Rabi and Ramsey magnetic resonance line shapes, *Phys. Rev. A* **59**, R922 (1999).
- [32] A. D. Cronin, J. Schmiedmayer, and D. E. Pritchard, Optics and interferometry with atoms and molecules, *Rev. Mod. Phys.* **81**, 1051 (2009).
- [33] V. Meyer, M. Rowe, D. Kielpinski, C. Sackett, W. Itano, C. Monroe, and D. Wineland, Experimental demonstration of entanglement-enhanced rotation angle estimation using trapped ions, *Phys. Rev. Lett.* **86**, 5870 (2001).
- [34] G. Smith, S. Chaudhury, A. Silberfarb, I. Deutsch, and P. Jessen, Continuous weak measurement and nonlinear dynamics in a cold spin ensemble, *Phys. Rev. Lett.* **93**, 163602 (2004).
- [35] D. Döring, G. McDonald, J. E. Debs, C. Figl, P. A. Altin, H. A. Bachor, N. P. Robins, and J. D. Close, Quantum-projection-noise-limited interferometry with coherent atoms in a Ramsey-type setup, *Phys. Rev. A* **81**, 043633 (2010).
- [36] A. Sørensen and K. Mølmer, Spin-spin interaction and spin squeezing in an optical lattice, *Phys. Rev. Lett.* **83**, 2274 (1999).
- [37] N. Bigelow, Quantum engineering-squeezing entanglement, *Nature (London)* **409**, 27 (2001).
- [38] A. Andre, A. Sørensen, and M. Lukin, Stability of atomic clocks based on entangled atoms, *Phys. Rev. Lett.* **92**, 230801 (2004).
- [39] D. Oblak, P. Petrov, C. Garrido Alzar, W. Tittel, A. Vershovski, J. Mikkelsen, J. Sørensen, and E. Polzik, Quantum-noise-limited interferometric measurement of atomic noise: Towards spin squeezing on the Cs clock transition, *Phys. Rev. A* **71**, 043807 (2005).
- [40] E. S. Polzik, Quantum physics: The squeeze goes on, *Nature (London)* **453**, 45 (2008).
- [41] D. Meiser, J. Ye, and M. J. Holland, Spin squeezing in optical lattice clocks via lattice-based QND measurements, *New J. Phys.* **10**, 073014 (2008).
- [42] I. D. Leroux, M. H. Schleier-Smith, and V. Vuletic, Orientation-dependent entanglement lifetime in a squeezed atomic clock, *Phys. Rev. Lett.* **104**, 250801 (2010).
- [43] D. F. Walls and P. Zoller, Enhanced sensitivity of a gravitational-wave detector, *Phys. Lett. A* **85**, 118 (1981).
- [44] J. Dunningham and K. Burnett, Sub-shot-noise-limited measurements with Bose-Einstein condensates, *Phys. Rev. A* **70**, 033601 (2004).
- [45] K. Goda, O. Miyakawa, E. E. Mikhailov, S. Saraf, R. Adhikari, K. McKenzie, R. Ward, S. Vass, A. J. Weinstein, and N. Mavalvala, A quantum-enhanced prototype gravitational-wave detector, *Nat. Phys.* **4**, 472 (2008).
- [46] J. Ma, X. Wang, C. P. Sun, and F. Nori, Quantum spin squeezing, *Phys. Rep.* **509**, 89 (2011).
- [47] M. Fadel, T. Zibold, B. Dccamps, and P. Treutlein, Spatial entanglement patterns and Einstein-Podolsky-Rosen steering in Bose-Einstein condensates, *Science* **360**, 409 (2018).
- [48] G. S. Agarwal, R. R. Puri, and R. P. Singh, Atomic Schrödinger cat states, *Phys. Rev. A* **56**, 2249 (1997).
- [49] G. Liu, Y. N. Wang, L. F. Yan, N. Q. Jiang, W. Xiong, and M. F. Wang, Spin squeezing via one- and two-axis twisting induced by a single off-resonance stimulated Raman scattering in a cavity, *Phys. Rev. A* **99**, 043840 (2019).

- [50] Z. Hu, Q. Li, X. Zhang, L. G. Huang, H. B. Zhang, and Y. C. Liu, Spin squeezing with arbitrary quadratic collective-spin interactions, *Phys. Rev. A* **108**, 023722 (2023).
- [51] S. D. Bennett, N. Y. Yao, J. Otterbach, P. Zoller, P. Rabl, and M. D. Lukin, Phonon-induced spin-spin interactions in diamond nanostructures: Application to spin squeezing, *Phys. Rev. Lett.* **110**, 156402 (2013).
- [52] P. B. Li, Y. Zhou, W. B. Gao, and F. Nori, Enhancing spin-phonon and spin-spin interactions using linear resources in a hybrid quantum system, *Phys. Rev. Lett.* **125**, 153602 (2020).
- [53] A. Banerjee, Generation of atomic-squeezed states in an optical cavity with an injected squeezed vacuum, *Phys. Rev. A* **54**, 5327 (1996).
- [54] M. Ueda, T. Wakabayashi, and M. Kuwata-Gonokami, Synchronous collapses and revivals of atomic dipole fluctuations and photon Fano factor beyond the standard quantum limit, *Phys. Rev. Lett.* **76**, 2045 (1996).
- [55] A. Kuzmich, K. Mølmer, and E. Polzik, Spin squeezing in an ensemble of atoms illuminated with squeezed light, *Phys. Rev. Lett.* **79**, 4782 (1997).
- [56] J. Hald, J. Sørensen, C. Schori, and E. Polzik, Entanglement transfer from light to atoms, *J. Mod. Opt.* **47**, 2599 (2000).
- [57] A. Kuzmich, N. Bigelow, and L. Mandel, Atomic quantum non-demolition measurements and squeezing, *Europhys. Lett.* **42**, 481 (1998).
- [58] K. Mølmer, Twin-correlations in atoms, *Eur. Phys. J. D* **5**, 301 (1999).
- [59] Y. Takahashi, K. Honda, N. Tanaka, K. Toyoda, K. Ishikawa, and T. Yabuzaki, Quantum nondemolition measurement of spin via the paramagnetic Faraday rotation, *Phys. Rev. A* **60**, 4974 (1999).
- [60] A. Kuzmich, L. Mandel, J. Janis, Y. Young, R. Egnisman, and N. Bigelow, Quantum nondemolition measurements of collective atomic spin, *Phys. Rev. A* **60**, 2346 (1999).
- [61] A. Kuzmich, L. Mandel, and N. Bigelow, Generation of spin squeezing via continuous quantum nondemolition measurement, *Phys. Rev. Lett.* **85**, 1594 (2000).
- [62] L. Thomsen, S. Mancini, and H. Wiseman, Continuous quantum nondemolition feedback and unconditional atomic spin squeezing, *J. Phys. B: At. Mol. Opt. Phys.* **35**, 4937 (2002).
- [63] L. Thomsen, S. Mancini, and H. Wiseman, Spin squeezing via quantum feedback, *Phys. Rev. A* **65**, 061801(R) (2002).
- [64] J. Zhang, K. Peng, and S. Braunstein, Backaction-induced spin-squeezed states in a detuned quantum-nondemolition measurement, *Phys. Rev. A* **68**, 035802 (2003).
- [65] M. Auzinsh, D. Budker, D. F. Kimball, S. M. Rochester, J. E. Stalnaker, A. O. Sushkov, and V. V. Yashchuk, Can a quantum nondemolition measurement improve the sensitivity of an atomic magnetometer? *Phys. Rev. Lett.* **93**, 173002 (2004).
- [66] I. Teper, G. Vrijsen, J. Lee, and M. A. Kasevich, Backaction noise produced via cavity-aided nondemolition measurement of an atomic clock state, *Phys. Rev. A* **78**, 051803(R) (2008).
- [67] P. J. Windpassinger, D. Oblak, U. B. Hoff, A. Louchet, J. Appel, N. Kjaergaard, and E. S. Polzik, Squeezing of atomic quantum projection noise, *J. Modern Opt.* **56**, (1993) (2009).
- [68] M. Saffman, D. Oblak, J. Appel, and E. S. Polzik, Spin squeezing of atomic ensembles by multicolor quantum non-demolition measurements, *Phys. Rev. A* **79**, 023831 (2009).
- [69] V. Shah, G. Vasilakis, and M. V. Romalis, High bandwidth atomic magnetometry with continuous quantum nondemolition measurements, *Phys. Rev. Lett.* **104**, 013601 (2010).
- [70] W. Jiang, H.-X. Zang, S.-C. Zhang, X.-D. Gao, C. Yu, Y. Liu, Z.-W. Liu, X.-D. Chen, Y. Dong, G.-C. Guo, and F.-W. Sun, Room-temperature experimental implementation of quantum logic on a hybrid magnon-spin system, *Phys. Rev. Appl.* **22**, 054048 (2024).
- [71] D. Xu, X.-K. Gu, H.-K. Li, Y.-C. Weng, Y.-P. Wang, J. Li, H. Wang, S.-Y. Zhu, and J. You, Quantum control of a single magnon in a macroscopic spin system, *Phys. Rev. Lett.* **130**, 193603 (2023).
- [72] Z.-Q. Wang, Y.-P. Wang, J. Yao, R.-C. Shen, W.-J. Wu, J. Qian, J. Li, S.-Y. Zhu, and J. You, Giant spin ensembles in waveguide magnonics, *Nat. Commun.* **13**, 7580 (2022).
- [73] J. Xu, C. Zhong, X. Han, D. Jin, L. Jiang, and X. Zhang, Coherent gate operations in hybrid magnonics, *Phys. Rev. Lett.* **126**, 207202 (2021).
- [74] J. Xu, C. Zhong, X. Han, D. Jin, L. Jiang, and X. Zhang, Floquet cavity electromagnonics, *Phys. Rev. Lett.* **125**, 237201 (2020).
- [75] Y.-P. Wang, G.-Q. Zhang, D. Zhang, T.-F. Li, C.-M. Hu, and J. You, Bistability of cavity magnon polaritons, *Phys. Rev. Lett.* **120**, 057202 (2018).
- [76] X. Zhang, N. Zhu, C.-L. Zou, and H. X. Tang, Optomagnonic whispering gallery microresonators, *Phys. Rev. Lett.* **117**, 123605 (2016).
- [77] Z. Shen, G.-T. Xu, M. Zhang, Y.-L. Zhang, Y. Wang, C.-Z. Chai, C.-L. Zou, G.-C. Guo, and C.-H. Dong, Coherent coupling between phonons, magnons, and photons, *Phys. Rev. Lett.* **129**, 243601 (2022).
- [78] F.-X. Sun, S.-S. Zheng, Y. Xiao, Q. Gong, Q. He, and K. Xia, Remote generation of magnon Schrödinger cat state via magnon-photon entanglement, *Phys. Rev. Lett.* **127**, 087203 (2021).
- [79] X. Li, I. Lekavicius, and H. Wang, Diamond nanomechanical resonators protected by a phononic band gap, *Nano Lett.* **22**, 10163 (2022).
- [80] S. Liu, A. Granados del Águila, D. Bhowmick, C. K. Gan, T. Thu Ha Do, M. A. Prosnikov, D. Sedmidubský, Z. Sofer, P. C. M. Christianen, P. Sengupta, *et al.*, Direct observation of magnon-phonon strong coupling in two-dimensional antiferromagnet at high magnetic fields, *Phys. Rev. Lett.* **127**, 097401 (2021).
- [81] M. C. Kuzyk and H. Wang, Scaling phononic quantum networks of solid-state spins with closed mechanical subsystems, *Phys. Rev. X* **8**, 041027 (2018).
- [82] M. Fukami, D. R. Candido, D. D. Awschalom, and M. E. Flatté, Opportunities for long-range magnon-mediated entanglement of spin qubits via on- and off-resonant coupling, *PRX Quantum* **2**, 040314 (2021).
- [83] G.-Q. Zhang, Z. Chen, D. Xu, N. Shammah, M. Liao, T.-F. Li, L. Tong, S.-Y. Zhu, F. Nori, and J. You, Exceptional point and cross-relaxation effect in a hybrid quantum system, *PRX Quantum* **2**, 020307 (2021).
- [84] Y. Tabuchi, S. Ishino, A. Noguchi, T. Ishikawa, R. Yamazaki, K. Usami, and Y. Nakamura, Coherent coupling between a ferromagnetic magnon and a superconducting qubit, *Science* **349**, 405 (2015).

- [85] C. A. Potts, E. Varga, V. A. S. V. Bittencourt, S. Viola Kusminskiy, and J. P. Davis, Dynamical backaction magnomechanics, *Phys. Rev. X* **11**, 031053 (2021).
- [86] D. Lachance-Quirion, Y. Tabuchi, A. Gloppe, K. Usami, and Y. Nakamura, Hybrid quantum systems based on magnonics, *Phys. Express* **12**, 070101 (2019).
- [87] N. Bar-Gill, L. Pham, A. Jarmola, D. Budker, and R. Walsworth, Solid-state electronic spin coherence time approaching one second, *Nat. Commun.* **4**, 1743 (2013).
- [88] G. Balasubramian, P. Neumann, D. Twitchen, M. Markham, R. Koselov, N. Mizuochi, J. Isoya, J. Achard, J. Beck, J. Tissler, V. Jacques, P. R. Hemmer, F. Jelezko, and J. Wrachtrup, Ultra-long spin coherence time in isotopically engineered diamond, *Nat. Mater.* **8**, 383 (2009).
- [89] M. W. Doherty, N. B. Manson, P. Delaney, F. Jelezko, J. Wrachtrup, and L. C. L. Hollenberg, The nitrogen-vacancy in diamond, *Phys. Rep.* **528**, 1 (2013).
- [90] C. E. Bradley, J. Randall, M. H. Aboeih, R. C. Berrevoets, M. J. Degen, M. A. Bakker, M. Markham, D. J. Twitchen, and T. H. Taminiau, A ten-qubit solid-state spin register with quantum memory up to one minute, *Phys. Rev. X* **9**, 031045 (2019).
- [91] T. Neuman, D. S. Wang, and P. Narang, Nanomagnonic cavities for strong spin-magnon coupling and magnon-mediated spin-spin interactions, *Phys. Rev. Lett.* **125**, 247702 (2020).
- [92] E. G. Dalla Torre, J. Otterbach, E. Demler, V. Vuletic, and M. D. Lukin, Dissipative preparation of spin squeezed stomic ensembles in a steady state, *Phys. Rev. Lett.* **110**, 120402 (2013).
- [93] W. Qin, Y. H. Chen, X. Wang, A. Miranowicz, and F. Nori, Strong spin squeezing induced by weak squeezing of light inside a cavity, *Nanophotonics* **9**, 4853 (2020).
- [94] J. Q. Chen, Y. F. Qiao, X. L. Dong, X. L. Hei, and P. B. Li, Dissipation-assisted preparation of steady spin-squeezed states of SiV centers, *Phys. Rev. A* **103**, 013709 (2021).
- [95] P. Groszkowski, M. Köppenhöfer, H. K. Lau, and A. A. Clerk, Reservoir-engineered spin squeezing: Macroscopic even-odd effects and hybrid-systems implementations, *Phys. Rev. X* **12**, 011015 (2022).
- [96] A. Feng, J. Xu, and X. Hu, Coherent population trapping for reservoir engineering and spin squeezing, *Phys. Rev. Res.* **6**, 033256 (2024).
- [97] G. S. Agarwal and R. R. Puri, Nonequilibrium phase transitions in a squeezed cavity and the generation of spin states satisfying uncertainty equality, *Opt. Commun.* **69**, 267 (1989).
- [98] G. S. Agarwal and R. R. Puri, Cooperative behavior of atoms irradiated by broadband squeezed light, *Phys. Rev. A* **41**, 3782 (1990).
- [99] M. O. Scully and M. S. Zubairy, *Quantum Optics* (Cambridge University Press, Cambridge, UK, 1997).
- [100] C. Gerry and P. Knight, *Introductory Quantum Optics* (Cambridge University Press, Cambridge, UK, 2005).
- [101] H. Huebl, C. W. Zollitsch, J. Lotze, F. Hocke, M. Greifenstein, A. Marx, R. Gross, and S. T. B. Goennenwein, High cooperativity in coupled microwave resonator ferrimagnetic insulator hybrids, *Phys. Rev. Lett.* **111**, 127003 (2013).
- [102] X. Zhang, C.-L. Zou, N. Zhu, F. Marquardt, L. Jiang, and H. X. Tang, Magnon dark modes and gradient memory, *Nat. Commun.* **6**, 8914 (2015).
- [103] Y. Li, T. Polakovic, Y.-L. Wang, J. Xu, S. Lendinez, Z. Zhang, J. Ding, T. Khair, H. Saglam, R. Divan, J. Pearson, W.-K. Kwok, Z. Xiao, V. Novosad, A. Hoffmann, and W. Zhang, Strong coupling between magnons and microwave photons in on-chip ferromagnet-superconductor thin-film devices, *Phys. Rev. Lett.* **123**, 107701 (2019).
- [104] E. D. Herbschleb, H. Kato, Y. Maruyama, T. Danjo, T. Makino, S. Yamasaki, I. Ohki, K. Hayashi, H. Morishita, M. Fujiwara, and N. Mizuochi, Ultra-long coherence times amongst room-temperature solid-state spins, *Nat. Commun.* **10**, 3766 (2019).
- [105] S. Han, X. Ye, X. Zhou, Z. Liu, Y. Guo, M. Wang, W. Ji, Y. Wang, and J. Du, Solid-state spin coherence time approaching the physical limit, *Sci. Adv.* **11**, eadr9298 (2025).