Letter

# Thermalization is typical in large classical and quantum harmonic systems

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We establish an analytical criterion for dynamical thermalization within harmonic systems, applicable to both classical and quantum models. Specifically, we prove that thermalization of various observables—such as particle energies in physically relevant random quadratic Hamiltonians—is typical for large systems ($N \gg 1$) with initial conditions drawn from the microcanonical distribution. Moreover, we show that thermalization can also arise from nontypical initial conditions, where only a finite fraction of the normal modes is excited. A different choice of initial conditions, such as all the initial energy localized in a single particle, instead leads to energy equipartition without thermalization. Since the models we consider are integrable, our findings provide a general dynamical basis for an approach to thermalization that bypasses chaos and ergodicity, focusing instead on the physical requirement that thermodynamic observables depend on a large number of normal modes, and they build a bridge between the classical and quantum theories of thermalization.

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*Introduction*. There is unanimous consensus that statistical mechanics (SM) captures in a precise way the behavior of macroscopic observables in both classical (CM) and quantum (QM) mechanics [1–4]. However, there is no complete agreement on the underlying reasons for this success. For instance, whether or not chaos in the system dynamics and ergodicity are necessary ingredients for thermalization in CM is still a matter of debate [5–12]. Thermalization in QM is usually explained through the eigenstate thermalization hypothesis (ETH) [13–17], which has not been proven analytically for general interacting quantum systems, or through *typicality* [18–26], i.e., the concept that thermalization is overwhelmingly likely to occur for most configurations or states, given a sufficiently large number of degrees of freedom.

Integrable systems [27] are not expected to thermalize, and their equilibration can be understood through generalized Gibbs ensembles [28–31], which can be combined with the ETH in the quantum case; see, e.g., [32–38] (see also [39] for a different open system perspective). In CM we can address their thermalization, despite the lack of ergodicity, by following the lines of the seminal works of Khinchin [40], and then Mazur and van der Linden [41], based on the idea that the very basic ingredients of SM are the large number of degrees of freedom and the global nature of the observables, regardless of the presence of chaos [8]. This approach is further connected to the central limit theorem for the sum of trigonometric functions [42]. The dynamical variables [43] that can display thermalization must be functions of many independent degrees of freedom, while the conserved quantities of the system are not. These concepts have influenced some studies of SM for specific integrable systems of classical harmonic oscillators [44–46], as well as numerical analyses for the linear chain [47,48] and the Toda model [49].

Inspired by these ideas, in this Letter we focus on thermalization in both classical and quantum harmonic systems of $N$ particles, which are a paradigmatic example of separable (and classically integrable) models whose motion is described by a collection of free *normal modes*. Our approach is rooted in the system dynamics, as we address the time average of observables $O$ defined as

$$\overline{O}_t = \frac{1}{t}\int_0^t ds\, O(s), \tag{1}$$

where in the quantum case $O(s)$ corresponds to the mean value of the observable on the system state at time $s$. If the long-time limit $\overline{O}_\infty$, for almost all initial conditions, is equal to the ensemble average of $O$ over the microcanonical ensemble [1,2], which we define as $\langle O \rangle$, then we say that $O$ thermalizes (in QM the time average is not always required; see, e.g., [50]).

In CM, the time variance of *microscopic* observables—such as single-particle energies—does not vanish. The time variance of *macroscopic* observables, in contrast, is expected to vanish after the thermalization time [3,4]. In QM, also the time fluctuations of microscopic observables vanish if the system is large, provided the latter satisfies the ETH [16]. In this work, whenever we refer to "thermalization" or

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"equilibration" we also check that the above prescriptions for the time variance hold.

The observables we are interested in are the energies of some modes of the system, which are functions of many normal modes. As we deal with harmonic systems, we obtain the same expression for the long-time average of these energies in both CM and QM. We prove that the conditions for their thermalization are overwhelmingly likely to occur for $N \gg 1$ assuming we draw the initial conditions from the microcanonical distribution. In other words, we establish that thermalization is *typical* in both classical and quantum large harmonic systems. Interestingly, we prove that thermalization still occurs even if the initial conditions are not typical, specifically when only a fraction $N^*$ of the normal modes is initially excited, provided $N^* \gg 1$. Different initial conditions, such as initial excitations uniformly distributed over the normal modes, irrespective of their energies, also lead to energy equipartition for large $N$. However, in these cases, the equipartition value may not correspond to the one predicted by the microcanonical ensemble average.

Our results provide a solid analytical foundation for Khinchin's approach to thermalization, grounded in the system dynamics. To the best of our knowledge, this is the first time these ideas have been extended to the quantum regime, demonstrating their applicability to both classical and quantum systems in a comparable manner. Moreover, our findings illustrate in a clear and rigorous way how thermalization and energy equipartition are very general concepts, even in physical models where chaos does not play any role, such as harmonic systems, assuming we deal with a large number of degrees of freedom and avoid pathological initial conditions.

*Model.* Any Hamiltonian of an $N$-particle harmonic system can be diagonalized on the basis of the normal modes and written as (all the summations go from 1 to $N$)

$$H = \sum_k \omega_k o_k^* o_k. \tag{2}$$

$\omega_k$ are the *eigenfrequencies* of the normal modes. For simplicity, we assume $\omega_k > 0$, and that the normal modes are nondegenerate: there is no $j \neq k$ such that $\omega_j = \omega_k$. This is a standard assumption, for instance, in the theory of quantum thermalization [16,51] and is usually justified by (even tiny) disorder in many-body systems. We point out that the incommensurability of the frequencies, which may be used to prove that the motion of harmonic oscillators is ergodic [52], is not a requirement of our model. $o_k$ and $o_k^*$ are symbols in a notation that is agnostic with respect to CM or QM. Specifically, in CM, $o_k = z_k$, $o_k^* = z_k^*$, where $z_k = (\omega_k q_k + i p_k)/\sqrt{2\omega_k}$ and $z_k^* = (\omega_k q_k - i p_k)/\sqrt{2\omega_k}$ are the complex canonical variables of the normal modes [53]. In QM, $o_k = a_k$, $o_k^* = a_k^\dagger$, where $a_k^\dagger$ and $a_k$ are the standard bosonic or fermionic ladder operators. Crucially, in both CM and QM the evolution of these quantities is given by $o_k(t) = e^{-i\omega_k t} o_k(0)$, $o_k^*(t) = e^{i\omega_k t} o_k^*(0)$.

Harmonic systems are *separable* [5], i.e., $H$ can be divided into the sum of $N$ constants of motion: $H = \sum_k E_k$. The full state of this system will never thermalize, as energy cannot be exchanged between the normal modes. Our goal, however, is to focus on dynamical variables that are a function of many normal modes.

*Randomly rotated modes*. We first consider a rotation of the normal modes:

$$\tilde{o}_j = \sum_k C_{jk} o_k, \quad \tilde{o}_j^* = \sum_k C_{jk}^* o_k^*. \tag{3}$$

$C_{jk}$ is a generic orthogonal or unitary matrix. The Hamiltonian written in the basis of rotated modes is not separable anymore:

$$H = \sum_k \tilde{E}_k^{(\mathrm{rot})} + \sum_{k,k':k\neq k'} \mu_{kk'} \tilde{o}_k^* \tilde{o}_{k'}, \tag{4}$$

with $\tilde{E}_k^{(\mathrm{rot})} = \mu_{kk} \tilde{o}_k^* \tilde{o}_k$ and $\mu_{kk'} = \sum_j C_{kj} C_{k'j}^* \omega_j$. Intuitively, energy can be exchanged between the rotated modes, so we are interested in studying the thermalization of the energies $\tilde{E}_k^{(\mathrm{rot})}$ in the regime $N \gg 1$. We consider their long-time average and obtain

$$O_k^{(\mathrm{rot})} := \overline{\tilde{E}_k^{(\mathrm{rot})}}_\infty = \sum_j |C_{kj}|^2 p_{0,j} \sum_{j'} |C_{kj'}|^2 \omega_{j'}, \tag{5}$$

where the initial *populations* are defined as $p_{0,j} = |z_j(0)|^2$ for CM and $p_{0,j} = \mathrm{Tr}[a_j^\dagger a_j \rho_0]$ for QM. For the derivations of all the results in the main text, we refer the reader to the Supplemental Material [54]. Next, we analyze three different scenarios for the initial conditions of the normal modes.

*Case 1: Initial conditions from the microcanonical distribution.* If we now assume that (i) the initial populations are extracted according to the microcanonical distribution (both in CM and QM) and (ii) the rotation matrix $C_{jk}$ is drawn uniformly over the sphere, it can be shown that thermalization is typical. First one easily notices that [54]

$$\left\langle O_k^{(\mathrm{rot})} \right\rangle = \left\langle \tilde{E}_k^{(\mathrm{rot})} \right\rangle, \tag{6}$$

where $\langle\cdot\rangle$ denotes a microcanonical average. On the left-hand side, this average is taken over the initial conditions, while on the right-hand side it refers to the standard ensemble average of $\tilde{E}_k^{(\mathrm{rot})}$. This result alone, however, is not enough: we also have to check that the fluctuations around the microcanonical average over the initial conditions vanish for large $N$, i.e., that thermalization is reached for almost any initial set of populations drawn from the microcanonical ensemble. To understand why this check is necessary, notice that the energies of the normal modes are, trivially, thermalized on average when the initial conditions are drawn from the microcanonical distribution. However, fluctuations around this average do not vanish for large $N$, so the fraction of initial conditions leading to nonthermalized normal modes remains finite, and thermalization is therefore nontypical.

The fluctuations are captured by

$$\epsilon_k := \frac{\left\langle \left(O_k^{(\mathrm{rot})}\right)^2 \right\rangle - \left\langle \tilde{E}_k^{(\mathrm{rot})} \right\rangle^2}{\left\langle \tilde{E}_k^{(\mathrm{rot})} \right\rangle^2}. \tag{7}$$

Within our assumptions, the rows of $C$ can be thought of as the vectors of an orthonormal basis in $\mathbb{R}^N$ [55], which for large $N$ are uniformly distributed over the sphere [56]. Using their probability distribution [54], we can compute the moments of the random elements $|C_{kj}|^2$. We obtain

$$\mathbb{E}[\epsilon_k] \to 0 \text{ for } N \gg 1 \tag{8}$$

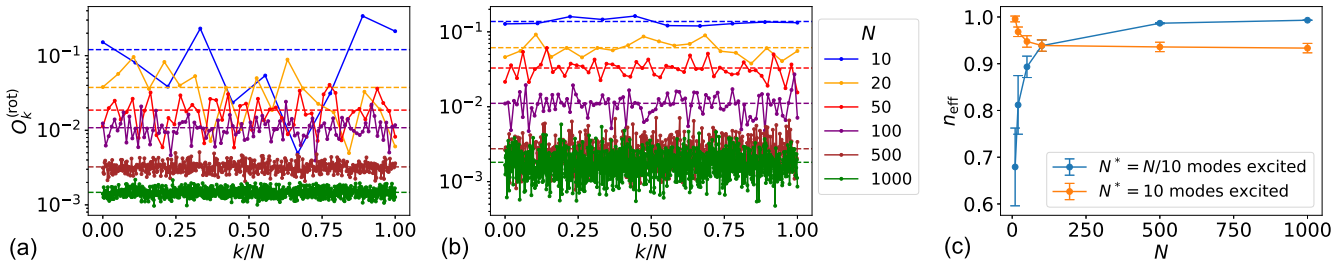


FIG. 1. Numerical results for the energy of randomly rotated modes, with random eigenfrequencies $\omega_j$ sampled uniformly over [1, 10]. (a) Distribution of $O_k^{(\mathrm{rot})}$ over $k$ for a single realization of $C$ [Eq. (5)]. The initial conditions are chosen in such a way that the first $N^*$ modes have the same energy, while the others are not excited. This is equivalent to the restricted classical microcanonical distribution discussed in [54]. The dashed lines are the theoretical predictions given by Eq. (10). (b) Same as (a), but only 10 normal modes are excited in the initial state independently of $N$. (c) $n_{\mathrm{eff}}$ [Eq. (11)] as a function of $N$ for the two kinds of initial conditions depicted in (a) and (b) averaged over 50 different realizations of $C$ (error bars = 1 std. dev.). Data points are joined by a line for visual guidance only.

and

$$\frac{\mathrm{Var}\,[\epsilon_k]}{(\mathbb{E}[\epsilon_k])^2} \to 0 \text{ for } N \gg 1, \tag{9}$$

where $\mathbb{E}[\cdot]$ and $\mathrm{Var}\,[\cdot]$ refer, respectively, to the mean value and variance over "disorder," i.e., different realizations of the random rotation matrix. In other words, for almost any set of modes obtained through the random rotation $C$, the fluctuations $\epsilon_k$ of the quantity $O_k^{(\mathrm{rot})}$ vanish in the large-$N$ limit, hence the result.

Moreover,

$$\mathbb{E}\left[\langle \tilde{E}_k^{(\mathrm{rot})} \rangle\right] = \omega_{\mathrm{av}} \langle p_0 \rangle_{\mathrm{av}}, \tag{10}$$

with $\langle p_0 \rangle_{\mathrm{av}} = \frac{E_{\mathrm{tot}}}{N} (\omega^{-1})_{\mathrm{av}}$ for CM and $\langle p_0 \rangle_{\mathrm{av}} = \sum_j 1/[N(1 \pm e^{-\beta\omega_j})]$ for, respectively, fermions and bosons in QM ($\beta$ is the inverse temperature in the microcanonical ensemble [54]). We have also introduced the total energy $E_{\mathrm{tot}} = \sum_j \omega_j p_{0,j}$, $\omega_{\mathrm{av}} = \sum_j \omega_j/N$, and $(\omega^{-1})_{\mathrm{av}} = \sum_j 1/(N\omega_j)$. Equation (10) is independent of $k$, i.e., the energy of the rotated modes is equipartitioned. However, note that the energy stored in the rotated modes at infinite time may not be equal to $E_{\mathrm{tot}}$, as some may be trapped in the interaction part of Eq. (4). Note that, while QSM does not predict energy equipartition of the normal modes at low temperatures [1], Eq. (10) predicts equipartition for the rotated modes also in QM, as $\mu_{kk}$ in Eq. (4) does not depend on $k$ for $N \gg 1$.

*Case 2: Only a fraction of the normal modes is initally excited.* Interestingly, it can be shown that Eqs. (6), (9), and (10) hold true even if we consider microcanonical averages over a finite subset $1 \ll N^* \ll N$ of the normal modes, showing that the initial conditions do not have to be typical on the full space of configurations [54]. We validate the latter result through numerical simulations shown in Fig. 1, which refer to classical systems. We consider rotations through some complex random unitary $C$. We plot the final distribution of $O_k^{(\mathrm{rot})}$ over $k$ for different $N$ in Figs. 1(a) and 1(b), and we compare them with Eq. (10), respectively, when we initially excite a fraction $N^* = N/10$ of normal modes with populations given by the (restricted) classical microcanonical average, and for a scenario in which only the first 10 normal modes are excited independently of $N$. In (a) we observe that the fluctuations around the mean value vanish for large $N$ and equipartition emerges. In (b) we are violating the condition $N^* \gg 1$, so the fluctuations do not vanish. This behavior can be captured by the normalized *effective number of degrees of freedom* [49]:

$$n_{\mathrm{eff}} = \frac{\exp\left(-\sum_k u_k \ln u_k\right)}{N}, \tag{11}$$

with $u_k = O_k^{(\mathrm{rot})} / \sum_j O_j^{(\mathrm{rot})}$. The energy is equipartitioned if $n_{\mathrm{eff}} = 1$. Figure 1(c) shows that $n_{\mathrm{eff}} \to 1$ for $N \gg 1$ only if a fraction $N^* \gg 1$ of the normal modes is initially excited.

*Case 3: Roughly equal initial populations.* Let us now discuss a case in which the populations are not drawn from the microcanonical ensemble, but they satisfy a different notion of "typicality." In particular, we assume that all populations are initially excited with a similar magnitude. In other words, if we introduce the average population $p_{\mathrm{av}} = \sum_j p_{0,j}/N$, we assume $p_{0,j}/p_{\mathrm{av}} = O(1)$ for all $j$. This is a common initial condition describing, for instance, a situation in which only one single rotated mode is initially excited. Indeed, if we initially excite only the $\alpha$th rotated mode with some energy $E_0$, $\tilde{z}_j^*(0)\tilde{z}_{j'}(0) = E_0 \delta_{j\alpha}\delta_{j'\alpha}$ (and analogously for QM), from Eq. (3) we obtain $|z_k(0)|^2 = E_0 |C_{k\alpha}|^2$, which does not depend on $k$ for large $N$.

Moreover, let us also assume that the magnitude of the eigenfrequencies is comparable, i.e., $\omega_j/\omega_{\mathrm{av}} = O(1)$ for all $j$. Then, it can be shown that for $N \gg 1$ the energies of the rotated modes are typically equipartitioned for both CM and QM. However, the equipartition value does not correspond to the microcanonical one, unless all the eigenfrequencies become identical for large $N$.

In other words, when the initial populations of the normal modes are comparable, we typically observe equilibration towards energy equipartition without thermalization. In this scenario, thermalization cannot occur unless the system dynamics exhibit nonintegrability. This is because these initial conditions constitute a measure-zero (although physically relevant) subset in the microcanonical ensemble for $N \gg 1$. This scenario describes initial states where the observables we are studying deviate more significantly from equilibrium compared to those sampled from the microcanonical distribution. Further discussions and details on this topic can be found in *Appendix A* (see also [54]).

Finally, we have verified that the time fluctuations of $\tilde{E}_k^{(\mathrm{rot})}$ do not vanish in CM, while they do disappear for fermions and hard-core bosons in QM, provided $N \gg 1$ and for all sets of initial conditions discussed above. Moreover, time fluctuations correctly vanish also for classical systems if we consider a macroscopic observable, e.g., the sum of several $\tilde{E}_k^{(\mathrm{rot})}$. More details can be found in *Appendix B*.

*Application to Hamiltonians with delocalized eigenvectors.* Let us now analyze a different set of physically motivated observables. We consider the Hamiltonian

$$H = \sum_{j,k=1} M_{jk} \tilde{o}_j^* \tilde{o}_k, \tag{12}$$

with $M_{jk} = M_{kj}^*$. $M_{kk}$ is the self-energy of the "local mode" $\tilde{o}_k$ of the $k$th particle, while $M_{jk}$ is the interparticle interaction. Equation (12) can describe many different physical systems of interest, such as a collection of interacting harmonic oscillators or an ensemble of superconducting qubits [57] coupled in a structured way (e.g., in the topology of a quantum computer [58]), or interacting with a spin bath [59]. $H$ can be diagonalized through a rotation $C$, which is the matrix of the normalized eigenvectors of $M$, and written in the basis of the normal modes as in Eq. (2), whose relation with the local modes is still given by Eq. (3). We again assume that the eigenfrequencies of the normal modes are nondegenerate.

We are interested in the long-time average of the energy of each local particle $\tilde{E}_k^{(\mathrm{loc})} = M_{kk} \tilde{o}_k^* \tilde{o}_k$, as this quantity may be easily accessible in an experiment [54]:

$$O_k^{(\mathrm{loc})} := \overline{\tilde{E}_k^{(\mathrm{loc})}}_{\infty} = M_{kk} \sum_j |C_{kj}|^2 p_{0,j}. \tag{13}$$

$p_{0,j}$ are the initial conditions, as previously defined.

Let us now assume that the eigenvectors of the matrix $M$ get *delocalized* for large $N$ [60], that is, we can effectively think of them as uniformly distributed over the unit sphere up to fluctuations vanishing with $N$. Then, the rows of $C$ are distributed uniformly over the sphere, and we can calculate the moments of $|C_{kj}|^2$ [54]. Different classes of physically relevant random matrices have been proven to have delocalized eigenvectors for large $N$, such as the GOE and GUE ensembles [60,61], the Rosenzweig-Porter model [62,63], symmetric and Hermitian matrices [64,65], and more [66,67]. So, it is conjectured that eigenvector delocalization is a typical property of random matrices for large $N$ [68].

Then, using $M_{kk} = \sum_j |C_{kj}|^2 \omega_j$, $O_k^{(\mathrm{loc})}$ can be easily transformed into the expression for $O_k^{(\mathrm{rot})}$ in Eq. (5), and it can be shown that the thermalization of $O_k^{(\mathrm{loc})}$ is also typical for initial conditions drawn uniformly from the microcanonical distribution: Eq. (9) still holds [54]. Similar results can be found also in the case of a single initially excited local mode, assuming all the self-energies are equal for large $N$. We refer the interested readers to the discussion in *Appendix C*.

Finally, we remark that a similar reasoning applies to the analogous case of quadratic Hamiltonians of the form $H = \sum_j p_j^2/2 + \sum_{jk} v_{jk} q_j q_k$, which is of great relevance for CM [44,45]. For further details, we refer to [54], where we also prove that the eigenvectors of the linear chain [46,48] get delocalized for $N \gg 1$. This holds independently of the boundary conditions of the chain, and it is true for both classical and quantum modes. At the same time, our results do not hold anymore for Hamiltonians with localized eigenvectors, such as lattices with disorder inducing Anderson-like localization.

*Conclusions.* In this Letter, we have shown that thermalization and energy equipartition are overwhelmingly likely to occur for the energies of almost any set of modes in both classical and quantum harmonic systems with a sufficiently large number of degrees of freedom and nonpathological initial conditions. These modes include those derived from random rotations of the normal modes and local modes of quadratic random Hamiltonians, which describe several paradigmatic models in both classical and quantum mechanics. We have considered two different notions of "typical" initial conditions. The first involves "populations" of the normal modes (or of a subset thereof) drawn from the microcanonical distribution, representing a scenario where the total energy of the system is fixed before being distributed among the normal modes. The second notion involves populations drawn from a peaked distribution, representing, for example, a scenario where only a single rotated mode is initially excited. In the first scenario, thermalization and energy equipartition are typical [see Eqs. (6), (9), and (10)]. In the second case, energy equipartition is typical at a value that may differ from that predicted by the microcanonical average (see *Appendix A*).

Our results, supported by various numerical simulations, offer robust analytical evidence for a conceptualization of thermalization based solely on the large number of degrees of freedom and typical initial conditions. Additionally, we have shown for the first time that this concept holds true for both classical and quantum mechanics fundamentally due to the same reasons. Our findings may thus provide an alternative way to understand thermalization, with significant implications, for instance, for the quantum-to-classical transition in statistical mechanics. For example, one may view the general concepts and ideas behind the ETH as exhibiting a similar spirit to Khinchin's approach to thermalization [69], which we have consolidated and demonstrated for a broad class of classical and quantum models. Then, a natural and promising direction for future work would be to investigate the connections between our framework and various results on the emergence of the ETH at different levels and with different implications in integrable systems [33,37,38,70–72]. This may provide a more unified understanding of thermalization across the classical and quantum domains, which have traditionally been treated through very distinct approaches. Moreover, it would be interesting to connect our results with other recent works that address thermalization in integrable systems relevant to current quantum technologies [39,59,73].

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*Data availability.* Data supporting the results reported in this article are available from the authors upon reasonable request.

*Appendix A: Initial conditions with roughly equal populations*. Given $p_{\rm av} = \sum_j p_{0,j}/N$, we assume $p_{0,j}/p_{\rm av} = O(1)$ for all $j$. Moreover, $\omega_j/\omega_{\rm av} = O(1)$ for all $j$. Then, for $N \gg 1$, we obtain [54]

$$\mathbb{E}\big[O_k^{(\rm rot)}\big] = \omega_{\rm av} p_{\rm av}[1 + O(1/N)]. \tag{A1}$$

The above expression does not depend on $k$, i.e., the energy is on average equipartitioned among the rotated modes. Furthermore, we find $\mathrm{Var}\,[O_k^{(\rm rot)}] = O(\omega_{\rm av}^2 p_{\rm av}^2/N)$, which immediately implies

$$\frac{\mathrm{Var}\big[O_k^{(\rm rot)}\big]}{\big(\mathbb{E}\big[O_k^{(\rm rot)}\big]\big)^2} = O\left(\frac{1}{N}\right) \to 0 \text{ for } N \to \infty, \tag{A2}$$

proving that energy equipartition is typical. It is also possible to show that

$$\mathbb{E}\big[O_k^{(\rm rot)}\big] \to \frac{E_{\rm tot}}{N} \text{ for } N \to \infty \tag{A3}$$

for typical initial conditions [54]. We have introduced the total energy in the system $E_{\rm tot} = \sum_j \omega_j p_{0,j}$.

We now compare Eq. (A1) with the corresponding value obtained for initial conditions drawn from the classical microcanonical distribution, Eq. (10). Using Eq. (A3), for CM their difference can be written as

$$\frac{E_{\rm tot}}{N}(\omega_{\rm av}(\omega^{-1})_{\rm av} - 1). \tag{A4}$$

Therefore, in CM the two values coincide only if $\omega_{\rm av} \approx 1/(\omega^{-1})_{\rm av}$, i.e., the distribution of the eigenfrequencies must be peaked, with relative fluctuations vanishing for large $N$.

The key difference between the microcanonical distribution and the "roughly equal" initial conditions $p_{0,j}$ lies in the weighting factor $1/\omega_j$. Suppose the eigenfrequencies are uniformly distributed over the interval $[1, 2]$. Setting $p_{0,j} = p_0$ independently of $j$ neglects the factor $1/\omega_j$, which is nonetheless within the standard microcanonical fluctuations of the $j$th mode population [54]. However, this choice of $p_{0,j}$ for all the modes systematically biases the initial conditions relative to the microcanonical distribution: modes with higher frequencies ($\omega_j \geqslant 1.5$) will, on average, be more populated than in the microcanonical ensemble, while lower-frequency modes will be underpopulated. This is why this choice represents a zero-measure subset of the microcanonical ensemble in the limit $N \gg 1$.

*Appendix B: Time fluctuations*

*1. Microscopic observables.* In this subsection, we study the time variance of the energies of each rotated mode. We make the additional assumption that there are no gap degeneracies in the spectrum of the eigenfrequencies, which is usual in the context of the ETH [16,72]. Using the definition in Eq. (4), for CM we observe [54]

$$\overline{\big(\tilde{E}_k^{(\rm rot)}\big)^2}_\infty \to 2\big(\overline{\tilde{E}_k^{(\rm rot)}}_\infty\big)^2 \quad \text{for} \quad N \to \infty. \tag{B1}$$

Therefore, the time fluctuations do not vanish in the classical regime.

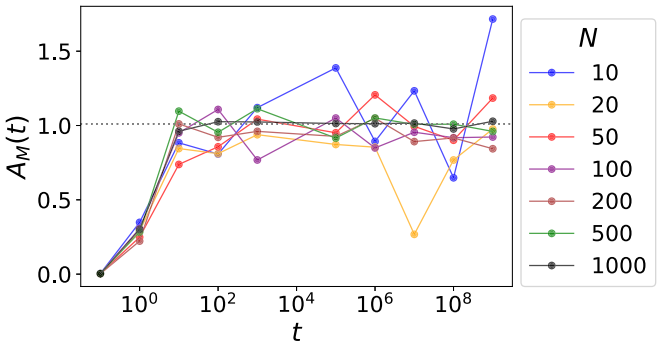


FIG. 2. $A_M$ from Eq. (B3) as a function of time, with $\mathsf{K} = (N/2, \ldots, N)$ for different values of $N$. The initial energy is concentrated in the first rotated mode. The eigenfrequencies are sampled uniformly from $[1, 3]$. The gray dotted line depicts the theoretical prediction for the equilibration value derived from Eq. (A3).

In contrast, for fermions and hard-core bosons in QM we obtain [54]

$$\overline{\big(\tilde{E}_k^{(\rm rot)}\big)^2}_\infty \leqslant \big(\overline{\tilde{E}_k^{(\rm rot)}}_\infty\big)^2[1 + O(1/N)] \tag{B2}$$

for large $N$. As a consequence, the time fluctuations vanish in the quantum case for $N \gg 1$.

*2. Macroscopic observables.* As a macroscopic observable, we consider any quantity that can be written as a sum of single-particle energies of the rotated modes, or single-particle occupation number of the rotated modes [76]. Let us introduce the observable

$$A_M = \sum_{k \in \mathsf{K}} \tilde{E}_k^{(\rm rot)}, \tag{B3}$$

where $\mathsf{K}$ is a set containing $N^*$ rotated modes, which for large $N$ is a finite fraction of all the modes: $N^* \gg 1$ if $N \gg 1$. Then, for both CM and QM it can be shown that [54]

$$\overline{A_M^2}_\infty = \overline{A_M}_\infty^2[1 + O(1/N^*)] \quad \text{for} \quad N^* \gg 1. \tag{B4}$$

Time fluctuations of macroscopic observables thus correctly vanish in the thermodynamic limit. Note that we would obtain the same results if we replaced $\tilde{E}_k^{(\rm rot)}$ with the occupation number $\tilde{o}_k^* \tilde{o}_k$ in Eq. (B3).

To test this prediction numerically, we study $A_M$ as the sum of the energies from mode $k = N/2$ to $k = N$ ($N^* = N/2$). We consider a scenario where only the first rotated mode is initially excited, thus $A_M(0) = 0$, and we study $A_M(t)$ for different $N$. Its time evolution is plotted in Fig. 2. $A_M(t)$ grows in time until approaching the equilibration value, and time fluctuations are clearly suppressed for larger $N$. If, instead, we drew the initial conditions from the microcanonical ensemble, $A_M(0)$ would be closer to the thermalization value, and fluctuations around it would vanish for $N \gg 1$.

This observation is consistent with Khinchin's original idea in classical SM, where thermalization arises solely from the choice of observable and the structure of the microcanonical ensemble, independently of time evolution. If an observable meets the requirements for thermalization, it will thermalize for most initial conditions (with probability approaching 1 in the microcanonical ensemble) at any time. The notion of typicality in QM is also based on similar ideas [12]. For

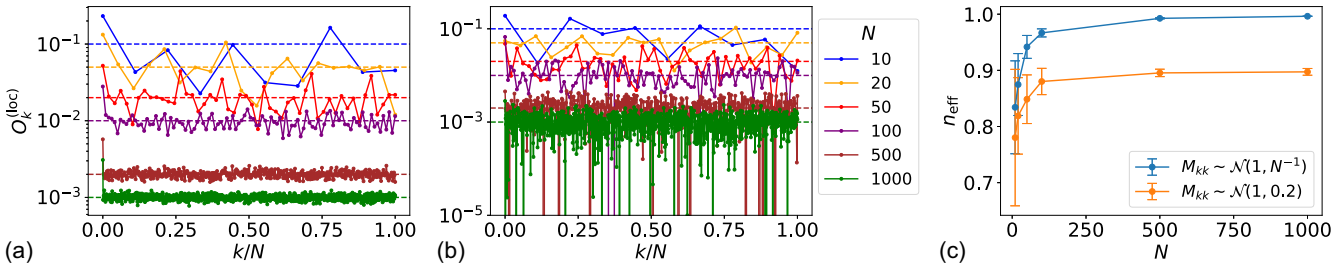


FIG. 3. Numerical results for the energy of the local particles of a random $H$ in Eq. (12). We initially excite a single local mode ($k = 1$) in a Fermi-Pasta-Ulam-Tsingou (FPUT) fashion [74,75], so that the $p_{0,j}$ are (roughly) randomly distributed among the normal modes. (a) Distribution of $O_k^{(\mathrm{loc})}$ over $k$ for a single realization of $M$ and with $M_{jk} \sim \mathcal{N}(0, 0.1)$, $M_{kk} \sim \mathcal{N}(1, N^{-1})$. (b) Same as (a) but with $M_{kk} \sim \mathcal{N}(1, 0.2)$. (c) $n_{\mathrm{eff}}$ [Eq. (11)] as a function of $N$ in the two scenarios depicted in (a) and (b) averaged over 50 different realizations of $M$ (error bars = 1 std. dev.). Data points are joined by a line for visual guidance only.

technical details on the connection between Khinchin's results and our findings, see [54].

*Appendix C: Energy equipartition for a random quadratic Hamiltonian.* In this Appendix, we study the long-time average of the observables $\tilde{E}_k^{(\mathrm{loc})}$ introduced in Eq. (13) for a random quadratic Hamiltonian given by Eq. (12). Note that if $M_{kk} = M_0$ for all $k$ ("identical particles," e.g., a collection of resonators with equal frequency [57]) and, crucially, $|C_{jk}|^2 \approx 1/N$, energy is equipartitioned among the particles.

Instead of drawing the initial conditions from the microcanonical ensemble, we now assume a peaked distribution of the populations, i.e., $p_{0,j}/p_{\mathrm{av}} = O(1)$ with $p_{\mathrm{av}} = \sum_j p_{0,j}/N$. We obtain [54]

$$\mathbb{E}\left[O_k^{(\mathrm{loc})}\right] = \mathbb{E}[M_{kk}]\, p_{\mathrm{av}}, \tag{C1}$$

$$\mathrm{Var}\left[O_k^{(\mathrm{loc})}\right] = p_{\mathrm{av}}^2 \left[\mathrm{Var}\,[M_{kk}] + O\left(\frac{\mathrm{Var}\,[M_{kk}]}{N}\right) + O\left(\frac{(\mathbb{E}[M_{kk}])^2}{N}\right)\right]. \tag{C2}$$

Next, we assume $\mathbb{E}[M_{kk}] = M_0$ for all $k$, and the fluctuations around the mean value vanish for large $N$, i.e., $\mathrm{Var}\,[M_{kk}] = 1/N^{\alpha}$ with $\alpha > 0$. This means that the particles must be identical for $N \to \infty$. Note that, since we assume that all the $\omega_j$ are of the same order, this is consistent with delocalized eigenvectors for large $N$, as $M_{kk} = \sum_j |C_{kj}|^2 \omega_j$. Then,

$$\frac{\mathrm{Var}\left[O_\infty^{(k)}\right]}{\left(\mathbb{E}\left[O_\infty^{(k)}\right]\right)^2} = O\left(\frac{1}{N^{\alpha}}\right) \to 0 \quad \text{for} \quad N \to \infty, \tag{C3}$$

that is, energy is equipartitioned among the particles.

We numerically illustrate the emergence of equipartition in the latter scenario by first generating a random matrix with elements drawn according to $M_{kk} \sim \mathcal{N}(1, N^{-1})$ and $M_{jk} \sim \mathcal{N}(0, 0.1)$. Then, we study the same problem but with $M_{kk} \sim \mathcal{N}(1, 0.2)$. The results are depicted, respectively, in Figs. 3(a) and 3(b): only in the first case do we obtain energy equipartition for $N \gg 1$, as also captured by the behavior of $n_{\mathrm{eff}}$ shown in Fig. 3(c). Note that, while $M$ may have negative eigenvalues, we can set them positive by shifting $M$ through $\delta\mathbb{I}$ with $\delta = O(\sqrt{N})$ [77]. We have also verified that the equipartition emerges for the non-Wigner matrix $MM^{\dagger}$, which is positive-semidefinite by construction and displays eigenvector delocalization for large $N$.


[1] L. Peliti, *Statistical Mechanics in a Nutshell* (Princeton University Press, Princeton, NJ, 2011).

[2] K. Huang, *Statistical Mechanics* (Wiley, Hoboken, New Jersey, 2008).

[3] L. Landau and E. Lifshitz, *Statistical Physics. Part 1* (Pergamon, Oxford, 1980).

[4] S.-K. Ma, *Statistical Mechanics* (World Scientific, Singapore, 1985).

[5] M. C. Gutzwiller, *Chaos in Classical and Quantum Mechanics*, Interdisciplinary Applied Mathematics Vol. 1 (Springer, New York, 1990).

[6] P. Gaspard, *Chaos, Scattering and Statistical Mechanics*, Cambridge Nonlinear Science Series (Cambridge University Press, Cambridge, 1998).

[7] Edited by J. Bricmont, D. Dürr, M. C. Galavotti, G. Ghirardi, F. Petruccione, and N. Zanghi, *Chance in Physics–Foundations and Perspectives* (Springer-Verlag, Berlin/Heidelberg, 2001).

[8] P. Castiglione, M. Falcioni, A. Lesne, and A. Vulpiani, *Chaos and Coarse Graining in Statistical Mechanics* (Cambridge University Press, Cambridge, 2008).

[9] L. Sklar, *Physics and Chance: Philosophical Issues in the Foundations of Statistical Mechanics* (Cambridge University Press, Cambridge, 1993).

[10] J. Bricmont, *Making Sense of Statistical Mechanics*, Undergraduate Lecture Notes in Physics (Springer Nature, Switzerland, 2022).

[11] Edited by A. Bassi, S. Goldstein, R. Tumulka, and N. Zanghi, *Physics and the Nature of Reality* (Springer, Berlin, 2024).

[12] M. Baldovin, G. Gradenigo, A. Vulpiani, and N. Zanghì, On the foundations of statistical mechanics, Phys. Rep. **1132**, 1 (2025).

[13] J. M. Deutsch, Quantum statistical mechanics in a closed system, Phys. Rev. A **43**, 2046 (1991).

[14] M. Srednicki, Chaos and quantum thermalization, Phys. Rev. E **50**, 888 (1994).

[15] M. Rigol, V. Dunjko, and M. Olshanii, Thermalization and its mechanism for generic isolated quantum systems, Nature (London) **452**, 854 (2008).

[16] L. D'Alessio, Y. Kafri, A. Polkovnikov, and M. Rigol, From quantum chaos and eigenstate thermalization to statistical mechanics and thermodynamics, Adv. Phys. **65**, 239 (2016).

[17] J. M. Deutsch, Eigenstate thermalization hypothesis, Rep. Prog. Phys. **81**, 082001 (2018).

[18] H. Tasaki, From quantum dynamics to the canonical distribution: General picture and a rigorous example, Phys. Rev. Lett. **80**, 1373 (1998).

[19] J. Gemmer and G. Mahler, Distribution of local entropy in the Hilbert space of bi-partite quantum systems: Origin of Jaynes' principle, Eur. Phys. J. B **31**, 249 (2003).

[20] S. Goldstein, J. L. Lebowitz, R. Tumulka, and N. Zanghì, Canonical typicality, Phys. Rev. Lett. **96**, 050403 (2006).

[21] S. Popescu, A. J. Short, and A. Winter, Entanglement and the foundations of statistical mechanics, Nat. Phys. **2**, 754 (2006).

[22] P. Reimann, Foundation of statistical mechanics under experimentally realistic conditions, Phys. Rev. Lett. **101**, 190403 (2008).

[23] N. Linden, S. Popescu, A. J. Short, and A. Winter, Quantum mechanical evolution towards thermal equilibrium, Phys. Rev. E **79**, 061103 (2009).

[24] P. Reimann and M. Evstigneev, Quantum versus classical foundation of statistical mechanics under experimentally realistic conditions, Phys. Rev. E **88**, 052114 (2013).

[25] M. P. Müller, E. Adlam, L. Masanes, and N. Wiebe, Thermalization and canonical typicality in translation-invariant quantum lattice systems, Commun. Math. Phys. **340**, 499 (2015).

[26] P. Correia, G. Dias Carvalho, and T. R. de Oliveira, Is entanglement necessary for the typicality argument in statistical mechanics? arXiv:2504.21090.

[27] Note that the definition of "integrable system" is problematic in QM [51], while in CM we refer to systems with a maximal number of conserved quantities.

[28] M. Rigol, A. Muramatsu, and M. Olshanii, Hard-core bosons on optical superlattices: Dynamics and relaxation in the superfluid and insulating regimes, Phys. Rev. A **74**, 053616 (2006).

[29] M. Rigol, V. Dunjko, V. Yurovsky, and M. Olshanii, Relaxation in a completely integrable many-body quantum system: An *ab initio* study of the dynamics of the highly excited states of 1D lattice hard-core bosons, Phys. Rev. Lett. **98**, 050405 (2007).

[30] L. Vidmar and M. Rigol, Generalized Gibbs ensemble in integrable lattice models, J. Stat. Mech. (2016) 064007.

[31] H. Spohn, Generalized Gibbs ensembles of the classical toda chain, J. Stat. Phys. **180**, 4 (2020).

[32] V. Alba, Eigenstate thermalization hypothesis and integrability in quantum spin chains, Phys. Rev. B **91**, 155123 (2015).

[33] J. M. Magán, Random free fermions: An analytical example of eigenstate thermalization, Phys. Rev. Lett. **116**, 030401 (2016).

[34] S. Nandy, A. Sen, A. Das, and A. Dhar, Eigenstate Gibbs ensemble in integrable quantum systems, Phys. Rev. B **94**, 245131 (2016).

[35] T. LeBlond, K. Mallayya, L. Vidmar, and M. Rigol, Entanglement and matrix elements of observables in interacting integrable systems, Phys. Rev. E **100**, 062134 (2019).

[36] M. Gluza, J. Eisert, and T. Farrelly, Equilibration towards generalized Gibbs ensembles in non-interacting theories, SciPost Phys. **7**, 038 (2019).

[37] P. Łydżba, R. Świetek, M. Mierzejewski, M. Rigol, and L. Vidmar, Normal weak eigenstate thermalization, Phys. Rev. B **110**, 104202 (2024).

[38] H. Tasaki, Macroscopic irreversibility in quantum systems: Free expansion in a fermionic chain, Phys. Rev. E **111**, L062103 (2025).

[39] K. Ptaszyński and M. Esposito, Open-system eigenstate thermalization in a noninteracting integrable model, Phys. Rev. E **111**, 014129 (2025).

[40] A. Y. Khinchin, *Mathematical Foundations of Statistical Mechanics* (Dover, New York, 1949).

[41] P. Mazur and J. Van Der Linden, Asymptotic form of the structure function for real systems, J. Math. Phys. **4**, 271 (1963).

[42] M. Kac, On the distribution of values of sums of the type $\sum f(2^k t)$, Ann. Math. **47**, 33 (1946).

[43] "Observables" in the language of quantum theory; we will use these terms interchangeably throughout the work.

[44] P. Mazur and E. Montroll, Poincaré cycles, ergodicity, and irreversibility in assemblies of coupled harmonic oscillators, J. Math. Phys. **1**, 70 (1960).

[45] G. W. Ford, M. Kac, and P. Mazur, Statistical mechanics of assemblies of coupled oscillators, J. Math. Phys. **6**, 504 (1965).

[46] M. Baldovin, R. Marino, and A. Vulpiani, Ergodic observables in non-ergodic systems: The example of the harmonic chain, Physica A **630**, 129273 (2023).

[47] F. Jin, T. Neuhaus, K. Michielsen, S. Miyashita, M. A. Novotny, M. I. Katsnelson, and H. De Raedt, Equilibration and thermalization of classical systems, New J. Phys. **15**, 033009 (2013).

[48] N. Cocciaglia, A. Vulpiani, and G. Gradenigo, Thermalization without chaos in harmonic systems, Physica A **601**, 127581 (2022).

[49] M. Baldovin, A. Vulpiani, and G. Gradenigo, Statistical mechanics of an integrable system, J. Stat. Phys. **183**, 41 (2021).

[50] M. C. Bañuls, J. I. Cirac, and M. B. Hastings, Strong and weak thermalization of infinite nonintegrable quantum systems, Phys. Rev. Lett. **106**, 050405 (2011).

[51] C. Gogolin and J. Eisert, Equilibration, thermalisation, and the emergence of statistical mechanics in closed quantum systems, Rep. Prog. Phys. **79**, 056001 (2016).

[52] A. Giorgilli, *Notes on Hamiltonian Dynamical Systems* (Cambridge University Press, Cambridge, 2022).

[53] F. Strocchi, Complex coordinates and quantum mechanics, Rev. Mod. Phys. **38**, 36 (1966).

[54] See Supplemental Material at http://link.aps.org/supplemental/10.1103/h9fj-ylgm for the proofs of all results presented in the main text, additional technical details, a discussion of the relation between our work and Khinchin's approach, and the example of the linear chain, which includes Refs. [78–83].

[55] For simplicity, we provide all the results for a real $C$; generalizations to a complex unitary are trivial.

[56] S. Goldstein, J. L. Lebowitz, R. Tumulka, and N. Zanghì, Any orthonormal basis in high dimension is uniformly distributed over the sphere, Ann. Inst. H. Poincaré Probab. Statist. **53**, 701 (2017).

[57] A. Blais, A. L. Grimsmo, S. M. Girvin, and A. Wallraff, Circuit quantum electrodynamics, Rev. Mod. Phys. **93**, 025005 (2021).
[58] Y. Kim, A. Eddins, S. Anand, K. X. Wei, E. van den Berg, S. Rosenblatt, H. Nayfeh, Y. Wu, M. Zaletel, K. Temme, and A. Kandala, Evidence for the utility of quantum computing before fault tolerance, Nature (London) **618**, 500 (2023).
[59] J. P. Pekola, B. Karimi, M. Cattaneo, and S. Maniscalco, Long-time relaxation of a finite spin bath linearly coupled to a qubit, Open Syst. Inf. Dyn. **30**, 2350009 (2023).
[60] S. O'Rourke, V. Vu, and K. Wang, Eigenvectors of random matrices: A survey, J. Comb. Theor. Ser. A **144**, 361 (2016).
[61] G. W. Anderson, A. Guionnet, and O. Zeitouni, *An Introduction to Random Matrices* (Cambridge University Press, Cambridge, 2009).
[62] A. Altland, M. Janssen, and B. Shapiro, Perturbation theory for the Rosenzweig-Porter matrix model, Phys. Rev. E **56**, 1471 (1997).
[63] D. Venturelli, L. F. Cugliandolo, G. Schehr, and M. Tarzia, Replica approach to the generalized Rosenzweig-Porter model, SciPost Phys. **14**, 110 (2023).
[64] L. Erdős, B. Schlein, and H.-T. Yau, Semicircle law on short scales and delocalization of eigenvectors for Wigner random matrices, Ann. Probab. **37**, 815 (2009).
[65] L. Erdős, B. Schlein, and H.-T. Yau, Local semicircle law and complete delocalization for wigner random matrices, Commun. Math. Phys. **287**, 641 (2009).
[66] M. Rudelson and R. Vershynin, Delocalization of eigenvectors of random matrices with independent entries, Duke Math. J. **164**, 2507 (2015).
[67] L. Benigni and P. Lopatto, Optimal delocalization for generalized Wigner matrices, Adv. Math. **396**, 108109 (2022).
[68] M. Rudelson, Delocalization of eigenvectors of random matrices. Lecture notes, arXiv:1707.08461.
[69] S. Pappalardi, Quantum (chaotic) dynamics. Lecture notes (2024).
[70] M. Rigol and M. Fitzpatrick, Initial-state dependence of the quench dynamics in integrable quantum systems, Phys. Rev. A **84**, 033640 (2011).
[71] P. Łydżba, M. Rigol, and L. Vidmar, Eigenstate entanglement entropy in random quadratic Hamiltonians, Phys. Rev. Lett. **125**, 180604 (2020).
[72] P. Łydżba, M. Mierzejewski, M. Rigol, and L. Vidmar, Generalized thermalization in quantum-chaotic quadratic Hamiltonians, Phys. Rev. Lett. **131**, 060401 (2023).
[73] J. P. Pekola and B. Karimi, Quantum thermalization via multi-wave mixing, Phys. Rev. Res. **6**, L042023 (2024).
[74] G. P. Berman and F. M. Izrailev, The Fermi–Pasta–Ulam problem: Fifty years of progress, Chaos **15**, 015104 (2005).
[75] Edited by G. Gallavotti, *The Fermi-Pasta-Ulam Problem* (Springer, Berlin, 2008).
[76] S. Goldstein, J. L. Lebowitz, C. Mastrodonato, R. Tumulka, and N. Zanghi, Approach to thermal equilibrium of macroscopic quantum systems, Phys. Rev. E **81**, 011109 (2010).
[77] G. Livan, M. Novaes, and P. Vivo, *Introduction to Random Matrices*, SpringerBriefs in Mathematical Physics Vol. 26 (Springer International, Cham, 2018).
[78] G. Marsaglia, Choosing a Point from the Surface of a Sphere, Ann. Math. Stat. **43**, 645 (1972).
[79] J. Zinn-Justin, *Path Integrals in Quantum Mechanics*, Oxford Graduate Texts (Oxford University Press, Oxford, UK, 2010).
[80] W. Greiner, L. Neise, and H. Stöcker, *Thermodynamics and Statistical Mechanics* (Springer, New York, 1995).
[81] O. E. Lanford and J. L. Lebowitz, Time evolution and ergodic properties of harmonic systems, in *Dynamical Systems, Theory and Applications* (1975), pp. 144–177.
[82] S. Graffi and A. Martinez, Ergodic properties of infinite quantum harmonic crystals: An analytic approach, J. Math. Phys. **37**, 5111 (1996).
[83] G. Jona-Lasinio and C. Presilla, Chaotic properties of quantum many-body systems in the thermodynamic limit, Phys. Rev. Lett. **77**, 4322 (1996).