

Certifying steady-state properties of open quantum systems

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Estimating the steady-state properties of open many-body quantum systems is a fundamental challenge in quantum science and technologies. In this work, we present a scalable approach based on semidefinite programming to derive certified bounds on the expectation value of an arbitrary observable in the steady state of Lindbladian dynamics. We illustrate our method on a series of many-body systems, including paradigmatic spin-1/2 chains and two-dimensional ladders, considering both equilibrium and nonequilibrium steady states. We benchmark our method with state-of-the-art tensor-network approaches that, unlike our method, are only able to provide estimates, with no guarantee, on steady-state quantities. For the tested models, only modest computational effort is needed to obtain certified nontrivial bounds for system sizes intractable by exact methods. Our method introduces the first general numerical tool for bounding steady-state properties of open quantum dynamics, opening an avenue in the understanding of stable configurations in many-body systems.

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I. INTRODUCTION

Open quantum systems, which interact continuously with their surrounding environment, are fundamental to understanding a vast array of physical phenomena and technological applications, including quantum optics, condensed matter physics, and quantum information processing [1–3]. These interactions often lead to dissipative dynamics that can significantly alter the system's behavior over time. Accurately determining the steady-state properties of such systems is crucial for predicting long-term behavior and for the design of quantum devices that can operate reliably under real-world conditions [4,5]. From a general perspective, there has been increasing recent interest [6,7] in finding techniques to compute the stationary state and its properties [8–10], especially with variational methods [11–14], in characterizing multiple steady states [15,16] and in designing Lindblad generators for target steady states [17,18]. On the experimental side, a growing effort was devoted to demonstrate stable quantum-correlated many-body states [19], steady-state superradiance in free space [20] and propose Dicke superradiant enhancement of heat current in circuit quantum electrodynamics [21].

A common framework for modeling the dynamics of open quantum systems is through the Lindblad master equation [22,23],

$$\dot{\rho} = \mathcal{L}(\rho), \quad (1)$$

which provides a Markovian description of the time evolution of the system's density matrix [3,24]. The Lindbladian superoperator has the form (in natural units [25])

$$\mathcal{L}(\rho) = -i[H, \rho] + \sum_i \gamma_i \left(L_i \rho L_i^\dagger - \frac{1}{2} \{L_i^\dagger L_i, \rho\} \right), \quad (2)$$

which encapsulates both the unitary evolution governed by the system's Hamiltonian H and the dissipative processes arising from environmental interactions. Finding the steady state ρ_{ss} involves solving for the density matrix that remains static under the Lindbladian dynamics,

$$\mathcal{L}(\rho_{ss}) = 0. \quad (3)$$

The exact computation of the steady state is possible for small systems (in the case of qubits, number of qubits $n \lesssim 8$; see the Supplemental Material for more details [26]), but soon becomes intractable when increasing the system size due to exponential scaling. In fact, the situation is harder than what is observed for other relevant many-body problems, such as the computation of ground states, as here one has to deal with mixed states. To reach larger sizes, one usually employs approximate variational approaches, such as tensor network (TN) methods [27]. Unfortunately, these methods only provide an estimate for the steady state, without guarantees on how close its observables are to the exact values. Instead, the method in this work allows deriving upper and lower bounds that determine an interval within which the true value lies with absolute certainty.

In many situations, one does not need a full description of the steady state as one is only interested in knowing a few relevant observables. In some cases, such as that of equilibrium states of local Hamiltonians, it has been shown that one can obtain results in quasipolynomial time by

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targetting local observables [28]. Determining the complete state may therefore be an unnecessarily complex task if, in the end, one can obtain similar results using a much smaller (i.e., scaling polynomially in the system size) number of parameters. Thus, it may be more practical to search for methods that target the observable(s) of interest without requiring the full reconstruction, or approximation, of the steady state.

In this work, we follow this approach and provide a scalable method to derive bounds on expectation values of any observable of interest at the steady state of open-system dynamics described by a Lindblad master equation. To do so, we exploit relaxations for polynomial optimization introduced in the context of quantum information theory [29–31] that make use of semidefinite programming (SDP). We apply our approach to paradigmatic models and compare them with TN methods, seeing that they attain similar performance but with certificates, something difficult, if not impossible, to obtain in variational methods [32]. Furthermore, while variational methods generally require the steady state to be unique [14], our approach also applies beyond this instance providing bounds over the whole convex set of steady states. Upon the finalization of this work, a similar method based on semidefinite programming to bound expectation values of steady-state observables was presented in Ref. [33] for few-mode bosonic systems, while in this work we focus on fermionic systems.

We are thus interested in bounding the expectation value of an observable O for the steady state of the system, i.e., minimizing/maximizing $\text{tr}(O\rho)$ when $\mathcal{L}(\rho) = 0$. Since this is linear in ρ , combined with the fact that a density matrix should be positive and have trace one, one can quite naturally formulate such a problem as an SDP (see the Supplemental Material for more details [26]). The full optimization, however, requires an explicit form of ρ , which scales exponentially with the number of constituents, n , typically qubits.

To avoid the exponential growth, in what follows we restrict positivity constraints to moment matrices made of subsets of all possible n -qubit operators. While the operators to construct moment matrices are arbitrary, a very natural choice for n -qubit systems consists of the “Pauli strings,” made of tensor products of Pauli matrices and the identity operator for each qubit. In this work, we will use the notation as follows: Pauli strings are denoted by $P_{\vec{\alpha}}$, where $\vec{\alpha}$ is a vector of n components that can take values in $\{0, 1, 2, 3\}$. For instance, P_{310} refers to $\sigma_z \otimes \sigma_x \otimes \mathbb{I}$, or even simpler $Z_1 X_2$. Such Pauli strings have an “order” equal to the number of nonidentity operators in the string (e.g., the previous Pauli string has order 2). Any generic observable can be written as a linear combination of Pauli strings with known real coefficients, for instance, in the case of our objective, $O = \sum_{\vec{\alpha}} o_{\vec{\alpha}} P_{\vec{\alpha}}$.

II. METHOD

In what follows, we provide several constraints that can be used to bound expectation values of steady-state observables through convex optimization techniques [34].

A. Moment matrix positivity constraints

For a given state ρ and vector of $s \geq 1$ operators $(\theta_i)_{i=1}^s$, we can define M as the corresponding moment matrix with elements

$$M_{ij} := \text{tr}(\rho \theta_i^\dagger \theta_j). \quad (4)$$

It can be shown [29] that such a moment matrix is positive semidefinite for any s and choice of operators $(\theta_i)_{i=1}^s$. As an example of such a moment matrix constraint, we can consider M as in Eq. (4) with $s = 7$, $(\theta_i)_{i=1}^7 \equiv (\mathbb{I}, X_1, Y_1, Z_1, X_2, Y_2, Z_2)^T$, and impose $M \succcurlyeq 0$. We refer to this as the level 1 moment matrix, as the top row contains only first-order expectation values. The level 2 moment matrix would contain all first- and second-order expectation values as the top row, and so on. Here, we also use the commutation rules between different spins (e.g., $[X_i, X_j] = 0$ for $i \neq j$) as well as the Pauli reduction rules ($X_i X_i = \mathbb{I}$, $X_i Y_i = iZ_i$, etc.) to simplify the matrix. In general, the size of moment matrix of level k grows like n^k , polynomially with the number of qubits.

B. State reconstruction positivity constraints

While it is completely intractable to constrain the positivity of the whole density matrix, what remains feasible is imposing positivity of all the reduced density matrices of a given, smaller, number of qubits, $m \ll n$. These positivity constraints are a function of the Pauli strings acting on the corresponding m spins. By reconstructing all m -site density matrices and enforcing their positivity, we constrain the set of feasible moments and improve the tightness of our bounds, often significantly, at minimal computational cost.

C. Linear Lindbladian constraints

While moment matrix optimization has been well documented in the literature [35,36], including the case of bounding observables for many-body ground states [37], the case of steady-state optimization offers an extra set of constraints. Based on the definition of the steady state, we have the extra constraint from Eq. (3) that the Lindbladian acting on our state should be zero. We can then use the adjoint Lindbladian [3] in order to derive constraints in terms of moments of observables acting on ρ . Considering a generic operator G and the definition of adjoint Lindbladian $\mathcal{L}^\dagger$, we have that

$$0 = \text{tr}(G\mathcal{L}(\rho_{\text{ss}})) = \text{tr}(\mathcal{L}^\dagger(G)\rho_{\text{ss}}) = \langle \mathcal{L}^\dagger(G) \rangle_{\text{ss}},$$

$$\mathcal{L}^\dagger(G) = i[H, G] + \sum_i \gamma_i \left(L_i^\dagger G L_i - \frac{1}{2} \{L_i^\dagger L_i, G\} \right). \quad (5)$$

The result is an equation into which we can place any operator to give us a linear constraint on the system. There is at least one steady state for a master equation in Lindblad form [3]. In the case of a single steady state, placing all possible $4^n - 1$ nontrivial Pauli strings (i.e., all except the identity) provides a fully constrained system of 4^n independent linear equations when combined with the trace-one condition on the state, $\langle \mathbb{I} \rangle_{\text{ss}} = 1$, which can then be solved to retrieve the steady state from its Pauli decomposition. As anticipated, such a problem is only solvable at small values of n , say $n \lesssim 8$. In

what follows, we will consider scalable relaxations involving only a subset of these linear constraints.

D. “Automatic” constraint generation

The previous ideas relax the exact treatment by restricting the positivity or Lindbladian constraints of the unknown steady state to a subset of Pauli strings. The choice of these strings is arbitrary, but, with the goal of achieving tighter bounds with as few variables as possible, we choose to apply the following algorithm to generate a subset of Pauli strings that should be relevant to our system and observable. We begin by placing the Pauli strings from the decomposition of the observable of interest O into the adjoint Lindbladian, as in Eq. (5), to generate linear constraints. We then take the Pauli strings that appear in the resulting constraints and plug them back into the adjoint Lindbladian. This process is repeated until we have generated a sufficiently large set of Pauli strings and corresponding linear constraints. In some cases, we note that this eventually reaches a “cycle,” in that no operators are generated by the insertion in Eq. (5) of the previous operators. In this case, the resulting constraints can form a fully constrained linear system, which can then be solved as before. The set of Pauli strings resulting from the whole iterative process is in turn also used to construct the moment matrix. We refer to this method of selecting a subset of informative Pauli strings as “automatic.” This approach appears more effective than, e.g., using randomly chosen Pauli strings or considering all Pauli strings of a given order.

E. Bounding expectation values of steady-state observables

By following the previous recipe, a subset of Pauli strings S_M is identified, which contains all those needed to specify the observable and the different constraints. We can now formulate our optimization problem in its general form: the lower/upper bound is given by

$$\begin{aligned}
 \langle O \rangle_{\text{lb/ub}} &= \min/\max_{\{ \langle P_{\vec{\alpha}} \rangle \}_{\vec{\alpha} \in S_M}} \sum_{\vec{\alpha}} o_{\vec{\alpha}} \langle P_{\vec{\alpha}} \rangle \\
 \text{s.t. } -1 &\leq \langle P_{\vec{\alpha}} \rangle \leq 1, \quad \forall \vec{\alpha} \in S_M, \\
 \langle \mathcal{L}^\dagger(P_{\vec{\alpha}}) \rangle &= 0, \quad \forall \vec{\alpha} \in S_L, \\
 \sum_{\vec{\alpha}} A_{k,\vec{\alpha}} \langle P_{\vec{\alpha}} \rangle &\geq 0, \quad \forall k \in \{1, \dots, m_p\}, \\
 \sum_{\vec{\alpha}} b_{k,\vec{\alpha}} \langle P_{\vec{\alpha}} \rangle &= 0, \quad \forall k \in \{1, \dots, m_s\}.
 \end{aligned} \quad (6)$$

These two SDPs define an interval

$$\langle O \rangle_{\text{lb}} \leq \langle O \rangle_{\text{ss}} \leq \langle O \rangle_{\text{ub}}, \quad (7)$$

excluding the possibility that $\langle O \rangle_{\text{ss}}$ lies outside it. The bounds “ $-1 \leq \langle P_{\vec{\alpha}} \rangle \leq 1, \forall \vec{\alpha} \in S_M$ ” are not always necessary, but provide a “safety net” to prevent the problem being unbounded in the case that one of the moments is not sufficiently constrained. The set S_L identifies those Pauli strings used in the linear constraints. The m_p linear matrix inequality constraints defined by known matrices $A_{k,\alpha}$ can correspond either to moment matrices (obeying some Pauli replacement rules) or to reconstructed reduced density matrix positivity (see the Supplemental Material for more details [26]). Finally,

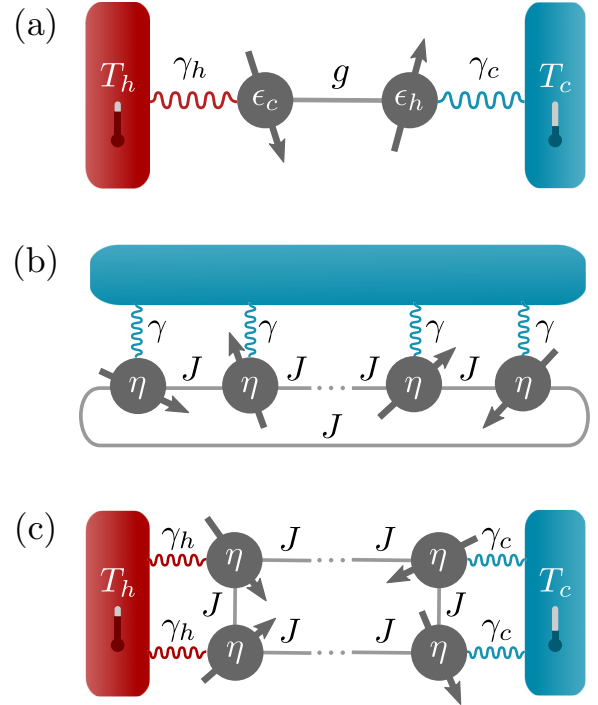


FIG. 1. (a) Two-qubit example: Here, the left qubit is connected to a hot bath and the right qubit is connected to a cold bath. The system is defined by parameters γ_h, γ_c , coupling of the qubits with their respective baths, ϵ_h, ϵ_c , energy gaps of the qubits, T_c, T_h , the temperatures of the baths, and g , coupling between the two qubits. (b) One-dimensional (1D) periodic chain example: Here, each qubit is coupled to the dissipative bath with coefficient γ and coupled to its nearest neighbors with coefficient J . The final qubit is also coupled to the first qubit. A field of coefficient η is also applied to each qubit in the X direction. (c) Two-dimensional (2D) ladder example: Here, the end qubits are connected to the respective heat bath with coefficients γ_h and γ_c , and each qubit is coupled to its nearest neighbors with coefficient J . A field of coefficient η is also applied to each qubit in the Z direction.

the last m_s linear equality constraints defined by coefficients $b_{k,\alpha}$ represent known symmetries specific to the problem. Such constraints add negligible computational cost, but often improve the bounds by a non-negligible amount (see the Supplemental Material for more details [26]). The data (and code generating said data) that support the findings of this work are available as an archived GitHub repository [38].

III. RESULTS

A. Two-qubit test case

We first consider the simple test case discussed, e.g., by Refs. [41–43]: a two-qubit system, in which the left qubit is connected to a hot bath and the right qubit is connected to a cold bath, as in Fig. 1(a). Choosing the local master equation, the open-system dynamics is described by the following Lindbladian:

$$\mathcal{L}(\rho) = -i[H, \rho] + \sum_{j \in \{h, c\}} (\gamma_j^+ D[\sigma_+^{(j)}] \rho + \gamma_j^- D[\sigma_-^{(j)}] \rho), \quad (8)$$

where $D[A]B := ABA^\dagger - (1/2)\{A^\dagger A, B\}$.

The Hamiltonian is $H := H_S + H_{\text{int}}$, where $H_S = \sum_{j \in \{h,c\}} \epsilon_j \sigma_+^{(j)} \sigma_-^{(j)}$ and $H_{\text{int}} = g(\sigma_+^{(h)} \sigma_-^{(c)} + \sigma_-^{(h)} \sigma_+^{(c)})$, with ϵ_h and ϵ_c being the energy gaps of the qubits, and g the coupling constant. Here, $\sigma_+^{(j)} = (1/2)(X_j + iY_j)$ and $\sigma_-^{(j)} = (1/2)(X_j - iY_j)$. The kinetic coefficients $\gamma_j^+ = \gamma_j n_B^j$ and $\gamma_j^- = \gamma_j(1 + n_B^j)$ account, respectively, for absorption from and emission to bath j and $n_B^j = 1/(e^{\epsilon_j/T_j} - 1)$ are the Bose factors. Furthermore, γ_h and γ_c are the couplings of the qubits with their respective baths at temperatures T_h and T_c . In Ref. [42], the authors focus on the heat current in the nonequilibrium steady state defined as $J_{\text{ss}} := Q_h^{\text{ss}} - Q_c^{\text{ss}}$, with

$$Q_j^{\text{ss}} = \text{tr}[H_S(\gamma_j^+ D[\sigma_+^{(j)}] + \gamma_j^- D[\sigma_-^{(j)}])\rho_{\text{ss}}], \quad (9)$$

providing, for our example, an objective function $\langle O \rangle \equiv J_{\text{ss}}$ in terms of Pauli strings,

$$J_{\text{ss}} = \frac{\epsilon_h}{2}(\gamma_h^+ - \gamma_h^-) - \frac{\epsilon_c}{2}(\gamma_c^+ - \gamma_c^-) - \frac{\epsilon_h}{2}(\gamma_h^+ + \gamma_h^-)\langle \sigma_h^z \rangle_{\text{ss}} + \frac{\epsilon_c}{2}(\gamma_c^+ + \gamma_c^-)\langle \sigma_c^z \rangle_{\text{ss}}. \quad (10)$$

This system serves as the first example of how simply “brute-forcing” all possible insertions into the adjoint Lindbladian is often not necessary, as solving with six *automatic* linear constraints results in the exact solution (i.e., we obtain $\langle O \rangle_{\text{lb}} = \langle O \rangle_{\text{ub}}$), rather than needing to insert all 15 nontrivial second-order Pauli strings (see the Supplemental Material for more details [26]) [44].

Extending this by adding extra qubits between the hot and cold qubits with the same coupling results in a system that appears to be efficiently solvable. The automatic linear constraint generation always appears to reach a cycle after $2n^2 - n$ constraints, resulting in a full-rank linear system that can be solved exactly. This has been tested up to 100 spins (19900 linear constraints). As such, it was assumed that this problem is therefore in P , since it appears to be solvable in polynomial time. It demonstrates a possible advantage of our constraint set, such that some systems may be shown as being efficiently solvable [45]. Note, however, that estimating steady states is known to be hard [46] and it is possible to encode the output of any quantum circuit on the steady state of a local master equation [47].

B. Periodic 1D chain

The next system we consider is that of a periodic n -site 1D chain. The Hamiltonian used is that of Ref. [14], the transverse Ising model, specifically a 1D chain such that the final qubit is also coupled to the first qubit and each qubit to a dissipative bath. The qubits are coupled to each other with a coupling strength J , undergo a transverse field η , and are connected to the dissipative bath with coefficient γ . A diagram of this system is given as Fig. 1(b). The Hamiltonian is as follows:

$$H = J \sum_{\langle i,j \rangle} Z_i Z_j + \eta \sum_i X_i, \quad (11)$$

where $\langle i, j \rangle$ denotes nearest neighbors. To form the full Lindbladian, we then use the spin decay operator

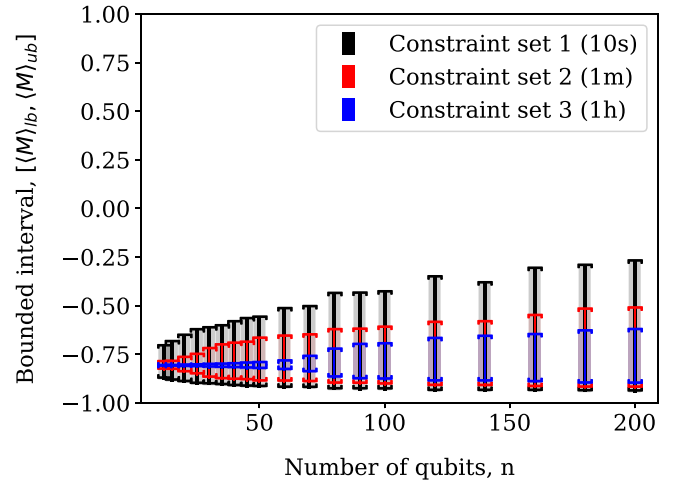


FIG. 2. Upper and lower bounds on the magnetization at the steady state as a function of the system size for the periodic linear chain. The plotted intervals represent the certified region in which the true value of the observable lies. The y axis is scaled to $[-1, 1]$ to demonstrate the fraction of the trivial bounds that we restrict to. We apply method (6) imposing a limited number of linear constraints and moment matrix size (set 1:10 000 linear constraints and a 40×40 moment matrix; set 2:30 000 constraints and a 100×100 matrix; and set 3:70 000 constraints and a 250×250 matrix). Symmetry between all qubits is also imposed. The time to optimize per point was roughly the same within each constraint set (set 1 ≈ 10 s, set 2 ≈ 1 min, and set 3 ≈ 1 h).

$\Gamma_k = (\sqrt{\gamma}/2)(X_k - iY_k)$ such that our full Lindbladian is

$$\mathcal{L}(\rho) = -i[H, \rho] + \sum_k^n \left(\Gamma_k \rho \Gamma_k^\dagger - \frac{1}{2} \{ \Gamma_k^\dagger \Gamma_k, \rho \} \right). \quad (12)$$

We then consider the case of $n = 12$, $J/\gamma = 0.5$, and $\eta/\gamma = 0.5$, as given in Ref. [14]. In their work, they utilize a matrix-product ansatz combined with a Monte Carlo-style optimization. For the 12-site chain, they obtain an estimate for the average magnetization $\langle M \rangle = (1/n) \sum_i \langle Z_i \rangle$ within 1% of the known optimum in approximately 22 s (based on their Fig. 5), while we obtain bounds of the optimum $\pm 1\%$ in approximately 50 s: $[-0.821\,151, -0.801\,716]$. While their approximation is faster, our method provides bounds on the quantity in a time of a similar order of magnitude. A table detailing the bounded intervals for the various constraints is given as Table I.

To demonstrate the tightness of our bounds irrespective of the observable, we sometimes represent the differences as percentages. These refer to the fraction of the *trivial bounds* that we restrict to. Such bounds are obtained by finding the minimum/maximum of our objective using only the knowledge that each Pauli string should be bounded to $[-1, 1]$. For instance, for the magnetization, the trivial bounds are $[-1, 1]$.

To demonstrate how the method scales with system size for this specific problem, we plot in Fig. 2 the bounds obtained using a fixed number of linear constraints and fixed moment matrix size for a variable number of qubits n , ranging from 12 to 200. Although the bounds become worse with increased system size, they still remain informative in the sense that they significantly restrict the range of possible values. It should

TABLE I. Bounds on the average magnetization for the 12-site periodic 1D chain coupled to a bath [Fig. 1(b) and Eqs. (11) and (12)], for a variety of possible constraints (the first four columns). The bounds given are the strict lower and upper bounds for the magnetization in the steady state, while the percentages represent the fraction of the full $[-1, 1]$ region that the bounds cover. The other values in parentheses are either the size of matrix or number of constraints added. The system is described by Eqs. (11) and (12), with parameters of $J/\gamma = 0.5$, $\eta/\gamma = 0.5$.

Moment matrix	Linear	State reconstruction	Symmetry	Bounded interval	Time
None	Auto (1000)	None	None	$[-1.000\,000, -0.274\,680]$ (36.27%)	0.38 s
None	Auto (10 000)	None	None	$[-0.926\,167, -0.670\,898]$ (12.76%)	22.87 s
Level 1 (37×37)	Auto (10 000)	None	None	$[-0.882\,682, -0.697\,738]$ (9.25%)	9.97 s
Auto (51×51)	Auto (10 000)	None	None	$[-0.880\,354, -0.702\,399]$ (8.90%)	11.70 s
None	Auto (10 000)	All 4-site ($794 \times 16 \times 16$)	None	$[-0.821\,151, -0.801\,716]$ (0.97%)	49.49 s
Auto (51×51)	Auto (10 000)	All 4-site ($794 \times 16 \times 16$)	None	$[-0.821\,150, -0.801\,713]$ (0.97%)	1 m 13 s
None	Auto (10 000)	All 4-site ($794 \times 16 \times 16$)	Yes (66)	$[-0.820\,893, -0.802\,149]$ (0.94%)	59.76 s
None	Auto (30 000)	All 4-site ($794 \times 16 \times 16$)	Yes (66)	$[-0.817\,397, -0.805\,514]$ (0.59%)	2 m 43 s

be noted that, for instance, the sign of the observable is still often certified for larger systems. The computation of all these points required very modest effort and, in particular, the time to optimize is roughly constant within any given constraint set. For instance, for the largest constraint set, in blue, each point takes around 1 h on a desktop computer¹ independently of the number of qubits.

C. 2D ladder

To explore how our method works in the 2D case, we now consider a 2D ladder system, as shown in Fig. 1(c); effectively we have two 1D chains linked at each qubit, coupled to a hot and a cold bath at either end of the chain. We use an extension of the Lindbladian and Hamiltonian from Eq. (8):

$$\mathcal{L}(\rho) = -i[H, \rho] + \sum_j^n (\gamma_j^+ D[\sigma_+^{(j)}] \rho + \gamma_j^- D[\sigma_-^{(j)}] \rho), \quad (13)$$

$$H = \sum_i^n \varepsilon_i \sigma_+^{(i)} \sigma_-^{(i)} + g \sum_{(i,j)}^n (\sigma_+^{(i)} \sigma_-^{(j)} + \sigma_-^{(i)} \sigma_+^{(j)}). \quad (14)$$

Here, the coefficients γ_j^+ and γ_j^- are only nonzero for the spins connected to the baths. Even adding this single extra layer to the system appears to add significant complexity. For the case of bounding the average magnetization of a 5×2 grid, with the same computational effort that for the 12-qubit linear chain gave a bound difference of 1%, here gives a bound difference of 36%. The largest constraint set tested also reached a bound difference of 36% after 2.5 h (see the Supplemental Material for more details [26]). As the bounds do not improve even with larger constraint sets, it seems to suggest that the system has multiple steady states and thus these bounds may be tight.

IV. DISCUSSION

We have introduced an approach to bound the expectation values of observables in the steady state of open quantum systems. We formulate the problem as an SDP and incorporate various enhancements—linear constraints from the adjoint Lindbladian, moment matrix generation, partial state positivity, and the exploitation of system symmetries. Our method

has been tested on relevant models, including a two-qubit chain, a one-dimensional chain, and a two-dimensional ladder. We achieved tight bounds for a 12-site linear chain, as well as informative bounds for systems composed of hundreds of qubits. These findings underscore the effectiveness of our method in providing bounds efficiently, making it an avenue for analyzing and certifying steady-state properties in open quantum systems.

Looking ahead, there are several directions for future research. A promising application is to use the behavior of our bounds in terms of a Liouvillian parameter to detect dissipative phase transitions. Through the application of a fermion-to-qubit mapping, one could also apply our method to fermionic systems. Finally, investigating the combination of our SDP method with other numerical techniques, such as quantum Monte Carlo, machine learning [48], or advanced TN methods, may yield hybrid approaches that capitalize on the strengths of multiple strategies.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [38].

¹PC used: Intel Xeon W-2295 3.00 GHz, 32 GB RAM.

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